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POTENTIAL OF A SPHEROID WITH GENERALIZED EXPONENTIAL DENSITY DISTRIBUTION

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Abstract. The internal potential of an inhomogeneous layered spheroid with a small ellipticity and general exponential density distribution is derived in an analytical form. The results are presented in the form of standard Gauss hypergeometric function and validated numerically. The computing time when using this formula is noticeably smaller than the time required by numerical integration.

Key words: gravitation – layered inhomogeneous ellipsoid, gravitational potential

1. INTRODUCTION

Consider an oblate spheroid with the boundary surface

$$\frac{r^2}{a_1^2} + \frac{x_3^2}{a_3^2} = 1, r^2 = x_1^2 + x_2^2 \tag{1}$$

and density distribution

$$\rho = \rho_0 (1 - m^2)^n, \quad m^2 = \frac{r^2}{a_1^2} + \frac{x_3^2}{a_3^2}. \tag{2}$$

Here $a_1 \geq a_3$ are the semiaxes of the spheroid boundary surface and n is a positive exponent. This spheroid consists of a family of geometrically similar homothetic equidensity layers. Parameter m takes values in the range $0 \leq m \leq 1$, and for each m there is the corresponding layer. At $m = 0$, the layer degenerates into the central point.

The potential at an internal point x of a spheroid with density law (2) is given by the following single integral (Chandrasekhar 1969; Kondratyev 2007):

$$\varphi(x) = \pi G \frac{\rho_0}{n+1} a_1^2 a_3 \int_0^\infty \frac{du}{(a_1^2 + u)\sqrt{a_3^2 + u}} \left(1 - \frac{r^2}{a_1^2 + u} - \frac{x_3^2}{a_3^2 + u}\right)^{n+1}, \tag{3}$$

where u is the integration variable.

Note that in the case of integer $n = 0, 1, 2, 3...$ a computer can evaluate general integrals of the form (3) very rapidly and with sufficient accuracy, whereas the computing time sharply increases in the case of noninteger n. Therefore, in practice, it would be appropriate to derive potential (3) in a finite analytical form. Moreover, the analytical form of the potential would also be more convenient to use in theoretical studies.

However, in the general case, integral (3) cannot be expressed in an analytical form. We therefore limit our analysis to the special case where spheroidal layers differ little from spherical shape. The problem remains relevant even in such a formulation because configurations of this kind are common in astronomy (planets and their satellites, stars and some galaxies, and the almost spherical dark matter halo).

2. SOLUTION OF THE PROBLEM

Let us expand the integrand in (3) in powers of a small parameter – eccentricity e of the meridian section of the spheroid. We introduce the eccentricity of the layers in the form $a_3^2 = a_1^2(1 - e^2)$. We can show, omitting simple but unwieldy calculations, that, up to a term proportional to e^2, the decomposition takes the form

$$\varphi(x) = \pi G \frac{\rho_0}{n+1} a_1^2 a_3 \left\{ I_0 + e^2 \left[\frac{a_1^2}{2} I_2 - (n+1) a_1^2 x_3^2 I_1 \right] \right\}, \tag{4}$$

where we introduce the following auxiliary integrals:

$$I_0 = \int_0^\infty \left(1 - \frac{r^2 + x_3^2}{a_1^2 + u}\right)^{n+1} \frac{du}{(a_1^2 + u)^{\frac{3}{2}}}; \tag{5}$$

$$I_1 = \int_0^\infty \left(1 - \frac{r^2 + x_3^2}{a_1^2 + u}\right)^{n} \frac{du}{(a_1^2 + u)^{\frac{7}{2}}}; \tag{6}$$

$$I_2 = \int_0^\infty \left(1 - \frac{r^2 + x_3^2}{a_1^2 + u}\right)^{n+1} \frac{du}{(a_1^2 + u)^{\frac{5}{2}}}. \tag{7}$$

The first term I_0 in equation (4) is, up to a factor of $\pi G(\rho_0/(n+1))a_1^2 a_3$, equal to the potential inside an inhomogeneous sphere.

We then use the substitution,

$$y = \frac{r^2 + x_3^2}{a_1^2 + u},$$

to write the auxiliary integrals (5–7) in the following form:

$$I_0 = \frac{1}{\sqrt{r^2 + x_3^2}} \int_0^\infty (1 - y)^{n+1} \frac{dy}{\sqrt{y}}, \tag{8}$$

$$I_1 = \frac{1}{(r^2 + x_3^2)^{\frac{5}{2}}} \int_0^\infty (1-y)^n y^{\frac{3}{2}} dy, \tag{9}$$

$$I_2 = \frac{1}{(r^2 + x_3^2)^{\frac{3}{2}}} \int_0^\infty (1-y)^{n+1} \sqrt{y} dy. \tag{10}$$

The goal is accomplished, because the integrals in equations (8-10) can be calculated in the closed form:

$$I_0 = \frac{2}{a_1} {}_2F_1 \left(\left[\frac{1}{2}, -n-1 \right], \frac{3}{2}, z \right), \tag{11}$$

$$I_1 = \frac{2}{5a_1^5} {}_2F_1 \left(\left[\frac{5}{2}, -n \right], \frac{7}{2}, z \right), \tag{12}$$

$$I_2 = \frac{2}{3a_1^3} {}_2F_1 \left(\left[\frac{3}{2}, -n-1 \right], \frac{5}{2}, z \right), \tag{13}$$

where

$$z = \frac{r^2 + x_3^2}{a_1^2} \tag{14}$$

is the normalized squared radius-vector of the sampled points, and ${}_2F_1\left([a,b], c, z\right)$ is the standard Gauss hypergeometric function.

