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The Simplest Non-Expected Utility Model for Lottery and Portfolio Choices

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Abstract: This paper explores a particularly simple model of choice under risk, based on geometric means and entropy. Despite its simplicity, it satisfies various prudence and risk aversion conditions, is consistent with the Allais paradox, and generates various insurance-related results. Within a portfolio framework with compounded reinvestments, our index fits the risks/rewards data from post-war US stock market returns and recent international markets, at least as well as does the standard deviation measures more typically used. It also generates returns that are consistent with the equity premium puzzle.

Keywords: empirical portfolio choice, theoretical asset choice, non-expected utility model

1 Introduction

When outcomes are uncertain, consumers need a way to place a relative value on alternatives whether those choices come from a set of lotteries or assembling a portfolio of assets with risky returns. Because of its mathematical elegance, the expected utility (EU) model is most often used to provide such evaluations (von Neumann and Morgenstern 1944). The EU model (or at least one version of the EU model) dates back to at least 1738 (Bernoulli, translated into English, 1954). Valuations with the EU model depend upon the type of utility function employed in the evaluation, making the EU model flexible, as well as mathematically tractable. But even with that flexibility, the EU model has been shown to be systematically inconsistent with some other commonly employed models of choice behavior and observed choices over risky outcomes.

For example in finance, modern portfolio theory usually begins by defining as the ‘efficient’ market opportunity set those asset allocations that avoid risk while yielding the highest expected returns (Elton and Gruber 1995, p. 70, the efficient frontier). Portfolio risk is usually defined as the standard deviation (or variance) of

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the portfolio, after accounting for correlation among the assets. Expected returns are usually based on the arithmetic mean of the portfolio. More often than not, asset returns are hypothesized to be normally distributed.

The efficient frontier yields the greatest return for a given risk under these criteria, or equivalently, the lowest risk for a given level of return to the portfolio. The crucial question then becomes as to where to locate on the efficient frontier. In undergraduate textbooks, choice among risky possibilities are generally addressed by assuming that the consumer's preferences follow the EU characterization of choices. Borch (1969) had a fundamental argument against applying mean/variance efficient frontiers with expected utility choices. He pointed out that the efficient frontier characterization of choice was inconsistent with the expected utility model unless the returns were normally distributed and/or the utility function from which EU was constructed was quadratic (Johnstone and Lindley 2013). Borch (1969) liked the elegance of the EU model and so tended to reject the efficient frontier model of consumer's portfolio possibilities. Others like the simplicity of the efficient frontier models, and so tend to limit the EU model as a basis for choice along the frontier.

Even setting aside these efficient frontier/EU model tensions, actual choices among assets (even without the portfolio-type analysis), are sometimes inconsistent with expected utility (EU). The EU model has to satisfy various prudence and risk aversion conditions to be consistent with actual choices. Even in its most general forms when all those other conditions hold, EU is still found to be inconsistent with some lottery choices (the Allais paradox, as just one example) and portfolio risk/returns (the equity premium puzzle).

Non-EU models vary greatly in their mathematical form and implications.¹ Like Bernoulli whose utility model was logarithmic in income and so his version of EU was fully explicit (that is, there was no general $U(X)$ modeling where only some of the signs of the derivatives of U were restricted), we propose a non-EU model that is also fully explicit with one real parameter (or no parameters, if tradeoff parameter, λ , is set equal to one). That is, like Bernoulli for EU theory, we suggest a fully explicit model as a starting point for non-EU model analysis. In this regard, we embrace Ockham's razor in seeking the simplest non-EU model which explains a lot of observed choice behavior under uncertainty. Our choice criteria differs from most of the non-EU literature, is simple, and applies equally to lotteries and portfolios of risky assets. We think this is important, as

¹ A partial list includes (Blavatskyy 2010; Blavatskyy 2014; Blavatskyy 2013; Carlin 1992; Conlisk 1989; Humphrey and Verschoor 2004; Masatlioglu and Raymond 2015; Starmer 1992; Tversky and Kahneman 1981; Yitzhaki 1982).

lotteries and portfolios are usually treated separately in the literature, and so subject to Borch-like concerns with aggregation from a model of choice among individual assets to a choice among portfolios.

In Section 2, we present the simplest non-EU model, which we call geomentropic preferences. But without an axiomatic foundation, nor a portfolio counterpart, our model of geomentropic preferences would be subject to Borch-like criticisms. In Section 3, we provide that axiomatic foundation and portfolio equivalence. Section 4 analyses the implication of our simplest model for commonly recognized results in the literature. Section 5 addresses financial market equilibrium. Section 6 uses the results in Section 5 to compare Asian economies' discounting of information uncertainty against other world economies (Table 1). Section 6 also provides empirical comparisons of the use of entropy vs. standard deviations to explain US stock market returns since 1927 (Table 2), and information discounting uncertainty over time and the implied relative risk aversion in US stock market data (equity premium puzzle, Table 3). Section 7 provides some concluding comments.

2 Geomentropic Preferences

“Geomentropic” is derived from geometric mean and entropy. Our simple non-EU choice model balances out geometric mean of returns against the entropy of returns (to measure uncertainty). The geometric mean of returns is arguably a better measure of reward than the arithmetic. Latané (1959) pointed out that for a stationary process where gains are reinvested proportionately, the portfolio with the greatest geometric mean results asymptotically in the highest terminal wealth.² Latané called this the maximum chance principle, and asserted that risking the same proportions on the same terms, even in a world experiencing some change, was at least as sensible as applying mean-variance analysis and expected utility theory. If agents implicitly adopt geometric mean strategies that have worked well in the past, there may be a tendency for geometric mean maximization to persist. (Samuelson 1971, p. 2496), by contrast, argued that even though maximizing the geometric mean leads to higher terminal wealth, it is still an inferior criterion: “These remarks critical of the criterion of maximum expected average compound growth do not deny that this criterion, arbitrary as it is, still avoids some of the even greater arbitrariness of conventional mean-variance

2 See (Latané 1959, p. 147) and (Samuelson 1971, p. 2494).

Table 1: International Estimates, Monthly Market Returns less Risk Free Rates, Regressed on Entropy, 1988–2013 by nation, (Probability Significance).

1 year market window

Country	Entropy	R^2	F-Stat	N
Australia	0.8423 (0.3234)	0.0034	0.98	287
Brazil	2.1906 (.2438)	0.0022	1.37	190
Canada	1.8846 (0.0351)	0.0154	4.48	287
China	2.0345 (0.4826)	0.0030	0.50	165
France	1.4130 (0.2519)	0.0046	1.32	287
Germany	0.4870 (0.5498)	0.0013	0.36	287
Hong Kong	3.4881 (0.0778)	0.0108	3.13	287
India	1.9971 (0.3956)	0.0032	0.72	225
Italy	0.1995 (0.8914)	0.0001	0.02	285
Japan	0.0999 (0.9403)	0.0000	0.01	285
Singapore	2.4874 (0.0890)	0.0101	2.91	287
South Korea	0.6882 (0.7685)	0.0004	0.09	238
Sweden	2.6209 (0.0945)	0.0097	2.81	285
Switzerland	1.5643 (0.1611)	0.0070	1.95	283
United Kingdom	1.0274 (0.2942)	0.0038	1.10	286
United States	2.0272 (0.0539)	0.0130	3.75	286

Note: The intercept is restricted=0. The entropy and standard deviation (SD) were constructed using lags of the monthly values over the last year. Because only 12 observations were used to construct the entropy values, we report the results using a 4-support point probability distribution. Results using a 6-point support instead yielded the same results except for Japan (whose entropy coefficient became negative). Correcting for first order autocorrelation yielded virtually the same estimates.

Table 2: Monthly Normalized Market Returns less Risk Free Rates, Regressed on Entropy or Standard Deviations of Market Rates, 1927–2013, (Probability Significance).

4-week market window							
Year	Entropy	R ²	F-Stat	SD	R ²	F-Stat	N
Overall (1927–2013)	1.5216 (0.0045)	0.0078	8.10	36.340 (0.2116)	0.0015	1.56	1038
Overall less WWII	1.3413 (0.0096)	0.0067	6.73	32.611 (0.2708)	0.0012	1.21	994
Post-WWII	1.5791 (0.0015)	0.0124	10.19	34.399 (0.2470)	0.0016	1.34	814
Post-WWII to ERISA	1.5206 (0.0566)	0.0103	3.66	56.744 (0.2999)	0.0031	1.08	353
ERISA to 2013	1.6092 (0.0019)	0.0137	6.38	26.601 (0.4613)	0.0012	0.54	461
6-week market window							
Year	Entropy	R ²	F-Stat	SD	R ²	F-Stat	N
Overall (1927–2013)	1.6159 (0.0018)	0.0093	9.74	57.372 (0.0470)	0.0038	3.95	1037
Overall less WWII	1.5140 (0.0049)	0.0081	8.15	53.230 (0.0702)	0.0033	3.28	993
Post-WWII	1.6957 (0.0004)	0.0152	12.56	53.327 (0.0710)	0.0040	3.27	814
Post-WWII to ERISA	1.7964 (0.0260)	0.0153	5.46	93.058 (0.9738)	0.0083	2.94	353
ERISA to 2013	1.6438 (0.0079)	0.0153	7.11	39.520 (0.2711)	0.0026	1.21	461

Note: The intercept is restricted=0. The entropy and standard deviation (SD) were constructed using lags of the indicated market window length. Using a 2 week window also resulted in virtually the same pattern of results, as did constructing the entropy on 4- or 10-support point probability distributions instead of the 6-point support used here. Correcting for first order autocorrelation yielded virtually the same estimates. Using French, Fama/French factors.

analysis. Its essential defect is that it attempts to replace asymptotically sufficient parameters [E log(X), variance log(X)] by the first of these alone.”³

3 Employing a slightly different approach, Jean (1980) showed that geometric mean ranking is a necessary, but not generally sufficient, condition for any degree of stochastic dominance ranking. Often critiques of Latané’s maximum chance principle have simply pointed out that it does not conform to expected utility maximization. (See Elton and Gruber 1995, p. 232). Since expected utility theory has long been criticized as inconsistent with experimental evidence, this is not a compelling critique.

