

Research Article

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Mass Distribution and Gravitational Potential of the Milky Way

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Abstract: Models of mass distribution in the Milky Way are discussed where those yielding the potential analytically are preferred. It is noted that there are three main contributors to the Milky Way potential: bulge, disc and dark halo. In the case of the disc the Miyamoto-Nagai formula, as simple enough, has shown as a very good solution, but it has not been able to satisfy all requirements. Therefore, improvements, such as adding new terms or combining several Miyamoto-Nagai terms, have been attempted. Unlike the disc, in studying the bulge and dark halo the flattening is usually neglected, which offers the possibility of obtaining an exact solution of the Poisson equation. It is emphasized that the Hernquist formula, used very often for the bulge potential, is a special case of another formula and the properties of that formula are analysed. In the case of the dark halo, the slopes of its cumulative mass for the inner and outer parts are explained through a new formalism presented here for the first time.

Keywords: Galaxy: kinematics and dynamics, structure

1 Introduction

Our own Galaxy, the Milky Way (MW), is known to consist of several subsystems. This must be taken into account when one wants to construct an MW model. Here a mass distribution model is borne in mind. The history of MW models is rich, the two models proposed by Schmidt (1956, 1965) are well known. In the subsequent years our knowledge of the mass distribution in MW was significantly enriched. Consequently, in recent times among astronomers, there is almost a consensus that in MW there are three essential subsystems: bulge, disc (thin) and halo (dark) (e.g. Rohlfs & Kreitschmann (1981); Carlberg & Innanen (1987); Dehnen & Binney (1998)). They are not only the most massive among the MW subsystems, but also their contribution to the gravitation field is substantial; each of them has its part of space wherein it dominates in the contribution to the gravitation field. For instance, the bulge is dominant in the central part, the disc at longer distances to the MW centre (especially in the midplane) and finally, far from the centre the domination belongs to the dark halo.

Among other things a mass model can be used for the purpose of calculating the galactocentric orbits. Then, it is very desirable for the MW potential to be given analytically. Such models are known to exist (e.g. Kutuzov *et al.*

(1989); Allen *et al.* (1991); Dinescu *et al.* (1999)). Their common characteristic is that the potential is assumed to be stationary and axially symmetric. The spherical symmetry is a special case of the axial symmetry and, as more simple, it has been frequently assumed for those subsystems where there is no relevant evidence in favour of a substantial flattening, for instance for the bulge and halo (e.g. Sellwood & Sanders (1956); Battaglia *et al.* (2005)). Then one can expect to obtain exact solutions of the Poisson equation, which relates the density and the potential. If the subsystem under study is also observable (bulge, unlike dark halo), then its density relies on the observational facts.

In the case of the disc, clearly, the spherical symmetry assumption is not acceptable. The unavoidable generalisation towards axial symmetry has resulted in seeking approximate solutions only.

In what follows a discussion will be presented, which concerns the formulae usually used for the purpose of describing either the density or the potential in the case of the main MW subsystems. The discussion also contains original ideas of the present author where some of them are presented for the first time.

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2 Spherical Subsystems

2.1 Density and Potential Formulae

In the case of a spherically symmetric subsystem, one defines a central part known as core. Generally within the core the density can be almost constant or have a substantial gradient known as cusp. Usually a cusp is mathematically described by a function which has a central singularity. As an example it is possible to present the following formula for the potential

$$\Pi = \frac{GM}{r_a + \sqrt{r_b^2 + r^2}}. \quad (1)$$

Here Π is the potential, G is the universal gravitation constant, M is the total mass of the system (subsystem), r_a and r_b are two constants having dimension of length and r is the only variable, distance to the centre. Formula (1), known as generalised isochrones potential formula, was proposed in 1973 (Kuzmin *et al.* (1973, p. 301)). Its special case was rediscovered in 1990 (Hernquist (1990)) so that it has often been cited as the Hernquist formula. The density expression corresponding to (1) in the special case $r_b \rightarrow 0$ is

$$\rho = \frac{\rho_0}{\xi(1+\xi)^3}, \quad \xi = \frac{r}{r_a}, \quad \rho_0 = \frac{M}{2\pi r_a^3}. \quad (2)$$

As seen from (2), the density tends to infinity at the centre. The version of (1) wherein $r_b \rightarrow 0$ has often been used for the potential of the MW bulge (*e.g.* Law & Majewski (2010)).

