Research Article

Min Jiang* and Qingtian Ma

The dynamical environment of the primary in the triple asteroid (45) Eugenia

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Abstract: We investigated the dynamical behavior in the potential of the primary in the triple asteroid (45) Eugenia with the calculation of the full gravitational potential caused by its 3D irregular shape. We presented the whole structure of the gravitational potential and the effective potential of (45) Eugenia in the coordinate planes, and showed the surface height, surface gravitational force accelerations, and the surface effective potential. The surface gravitational environment has been discussed. The zero-velocity curves and the position of external equilibrium points are calculated and showed relative to the 3D shape of the asteroid to help compare the relationship of the characteristic of the gravitational potential and the shape of the asteroid. There are five equilibrium points in the gravitational potential (45) Eugenia. We presented the positions, eigenvalues, topological cases, and stability of these equilibrium points. To analyze the variety of the orbital parameters close to (45) Eugenia, we computed two different orbits and compared the results. The mechanical energy, the semi-major axis, and the eccentricity have two different periods: the long period and the short period. The inclination have three different periods, an intermediate period is occurred. The longitude of the ascending node and the argument of periapsis not only have two periodic terms, but also have a secular term.

Keywords: triple asteroidal system; (45) Eugenia; zero-velocity curves; orbits

1 Introduction

The study of dynamics of binary and triple asteroidal systems is helpful for us to understand the formation and evolution of minor celestial bodies and the early stage of the solar system. Recently, asteroid Florence is found to be a triple system and has two tiny moonlets with the orbit period 8h for the inner one and 22 h to 27 h for the outer one (Irizarry 2017). There are totally 13 triple asteroidal systems found till now, including (45) Eugenia (Merline *et al.* 1999; Beauvalet and Marchis 2014; Jiang *et al.* 2016a), (87) Sylvia (Beauvalet and Marchis 2014; Jiang *et al.* 2016a), (93) Minerva (Marchis *et al.* 2013), (107) Camilla (Hargrove *et al.* 2012), (130) Elektra (Yang *et al.* 2016), (216) Kleopatra (Descamps *et al.* 2011; Chanut *et al.* 2015), (2577) Litva (Lust and Britt 2014), (3749) Balam (Vokrouhlicky 2009), (6186) Zenon (Elst 2007), (47171) 1999 TC36 (Benecchi *et al.* 2012), (136617) 1994CC (Fang *et al.* 2011), (153591) 2001 SN263 (Fang *et al.* 2011; Jiang *et al.* 2017), and (3122) Florence (Irizarry 2017). Besides, the triple system (136108) Haumea (Dumas *et al.* 2011) and the sextuple system (134340) Pluto (Brozović *et*

al. 2015) are the dwarf planet systems, also the multiple trans-Neptunian objects in the solar system. The formation and evolution mechanisms of the triple asteroidal systems and the multiple trans-Neptunian objects (Beauvalet and Marchis 2014), the generation of triple craters or craters chains on the surface of planets and minor celestial bodies (Melosh and Whitaker 1994; Auger *et al.* 2015; Richardson and Hagen 2015; Vincent *et al.* 2015), as well as the acceleration and rotational breakup of asteroids (Kaasalainen *et al.* 2007; Walsh *et al.* 2008; Marchis *et al.* 2010; Araujo *et al.* 2017) make the dynamics around triple asteroidal systems attract great interest. Simplified model can be used to simulate some different kind special binary/triple asteroidal systems, such as synchronous binary asteroid (Davis and Scheeres 2020) and large-size-ratio binary/triple asteroidal systems (Wang and Xu 2018). The study of dynamical environment of binary/triple asteroidal systems can help human to design the mission trajectories around the systems (Jiang 2018; Guzzetti and Baoyin 2019; Guzzetti *et al.* 2019).

Most of these triple asteroidal systems are large-size-ratio triple systems, such as (45) Eugenia, (87) Sylvia, (216) Kleopatra, etc. (45) Eugenia is a main-belt triple asteroidal system with two moonlets Petit-Prince and Princesse. The diameter of the primary Eugenia is $304 \times 220 \times 146$ km, and

Corresponding Author: Min Jiang: Xi'an Satellite Control Center, Xi'an 710043, China; Email: mingmayi@163.com

Qingtian Ma: Xi'an Satellite Control Center, Xi'an 710043, China

the diameter of Petit-Prince and Princesse are 7km and 6km, respectively (Beauvalet and Marchis 2014). Previous literature has studied the variety of the orbital elements of the two moonlets Petit-Prince and Princesse relative to the primary Eugenia, and discussed the motion stability in this systems (Beauvalet and Marchis 2014; Jiang *et al.* 2016a; Marchis *et al.* 2010; Araujo *et al.* 2017).