Table 3: Reinvested Wealth: Arbitrage values for λ with a 30-year Reinvestment Horizon (λ^1) or Discounted Infinite Horizon (λ^2), with Implied Relative Risk Aversion Values (RRA).

year	ln(mktgeo)	ln(rfgeo)	entropy	λ^1	λ^2	mktmean	rfmean	RRA
1960	0.07087	0.012533	1.68301	1.03982	3.03154	0.11616	0.011433	110.118
1962	0.10304	0.012213	1.62457	1.67719	5.11964	0.14814	0.010920	164.007
1964	0.09790	0.013520	1.61668	1.56577	4.27572	0.13896	0.012207	196.526
1966	0.10055	0.015481	1.56934	1.62615	3.73570	0.13338	0.014510	232.165
1968	0.11073	0.018083	1.45854	1.90555	3.64837	0.13912	0.017410	268.761
1970	0.11073	0.021320	1.45854	1.83897	2.87653	0.13059	0.021310	285.465
1972	0.13331	0.024239	1.36492	2.39723	3.19418	0.14433	0.025017	355.489
1974	0.11979	0.026838	1.43879	1.93815	2.27991	0.13107	0.028337	262.633
1976	0.10628	0.030075	1.49265	1.53151	1.56245	0.11648	0.032673	189.669
1978	0.11073	0.032995	1.45854	1.59884	1.49187	0.12318	0.035723	195.741
1980	0.11073	0.035571	1.45854	1.54584	1.26791	0.12352	0.040640	183.811
1982	0.10294	0.038148	1.48594	1.30809	0.90344	0.11437	0.048263	150.138
1984	0.11073	0.040725	1.45854	1.43983	0.89603	0.12298	0.053563	166.499
1986	0.10114	0.043632	1.45854	1.18280	0.67523	0.10705	0.058390	116.973
1988	0.11465	0.045231	1.40211	1.48537	0.82001	0.11616	0.060380	134.150
1990	0.11465	0.047158	1.40211	1.44415	0.75745	0.11357	0.063553	122.730
1992	0.11646	0.048757	1.43208	1.41817	0.71444	0.11364	0.066167	126.150
1994	0.11534	0.048757	1.46619	1.36236	0.68460	0.11486	0.066333	142.457
1996	0.11603	0.049395	1.50682	1.32661	0.65968	0.11809	0.067033	159.733
1998	0.12895	0.050030	1.49453	1.58423	0.78427	0.13208	0.067333	203.362
2000	0.13674	0.049395	1.45781	1.79746	0.90134	0.14763	0.066473	241.747
2002	0.11989	0.049075	1.52657	1.39162	0.70419	0.13474	0.065873	200.110
2004	0.11989	0.047136	1.52657	1.42973	0.75396	0.13539	0.063210	207.834
2006	0.12381	0.045210	1.47852	1.59494	0.88558	0.13688	0.060033	223.867
2008	0.12715	0.044892	1.45303	1.69836	0.94711	0.13829	0.059773	232.212
2010	0.10850	0.041985	1.55350	1.28442	0.78233	0.12641	0.054727	227.762
2012	0.10669	0.038748	1.49206	1.36616	0.98412	0.12304	0.046273	279.719

Note: All calculations from annual data, employing the previous 30 years of values of the respective variables in their calculations (S&P 500 annual returns for as a proxy for risky returns-mkt; 3-month treasury t-bill rate annualized for the proxy for risk-free returns-rf, and real gross domestic product per capita in the calculation of the relative risk aversion parameter (using the simplified model of Gollier (2001, Section 5.2). Calculations for λ^2 used an infinite horizon with discounting by the risk-free rate (mean risk-free returns over the relevant 30 year period). Calculations for odd years yielded similar values.

To include risk in models derived from expected utility, most researchers choose some function of the whole distribution of wealth (such as the lottery variance), or some function of the conditional distribution of wealth (such as value-at-risk measures, based on various quintiles of the distribution). Consistent with Latané's maximization approach, we develop a "maximum choice" model by generalizing his maximum chance principal. Specifically, we integrate uncertainty

by considering optimal betting on wealth outcomes. We assume that the value of such bets is a power function of wealth in state i , $w_i^{1/\lambda}$, where λ is an unknown parametric constant, and that investors optimize the log of geometric mean value of those bets; such an approach yields a risk measure arising naturally from the Latané geometric mean reinvestment framework.

More specifically, suppose an investor chooses to invest, for each i , a share $a_i \geq 0$ of his or her wealth in an asset that pays w_i if the state i is realized and nothing otherwise, with the objective to maximize

$$\sum_i p_i \ln(a_i w_i^{1/\lambda}), \quad (1)$$

or, equivalently, the log of the geometric mean of their value of lottery bets,

$$(1/\lambda) \sum_i p_i \ln w_i + \sum_i p_i \ln a_i \quad (2)$$

subject to $\sum a_i = 1$. The a_i only appear in the second term, and constrained optimization yields $a_i = p_i$ for each i . Hence, lotteries are chosen on the basis of maximizing the following (choosing a convenient re-normalization):

$$\sum_i p_i \ln w_i + \lambda \sum_i p_i \ln p_i = \sum_i p_i \ln w_i - \lambda \left(- \sum_i p_i \ln p_i \right) \quad (3)$$

The term $(-\sum_i p_i \ln p_i)$ is the entropy discount for event uncertainty.⁴ So unlike Latané, and consistent with Samuelson's critique, we do not make the geometric mean do all the work. We use the geometric mean of wealth to measure reward, but we subtract the log of the geometric mean of probabilities to measure risk, that is, we use entropy as our risk measure.

Entropy has been useful in information theory (Shannon 1948), inequality measures (Theil 1972), and psychology (Norwich 1993), but has yet to gain any traction—parallel with Latané's geometric mean criteria—in the economics of uncertainty and finance.⁵ By trading off entropy against the log of the geometric mean, we develop a simple one-parameter index for lotteries. The entropy discount for a lottery's uncertainty,

$$-\lambda \left(\sum_{\sigma \in \mathbb{S}} -p(\sigma) \ln(p(\sigma)) \right), \quad (4)$$

⁴ Arrow 1971.

⁵ For one partial exception, see Yang and Qiu (2005).

where λ is a positive parameter, implies a probability-partitioning effect in lotteries. Indeed, we view the partition of the state space as a central feature of the consumer's perceptual apparatus regarding uncertain outcomes. A lottery initially presented to a consumer is perceived as different from that same initial lottery but where lottery branches have been perturbed with respect to probabilities: whether branches with the same outcome have been merged across probabilities, or whether new lottery branches are generated by dividing up a given probability for a given outcome into several new branches—such perturbations are new lotteries as far as the consumer is concerned. In particular, any division of $p(\sigma)$ into smaller probabilities with the same outcome (e.g. $(p_1(\sigma), \dots, p_N(\sigma))$ into $(\sum_{i=1}^{N_1} p_1^i(\sigma), \dots, \sum_{i=1}^{N_N} p_N^i(\sigma))$ such that $\sum_{i=1}^N p_j^i(\sigma) = p_j(\sigma)$) increases entropy and lowers the ranking of the lottery. Lotteries thus “simplified” or “expanded” across probabilities (for given outcomes) are different lotteries given consumers geomentropic tastes (and the partitioning effect due to entropy), and thus ranking formulas implicitly assuming such simplifications, such as the first order stochastic dominance criteria, will not provide a reliable guide to individuals' actual lottery rankings. In particular, a lottery with (wealth, probability) pair $(w(\sigma); p(\sigma))$ will be preferred to a lottery in which $(w(\sigma); p(\sigma))$ is replaced with a lottery containing the pair $(w(\sigma) + \epsilon; p(\sigma)/2)$, $(w(\sigma) + \epsilon; p(\sigma)/2)$ for some $\epsilon > 0$ even though such a choice violates first order stochastic dominance.⁶

It follows that the independence axiom, which drives many of the results from expected utility theory, does not hold as it depends on the simplification property holding. In particular, merging lottery branches with the same outcome by adding their probabilities, or dividing an outcome into several new lottery branches by dividing up its probability, changes lottery ordering in our model but not in the expected utility model. So our model explicitly departs from the expected utility framework.⁷ We show the index is empirically relevant in explaining real world lottery choices. We also extend the analysis to create an index for portfolios.

Section 2 formally introduces our index. It also extends the analysis to apply to endogenously constructed portfolios. In Section 3, we show that many of the desirable properties of a lottery index hold for our index, including some of the basic insurance results from expected utility theory. Market equilibrium

⁶ See Birnbaum (2005) for many examples of choices that violate the first order stochastic dominance (FOSD) criteria. We are grateful to Mark Machina for pointing this out to us. Note that the probability partitioning property of entropy implies that not only is FOSD an inappropriate metric for consumer's ranking of risky choices, but all moment indices of risk where probabilities might be so partitioned.

⁷ For prior departures see, for example, Machina (1989), Starmer (2000), Kahneman and Tversky (1979), and Tversky and Kahneman (1992).

is discussed in Section 4. Section 5 provides the estimation framework for the trade-off between geometric means and entropy. Section 6 concludes.

3 Geomentropic Preferences

3.1 Geomentropic Preferences over Lotteries

A lottery is a list $L = (S, \mathbb{S}, p, w)$ where S is a finite state space, $\mathbb{S}$ is a partition of S , $p : \mathbb{S} \rightarrow [0, 1]$ assigns probabilities to partition sets, and $w : \mathbb{S} \rightarrow R_{++}$ assigns pay-offs to partition sets. The geometric mean of the lottery L is $G(L) = \prod_{\sigma \in \mathbb{S}} w(\sigma)^{p(\sigma)}$ and its entropy is $N(L) = - \sum_{\sigma \in \mathbb{S}} p(\sigma) \ln p(\sigma)$.⁸

Let $\geq$ be a preference ordering over lotteries. If for two lotteries L and L' , both $L \geq L'$ and $L' \geq L$ then we write $L \sim L'$. If for two lotteries L and L' , $L \geq L'$ but it is not the case that $L' \geq L$ then we write $L > L'$.

Our lottery index can be derived from preference axioms in straightforward fashion. The first two axioms, A1 and A2, are standard:

A1. (Completeness and Transitivity) For all lotteries L and L' , either $L \geq L'$ or $L' \geq L$. Furthermore, for all lotteries L, L' , and L'' , if $L \geq L'$ and $L' \geq L''$ then $L \geq L''$.

By a slight abuse of notation, let $(S, S, 1, w)$ denote the the zero-entropy lottery that assigns w to every state. Specifically, $\mathbb{S} = S$ is meant to suggest that the partition contains only one set: the entire state space, S . This lottery is analogous to a risk-free lottery in the mean-variance framework.

A2. (Continuity) For any lottery L , the sets $\{(S, S, 1, w) \geq L\}$ and $\{L \geq (S, S, 1, w)\}$ are closed.