For the purpose of describing also the bulge contribution to the potential of a spiral galaxy, like MW, the present author has proposed another formalism (Ninković (2014)). This formalism also comprises a cuspy distribution, but it can be more diverse than just a simple power law near the centre. For instance, near the centre expression (2) is approximately reduced to an r^{-1} dependence.

In the case of the dark halo a cuspy formula is the well-known Navarro-Frenk-White expression (NFW – Navarro *et al.* (1997)) having the form

$$\rho = \frac{\rho_0}{\xi(1+\xi)^2}, \quad \xi = \frac{r}{r_c}, \quad \xi \in (0, \xi_l], \quad \xi_l = \frac{r_l}{r_c}. \quad (3)$$

As seen, expression (3) is similar to (2), the difference is in the exponent value, this time 2, instead of 3 in (2); also due to such an exponent value expression (3) is defined over a finite volume, inside the sphere $r = r_l$, because otherwise the total mass would be infinite.

The dark halo of MW has been also modelled by using density functions with no cusp, for instance variants

of the quasi-isothermal profile (*e.g.* Einasto & Haud (1989); Ninković (1992)) wherein the density attains zero at a distance $r = r_l$, but unlike (3) without density discontinuity at the boundary. The following example,

$$\rho = \rho_0 \left[\frac{1}{(1+\xi^2)^a} - \frac{1}{(1+\xi_l^2)^a} \right], \quad 0 < a \leq \frac{3}{2}, \quad (4)$$

$$\xi = \frac{r}{r_c}, \quad \xi \in [0, \xi_l], \quad \xi_l = \frac{r_l}{r_c},$$

satisfies these conditions. Strictly speaking, the simple density profile $\rho \propto r^{-2}$ should be referred to as the “isothermal” one, so that if the variable square is added to the softening parameter (in (4) $1 + \xi^2$), then it is only quasi-isothermal. In (4) the density is zero from both sides of the boundary sphere $r = r_l$. Typical values to be substituted for a are: $a = 1$ and $a = 3/2$. The former one is “classical”, the latter one can in some way connect model (4) with model (3).

2.2 Mass Distribution in the Dark Halo

Here, for the first time, a density formula similar to (4) will be considered. Its form is

$$\rho = \rho_0 \left[(1+\xi^k)^{-1} - (1+\xi_l^k)^{-1} \right]. \quad (5)$$

In (5) k is supposed to be a natural number; ξ_l depends on k , if $k \leq 3$, ξ_l must be finite, if $k > 3$, $\xi_l \rightarrow \infty$. Consequently, if $k > 3$, ρ_0 becomes equal to the density at the centre. For practical reasons, which will be given below, the interval for k should be limited to $2 \leq k \leq 4$. Substituting $a = 1$ in (4) and $k = 2$ in (5) one finds the case in common.

By increasing the value for the exponent in (5) ($k > 2$) compared to (4) ($a = 1$) it is possible to obtain a more rapid increase of the cumulative mass within the core ($r \leq r_c$). The core radius of the dark halo is usually expected to exceed the galactocentric distance of the Sun (*e.g.* Rohlfs & Kreitschmann (1981); Allen *et al.* (1991); Ninković (1992)). The choice of the parameter values depends on the agreement with an assumed rotation curve. In the case of the dark halo, this means that we assume a value for the exponent of the cumulative mass, *i.e.* density, on the basis of the achieved agreement. For instance, if NFW (expression (3)) is assumed for the dark halo density, near the centre, the cumulative mass will increase as r^2 , whereas if assumed (4) or (5), the cumulative mass near the centre will follow r^3 , but by increasing k in (5) the density decrease within the core becomes less intense. However, this does not mean that the parameter k should have an arbitrarily high value, even if the fit to an MW rotation curve

suggested a very intense increase of the cumulative mass within the core of the dark halo. This statement can be illustrated by means of the following analysis.