Marchis *et al.* (2010) presented the dynamical solution of the triple asteroidal system (45) Eugenia, including the orbital elements of the two moonlets Petit-Prince and Princesse in the EME2000 (Earth Mean Equator and Equinox of Epoch J2000), the pole solution of Eugenia in EME2000, as well as the gravitational coefficient J2 of Eugenia. Beauvalet and Marchis (Beauvalet and Marchis 2014) considered the spheroidal gravity and the J2 perturbation of the primary and discussed the inner structure of the primary in the triple systems (45) Eugenia and (87) Sylvia, the result is compatible with Sylvia, but fails to explain the higher density for the core of Eugenia. They guess that the Eugenia has a dense and spherical core with a less dense layer. Jiang *et al.* (Jiang *et al.* 2016a) used the discrete element model to fill in the polyhedron model of Eugenia, and considered the full gravitational potential of the 3D shape of Eugenia and the gravitational potential two moonlets with the sphere model, then calculated the dynamics behaviors of this triple system. The polyhedron model for Eugenia include 1022 vertices and 2040 faces, and the discrete element model include 6614 spheres with the radius 3967.2 m. The variety of the semi-major axis, eccentricity, inclination, and the longitude of the ascending node for the two moonlets have been presented. Araujo *et al.* (Araujo *et al.* 2017) used the spheroidal gravity and the J2 perturbation of the primary as well as the spheroidal gravity of the two moonlets to simulate the dynamics of this triple, and simulated a massless particle orbiting in the system. They presented the stable and unstable regions for the particle, and found that there exist larger stable regions of the prograde and retrograde orbits for particles, and suggested it is because the system is a large-mass-ratio system, and the moonlets are not so close to the primary.

However, the dynamical environment close to the primary in the triple asteroidal system (45) Eugenia is still unknown, including the zero-velocity curves and the relative equilibrium points. The variety of orbits close to the primary which is mainly affected by the full gravitational potential of the (45) Eugenia caused by the 3D irregular shape also needs to be studied. We propose in this work to investigate the dynamics in the gravitational potential of the primary in the triple asteroid (45) Eugenia. The calculation of gravitational potential and effective potential are presented in Section 2. We calculated the full gravita-

tional potential, effective potential, zero-velocity curves, and equilibrium points caused by this rotating body's 3D irregular shape using the polyhedron model method with 1022 vertices and 2040 faces. In Section 3, we show the dynamics evolution of orbits close to (45) Eugenia. A brief discussion is presented in Section 4.

1.1 Gravitational Potential and Effective Potential

For an asteroid with the uniform density, the gravitational potential U can be computed by the polyhedron model (Werner and Scheeres 1997):

$$U = -\frac{1}{2}G\sigma \sum_{e \in \text{edges}} \mathbf{r}_e \cdot \mathbf{E}_e \cdot \mathbf{r}_e \cdot L_e + \frac{1}{2}G\sigma \sum_{f \in \text{faces}} \mathbf{r}_f \cdot \mathbf{F}_f \cdot \mathbf{r}_f \cdot \omega_f. \quad (1)$$

In addition, the gravitational force acceleration ∇U of the asteroid and its gravity gradient matrix $\nabla(\nabla U)$ can be computed by

$$\nabla U = G\sigma \sum_{e \in \text{edges}} \mathbf{E}_e \cdot \mathbf{r}_e \cdot L_e - G\sigma \sum_{f \in \text{faces}} \mathbf{F}_f \cdot \mathbf{r}_f \cdot \omega_f \quad (2)$$

and

$$\nabla(\nabla U) = -G\sigma \sum_{e \in \text{edges}} \mathbf{E}_e \cdot L_e + G\sigma \sum_{f \in \text{faces}} \mathbf{F}_f \cdot \omega_f \quad (3)$$

Here all the vectors are in the body-fixed frame of the asteroid. $\mathbf{r} = (x, y, z)$ represents the particle's position vector, $G=6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ represents Newtonian gravitational constant, σ means the density of the asteroid body, the subscripts e and f means the parameters of edge e and face f of the polyhedron model, respectively. $\mathbf{r}_e$ and $\mathbf{r}_f$ are vectors from a field point, $\mathbf{r}_e$ points to the edge e , and $\mathbf{r}_f$ points to the face f . $\mathbf{E}_e$ is a dyad calculated using two face- and edge-normal vectors, while $\mathbf{F}_f$ is a dyad calculated using the normal vector of the surface of the polyhedron. L_e represents an integrate factor related to the general edge e , and ω_f represents a signed angle of the planar region.