A consequence of A1 and A2 is the existence of a zero-entropy equivalent, that is, $G^*(L) \in R_{++}$ such that $(S, S, 1, G^*(L)) \sim L$. A3 makes explicit the trade-off between geometric mean and entropy.

A3(a). (Geometric Mean Monotonicity) If $N(L) = N(L')$ and $G(L) > G(L')$ then $L > L'$.

A3(b). (Entropy Monotonicity) If $G(L) = G(L')$ and $N(L) < N(L')$ then $L > L'$.

Under A1–A3, lotteries can be ranked by their zero-entropy equivalents, as shown in the following proposition:

Proposition 1: If preferences satisfy A1–A3 then $L > L'$ iff $G^*(L) > G^*(L')$.

⁸ Shannon (1948) is the most significant early advocate for this entropy measure.

Proof. First suppose $L > L'$. Let K and K' be the respective zero-entropy lotteries that are equally preferred to L and L' . (These exist by A2.) By A1, $K > K'$. Since $N(K) = N(K') = 0$, A3 implies $G(K) > G(K')$ and hence $G^*(L) > G^*(L')$. ■

Now suppose $G^*(L) > G^*(L')$. Recall that $G^*(L)$ and $G^*(L')$ are the geometric means of the risk-free lotteries K and K' that are equally preferred to L and L' , respectively. Since K and K' are risk-free, and $G^*(L) > G^*(L')$, by A3, $L > L'$. QED

Although more flexibility is possible, a constant rate of substitution between risk and reward is a reasonable benchmark, and can be thought of as a first order approximation to actual preferences. It will also be seen to be general enough for many purposes.

A4. (Constant Marginal Rate of Substitution) There exists a constant $\lambda > 0$ such that, for any two lotteries L and L' , if $L \sim L'$ then $\ln G(L) - \ln G(L') = \lambda[N(L) - N(L')]$.

Proposition 2: If preferences satisfy A1-A4 then there exists $\lambda > 0$ such that $L > L'$ iff $\ln G(L) - \lambda N(L) > \ln G(L') - \lambda N(L')$.

Proof. For any two lotteries L and L' , $G^*(L)$ and $G^*(L')$ are well defined by A2. By A4, there exists $\lambda > 0$ such that $G^*(L) = \ln G(L) - \lambda N(L)$ and $G^*(L') = \ln G(L') - \lambda N(L')$. Proposition 1 ensures $G^*(L) > G^*(L')$ iff $L > L'$. So, by A1, $\ln G(L) - \lambda N(L) > \ln G(L') - \lambda N(L')$ iff $L > L'$. QED ■

We will refer to the index constructed in Proposition 2 as the *geomentropic* index and the preferences described by A1–A4 as *geomentropic* preferences. In contrast to expected utility preferences, geomentropic preferences do not imply an index that is linear in the probabilities of the various outcomes. This linearity is a consequence of A2 along with Neumann's and Morgenstern's independence axiom: $L \geq L'$ iff $\alpha L + (1 - \alpha)L'' \geq \alpha L' + (1 - \alpha)L''$ for all $\alpha \in [0, 1]$ and all lotteries L'' . The independence axiom has long had its critics,⁹ and we do not impose it here.

3.2 Geomentropic Preferences over Compound Lotteries

Let $\{S_k, \mathbb{S}_k, p_k, w_k\}_{k=1}^K$ be a list of lotteries. The associated compound lottery with probability weights $\alpha = \{\alpha_k\}_{k=1}^K$ is of the form $\{\cup_k S_k, \cup_k \mathbb{S}_k, \alpha p, w\}$, where αp assigns $\alpha_k p(\sigma)$ to all $\sigma \in S_k$ and w assigns $w_k(\sigma)$ to all $\sigma \in \mathbb{S}_k$. This compound lottery can be interpreted as first randomly choosing a component lottery and then randomly generating the outcome of the chosen lottery. The geometric mean of the

⁹ See, for example, Allais (1953), Machina (1982), and Machina (1989).

compound lottery L is $G(L) = \prod_k \prod_{\sigma \in \mathbb{S}_k} w_k(\sigma)^{\alpha_k p_k(\sigma)}$ and its entropy is defined as $N(L) = - \sum_k \sum_{\sigma \in \mathbb{S}_k} \alpha_k p_k(\sigma) \ln \alpha_k p_k(\sigma)$.

Thus the geomentropic index distributes lottery weights. However, it does not merge branches with the same outcomes by adding probabilities, nor does it divide one branch outcome into two or more by dividing the probabilities. Because it does not exhibit *scalability* over outcomes, it does not satisfy the reduction of compound lottery assumption implicit in the independence axiom. For example, an expected utility maximizer is indifferent between the compounding of two identical lotteries with probability weights α and $(1 - \alpha)$ and the same lottery with certainty. By contrast, the geomentropic agent prefers the simple lottery $(S, \mathbb{S}, w, p)$ to the compounding of two identical simple lotteries due to the higher entropy of the compound lottery (the probability-partitioning effect). For the non-expected utility maximizer, the two lotteries are not equivalent, a point made years ago by Mark Machina (1987). Geomentropic preferences assume a trade-off between the log of the lottery's geometric mean and its entropy, instead of the reduction of compound lottery assumption inherent in the independence axiom of expected utility.

In summary, under geomentropic preferences, history matters in the sense that lottery presentation matters, with individuals perceiving subsequent algebraic manipulations involving outcomes as different lotteries. Scalability over outcomes implies non-unique geomentropic preferences. Harrison, Martinex-Correna, and Swarthout (2015) find that when choosing among many alternatives, the choices of experimental subjects do not support the reduction of compound lotteries assumption in his random lottery incentive mechanism. (See also Hillel (1973), Kahneman and Tversky (1979), Bernasconi and Loomes (1992), Miao and Zhong (2012), Abdellaoui, Klibanoff, Placido (2015)).

3.3 Geomentropic Preferences over Portfolios

A portfolio is a non-negative unit vector $v = (v_1, \dots, v_K)$ where v_k is interpreted as the fraction of the portfolio value W invested in the k^{th} asset. The k^{th} asset is a lottery $(S_k, \mathbb{S}_k, p_k, w_k)$. Given $\sigma = (\sigma_1, \dots, \sigma_K)$, where $\sigma_k \in \mathbb{S}_k$ for all k , let $\gamma(\sigma)$ be the joint probability that $s_k \in \sigma_k$ for all $k \in \{1, \dots, K\}$. The geometric mean of the one-period payoff from a portfolio is

$$G_p(v) = \prod_{\sigma \in \times_k \mathbb{S}_k} \left[\sum_k \left(\frac{v_k W}{\pi_k} \right) w_k(\sigma_k) \right]^{\gamma(\sigma)} \quad (5)$$

where W is the wealth invested and π_k is the price of a unit of the k^{th} asset. Portfolio risk is derived from the risks of the individual assets, as measured by

the entropy of their marginal distributions, and their respective importance in the portfolio. Formally, define the marginal probabilities $p_k(\sigma_k)$ for the marginal distribution of asset k by

$$p_k(\sigma_k) = \sum_{\ell \neq k} \sum_{\sigma_\ell \in \mathbb{S}_\ell} \gamma(\sigma) \quad (6)$$

Define the entropy of the k^{th} asset as

$$N_k = - \sum_{\sigma_k \in \mathbb{S}_k} p_k(\sigma_k) \ln p_k(\sigma_k). \quad (7)$$

Then the entropy of the portfolio is defined as

$$N_p(v) = \sum_k v_k N_k \quad (8)$$

Specifically, portfolio risk consists of the arithmetic average of individual asset uncertainty (N_k), weighted by their presence in the portfolio. Unlike the standard portfolio model based on mean-variance analysis, unsystematic risk does not go to zero as the number of assets increases. Rather, the portfolio risk converges to the mean asset risk.

It is common in the finance literature to assume the existence of a risk-free asset. The geomentropic analog is an asset that exhibits only the trivial partition, that is, the partition $\mathbb{S}$ that contains only the entire space S . Such an asset has entropy of zero and exhibits one outcome in all states of the world. Denote the price of a unit of the risk-free asset (cash) as π_C . It is straightforward to show that as the share of the riskless asset increases, holding the proportions of the other assets constant, portfolio entropy falls. If only the risk-free asset is held in the portfolio, then portfolio entropy is zero.

3.4 Dynamic Geomentropic Preferences

Suppose the investor behaves as if the stochastic process governing the states of the world is ergodic, that is, there exists $\gamma : \times_k \mathbb{S}_k \rightarrow [0, 1]$, such that $\sum_{\sigma \in \times_k \mathbb{S}_k} \gamma(\sigma) = 1$, $\gamma(\sigma) > 0$ for all σ , and $\lim_{t \rightarrow \infty} p_{\alpha\sigma}^t = \gamma(\sigma)$ for all α and for all σ , where $p_{\alpha\sigma}^t$ are the transition probabilities from state α to σ after t periods.¹⁰ We assume the dynamic geomentropic investor acts as if investing to maximize relative to this ergodic distribution, and thus write the investor's objective

¹⁰ Shiriyayev (1984, pp. 116–118, the Ergodic Theorem and page 545, Theorem 3) presents conditions for existence of an ergodic distribution.

function as:

$$\ln G_p(v) - \lambda N_p(v) = \sum_{\sigma \in \times_k \mathbb{S}_k} T_{\gamma}(\sigma) \ln \left[\sum_k \left(\frac{v_k W}{\pi_k} \right) w_k(\sigma) \right] - \lambda N_p(v) \quad (9)$$

for the T-period horizon reinvestment problem when all available assets are risky. We assume that the appropriate entropy-metric for the portfolio, is the entropy of this ergodic distribution. Under these assumptions, dynamic geomentropic preferences when there is a risk-free asset (w^C) can be written as

$$\begin{aligned} \ln G_p(v) - \lambda N_p(v) = \ln \left\{ \prod_{\sigma \in \times_k \mathbb{S}_k} \left[\sum_k \left(v_k \frac{W}{\pi_k} w_k(\sigma) \right) + \left(1 - \sum_m v_m \right) \frac{W}{\pi_C} w^C \right]^{T_{\gamma}(\sigma)} \right\} \\ - \lambda \left[\sum_k v_k N_k \right] \end{aligned} \quad (10)$$

A lottery is a portfolio with one risky asset.

4 Implications for Individual Behavior

4.1 Geomentropic Risk Aversion

An expected utility maximizing agent who strictly prefers to receive the expected value of a lottery rather than the lottery itself is said to be risk averse. Analogously, we will define geomentropic risk aversion as a strict preference for receiving the geometric mean of a lottery rather than receiving the lottery itself. Given A3b, geomentropic agents exhibit geomentropic risk aversion. This follows directly from the entropy associated with a non-degenerate lottery.