As a result, the desired internal potential of the spheroid takes the form

$$\varphi(x) = 2\pi G\rho_0 \frac{a_1 a_3}{n+1} \Bigg\{ {}_2F_1 \left(\left[\frac{1}{2}, -n-1 \right], \frac{3}{2}, z \right) + e^2 \left[\frac{1}{6} {}_2F_1 \left(\left[\frac{3}{2}, -n-1 \right], \frac{5}{2}, z \right) - \frac{n+1}{5}(1-e^2)\frac{x_3^2}{a_3^2} {}_2F_1 \left(\left[\frac{5}{2}, -n \right], \frac{7}{2}, z \right) \right] \Bigg\}. \tag{15}$$

The first term in (15),

$$\varphi(x) = 2\pi G\rho_0 \frac{R^2}{n+1} {}_2F_1 \left(\left[\frac{1}{2}, -n-1 \right], \frac{3}{2}, \frac{r^2}{R^2} \right), \tag{16}$$

is of special interest because it represents an analytical potential of inhomogeneous sphere of radius R for a wide range of density distributions (Kondratyev 2007):

$$\rho = \rho_0 \left(1 - \frac{r^2}{R^2} \right)^n. \tag{17}$$

The other two terms in formula (15) take into account the ellipticity of the layers of the spheroid with an accuracy up to the terms of order e^2 inclusive.

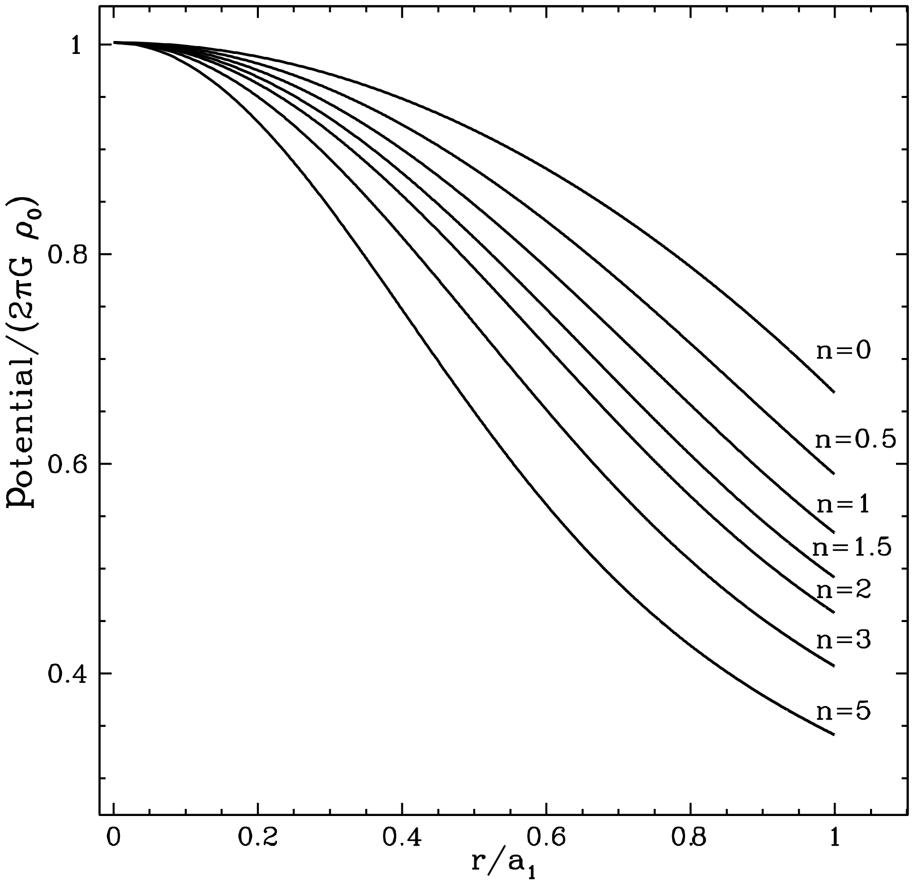

Fig. 1. Internal potential in the equatorial plane of a spheroid. The seven curves from top to bottom represent the potentials computed for different n in the ascending order this parameter: $n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3, 5$.

To test formula (15), we calculated the distribution of potential in the equatorial plane of the spheroid with eccentricity $e = 0.1$ for exponents $n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3, 5$. We show the results of our calculations in Fig. 1.

3. CONCLUSIONS

Potentials of gravitating systems have been a popular topic among the researchers. Note, for example, the studies by Kuzmin (1956), Miymoto & Nagai (1975), and Kutuzov & Osipkov (1980) who carefully analyzed the potentials of different models of galaxies. However, the results of the above investigations do not cover our work, whose results can also be used in the study of planets, satellites and dark-matter galactic halo.

In practice, it is very important to know the potential of the body in analytical form. Calculations show that formula (15) works perfectly both for integer ($n = 0, 1, 2, 3 \ldots$) and fractional exponents n. Furthermore, analytical form of the potential is also convenient to use in theoretical studies. Solving some problems also requires potentials of inhomogeneous ellipsoids with variably oblate layers to be known. For this purpose a method developed earlier (Kondratyev 1982) can be used.

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