We assume asteroid 45 Eugenia has the uniform density and rotates uniformly. The gravitational potential around asteroid 45 Eugenia has been presented in Figure 1. In Figure 1(a), we plotted the 3D view of the gravitational potential in the xy, yz, and zx planes, respectively. From Figure 1(a), one can see the whole structure of the gravitational potential caused by the irregular shape of asteroid 45 Eugenia. To compare with the structure of the gravitational potential and the shape of 45 Eugenia, we presented Figure 1(b), 1(c), and 1(d). One can see that the gravitational

Figure 1. Gravitational potential around asteroid 45 Eugenia. (a) 3D plot of the gravitational potential, the unit of the gravitational potential is m^2s^{-2} ; (b) The gravitational potential in the equatorial plane of the body-fixed frame; (c) The gravitational potential in the yz plane of the body-fixed frame; (d) The gravitational potential in the zx plane of the body-fixed frame.

potential is related to the irregular shape of 45 Eugenia. Thus if we want to study the dynamical environment close to 45 Eugenia, the full gravitational potential of 45 Eugenia cannot be neglected.

The surface parameters of 45 Eugenia are also calculated and showed in Figure 2. Figure 2(a) gives the surface height. From Figure 2(a), one can see that the maximum altitude region is near the equatorial plane of 45 Eugenia. There are two obvious local maximum altitude regions. The surface gravitational force accelerations have been illustrated in Figure 2(b). From Figure 2(b), one can see that the surface gravitational force accelerations on the North Pole and the South Pole are larger than on the regions near the equatorial plane. The distribution of the surface gravitational force accelerations is related to the surface height, but the structure of the contour plot of surface heights and the structure of the contour plot of surface gravitational force accelerations have some difference. In Figure 2(b),

we don't include the Coriolis force; only the surface gravitational force is calculated. Figure 2(c) shows the surface effective potential of 45 Eugenia. The local extreme point of the surface effective potential is the surface equilibrium point. There are more than 20 surface equilibria for the motion of a smooth grain on the surface.

We consider a particle orbiting around a rotating asteroid. Let $\boldsymbol{\omega}$ be the rotational angular velocity of the asteroid relative to inertial space, $\mathbf{r}$ be the radius vector of the particle relative to the asteroid's mass centre, the time derivatives of $\mathbf{r}$ are also relative to the body-fixed coordinate system. Then the velocity of the particle relative to the inertial space is

$$\mathbf{v}_I = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}. \quad (4)$$

Here $\mathbf{v}_I$ represents the velocity of the particle relative to the inertial space, $\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$ represents the velocity of the particle relative to the rotating asteroid. For a particle with the mass

Figure 2. Contour plot of the surface parameters of 45 Eugenia. (a) Surface height on the surface; (b) Gravitational force on the surface; (c) Effective potential on the surface.

m , the Lagrange function is

$$L = \frac{1}{2} m \mathbf{v}_1 \cdot \mathbf{v}_1 - U = \frac{1}{2} m (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \cdot (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) - U. \quad (5)$$

Thus

$$\begin{cases} \frac{\partial L}{\partial \dot{\mathbf{r}}} = m (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \\ \frac{\partial L}{\partial \mathbf{r}} = m [(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \times \boldsymbol{\omega}] - \nabla U \end{cases} \quad (6)$$

Using the Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = 0, \quad (7)$$

One can get the following motion equation of the particle

$$m [\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}] + \frac{\partial U(\mathbf{r})}{\partial \mathbf{r}} = 0. \quad (8)$$

Eq. (8) is expressed in the body-fixed frame of the asteroid. To understand Eq. (8), one can think for the motion of the

particle relative to the rotating asteroid, there are three fictitious forces acting on the particle, including the rotational inertia force $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$, the Coriolis force $-2m\boldsymbol{\omega} \times \dot{\mathbf{r}}$, and the centrifugal force $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. If the asteroid rotates uniformly, the rotational inertia force is zero, then Eq. (8) is simplified to

$$m [\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] + \frac{\partial U(\mathbf{r})}{\partial \mathbf{r}} = 0. \quad (9)$$