4.2 Geomentropic Complexity Aversion

We will say that an agent is complexity averse if $L' > L$ whenever $w(\sigma) = w'(\sigma)$ for all $\sigma \in \mathbb{S}$ and the partition $\mathbb{S}'$ is coarser than the partition $\mathbb{S}$. Then the definitions immediately imply the following theorem:

Theorem 1: Geomentropic agents are complexity averse.

Note, by contrast, that expected utility maximizers are not inclined to complexity aversion. This is because of the scalability properties of expected utility.

Thus compound lotteries are equally preferred to their expected utility equivalents, a property not shared by geomentropic preferences over lotteries.

Financially, investors with geomentropic preferences would rather hold a treasury bill than stock with an option that yields the same return.

4.3 Geomentropic Uncertainty Aversion

Geomentropic preferences exhibit a property that is analogous to the notion of uncertainty aversion introduced by Gilboa and Schmeidler (1989). Given two equally preferred lotteries, geomentropic individuals prefer a convex combination of the two lotteries over the chance of getting either one. The result relies on the distinction between taking a convex combination in which the weighting variable, call it v , is a choice variable in the sense that its value is realized a priori and hence treated as non-random, and a true lottery in which one of the lotteries is received at random. In this latter case say the chance of getting the first lottery is γ , and as a random variable it induces some entropy. Uncertainty aversion means that any portfolio of the two lotteries, whatever the value of v , is preferred to a lottery with those two lotteries as prizes, whatever the value of γ .

Proposition 3: For all $v \in (0, 1)$, for all $\gamma \in (0, 1)$, and for all pairs of lotteries satisfying $(X; p) \sim (Y; q)$, geomentropic preferences satisfy:

$$v(X; p) + (1 - v)(Y; q) > ((X; p), (Y; q); \gamma, (1 - \gamma)) = (X, Y; \gamma p, (1 - \gamma)q). \quad (11)$$

Proof. This is equivalent to $v\sum_i p_i \ln X_i + \lambda v\sum_i p_i \ln p_i + (1-v)\sum_i q_i \ln Y_i + \lambda(1-v)\sum_i q_i \ln q_i > \sum_i p_i \gamma \ln X_i + \lambda \sum_i p_i \gamma \ln(p_i \gamma) + \sum_i q_i (1 - \gamma) \ln Y_i + \lambda \sum_i q_i (1 - \gamma) \ln(q_i (1 - \gamma))$. ■

Subtracting terms on the right hand side with similar terms on the left hand side, we can rewrite the last inequality as $(v - \gamma)[\sum_i p_i \ln X_i + \lambda \sum_i p_i \ln p_i - \sum_i q_i \ln Y_i - \lambda \sum_i q_i \ln q_i] > \lambda(\gamma \ln \gamma + (1 - \gamma) \ln(1 - \gamma))$.

Since the two lotteries are equally preferred, the terms in the square brackets sum to zero, and the last inequality becomes $0 > \lambda(\gamma \ln \gamma + (1 - \gamma) \ln(1 - \gamma))$, which is always true because γ is bounded between 0 and 1. QED

4.4 Geomentropic Prudence

(Gollier 2001, p. 236) defines prudence, a concept due to Kimball (1992), by designating an agent as prudent if adding an uninsurable zero mean risk to his future

wealth raises his optimal level of saving. In other words, an increase in risk for a given arithmetic mean causes the prudent agent to increase holdings of the riskless asset. The geomentropic analogue is for an increase in the entropy of the risky asset for a given geometric mean to cause an agent to increase holdings of the risk-free asset.

Consider a two-asset world where asset 1 is a risky asset with price π_1 and asset 2 is a riskless asset with price π_C . The payoffs associated with the two assets are $w_1(\sigma)$ and w^C , respectively. The partition of the state space associated with the risk-free asset is the trivial partition containing only the state space and thus its entropy is $N(L_2) = -\ln(1) = 0$. If v is the share of the risky asset in the portfolio then the geomentropic index of the portfolio ($v, 1 - v$) is

$$\ln G_p(v) - \lambda N_p(v) = \sum_{\sigma \in \times_k \mathbb{S}_k} T_\gamma(\sigma) \ln \left[v \frac{W}{\pi_1} w_1(\sigma) + (1 - v) \frac{W}{\pi_C} w^C \right] - \lambda v N_1. \quad (12)$$

For any horizon length, T , the geomentropic index is concave in v . If real asset prices are such that there is an interior solution—that is, if the optimal v is in the interior of $(0, 1)$ —then the first order condition for the optimal choice of v is

$$\sum_{\sigma \in \times_k \mathbb{S}_k} T_\gamma(\sigma) \left[\left\{ \frac{W}{\pi_1} w_1(\sigma) - \frac{W}{\pi_C} w^C(\sigma) \right\} / \left\{ v \frac{W}{\pi_1} w_1(\sigma) + (1 - v) \frac{W}{\pi_C} w^C \right\} \right] = \lambda N_1 \quad (13)$$

Inspection of this first order condition for fixed asset prices and fixed distribution of returns reveals that a decrease in risk tolerance—that is, an increase in λ —must imply a decrease in $\{v \frac{W}{\pi_1} w_1(\sigma) + (1 - v) \frac{W}{\pi_C} w^C\}$ and (assuming the mean return of the risky asset exceeds that of the riskless asset) a decrease in v , the fraction of the portfolio invested in the risky asset, and thus an increase in risk-free investment.

4.5 Fanning and the Allais Paradox with Geomentropic Preferences

To explain some empirical anomalies (including the Allais Paradox) for the expected utility model, Machina (1992) employs a three outcome lottery model with $p = (p_1, p_2, p_3)$ and corresponding fixed wealth outcomes $w = (w_1, w_2, w_3)$, where $w_3 > w_2 > w_1$. In the probability triangle with p_3 on the vertical axis, and p_1 on the horizontal axis (with p_2 's value implicitly determined by the choice of the other two probabilities within the triangle), under expected utility sets of indifferent lotteries lie on parallel straight lines with slopes equal to

Expected Utility indifference curves slope =

$$\frac{-[u(w_1) - u(w_2)]}{[u(w_3) - u(w_2)]} \quad (14)$$

Given the linearity in probability assumptions under expected utility preferences (the result of the independence assumption in expected utility, see for example, (Gollier 2001, Section 1.3)), lotteries lying further to the northwest receive higher probability weights on the better outcome (p_3 is larger and p_1 is smaller, holding p_2 constant), and hence, are preferred to all lotteries on indifference curves to the southeast.

Geomentropic indifference curves exhibit more complex patterns within the Machina probability triangle than expected utility. Geomentropic indifference curves additionally depend on the log-odds ratios as follows:

Geomentropic indifference curves slope =

$$\frac{-[\ln(w_1) - \ln(w_2) + \ln(p_1/p_2)]}{[\ln(w_3) - \ln(w_2) + \ln(p_3/p_2)]} \quad (15)$$

Geomentropic indifference curves tend to “fan out” (that is, their slopes increase as p_3 increases, holding p_2 constant) when lottery prizes are skewed to the right (that is, when $(\ln(w_3) - \ln(w_2)) > (\ln(w_2) - \ln(w_1))$). Unlike the expected utility indifference curves, geomentropic indifference curves may be negative for low values of p_2 or high values of λ as they “fan out.”

In particular, this section shows that for values of λ that are consistent with empirical observation (roughly $3 > \lambda > .4$), geomentropic preferences are consistent with Allais paradox choices. This is in contrast to expected utility theory, which is inconsistent with such choices, regardless of the form of the utility function.

The Allais paradox considers choices between two paired lotteries (Machina, 1989), which can be appropriately regarded as choices between lotteries changing the wealth level of participant. (Recall that Latané’s maximum chance principle is essentially a wealth investment principle.) Given initial wealth $W > 0$, consider four lotteries:

L_1 :

1.00 chance of \$1,000,000 + W

L_2 :

0.10 chance of \$5,000,000 + W

0.89 chance of \$1,000,000 + W

0.01 chance of \$0 + W

L_3 :

0.10 chance of \$5,000,000 + W

0.90 chance of \$0 + W

L_4 :

0.11 chance of \$1,000,000 + W

0.89 chance of \$0 + W

Most people claim to prefer L_1 over L_2 while preferring L_3 over L_4 . However, whatever the utility function used, this is an inconsistent set of choices under the expected utility model because of the reduction of compound lotteries assumption (see Machina 1989, the diagram on the middle of the right hand column of p. 1629 and corresponding discussion).

By contrast, for feasible values of wealth W , and λ , geomontropic preferences are consistent with experimentally observed preferences of L_1 over L_2 , and at the same time, preferences for L_3 over L_4 .

L_1 is preferred to L_2 if, upon factoring out $\ln(W)$ from both sides,

$$\begin{aligned} \ln(\$1m/W + 1) &> 0.1 \ln(\$5m/W + 1) + 0.89 \ln(\$1m/W + 1) + \lambda(0.1 \ln(0.1) \\ &+ 0.89 \ln(0.89) + 0.01 \ln(0.01)) \end{aligned} \quad (16)$$

For geomontropic preferences, preference of L_1 over L_2 indicate the certain relative wealth term $\ln(\$1m/w+1)$ is preferred to the uncertain income $0.1 \ln(\$5m/W + 1) + 0.89 \ln(\$1m/W + 1)$, discounted by the entropy associated with the uncertainty $\lambda(0.1 \ln(0.1) + 0.89 \ln(0.89) + 0.01 \ln(0.01))$. Simulations indicate that for all values of wealth W , L_1 is preferred to L_2 whenever λ is 0.34 or greater. This critical value of λ falls monotonically as W increases. For $W = \$1m$, the critical value is $\lambda = 0.28$; for $W = \$5m$, the critical value is $\lambda = 0.13$; and for $W = \$10m$, $\lambda = 0.08$. When λ is above these critical values L_1 is always preferred to L_2 .

At the same time, L_3 is preferred to L_4 if

$$\begin{aligned} 0.1 \ln(\$5m/W + 1) + \lambda(0.1 \ln(0.1) + 0.9 \ln(0.9)) &> 0.11 \ln(\$1/W + 1) \\ &+ \lambda(0.11 \ln(0.11) + 0.89 \ln(0.89)) \end{aligned} \quad (17)$$

The upper area in Figure 1 indicates that for every wealth level, when $\lambda > 0.34$, geomontropic preferences are consistent with typical Allais preferences ($L_1 > L_2$ and $L_3 > L_4$). Low levels of wealth are consistent with Allais choices, even for very low values of λ as indicated. The highest value of the lower bound for λ of 0.34

Figure 1: Wealth- λ values consistent with Allais choices.