Let $k = 4$ be assumed in (5). The corresponding expression for the cumulative mass is easily obtained,

$$\mathcal{M}_r = 4\pi\rho_0 r_c^3 \frac{\sqrt{2}}{4} \left[\ln \sqrt{\frac{(\xi - \frac{\sqrt{2}}{2})^2 + \frac{1}{2}}{(\xi + \frac{\sqrt{2}}{2})^2 + \frac{1}{2}}} + \arctan(\sqrt{2}\xi + 1) + \arctan(\sqrt{2}\xi - 1) \right]. \quad (6)$$

From (6) one finds that the total mass ($\xi \rightarrow \infty$) is equal to $\sqrt{2}\pi^2\rho_0 r_c^3$. It depends on two parameters only, ρ_0 and r_c . The values for both of them can be established from a fit to the rotation curve, *i.e.*, bearing in mind the comment given above, the total mass will be completely determined by the conditions governing in the core of the dark halo and, therefore, any constraint due to the conditions in the outer MW parts would not be usable. However, a constraint due to the conditions in the outer MW parts may be relevant, for instance a value for the total mass of the dark halo inferred without taking such constraint into account may be too low. The purpose of this example is to show why substituting high values for k (say $k > 4$) in (5) does not seem recommendable.

In the next step the case $k = 3$ will be analysed. The cumulative mass is given by

$$\mathcal{M}_r = 4\pi\rho_0 r_c^3 \left[\frac{\ln(1 + \xi^3)}{3} - \frac{\xi^3}{3(1 + \xi^3)} \right]. \quad (7)$$

In (7), as earlier, it is $\xi = r/r_c$, $\xi_l = r_l/r_c$. Now, the total mass ($\xi = \xi_l$) depends on three parameters: ρ_0 , r_c and r_l (ξ_l). This can be an advantage compared to the case $k = 4$ because more constraints, those concerning the inner MW (say rotation curve), as well as those concerning the outer MW (say MW satellites), can be taken into account.

The present analysis allows a better understanding of the limitation given above $-2 \leq k \leq 4$, which is valid for expression (5). The lowest value, $k = 2$, is, as already said, a case in common with (4), the other two ($k = 3$ and $k = 4$) are rather marginal cases; because if assumed $k = 3$, the property of requiring an almost constant density core for the dark halo (of course, if supported by observational constraints) would be better reflected than in the case $k = 2$, but any constraint concerning the total mass of the dark halo imposed by the evidence from MW outer parts could be still taken into account. On the other hand, if assumed $k = 4$, the property of requiring an almost constant density core would be reflected even better than if $k = 3$, but any constraint due to the evidence for the outer MW parts would be no longer usable.

The choice $k = 3$ may have another consequence. It is possible to unify all the three: expression (3), expression (4), special case $a = 3/2$ and (5), $k = 3$. To achieve a full similarity, in (3) the constant term (like in (4) and (5)) should be added in order to remove the density discontinuity at the boundary ($\xi = \xi_l$). The following density formula is obtained

$$\rho = \rho_0 \left[\frac{1}{\xi^{l_1} (1 + \xi^{l_2})^{\frac{l_2}{2}}} - \frac{1}{\xi_l^{l_1} (1 + \xi_l^{l_2})^{\frac{l_2}{2}}} \right]. \quad (8)$$

In (8) l_1 , l_2 and l_3 are three non-negative integers. They satisfy the following conditions:

$$0 \leq l_1 \leq 2, \quad l_1 + \frac{1}{2}l_2l_3 = 3. \quad (9)$$

In this way, expressions (3), (4) and (5) become special cases of (8), in particular, $l_1 = l_2 = 1$, $l_3 = 4$ - modified expression (3), $l_1 = 0$, $l_2 = 2$, $l_3 = 3$ - special case of (4) and $l_1 = 0$, $l_2 = 3$, $l_3 = 2$ - a special case of (5). Also, a variant of the mass distributions proposed by Jaffe (1983) and Lynden-Bell & Lynden-Bell (1995), where $l_1 = 2$, is included. Thus if in forming an MW mass model one assumes the density of the dark halo to decrease approximately as ξ^3 in the outer parts, then by assuming generally (8) it will be possible to look for the best fit by selecting the values for the three integers - l_1 , l_2 and l_3 .

3 Rotation Curve for the Milky Way

In order to test the formulae proposed here for the mass distribution in the dark halo ((5)–(9)) a rotation curve of MW will be calculated. This means to calculate the circular speed being defined as

$$u_c = \sqrt{-R \frac{\partial \Pi}{\partial R}}, \quad z = 0. \quad (10)$$

In (10) R is the distance to the rotation axis, $|z|$ that to the midplane; the potential Π is a sum of components due to the subsystems (bulge, disc, dark halo). The calculated circular speed will be compared to an empirical rotation curve.