Using $\dot{\mathbf{r}}$ to make the dot product with Eq. (9), one will get

$$\frac{dH}{dt} = 0. \quad (10)$$

Here

$$H = \frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - \frac{1}{2} (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) + U(\mathbf{r}). \quad (11)$$

H is the relative energy of the particle, H is also called the totally energy or the Jacobi integral (Scheeres *et al.* 1996;

Figure 3. Effective potential of the asteroid 45 Eugenia. (a) 3D plot of the effective potential in the three coordinate planes of the body-fixed frame; (b) the effective potential in the equatorial plane and the 3D shape of 45 Eugenia; (c) the effective potential in the yz plane and the 3D shape of 45 Eugenia; (d) the effective potential in the zx plane and the 3D shape of 45 Eugenia.

Jiang and Liu 2019; Jiang *et al.* 2016b,c; Zeng *et al.* 2016; Yang *et al.* 2018). The mass of the particle is too small relative to the mass of the asteroid, thus we assume it is a massless particle.

Now we consider the energy

$$E = \frac{1}{2} m \mathbf{v}_I \cdot \mathbf{v}_I + U(\mathbf{r}), \quad (12)$$

Substituting Eq. (4) into Eq. (12) gives

$$E = \frac{1}{2} m (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \cdot (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) + U(\mathbf{r}) = H + \boldsymbol{\omega} \cdot \mathbf{L}, \quad (13)$$

where $\mathbf{L}$ represents the moment of momentum for the particle, and can be calculated by

$$\mathbf{L} = \mathbf{r} \times \mathbf{v}_I = \mathbf{r} \times (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}). \quad (14)$$

For an asteroid which rotates uniformly, the energy E and the moment of momentum $\mathbf{L}$ are not conservative, but the

relative energy $H = E - \boldsymbol{\omega} \cdot \mathbf{L}$ is conservative. We now analysis the relative energy, if we denote the effective potential as

$$W(\mathbf{r}) = -\frac{1}{2} (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) + U(\mathbf{r}). \quad (15)$$

Then the relative energy can be expressed by

$$H = \frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + W(\mathbf{r}). \quad (16)$$

The effective potential of asteroid 45 Eugenia in the xy , yz , and zx planes has been illustrated in Figure 3. The shapes of the effective potential in the yz plane and in the zx plane look similar; both of them are saddle surfaces. However, the shapes of the effective potential in the yz plane and in the zx plane have different orientation. The shape of the effective potential in the xy plane is quite different from the other two shapes.

The motion equation (9) can be expressed by

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \frac{\partial W(\mathbf{r})}{\partial \mathbf{r}} = 0, \quad (17)$$

Figure 4. Zero-velocity curves and equilibrium points around asteroid 45 Eugenia, the unit of the values on the zero-velocity curves is m^2s^{-2} . (a) Zero-velocity curves and projections of equilibrium points E1-E5 in the equatorial plane; (b) Zero-velocity curves and projections of equilibrium points E2, E4, and E5 in the yz plane; (c) Zero-velocity curves and projections of equilibrium points E1, E3, and E5 in the xz plane.

Table 1. Positions and Jacobi integrals of the equilibrium points around 45 Eugenia.

Equilibrium Points	x (km)	y (km)	z (km)	Jacobi integral (m^2s^{-2})
E1	140.358	-4.35472	-1.96322	2448.83
E2	5.82722	127.883	-0.871844	2295.66
E3	-139.847	-10.9377	0.0935185	2446.53
E4	3.27782	-128.014	0.125190	2295.73
E5	-0.0234455	1.00172	0.192504	3313.26

Table 2. Eigenvalues of the equilibrium points.