(achieved at a wealth level of about \$80,000) monotonically declines as wealth increases. This establishes the existence of an open set of parameters for which Allais-style preferences are consistent with geomentropic preferences.

4.6 Insurance Decisions with Geomentropic Preferences

The results regarding fair pricing and insurance demand from expected utility maximization also derive from geomentropic preferences.

Specifically, let I^* denote the amount of insurance an agent purchases at unit price s . The agent faces a loss of I with probability p ; with probability $(1 - p)$ there is no loss. The agent chooses I^* to maximize

$$p \ln(W - sI^* - I + I^*) + (1 - p) \ln(W - sI^*) + p \ln p + (1 - p) \ln(1 - p) \quad (18)$$

The first order condition is

$$p(1 - s)/(s(1 - p)) = (W - sI^* - I + I^*)/(W - sI^*) \quad (19)$$

With actuarially fair insurance, so $s = p$, the first order condition implies $I^* = I$; thus the agent purchases full coverage.

When $s > p$, so insurance prices are not actuarially fair, the left side of the first order condition above is less than one so the right side must be less than one at the optimum. Inspection reveals this to be true only if $I^* < I$, so the agent purchases only partial insurance. This differs somewhat from standard expected

utility, which can generate a demand for full insurance coverage, even with unfair prices, if the consumer is sufficiently risk adverse.

5 Dynamic Geomentropic Preferences and Asset Prices

5.1 Equilibrium and Re-Investment Certainty Equivalence

Assume that all investors have the same marginal rate of substitution, λ . As explained in Section 2.4, and consistent with Latané's wealth reinvestment principle, we assume geomentropic investors behave as if maximizing over their time horizon, T , with respect to the ergodic distribution, γ .

Investors' re-investment horizon has not been settled in the literature, though given Modigliani's (1966) *life cycle* saving/investing model, values for an individual's investment lifetime would plausibly be in the range of 20 to 40 years. The empirical sections below assume a 30-year horizon, but the linearity of key identifying restrictions implies that the estimates will be quite robust for the 20–40 year range. For long investment horizons, the value of a portfolio will grow close to its long-run geometric mean growth path with high probability. Hence, the compound growth of the risky portfolio over T periods less the λ -weighted entropy of the risky portfolio should roughly equal the compound growth of the risk-free portfolio. In particular, the index of risky and risk-free rates will satisfy the following equality because of re-investment certainty equivalence:

$$\left\{ T \sum_{\sigma \in \times_k S_k} \gamma(\sigma) \ln \left[\sum_k v_k \left(\frac{W}{\pi_k} w_k(\sigma) \right) \right] \right\} - T \ln \left[\left(\frac{W}{\pi_C} \right) w^C \right] = \lambda N_p \quad (20)$$

where w^C is the risk-free return. The empirical sections below report estimates of λ using the certainty equivalencies of risky and risk-free portfolios. These estimates lie between 1 and 2, values that account for the equity premium puzzle.

5.2 Financial Market Equilibrium with Homogeneous Agents

A financial market is a triple, (x, W, ϕ) , where $x = (x_1, \dots, x_K)$ is the exogenous supply of each asset, $W = (W^1, \dots, W^I)$ is the exogenous initial wealth of each investor, and $\phi(G, N)$ is the investors' geomentropic preference index. A financial equilibrium is a pair (π, v) such that v maximizes the geomentropic index and

the market clearing conditions hold for each asset: $v \in \arg_v \max \{\phi(G_p(v), N_p(v))\}$, $\sum_{i=1}^I \frac{v_k W^i}{\pi_k} = x_k$ for all risky assets k , and $\sum_{i=1}^I (1 - \sum_k v_k) W^i / \pi_C = x_C$ for the risk-free asset. The first order conditions associated with the maximization of

$$\ln G_p(v) - \lambda N_p(v) = \ln \left\{ \prod_{\sigma \in \times_k \mathbb{S}_k} \left[\sum_k \left(v_k \frac{W^i}{\pi_k} w_k(\sigma) \right) + \left(1 - \sum_m v_m \right) \frac{W^i}{\pi_C} w^C \right]^{T \gamma(\sigma)} \right\} - \lambda \left[\sum_k v_k N_k \right] \quad (21)$$

set the partial derivatives of the objective function equal to zero: namely,

$$T \sum_{\sigma \in \times_k \mathbb{S}_k} \gamma(\sigma) \left[\left\{ \frac{W^i w_j(\sigma)}{\pi_j} - \frac{W^i w^C}{\pi_C} \right\} / \left\{ \sum_k \left(v_k \frac{W^i w_k(\sigma)}{\pi_k} \right) + \left(1 - \sum_k v_k \right) \frac{W^i w^C}{\pi_C} \right\} \right] - \lambda N_j = 0 \quad (22)$$

for all $j \in \{1, \dots, K\}$. An *aggregate first order condition* can be derived by multiplying the j^{th} first order condition by v_j , summing over j , and canceling the common wealth terms:

$$T \sum_{\sigma \in \times_k \mathbb{S}_k} \gamma(\sigma) \sum_j v_j \left[\left\{ \frac{w_j(\sigma)}{\pi_j} - \frac{w^C}{\pi_C} \right\} / \left\{ \sum_k \left(v_k \frac{w_k(\sigma)}{\pi_k} \right) + \left(1 - \sum_m v_m \right) \frac{w^C}{\pi_C} \right\} \right] = \lambda N_p(v) \quad (23)$$

For the portfolio as a whole, the *net portfolio returns* (given on the right side of the previous equation) are the *average risky returns above the risk free return*, normalized by the weighted returns of the whole portfolio. Investors set portfolio returns equal to λ times portfolio entropy, $\lambda \sum_k v_k N_k = \lambda N_p(v)$. As shown in the appendix, geomentropic preferences are concave in shares (that is, concave in v).

Substituting the market clearing conditions into the first order conditions, and simplifying, yields

$$T \sum_{\sigma \in \times_k \mathbb{S}_k} \gamma(\sigma) \left[\left(\sum_j v_j \left[\left\{ \frac{W w_j(\sigma)}{\pi_j} - \frac{W w^C}{\pi_C} \right\} \right] / \left[\sum_k (w_k(\sigma) x_k + w^C x_C) \right] \right) \right] = \lambda N_p(v) \quad (24)$$

where $W = \sum_{i=1}^I W^i$, the sum of wealth invested in the market by all the I investors.

The representative investor holds less of every risky asset in equilibrium given an exogenous increase in λ —that is, the greater the investors' taste for certainty,

the more risk-free asset they hold in their equilibrium portfolio (the lower the share of risky assets, $\sum_k v_k$). This is the content of the following theorem:

Theorem 2: The share of each risky asset is decreasing in λ .

Proof. Substitute the market clearing condition into the first order conditions:

$$T \sum_{\sigma \in \times_k \mathbb{S}_k} \gamma(\sigma) \left[\left\{ \frac{W^m w_j(\sigma)}{\pi_j} - \frac{W^m w^C}{\pi_C} \right\} / \left\{ \sum_k (w_k(\sigma) x_k) + w^C x_C \right\} \right] = \lambda N_j \quad (25)$$

Designating the price of the risk-free asset as numeraire, as λ increases on the right side, the price of the j th asset, π_j , must fall to maintain equilibrium. From the market clearing condition for that asset, $\sum_{i=1}^I \frac{v_j W^i}{\pi_j} = x_j$, a decrease in the price of the risky asset must be accompanied by a proportional decrease in the share of the asset. QED ■

For an individual investor, any portfolio satisfying the portfolio maximization condition

$$T \sum_{\sigma \in \times_k \mathbb{S}_k} \gamma(\sigma) \sum_j v_j \left[\left\{ \frac{w_j(\sigma)}{\pi_j} - \frac{w^C}{\pi_C} \right\} / \left\{ \sum_k \left(v_k \frac{w_k(\sigma)}{\pi_k} \right) + \left(1 - \sum_m v_m \right) \frac{w^C}{\pi_C} \right\} \right] = \lambda N_p \quad (26)$$

is equivalent, and different investors may choose different portfolios as long as the compounded, normalized excess returns to the risky assets in the portfolio equaled λ times that portfolio's entropy. Portfolios with higher entropy must also have higher geometric returns.

To picture the equilibrium condition in eq. (26) geometrically, let the portfolio consist of three potential assets: a risk free asset, a riskier asset (with entropy N_1), and a less risky asset (with entropy N_2). The dotted simplex above the plane of shares of risky assets (v_1, v_2) gives the portfolio entropy as a function of those asset shares, assuming that $\lambda = 1$. (Different values of λ merely rescales the slope of the plane, without changing the intuition.) The equilibrium condition in eq. (26) indicates that the asset prices (and, hence, returns) do not increase linearly as entropy increases (even if asset shares are held constant). The left-hand side of eq. (26) is not a linear function of those returns (prices) as shares are held constant.

Figure 2: Portfolio net returns and asset shares. (author's calculation)

6 Estimating the Marginal Rate of Substitution

Recall the “first order” condition for the aggregate portfolio, with risky assets resulting in wealth levels $w_k(\sigma)$ per share for the k th asset, when the σ state of the world is realized, and wealth levels of w^C per share for the risk-free asset in all states of the world:

$$\sum_j v_j \sum_{\sigma \in \times_k S_k} T_{\gamma}(\sigma) \left[\left\{ \frac{w_j(\sigma)}{\pi_j} - \frac{w^C}{\pi_C} \right\} / \left\{ \sum_k \left(v_k \frac{w_k(\sigma)}{\pi_k} \right) + \left(1 - \sum_m v_m \right) \frac{w^C}{\pi_C} \right\} \right] = \lambda N_p \quad (27)$$

This is the equation we estimate by regressing the normalized excess returns (essentially, $\sum_k v_k (E(r_k) - r^C)$, where E is the expectation operator), normalized by the average portfolio returns across all states of the world ($\sum_k (v_k \frac{w_k(\sigma)}{\pi_k}) + (1 - \sum_k v_k) \frac{w^C}{\pi_C}$), on portfolio entropy in order to estimate the λ that optimally trades off asset uncertainty against asset returns.