In view of some recent empirical rotation curves (Li (2016); Locco *et al.* (2015); Smith *et al.* (2015)) it is possible to conclude that from about $0.25 R_\odot$ to about $2 R_\odot$ the circular speed seems to be approximately constant. In the very inner part it may have a prominent maximum (Li (2016); Smith *et al.* (2015)), which is, certainly, due to the bulge. Let it be here assumed that such a maximum really exists. For this reason the bulge is here described by

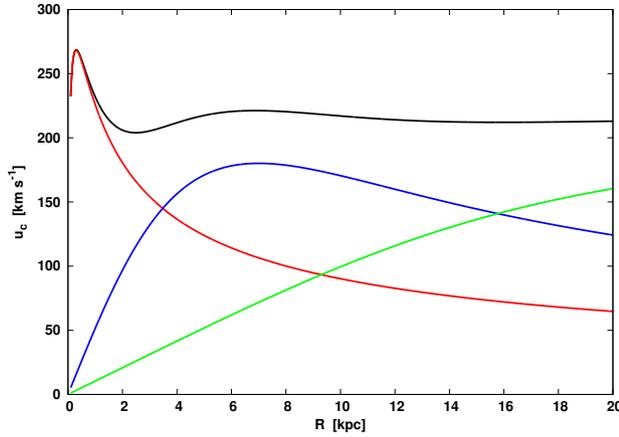


Fig. 1. Rotation curve of the Milky Way (black) with the curves due to the components: bulge (red), $\mathcal{M}_b = 19.16 \times 10^9 \mathcal{M}_\odot$, $r_b = 0.3$ kpc, disc (blue) $\mathcal{M}_d = 61 \times 10^9 \mathcal{M}_\odot$, $r_d = 3.3$ kpc, dark halo (green), $\rho_0 = 0.0058 \mathcal{M}_\odot \text{pc}^{-3}$, $r_c = 17$ kpc, $\xi_l = 5.665$.

means of expression (2), *i.e.* its potential obeys the special case, $r_b = 0$, of expression (1). For the disc the exponential model (Freeman 1970) is assumed. The dark halo is modelled assuming (8) with $l_2 = 3$, $l_3 = 2$, or according to (5), $k = 3$, which is the same. The curve is adapted to the values $R_\odot = 8.5$ kpc, $u_c(R_\odot) = 220 \text{ km s}^{-1}$ assumed a priori. The scale lengths of the three subsystems are then: $r_a = 0.035R_\odot$ (expression (1)) for the bulge, $r_d = 0.39R_\odot$ for the disc and $r_c = 2R_\odot$ (expressions (4)–(5)) for the dark halo; the other parameter concerning the dark halo, ξ_l (expressions (4)–(5)), is optional only. It could be relevant only if a reliable constraint for the total mass of the Milky Way were included. However, an analysis based on the rotation curve only is insufficient, the data concerning the motion of the Milky Way satellites should be also taken into account, but this is beyond the scope of the present paper. The values of all parameters can be found in the caption to Fig. 1 where the resulting rotation curve is presented.

4 Flat Subsystems

Bearing in mind what said above, there is only one flat subsystem, the disc. Discs of the spiral galaxies are known to obey an exponential dependence of the surface density on the distance to the axis of symmetry, R . There exist formulae (*e.g.* Cuddeford (1993)) yielding the dependence of the volume density, ρ , on the variables R and $|z|$ (density is even function of z). This dependence is given by means of transcendental functions (such as exponential, for instance) and there are no analytical solutions of the Poisson

equation, at least when elementary functions are understood. This has been the reason why to look for approximate solutions wherein the potential is expressed through algebraic functions, for instance the Ollongren interpolation polynomial (*e.g.* Allen *et al.* (1986)) and many papers where the Miyamoto-Nagai formula (Miyamoto & Nagai (1975)) is used. The present author recently published a paper (Ninković (2015)) wherein he indicated some problems following the application of the Miyamoto-Nagai formula and proposed certain improvements.