Equilibrium Points ($\times 10^{-3}\text{s}^{-1}$)	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
E1	0.378481i	-0.378481i	0.326735i	-0.326735i	0.249848	-0.249848
E2	0.334693i	-0.334693i	0.111246+	-0.111246+	0.111246-	-0.111246-
			0.223955i	0.223955i	0.223955i	0.223955i
E3	0.376862i	-0.376862i	0.328175i	-0.328175i	0.249287	-0.249287
E4	0.340158i	-0.340158i	0.128524+	-0.128524+	0.128524-	-0.128524-
			0.229029i	0.229029i	0.229029i	0.229029i
E5	0.799463i	-0.799463i	0.662875i	-0.662875i	0.177825i	-0.177825i

In Eq. (16), let $\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = 0$, then the particle is static relative to the uniformly rotating asteroid. Thus we get the zero velocity curves:

$$W(\mathbf{r}) = H. \quad (18)$$

In the body-fixed frame, the forbidden region of the particle can be calculated by $W(\mathbf{r}) > H$, and the allowed region can be calculated by $W(\mathbf{r}) < H$. The points on the curves $W(\mathbf{r}) = H$ mean that the particle is static in the body-fixed frame. The equilibrium points close to the asteroid satisfy

$$\frac{\partial W(\mathbf{r})}{\partial \mathbf{r}} = 0, \quad (19)$$

The motion of a massless particle opposite the equilibrium point can be linearised using Eq. (17) and written as

$$\frac{d^2}{dt^2} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} + \begin{pmatrix} 0 & -2\omega_z & 2\omega_y \\ 2\omega_z & 0 & -2\omega_x \\ -2\omega_y & 2\omega_x & 0 \end{pmatrix} \cdot \frac{d}{dt} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} + \begin{pmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{xy} & W_{yy} & W_{yz} \\ W_{xz} & W_{yz} & W_{zz} \end{pmatrix} \cdot \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = 0 \quad (20)$$

where $\boldsymbol{\omega} = [\omega_x, \omega_y, \omega_z]^T$, $W_{pq} = \left(\frac{\partial^2 W}{\partial p \partial q} \right)_E$ ($p, q = x, y, z$), $\delta \mathbf{r} = [\xi \quad \eta \quad \zeta]^T = \mathbf{r}_E - \mathbf{r}$ represents the position vector opposite the equilibrium point for the particle, $\mathbf{r}_E$ represents the vector from the mass centre of the asteroid to the equilibrium point. Thus the characteristic equation of the

equilibrium point satisfies

$$\begin{vmatrix} \lambda^2 + W_{xx} & -2\omega_z\lambda + W_{xy} & 2\omega_y\lambda + W_{xz} \\ 2\omega_z\lambda + W_{xy} & \lambda^2 + W_{yy} & -2\omega_x\lambda + W_{yz} \\ -2\omega_y\lambda + W_{xz} & 2\omega_x\lambda + W_{yz} & \lambda^2 + W_{zz} \end{vmatrix} = 0. \quad (21)$$

Here λ represents the eigenvalues of the equilibrium point. There are several topological cases for the equilibrium point (Jiang *et al.* 2016c). The most common cases are ordinary case O1, O2, O3 and O4. The distributions of the eigenvalues for these cases are: Case O1: $\pm i\beta_j$ ($\beta_j \in \mathbb{R}, \beta_j > 0; j = 1, 2, 3 \mid \forall k \neq j, k = 1, 2, 3, \text{ s.t. } \beta_k \neq \beta_j$); Case O2: $\pm \alpha_j$ ($\alpha_j \in \mathbb{R}, \alpha_j > 0, j = 1$) and $\pm i\beta_j$ ($\beta_j \in \mathbb{R}, \beta_j > 0; j = 1, 2 \mid \beta_1 \neq \beta_2$); Case O3: $\pm \alpha_j$ ($\alpha_j \in \mathbb{R}, \alpha_j > 0; j = 1, 2 \mid \alpha_1 \neq \alpha_2$) and $\pm i\beta_j$ ($\beta_j \in \mathbb{R}, \beta_j > 0, j = 1$); Case O4: $\pm i\beta_j$ ($\beta_j \in \mathbb{R}, \beta_j > 0, j = 1$) and $\pm \sigma \pm i\tau$ ($\sigma, \tau \in \mathbb{R}; \sigma, \tau > 0$). If an equilibrium point has eigenvalues on the right half plane, the equilibrium point is unstable. Thus an equilibrium point corresponding to Case O1 is linearly stable, and an equilibrium point corresponding to Case O2, Case O3, or Case O4 is unstable.

The 3D views of the position relation of the 3D shape, the zero-velocity curves, and outside equilibrium points around asteroid 45 Eugenia are shown in Figure 4. Positions and Jacobi integrals of the equilibrium points around 45 Eugenia are presented in Table 1. There are five equilibrium points close to asteroid 45 Eugenia. For a particle located in the equilibrium points, the resultant external force acting on the particle relative to the body of the asteroid is zero.