In somewhat more standard, financial notation, $w_k(\sigma) = 1 + r_k(\sigma)$, where $r_k(\sigma)$ is the percent return per share in the specified time period. This more usual, financial market notation corresponds better to reported data, so that the aggregate first order condition can be written as:

$$\sum_j v_j \sum_{\sigma \in \times_k \mathbb{S}_k} T\gamma(\sigma) \left[\left\{ \frac{1+r_j(\sigma)}{\pi_j} - \frac{1+r_C}{\pi_C} \right\} / \left\{ \sum_k \left(v_k \frac{1+r_k(\sigma)}{\pi_k} \right) + \left(1 - \sum_m v_m \right) \frac{1+r_C}{\pi_C} \right\} \right] = \lambda N_p \quad (28)$$

Letting R_k^* be the relative returns to the k th risky asset net of the risk-free rate, the above first order equation for the k th asset can be written as

$$R_k^* = \lambda N_k \quad (29)$$

and for the *portfolio*, the first order condition can be written as:

$$R_p^* = \lambda N_p \quad (30)$$

To approximate this first order condition for empirical purposes, we make three assumptions. Like other researchers, we assume that the past matters in the sense of indicating what the future entropy and geometric mean returns are likely to be. Related to the first assumption, we also assume that average market returns of risky assets, less the risk free rate (suitably normalized), approximates the left side term above. Finally, we assume that the denominator in the left side sum, which is the average portfolio return, can be approximated by the average returns over the whole of the market period, $\sum_k \sum_{\sigma \in \times_k \mathbb{S}_k} v_k \gamma(\sigma) ((1+r^k(\sigma)) + (1 - \sum_m v_m)(1+r^C))$. This assumes that the error in measurement of the left side variable (so this is not a case of attenuation bias) can be treated as approximately zero:

$$\begin{aligned} & \sum_k \sum_{\sigma \in \times_k \mathbb{S}_k} v_k T\gamma(\sigma) [(1+r^k(\sigma))/\pi_k - (1+r^C)/\pi_C] \\ & \left\{ 1 / \left[\left(\sum_m (v_m(1+r^m(\sigma))/\pi_m) + (1 - \sum_m v_m)(1+r^C)/\pi_C \right) \right] \right\} - \\ & 1 / \left[\left(\sum_m \sum_{\sigma \in \times_k \mathbb{S}_k} \gamma(\sigma) (v_m(1+r^m(\sigma))/\pi_m) + \left(1 - \sum_n v_n \right) (1+r^C)/\pi_C \right) \right] \right\} \quad (31) \end{aligned}$$

With these simplifications, we regress the monthly average market returns less the risk free returns on the entropy of the market returns for international comparisons (Table 1) and US time trends (Table 2), that is, we regress $r^{mkt} - r^C$, divided by the average portfolio returns over the respective period (and assuming $v_r = v_C = 0.5$), compounded for 30 years, upon the entropy for the market returns. (Tables using daily returns for the US yielded the same results, and are available from the authors upon request.)

The entropy coefficient estimates λ , as indicated by the first order condition above. We calculate entropy from the history of average recent returns, using different window lengths, and then again, using different numbers of support points in the entropy distribution. None of those alternative specifications differed much from the results reported here. Correcting for first order autocorrelation also had virtually no effect on the estimates presented in Tables 1 through 3.

Because we regress $r^{mkt} - r^C$ divided by a constant (average portfolio returns over the whole period) on entropy, we can simulate market ‘mean/variance’ regressions by regressing the same dependent variable on estimated standard deviation from market returns (this is akin to fitting the efficient frontier relationship as specified say, in Elton and Gruber 1995, p. 88, after suitable substitutions) for comparison. The coefficients on our normalized dependent variable may be different from a standard mean/variance model without the normalization, but the fit (regression F-statistics, probability significance for coefficients, and R-squares) statistics will be unaffected. This allows us to compare the explanatory power of entropy against the more traditional measure of uncertainty, standard deviation of portfolio returns. Again, because of the relatively high correlation between these uncertainty measures, it would not be surprising that they fit the data similarly in terms of regression R-square. This comparison we make for US data in Table 2.

6.1 Table 1: International Estimates Using Monthly Returns Data

This data comes from Marmi (2013), available for 16 developed economies. Again, the distinguishing feature of these data is that only monthly returns are available (where daily returns are available for US stock market data sets). As we have a relatively smaller time frame for the data in Table 1, all the entropy and standard deviations here are constructed from the last twelve months of market returns using only the monthly return data. This means that there is likely more noise in our uncertainty measures in Table 1 than there is in Tables 2 and 3. Nonetheless, the estimated λ s are surprisingly similar to those estimated with the US data, available over a much longer time period.

The starting time for each data set varies in Table 1. The data for “developed” markets starts in 1988, and “emerging” markets (South Korea, India, Brazil, and China) have later starting dates (1992, 1993, 1995, and 1998, respectively), as reflected in the sample sizes. Marmi (2013) used the 1000 largest companies in the market for at least some of the estimates. Excluding Japan (whose estimated λ value is very imprecise, even among all the imprecision in Table 1), estimates for λ range generally between 1.4 and 2.0, except for Australia, Germany, and South

Korea, which were a little bit lower (and hence, seem to discount information uncertainty a little bit less than other countries in this period).

Larger estimated entropy coefficients indicate greater ‘caution’ against market uncertainty for the respective country. Note that the estimated Chinese λ is virtually identical to that of the US (2.0345 vs. 2.0272) for this period, as are those λ values for Brazil (2.1906), Canada (1.8846), and India (1.9976). The most conservative economies in this analysis are Hong Kong (3.4881), Singapore (2.4874), and Sweden (2.6209)—relatively older, established market for their respective geographical regions.

6.2 Tables 2 and 3

The data came from the Fama/French factors available from the Wharton Business School. Data used in the first two tables was for the January 1, 1927 to July 31, 2013. The results in Table 2 are divided into two ‘columns’. The ‘Entropy R^2 F-Stat’ on the left hand side are the results for entropy estimates (regressing R_m^* on the market entropy, N_m , of results, calculated returns in the indicated window), the results under the headings ‘SD R^2 F-Stat’ on the right hand side are for standard deviations (regressing R_p^* on the standard deviations calculated from the respective market window). The data represent the returns from the value-weighted returns of all CRSP firms (Center for Research in Security Prices data base) incorporated in the US and listed on the NYSE, NASDAQ, or AMEX. It only reports information on ordinary common shares which either haven’t been or don’t need to be further defined (no firms incorporated outside the US, no trust components, no closed end funds).

The estimates in Table 2 and are OLS regressions without an intercept, as implied by the geomentropic model’s first order condition for portfolios (given in Section 5). The estimated value of λ using a 4-week market window (entropy and the standard deviation of market returns constructed from the last 20 days of trades) over the whole sample period from 1927 to 2013 is about 1.5, statistically significant at better than the one percent level (significance level is 0.0002). The overall fit using entropy as the dependent variable is not different statistically from the fit using the market standard deviation as the measure of risk/uncertainty: the R-square is the same on the left and right hand sides (0.0078 vs 0.0015), and the F-statistic for model fit are both close to significance (8.75 for the entropy model, and 1.56 for the standard deviation model).

Many things are apparent from the 4-week window models in the upper panel: very little of the variation in the normalized net returns is explained by either measure of uncertainty (one tenth of one percent at most), and entropy generally does as good a job at explaining the returns as does the standard deviation

(SD). Indeed, in the post-war models (last three lines), the entropy R-square and F-statistics are much better than the standard deviations for the data based on monthly market returns.

In the estimates, we examine the overall period with and without information from World War II (WWII), and also examine just changes in the post war period to see whether risk tolerance has changed over time. The slight increase in λ suggests less tolerance for risk after the war, but when placed in the context of a shift in a spline function, the upward shift in λ towards less tolerance for uncertainty is statistically insignificant at the 10 percent level (it is significant at the 15 percent level). Perhaps the slight postwar increase in λ is associated with lingering concerns resulting from the stock market crash of the 1930s.

Similarly, the division of the post war period into pre- and post-ERISA (Employee Retirement Income Act) periods was to test whether the expansion of stock market participation after the enactment of ERISA in 1974, significantly changed the observed λ : either because of a change in perceived market transparency—which may cause λ to fall, or a change in the marginal investor's tolerance for uncertainty—which may cause λ to fall or rise. ERISA encouraged corporate pension fund investments in stocks by requiring that pensions be appropriately funded, and prohibited plans from holding more than 10 percent of the corporation's own stock. Institutional stock ownership also increased as individuals shifted from direct investment in equities to holding mutual fund shares and as public pension funds and nonprofit endowments abandoned their traditional policies against investments in equities. For example, public funds in California and 15 other states did not invest in any stocks until 1968. After ERISA, there was a dramatic increase in stock ownership by public entities (Bhide 1993). Using either the 4-week window estimates in the upper panel (where λ increases from about 1.52 to 1.61) or the 6-week estimates in the lower panel (where λ increases from 1.61 to 1.64), the estimated λ is relatively unchanged.

Results for the entropy coefficient indicate reasonable values for λ , from about 1.5 to 1.8 when estimating first order conditions using daily returns. These results were virtually unchanged when the model was estimated by first order autocorrelation regressions. (The estimated value of λ is obviously sensitive to our assumption that half the portfolio is in the risk-free asset: if the share of the risk-free asset fell to zero, the estimated value of λ tends to one.)

Overall, the results suggest reasonable approximations for portfolio λ s given the first order conditions derived from our theoretical model, and, entropy measures of uncertainty fit the data as well as variance (standard deviation) measures of uncertainty.

6.3 Table 3: “Certainty Equivalencies” Over A Lifespan

Data on the annualized risky rates, risk free rates, as well as real disposal gross domestic product per capita are available from 1929. We assume, following the literature, that the ex post trends in the last 30 years of returns approximate the ex ante returns, and use 30-year lags to calculate the geometric mean of risky returns, the probability distribution of risky returns (we use a 6-bucket decomposition in Table 3) and entropy. (As a practical matter, we also employ the (log of) geometric mean for the riskfree rates over the prior 30 years. The results using the arithmetic mean instead over the period yielded virtually the same results.) The certainty equivalency of the risky, and risk-free, portfolios condition allows for a robustness check on the data, as it implies a particular value for λ (here, $p(\sigma)$ are derived from the observed aggregate returns of risky assets):

$$\lambda = \left[\ln \left(\prod_{\sigma \in S} (1 + r(\sigma))^{p(\sigma)30} \right) - \ln((1 + r^C)^{30}) \right] / N_p \quad (32)$$

Or,

$$\lambda = 30 \left[\ln \left(\prod_{\sigma \in S} (1 + r(\sigma))^{p(\sigma)} \right) - \ln((1 + r^C)) \right] / N_p \quad (33)$$

For example, by 1959, the previous 30 years of data indicated that the annual log of the geometric mean of the risky asset was 0.06308, and the annual log of the geometric mean of the risk-free asset was 0.012533. Hence, the implied λ consistent with the certainty equivalence condition is thirty times this difference ($30(0.06308 - 0.012533)$) divided by the entropy value for this period, 1.69337, which equals 0.89550, as indicated in the middle column of estimates.