Besides, we have met ideas of combining several Miyamoto-Nagai terms in order to achieve a better fit to the disc mass distribution. The Miyamoto-Nagai formula has the form

$$\Pi = \frac{GM}{\sqrt{R^2 + (a + \sqrt{b^2 + z^2})^2}}. \quad (11)$$

In (11) the designations G , \mathcal{M} and Π have the same meaning as in (1), whereas a and b are two constants having dimension of length; R and z are the variables. It is easy to see that expressions (1) and (11) have a common special case, $r_a = 0$ in (1), $a = 0$ in (11). In formula (11) there are three parameters: \mathcal{M} , a and b . One of the possibilities has been to express the disc potential as a sum of three terms like that in (11), as done *e.g.* by Smith *et al.* (2015), where one of them is negative. In such a case there would be generally nine parameters, but by putting equal some of them, this number can be reduced (just as done by Smith *et al.* (2015)). Another possibility is to use the denominator of the right-hand side in (11), let it be called the Miyamoto-Nagai function and denoted as \mathcal{F} . Then the disc potential may be expressed in the following way

$$\Pi = G\mathcal{M} \frac{\mathcal{F}_1^\alpha}{\mathcal{F}_2^{\alpha+1}}, \quad \alpha \geq 0. \quad (12)$$

This modification has been done by the present author (Ninković (2017)). Here the two Miyamoto-Nagai functions differ in the values of their parameters, a_1 and b_1 in the numerator, a_2 and b_2 in the denominator. The number of parameters will be five now, however this can be reduced, for instance by requiring equal ratios, $b_1/a_1 = b_2/a_2$, as done by Ninković (2017). Also, the ratio $(a_1 + b_1)/(a_2 + b_2)$, appearing in the calculation of the rotation curve, is not independent of α . So it becomes clear why the number of parameters is five (\mathcal{M} , a_1 , b_1 , a_2 and b_2).

5 Discussion and Conclusions

In the field of Milky Way (mass) models there is almost a consensus that there are three essential contributors to

the potential: the bulge, the disc and dark halo. For all of them, steady state has been assumed. In the case of the bulge spherical symmetry seems quite acceptable, the potential is usually of the form as given here (expression (1), $r_b \rightarrow 0$), though models with $r_b > 0$ cannot be rejected. On the other hand, the disc is generally described as extremely thin and exponential, but in this case the problem is how to obtain a suitable analytical potential expressed by means of elementary functions. Though very often applied, the Miyamoto-Nagai formula (given here – expression (11)) cannot satisfy all requirements, as indicated by Ninković (2015). For this reason various improvements (e.g. Ninković (2015); Smith *et al.* (2015); Ninković (2017)) have been attempted.

Unlike the bulge and disc, which are observable, in modelling the dark halo only constraints due to the seen matter can be used. If for it the spherical symmetry is assumed, as has often been the case, then the slopes of the cumulative mass for the inner parts (usually inferred from the rotation curve) and for the outer parts (motion of MW satellites) may be a problem. Here attention is drawn that a density function applicable for arbitrarily large argument values and capable of yielding a satisfactory explanation of the rotation curve (expression (5), $k = 4$ and expression (6) corresponding to it), may not result in a sufficiently high total mass. In such a case the exponent value in (5) (k) should be reduced. With the objective to comprise a variety of possibilities, which can arise for the case when the density of the dark halo approximately decreases as r^3 in the outer parts, here a general formula is proposed (expressions (8) and (9)).

The dark halo modelled following Expression (5) (case $k = 3$), i.e. (8) (case $l_1 = 0, l_2 = 3, l_3 = 2$), is used here to construct a rotation curve for the Milky Way (Fig. 1). The obtained curve should satisfy the requirement that there exists a prominent maximum near the centre due to the bulge, being a prerequisite for the bulge model. The present dark-halo model provides sufficiently high values for the circular speed in the outer parts ($R > R_\odot$), at $R = 2R_\odot$ it is the largest contributor to the circular speed square. This is one of the possibilities to explain such a high circular speed in the outer parts of the Milky Way. For instance, Li (2016) explains the phenomenon by introducing a gas component instead of the dark one. Such an approach is based on the fact that a Milky-Way mass of $(1 - 2) \times 10^{11} M_\odot$ inside about 20 kpc from its centre might be still acceptable without introducing an exotic kind of matter. The same effect is achieved in the present paper if in expression (5) $k = 4$, instead of 3, is substituted (or, in (8) $l_1 = 0, l_2 = 4, l_3 = 2$) because then the total mass of the dark halo (expression (6)) is determined with two