From Table 2, one can see the distribution of eigenvalues of the equilibrium points. Table 3 gives topological property and stability of these five equilibrium points. The

Table 3. The topological property and stability of the equilibrium points. LS: linearly stable; U: unstable; P: positive definite; N: non-positive definite.

Equilibrium Points	Topological Case	Stability	$\nabla^2 V$
E1	O2	U	N
E2	O4	U	N
E3	O2	U	N
E4	O4	U	N
E5	O1	LS	P

distributions of eigenvalues of the equilibrium points E1 and E3 are the same, and the distributions of eigenvalues of the equilibrium points E2 and E4 are the same. The topological cases of external equilibrium points E1, E2, E3, and E4 are Case O2, Case O4, Case O2, Case O4, respectively. Thus the external equilibrium points with different topological cases have a staggered distribution. There are two external equilibrium points belong to Case O2, which are E1 and E3, and they are at the opposite positions. Additionally, there are two external equilibrium points belong to Case O4, which are E2 and E4, and they are also at the opposite positions. The number of the external equilibrium points which belong to Case O2 is 2, and the number of the external equilibrium points which belong to Case O4 also equals 2. There is only one internal equilibrium point, denoted as E5. E5 belongs to Case O5, and is linearly stable. All the external equilibrium points are unstable.

1.2 Dynamics Evolution Close to 45 Eugenia

Here we calculate two different orbits close to asteroid 45 Eugenia to analyze the orbital stability around 45 Eugenia. The full gravitational force acceleration of 45 Eugenia generated by the 3D irregular shape of the body is considered. The initial position and velocity of the first orbit for a massless particle in the body-fixed coordinate system of the principal axis of inertia of 45 Eugenia is $[554590.0, 0.0, 0.0]$ m and $[-0.00166136, -151.033, -6.27843]$ m·s⁻¹. Figure 5 shows the orbital parameters of this orbit in the gravitational potential of 45 Eugenia. The 3D plot of the orbit relative to the body-fixed coordinate system of the principal axis of inertia has significant amplitude. However, the 3D plot of the orbit relative to the inertia coordinate system has minor amplitude. The mechanical energy for the orbit varies periodically, and has two different periods, one period is long, and the other one is short. The relative energy of the orbit changes from -351.47199130812 J·kg⁻¹ to -3351.47199127350 J·kg⁻¹. The first 10 significant digits are the same, the last 5 numbers are different because of the existence of inte-

gral error. Thus the Jacobi integral is a conserved quantity, which is the same as the theoretical analysis results. Each long period equals about 19 short periods. The semi-major axis also varies periodically, and has two different periods. The time for the long period of the semi-major axis equals the time for the long period of the mechanical energy, and the time for the short period of the semi-major axis equals the time for the short period of the mechanical energy. The qualitative conclusion for the eccentricity is similar with it for the semi-major axis. The eccentricity also has two periods, one has the same time length with the long period of the semi-major axis, and the other one has the same time length with the short period of the semi-major axis.

For the inclination, the qualitative conclusion is different. There are three different periods for the inclination. The long period and the intermediate period of the inclination look obviously, but the short period of the inclination is inconspicuous. The time length for the long period of the inclination is the same as the time length for the long period of the mechanical energy, the semi-major axis, and the eccentricity. The intermediate period equals a half of the long period.

The longitude of the ascending node in the inertia coordinate system has a secular term with two small periodic terms. In general, the longitude of the ascending node gradually decreases. The argument of periapsis has a secular term and two periodic terms. The time length for the long period of the argument of periapsis is the same as the time length for the long period of the mechanical energy, the semi-major axis, the eccentricity, and the inclination. And the time length for the short period of the argument of periapsis is the same as the time length for the short period of the mechanical energy, the semi-major axis, the eccentricity, and the inclination. However, the argument of periapsis gradually increases in general. The semi-major axis, eccentricity, and argument of periapsis determine the shape of orbit in the orbit plane, they have the same period of oscillations. The inclination and the longitude of the ascending node determine the normal direction of the orbit plane, they also have the same period of oscillations, but different from the period of oscillations of semi-major axis. If one define the eccentricity vector and the inclination vector as $e_x = e \cos(\omega)$, $e_y = e \sin(\omega)$ and $i_x = i \cos(\Omega)$, $i_y = i \sin(\Omega)$, respectively; they one can see that the secular terms of ω means the motion of the inclination vector, and the secular terms of Ω means the motion of the inclination vector.