The data for remaining years shows consistent values of λ with the empirical findings in Tables 1 and 2, which is also consistent with Allais preferences. Overall, λ values are around 1.5, generally bounded between 1 and 2.

To compare our reconciliation of the risky and risk-free rates with the expected utility model, we also present implied relative risk aversion coefficients (RRA) using the same data (prior 30 years means for the risky and risk-free rates, whose differences represent the equity premium for risk) using the simple model of Gollier (2001, Section 5.2). The “equity premium” for 1959, the difference in risky and risk-free rates that compensates the investor for holding the risky assets,

for example, is $0.109 - 0.011 = 0.098$. Employing Gollier's formula, we find the RRA factor is 100, an implausibly large value. Indeed for all years, these estimates of the RRA imply that an investor is so risk averse that she is willing to give up 29 percent of her wealth to avoid a gamble in which she receives a 30 percent wealth gain half the time, and a 30 percent wealth reduction half the time.

The four right hand columns suggest that as entropy rises, so does the equity premium. Indeed, a simple regression indicates as entropy goes up, so does the equity premium (a standard deviation increase in entropy results in almost half a standard deviation increase in the equity premium), with an R-square of .18.

7 Concluding Remarks

Extending Latané's (1959) maximum chance, or wealth reinvestment principle, we examine a very simple one-parameter "maximum choice" model, which we call geomentropic preferences. This non-expected utility model trades off the log of the geometric mean of wealth against entropy, weighted by the parameter λ . Geomentropic preferences explain many of the commonly observed characteristics of consumer behavior under uncertainty, including the Allais paradox.

When we extend this model to portfolio choice, and allow for reinvestment of the assets, we find geomentropic preferences fit the data as well as, and sometimes better than, standard mean-variance models. Robustness checks suggest that a geomentropic characterization of the financial markets tracks aggregate growth as predicted by analyses using daily and monthly returns. Moreover, certainty equivalence under wealth reinvestment, given geomentropic preferences, explains the equity premium puzzle of the the expected utility model, which implies unfeasibly high relative risk aversion.

Our model explains lottery and portfolio choices likely to be observed in the market place. In future research, it would be useful to test our model relative to other explicit choice models both in lab experiments and in simulations of optimal portfolio holdings at the individual level given real market data. However, like other non-expected utility choice models, we do not expect geomentropic preferences to describe the behavior of non-market outcomes or unfeasible lotteries, say of the type embedded in perpetual-wealth enhancing lotteries (Machina 1989).

Appendix

A. Geomentropic Preferences with Additive, Zero-Mean Errors

A.1 Geomentropic Prudence

We show that an increase in the uncertainty of returns to both the risky asset and the same asset with a zero-mean, symmetrically distributed, uninsurable risk, increases savings (via an increase in the share of the risk-free, cash asset). This extends geomentropic preferences to the usual analysis of risk changes employed using expected utility models.

Denote uninsurable risk as ϵ . We then examine what is required in terms of shift in savings share to maintain its geomentropic value relative to the case with no stochastic variation in the returns to future wealth: the returns for the risky asset goes from $(1 + r_i)/\pi$ to $((1 + r_i)/\pi + \epsilon)$, and the returns to the safe asset goes from $1 + r^C$ to $1 + r^C + \epsilon$ where ϵ is the uninsurable perturbation in the returns as noted. (The proof also works for the case where ϵ only affects the risky asset.) To ensure that wealth is non-negative, we assume that ϵ is small relative to the average portfolio returns, $v(1 + r_i)/\pi + (1 - v)(1 + r^C)$, where v is the share of the risky asset in the portfolio, π is the price of the risky asset, and the price of the risk-free asset has been set to 1. Though the consumer takes π as given here, the low-down theorem above is a prudence result arising from asset equilibrium.

Our approach here is to examine the change in the share of the risky asset, Δv , that equates the geomentropic values with and without the uninsurable risk (we assume that the consumer is not always at the same corner solution):

$$(A1) E_{\epsilon} \sum_{\sigma \in S} \{p(\sigma) \ln[(v + \Delta v)W((1 + r(\sigma))/\pi + \epsilon) + (1 - (v + \Delta v))W(1 + r^C + \epsilon)]\} - \lambda N_p = \sum_i \{p_i \ln[(vW(1 + r(\sigma))/\pi + (1 - v)W(1 + r^C))]\} - \lambda N_p$$

Canceling common terms and moving both expressions to the left hand side of the equation yields the equivalent condition that

$$(A2) E_{\epsilon} \sum_{\sigma \in S} \{p(\sigma) \ln[(v + \Delta v)W(1 + r(\sigma))/\pi + \epsilon) + (1 - (v + \Delta v))W(1 + r^C + \epsilon)] - \ln[(vW(1 + r(\sigma))/\pi + (1 - v)W(1 + r^C))]\} = 0$$

Noting that the left-hand, square-bracketed term can be rewritten as

$$(A3) \{\Delta v((1 + r(\sigma))/\pi - (1 + r^C)) + \epsilon\} + \{(vW(1 + r(\sigma))/\pi + (1 - v)W(1 + r^C))\}$$

Factoring out the common $vW(1+r(s))/\pi + (1-v)W(1+r^C)$ terms and canceling them, and then factoring out and canceling the W terms, the expression for geomentropic prudence becomes

$$(A4) E_{\epsilon} \sum_{\sigma \in S} \{p(\sigma) \ln[1 + [\Delta v((1+r(\sigma))/\pi - (1+r^C)) + \epsilon]/[(v(1+r(\sigma))/\pi + (1-v)(1+r^C))]]\} = 0$$

To take expectation of the left-hand term, we split up the symmetrically distributed, zero-mean ϵ random variable into intervals:

$$(A5) \int_{-\infty}^0 \sum_{\sigma \in S} \{p(\sigma) \ln[1 + [\Delta v((1+r(\sigma))/\pi - (1+r^C)) + \epsilon]/[(v(1+r(\sigma))/\pi + (1-v)(1+r^C))]]\} \phi(\epsilon) d\epsilon +$$

$$(A6) \int_0^{\infty} \sum_{\sigma \in S} \{p(\sigma) \ln[1 + [\Delta v((1+r(\sigma))/\pi - (1+r^C)) + \epsilon]/[(v(1+r(\sigma))/\pi + (1-v)(1+r^C))]]\} \phi(\epsilon) d\epsilon$$

Since ϵ is symmetric (hence, $\phi(-\epsilon) = \phi(\epsilon)$), we can replace these integrals with

$$(A7) \int_0^{\infty} \sum_{\sigma \in S} \{p(\sigma) \ln[1 + [\Delta v((1+r(\sigma))/\pi - (1-r^C)) - \epsilon]/[(v(1+r(\sigma))/\pi + (1-v)(1+r^C))]]\} \phi(\epsilon) d\epsilon +$$

$$(A8) \int_0^{\infty} \sum_{\sigma \in S} \{p(\sigma) \ln[1 + [\Delta v((1+r(\sigma))/\pi - (1-r^C)) + \epsilon]/[(v(1+r(\sigma))/\pi + (1-v)(1+r^C))]]\} \phi(\epsilon) d\epsilon$$

Combining these integrals, and letting $D = [(v(1+r(s))/\pi + (1-v)(1+r^C))]$, the defining geomentropic prudence equation becomes

$$(A9) \int_0^{\infty} \sum_{\sigma \in S} \{p(\sigma) \ln[1 + [2\Delta v((1+r(\sigma))/\pi - (1-r^C))]/D + [(\Delta v)^2(r(\sigma)/\pi - r^C)^2]/D^2 - [\epsilon^2]/D^2]\} \phi(\epsilon) d\epsilon = 0$$

By our requirement that ϵ be small relative to the returns (so that there be no negative wealth values), the $[\epsilon^2]/D^2$ term on the far right hand side will be small relative to the other terms, so that our equality will only be satisfied if the positive term middle term, $[(\Delta v)^2(r(\sigma)/\pi - r^C)^2]/D^2$, is offset by the left hand term $[2\Delta v((1+r(\sigma))/\pi - (1-r^C))]/D$, which given that on average the risky returns will be greater than the returns to cash, will only be the case if $\Delta v < 0$, that is, if the share of the risky asset falls and the share of savings increases. QED

From the form of the proof, it is obvious that it readily extends to several risky assets in our portfolio framework.

A.2 “Weak” Temperance

Eeckhoudt and Schlesinger (2006) define temperance (we call this “arithmetic-mean temperance”) to be $((W - \epsilon_1), (W + \epsilon_2); 0.5, 0.5) > ((W, W - \epsilon_1 + \epsilon_2); 0.5, 0.5)$.

They indicate that “...temperance shows a type of preference for disaggregation of two independent zero-mean random variables.” Geomentropic preferences are consistent with arithmetic-mean temperance, as we show below.

An analogous, alternative definition for “geometric-mean temperance” would be

$((W^{(1-\epsilon_1)}, (W^{(1+\epsilon_2)}); 0.5, 0.5) > ((W, W^{(1-\epsilon_1+\epsilon_2)}); 0.5, 0.5)$. But this last preference ordering doesn't hold for equivalent geometric means, like it will for additive, zero-mean errors. Rather, in every case where the geometric means are the same, the geomentropic evaluation for the two alternative lotteries is equivalent as entropy for the comparison lotteries are also equal. Hence, $((W^{(1-\epsilon_1)}, (W^{(1+\epsilon_2)}); 0.5, 0.5) \sim ((W, W^{(1-\epsilon_1+\epsilon_2)}); 0.5, 0.5)$, and there can be no geometric-mean temperance with geomentropic preferences.