parameters only (ρ_0 and r_c - (6)). The values for both are completely derivable from the rotation curve and, consequently, the resulting total mass may be too low, if any rather strong constraint suggesting a very high total mass for the Milky Way (say $\approx 1 \times 10^{12} M_\odot$) is found. Then a mass model involving the dark matter concentrated within an arbitrarily large radius, like that given here (expression (5), $k = 3$ allowing a third parameter, ξ_l) will be the only way out, unless an alternative physics is assumed. Such alternative hypotheses (e.g. Napolitano *et al.* (2012); Zhao *et al.* (2013)) should not be rejected a priori, but they must be confirmed in our experience. On the other hand, the nature of the dark matter must be disclosed. Future research will provide answers to these questions.

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References

- Allen, C., Martos, M. 1986, *Revista Mexicana Astron. Astrof.*, 13, 137-147.
- Allen, C., Santillan, A. 1991, *Revista Mexicana Astron. Astrof.*, 22, 255-263.
- Battaglia, G., Helmi, A., Morrison, H., Harding, P., Olszewski, E., Mateo, M. et al. 2005, *MNRAS*, 364, 433-422.
- Carlberg, R. G., Innanen, K. A. 1987, *AJ*, 94, 666-670.
- Cuddeford, P. 1993, *MNRAS*, 262, 1076-1086.
- Dehnen, W., Binney, J. 1998, *MNRAS*, 294, 429-438.
- Dinescu, D. I., Girard, T. M., van Altena, W. F. 1999, *AJ*, 117, 1792-1815.
- Einasto, J., Haud, U. 1989, *A&A*, 223, 89-94.
- Freeman, K.C. 1970, *ApJ*, 160, 811-830.
- Hernquist, L. 1990, *ApJ*, 356, 359-364.
- Jaffe, W. 1983, *MNRAS*, 202, 995-999.
- Kutuzov, S. A., Osipkov, L. P. 1989, *Astron. Zh.*, 66, 965-974 (in Russian).
- Kuzmin, G. G., Veltmann, Ü.-I. K. 1973, *Publ. Tartuskoj astrof. Observatorii*, 40, 281-323.
- Law, D. R., Majewski, S. R. 2010, *ApJ*, 714, 229-254.
- Li, E. 2016, *ArXiv:1612.07781*
- Locco, F., Pato, M., Bertone, G. 2015, *Nat Phys letters*, 11(3), 245-248.
- Lynden-Bell, D., Lynden-Bell, R. 1995, *MNRAS*, 275(2), 429-442.
- Miyamoto, M., Nagai, R. 1975, *PASJ*, 27(4), 533-543.
- Napolitano, N.R., Capozziello, S., Romanowsky, A.J., Capaccioli, M., Tortora, C. 2012, *ApJ*, 748(2), 87-92.
- Navarro, J. F., Frenk, C. S., White, S. D. M. 1997, *ApJ*, 490(2), 493-508.
- Ninković, S. 1992, *Astron. Nachr.*, 313(2), 83-89.
- Ninković, S. 2014, *Serbian AJ*, 188, 23-28.
- Ninković, S. 2015, *PASA*, 32, e032.

- Ninković, S. 2017, *Revista Mexicana Astron. Astrof.* (in press).
- Rohlf, K., Kreitschmann, J. 1981, *Ap&SS*, 79(2), 289-319.
- Schmidt, M. 1956, *Bull. Astron. Inst. Netherlands*, 13, 15-41.
- Schmidt, M. 1965, *Galactic Structure*, In: Blaauw A. and Schmidt M. (Eds.), University of Chicago Press, Chicago & London, 513-530.
- Sellwood, J. A., Sanders, R. H. 1988, *MNRAS*, 233, 611-620.
- Smith, R., Flynn, C., Candlish, G. N. et al 2015, *MNRAS*, 448(3), 2934-2940.
- Sofue, Y. 2012, *PASJ*, 64(4), 75-82.
- Zhao, H., Famaey, B., Lüghausen, F., Kroupa, P. 2013, *A&A*, 557(L3),1-4.