To compare with the results of the above example of orbit, we calculate another orbit. The initial position and velocity of the second orbit for a massless particle in the body-fixed coordinate system of the principal axis of inertia

Figure 5. The first orbit in the gravitational potential of 45 Eugenia. (a) 3D plot of the orbit relative to the body-fixed coordinate system of principal axis of inertia; (b) 3D plot of the orbit relative to the inertia coordinate system; (c) Mechanical energy for the orbit; (d) Semi-major axis; (e) Eccentricity; (f) Inclination in the inertia coordinate system (xy plane of the inertia coordinate system is the same as the xy plane of the body-fixed coordinate system of principal axis of inertia, similarly hereinafter); (g) Longitude of the ascending node in the inertia coordinate system; (h) Argument of periapsis in the inertia coordinate system; (i) Eccentricity vector, $e_x = e \cos(\omega)$, $e_y = e \sin(\omega)$.

of 45 Eugenia is $[1154590.0, 0.0, 0.0]$ m and $[-0.00374632, -340.574, -3.40574]$ $\text{m}\cdot\text{s}^{-1}$. Figure 6 shows the orbital parameters of the second orbit in the gravitational potential of 45 Eugenia. The amplitude of the orbit relative to the body-fixed coordinate system look much larger than the amplitude of the orbit relative to the inertia coordinate system.

The mechanical energy, the semi-major axis, and the eccentricity also have two different periods. The time length for the long period of the mechanical energy, the semi-major axis, as well as the eccentricity are the same, and the time length for the short period of the mechanical energy, the semi-major axis, as well as the eccentricity are also the

same. Each long period equals about 58 short periods. The Jacobi integral of the orbit changes from -4683.02953738718 $\text{J}\cdot\text{kg}^{-1}$ to -4683.02953726622 $\text{J}\cdot\text{kg}^{-1}$.

The inclination has three different periods, i.e. the long period, the intermediate period, and the short period. All the three different periods look obviously. Both of the longitude of the ascending node and the argument of periapsis have a secular term and two small periodic terms. The longitude of the ascending node gradually decreases, while the argument of periapsis gradually increases. The small periodic terms for the longitude of the ascending node and the argument of periapsis include a long period term and a short period term. The time length for the long period

Figure 6. The second orbit in the gravitational potential of 45 Eugenia. (a) 3D plot of the orbit relative to the body-fixed coordinate system of principal axis of inertia; (b) 3D plot of the orbit relative to the inertia coordinate system; (c) Mechanical energy for the orbit; (d) Semi-major axis; (e) Eccentricity; (f) Inclination in the inertia coordinate system; (g) Longitude of the ascending node in the inertia coordinate system; (h) Argument of periapsis in the inertia coordinate system; (i) Eccentricity vector, $e_x = e \cos(\omega)$, $e_y = e \sin(\omega)$.

of the inclination, the longitude of the ascending node, as well as the argument of periapsis are the same as the time length for the long period of the mechanical energy, the semi-major axis, as well as the eccentricity. And the time length for the short period of the inclination, the longitude of the ascending node, the argument of periapsis, the mechanical energy, the semi-major axis, and the eccentricity are also the same. Only the inclination has an intermediate term, the time length of the long period of the inclination is twice as the time length of its intermediate period.

Compare the second orbit with the first orbit, one can find some common points of the results. The mechanical energy, the semi-major axis, the eccentricity, the longitude of the ascending node, and the argument of periapsis have

two different periods, one is a long period, and the other one is a short period. The inclination has three different periods, *i.e.* long period, intermediate period, and short period. The time lengths for all the long periods of these parameters are the same, and the time lengths for all the short periods of these parameters are also the same. For the inclination, the intermediate period is a half of the long period. The mechanical energy, the semi-major axis, the eccentricity, and the inclination have no secular terms. Both of the longitude of the ascending node and the argument of periapsis have secular terms. The secular term makes the longitude of the ascending node decreases and the argument of periapsis gradually increases. The variation results of the orbital parameters are derived based on the given

Figure 7. Two periodic orbits around the surface of 45 Eugenia.

Table 4. Initial conditions of periodic orbits, here the length unit 232.409 km is the maxsize of 45 Eugenia, and the time unit 5.699 h is the rotational period of 45 Eugenia.