“Arithmetic-mean” temperance, on the other hand, leads people facing unavoidable risk to reduce exposure to another independent risk when preferences are geomentropic. We assume ϵ_1, ϵ_2 are independent random, symmetric variates with zero mean. Entropy terms cancel, so that arithmetic-mean temperance holds under our index if $E_{\epsilon_1} E_{\epsilon_2} (\ln(W + \epsilon_1) + \ln(W + \epsilon_2)) > E_{\epsilon_1} E_{\epsilon_2} (\ln(W) + \ln(W - \epsilon_1 + \epsilon_2))$ or if $E_{\epsilon_1} E_{\epsilon_2} (\ln((W + \epsilon_1)(W + \epsilon_2))) > E_{\epsilon_1} E_{\epsilon_2} (\ln((W)(W + \epsilon_1 + \epsilon_2)))$. Noting that $(W + \epsilon_1)(W + \epsilon_2) = (W)(W + \epsilon_1 + \epsilon_2) + \epsilon_1 \epsilon_2$, and canceling similar terms from both lottery indices, we get temperance if

$$(A10) E_{\epsilon_1} E_{\epsilon_2} \ln(1 + (\epsilon_1 \epsilon_2)/\gamma) > 0 \text{ where } \gamma = W^2 + \epsilon_1 W + \epsilon_2 W.$$

Since our variates are symmetric, $\phi(\epsilon) = \phi(-\epsilon)$ for all ϵ_1, ϵ_2 . Hence,

$$E_{\epsilon_1} E_{\epsilon_2} \ln(1 + (\epsilon_1 \epsilon_2)/\gamma) =$$

$$(A11) \int_{-\infty}^0 \int_{-\infty}^0 \ln(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 + \\ \int_{-\infty}^0 \int_0^{\infty} \ln(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 + \\ \int_0^{\infty} \int_{-\infty}^0 \ln(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 + \\ \int_0^{\infty} \int_0^{\infty} \ln(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 =$$

$$(A12) \int_0^{\infty} \int_0^{\infty} \ln(1 + ((-\epsilon_1)(-\epsilon_2))/(W^2 - \epsilon_1 W - \epsilon_2 W)) \phi(-\epsilon_1) d\epsilon_1 \phi(-\epsilon_2) d\epsilon_2 + \\ \int_0^{\infty} \int_0^{\infty} \ln(1 + (\epsilon_1(-\epsilon_2))/(W^2 + \epsilon_1 W - \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(-\epsilon_2) d\epsilon_2 + \\ \int_0^{\infty} \int_0^{\infty} \ln(1 + ((-\epsilon_1)\epsilon_2)/(W^2 - \epsilon_1 W + \epsilon_2 W)) \phi(-\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 + \\ \int_0^{\infty} \int_0^{\infty} \ln(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 =$$

$$(A13) \int_0^{\infty} \int_0^{\infty} \ln[(1 + (\epsilon_1 \epsilon_2)/(W^2 - \epsilon_1 W - \epsilon_2 W))(1 + (-\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W - \epsilon_2 W)) \\ (1 + (-\epsilon_1 \epsilon_2)/(W^2 - \epsilon_1 W + \epsilon_2 W))(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W))] \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2$$

There will be arithmetic-mean temperance with geomentropic preferences if this last integral is positive. Algebraic manipulations reduces the “ $\ln(\cdot)$ ” in the integral to $\ln[(1 + (\epsilon_1^4 \epsilon_2^4 + 2W^2 \epsilon_1^2 \epsilon_2^2 (3W^2 - \epsilon_1^2 - \epsilon_2^2))/denominator] > 0$, where $denominator = (W^2 - W(\epsilon_1 + \epsilon_2))(W^2 - W(\epsilon_2 - \epsilon_1))(W^2 + W(\epsilon_2 - \epsilon_1))(W^2 + W(\epsilon_1 + \epsilon_2))$ as both denominator and numerator terms in the ratio term are positive as long as wealth is large relative to the stochastic gains and losses. QED.

A.3 Losses Hurt More than Equivalent Gains Help

To show the general nature of the result, we place the random shift in a portfolio context. Specifically, suppose that our portfolio has one risky asset (with share v and asset price π) and one risk-free asset (with share $1 - v$ and nominal returns per share of $1 + r^C$). Consider the change in geomentropic outcomes when the wealth levels achieved from the risky asset go from $(vW)/\pi(1 + r(\sigma))$ if state s is realized to $(vW)/\pi(1 + r(\sigma) + \epsilon)$ if state σ is realized, where ϵ is a mean zero random variable (this could be the case for one or more states of the world). Then under geomentropic preferences, greater certainty in returns is preferred to uncertainty in returns for a portfolio (the results are obviously more general than this example of a 1 risky asset, 1 riskless asset portfolio), if

$$(A14) \sum_{\sigma \in S} p(\sigma) \ln[(vW)/\pi(1 + r(\sigma)) + (1 - v)W(1 + r^C)] + \lambda v \sum_{\sigma \in S} p(\sigma) \ln p(\sigma) > E_{\epsilon} \sum_{\sigma \in S} p(\sigma) \ln[(vW)/\pi(1 + r(\sigma) + \epsilon) + (1 - v)W(1 + r^C)] + \lambda v \sum_{s \in S} p(s) \ln p(s)$$

Factoring out common W terms, and common entropy terms from both sides, yields the equivalent condition

$$(A15) \sum_{\sigma \in S} p(\sigma) \ln((v/\pi)(1 + r(\sigma)) + (1 - v)(1 + r^C)) > E_{\epsilon} \sum_{s \in S} p(s) \ln((v/\pi)(1 + r(\sigma) + \epsilon) + (1 - v)(1 + r^C))$$

As

$$(A16) E_{\epsilon} \sum_{\sigma \in S} p(\sigma) \ln(((v/\pi)(1 + r(\sigma) + \epsilon) + (1 - v)(1 + r^C))) = \sum_{\sigma \in S} p(\sigma) \ln((v/\pi)(1 + r(\sigma)) + (1 - v)(1 + r^C)) + \sum_{\sigma \in S} p(\sigma) E_{\epsilon} \ln(1 + ((v/\pi)\epsilon)/\{(v/\pi)(1 + r(\sigma)) + (1 - v)(1 + r^C)\}),$$

We cancel the “certainty” terms (roughly, those not involving ϵ) from both sides of the above inequality, and greater certainty in returns is preferred to uncertainty in returns if

$$(A17) 0 > \sum_{\sigma \in S} p(\sigma) E_{\epsilon} \ln(1 + ((v/\pi)\epsilon)/\{(v/\pi)(1 + r(\sigma)) + (1 - v)(1 + r^C)\})$$

Since by Jensen’s inequality for concave functions,

$$(A18) E \ln(1 + ((v/\pi)\epsilon)/\{(v/\pi)(1 + r(\sigma)) + (1 - v)(1 + r^C)\}) < \ln(1 + (v/\pi)E\epsilon)/\{v(1 + r(\sigma)) + (1 - v)(1 + r^C)\} = \ln(1) = 0 \text{ as } E(\epsilon) = 0. \text{ QED.}$$

That is, more randomness in the returns to the risky asset reduces its geometric mean (G_r falls), lowering the geomentropic value of the portfolio.

A.4 Geomentropic Preferences are Concave in Shares

Proof of Concavity of Geomentropic Preferences

To show that geomentropic preferences are concave in the shares, rewrite $\ln G_p(v) - \lambda N_p(v)$ as

$$(A19) \ln \left\{ \prod_{\sigma \in \times_k \mathbb{S}_k} \left[\frac{W}{\pi_C} w^C + \sum_k v_k \left(\frac{W}{\pi_k} w_k(\sigma) - \frac{W}{\pi_C} w^C \right) \right]^{T\gamma(\sigma)} \right\} - \lambda \left[\sum_k v_k N_k \right]$$

Let $\frac{W}{\pi_C} w^C = \phi$ and $\frac{W}{\pi_k} w_k(\sigma) - \frac{W}{\pi_C} w^C = \theta_k$, so that geomentropic preferences can be written as

$$(A20) \ln \left\{ \prod_{\sigma \in \times_k \mathbb{S}_k} \left[\phi + \sum_k v_k \theta_k \right]^{T\gamma(\sigma)} \right\} - \lambda \left[\sum_k v_k N_k \right]$$

Factor out ϕ from both terms in the product, and take logarithms to get

$$(A21) \sum_{\sigma \in \times_k \mathbb{S}_k} T\gamma(\sigma) \ln \left(\phi \left[1 + \sum_k v_k \theta'_k \right] \right) - \lambda \left[\sum_k v_k N_k \right]$$

where $\theta'_k = \theta_k/\phi$. This is concave in shares if for $v_k \neq v'_k$, the following holds for all $0 < t < 1$

$$(A22) \quad \sum_{\sigma \in \times_k \mathbb{S}_k} T\gamma(\sigma) \ln \left(\phi \left[1 + \sum_k (tv_k + (1-t)v'_k) \theta'_k \right] \right) - \lambda \left[\sum_k (tv_k + (1-t)v'_k) N_k \right] > t \left\{ \sum_{\sigma \in \times_k \mathbb{S}_k} T\gamma(\sigma) \ln \left(\phi \left[1 + \sum_k v_k \theta'_k \right] \right) - \lambda \left[\sum_k v_k N_k \right] \right\} + (1-t) \left\{ \sum_{\sigma \in \times_k \mathbb{S}_k} T\gamma(\sigma) \ln \left(\phi \left[1 + \sum_k v'_k \theta'_k \right] \right) - \lambda \left[\sum_k v'_k N_k \right] \right\}$$

Canceling the common ϕ and entropy terms from both sides of the inequality, the inequality is seen to hold if

$$(A23) \quad \sum_{\sigma \in \times_k \mathbb{S}_k} T\gamma(\sigma) \ln \left(\left[1 + \sum_k (tv_k + (1-t)v'_k) \theta'_k \right] \right) > t \left\{ \sum_{\sigma \in \times_k \mathbb{S}_k} T\gamma(\sigma) \ln \left(\left[1 + \sum_k v_k \theta'_k \right] \right) \right\} + (1-t) \left\{ \sum_{\sigma \in \times_k \mathbb{S}_k} T\gamma(\sigma) \ln \left(\left[1 + \sum_k v'_k \theta'_k \right] \right) \right\}$$

Note that this inequality holds for any state of the world (that is, holding σ fixed) as the arithmetic mean exceeds the geometric mean:

$$(A24) \quad \ln \left(\left[1 + \sum_k (tv_k + (1-t)v'_k) \theta'_k \right] \right) > t \ln \left(\left[1 + \sum_k v_k \theta'_k \right] \right) + (1-t) \ln \left(\left[1 + \sum_k v'_k \theta'_k \right] \right)$$

since

$$(A25) \quad \left(t \left[1 + \sum_k v_k \theta'_k \right] + (1-t) \left[1 + \sum_k v'_k \theta'_k \right] \right) > \left[1 + \sum_k v_k \theta'_k \right]^t \left[1 + \sum_k v'_k \theta'_k \right]^{(1-t)}$$

Since the inequality holds for each state of the world, it holds for all states of the world when summed over both sides of the inequality. (QED)

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