	Periodic orbit in Figure 7(a)	Periodic orbit in Figure 7(b)
x	0.419040171412E+00	0.559587058533E+00
y	0.507981897167E+00	0.532537227772E+00
z	0.175030845129E+00	0.132879128942E+00
vx	0.466741576981E+01	0.333571913257E+01
vy	-0.434218126020E+01	-0.469382489299E+01
vz	0.269943120511E+01	0.287826155596E+01

Table 5. Characteristic multipliers of periodic orbits.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
Periodic orbit in Figure 7(a)	1	1	1.053303	0.949393	0.1290830+ 0.9916340i	0.129083- 0.9916340i
Periodic orbit in Figure 7(b)	1	1	0.9543243+ 0.2987730i	0.9543243- 0.2987730i	0.8640753+ 0.5033626i	0.8640753- 0.5033626i

time-scale evolution, these results still hold for the longer time-scale evolution of orbits because of the conservation of Jacobi integral as well as the periodical variety of the mechanical energy. For the motion of dust grains around (45) Eugenia, the Solar radiation should also be considered, then the Jacobi integral is no longer the conserved quantity, and the results would not hold for the longer time-scale evolution of orbits, which is not the scope of this article.

The above two orbits are not periodic orbits. We now consider the stability of periodic orbits near the surface of 45 Eugenia. The periodic orbits around asteroids can be calculated by the global search method with hierarchical parametrization (Yu and Baoyin 2012; Shang *et al.* 2017). The stability of periodic orbits can be distinguished by characteristic multipliers. For a periodic orbit, there are 6 characteristic multipliers, two of them equal 1, and the absolute value of the other four characteristic multipliers determines the stability of the periodic orbit. If all the other four charac-

teristic multipliers λ satisfy $|\lambda| = 1$, then the periodic orbit is linearly stable; otherwise the periodic orbit is unstable. For a small error of the unstable periodic orbit, the orbit will apart from the periodic motion. Figure 7 shows two periodic orbits near the surface of 45 Eugenia. Initial conditions and characteristic multipliers of these two periodic orbits are presented in Table 4 and Table 5, respectively. For the periodic orbit in Figure 7(a), $|\lambda_3| > 1$ and $|\lambda_4| > 1$, thus the periodic orbit in Figure 7(a) is unstable. For the periodic orbit in Figure 7(b), all the characteristic multipliers satisfy $|\lambda| = 1$, thus the periodic orbit in Figure 7(b) is stable.

2 Conclusions

This paper presents a complementary combination of dynamical behavior in the potential of the primary in the triple asteroid (45) Eugenia. We calculated the gravitational po-

tential around asteroid 45 Eugenia and showed the whole structure of the gravitational potential. The gravitational potential is related to the irregular shape of 45 Eugenia. The surface height, surface gravitational force accelerations, as well as the surface effective potential have been presented. There are two obviously local maximum altitude regions on the surface of 45 Eugenia. It is found that the surface gravitational force accelerations on the North Pole and the South Pole are significant larger than them on the regions near the equatorial plane. The structure of the contour plot of surface heights and the structure of the contour plot of surface gravitational force accelerations look similar, but also have some differences. It is found that there are more than 20 surface equilibria for the motion of a smooth grain on the surface of 45 Eugenia.

The 3D view of the effective potential in the yz plane and zx plane look to be saddle surfaces. We presented the zero-velocity curves and the position relation of the external equilibrium points relative to the 3D shape of the asteroid body. It is found that there are totally five equilibrium points in the potential of 45 Eugenia. The external equilibrium points, denoted as E1-E4, are all unstable; and the external equilibrium points with different topological cases have a staggered distribution.

We simulated two different orbits around asteroid 45 Eugenia with considering the full gravitational potential caused by its 3D complex shape. We use the mechanical energy and the orbital elements to analyze the variety of the orbits close to 45 Eugenia. The results reveal that the mechanical energy, the semi-major axis, and the eccentricity vary periodically, and have two different periods: the long period and the short period. The inclination also varies periodically, and has three different periods, the long one, the intermediate one, and the short one. Both of the longitude of the ascending node and the argument of periapsis have a secular term, a long periodic term, and a short periodic term. For all these parameters, the time lengths for all the long periodic term are the same, and the time lengths for all the short periodic term are the same. The intermediate periodic term of the inclination has a half time length of the long periodic term. Two periodic orbits near the surface of (45) Eugenia are calculated. The value of characteristic multipliers shows that one is unstable while the other one is stable.

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