Contributions

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The Taylor principle is valid under wage stickiness

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Abstract: We consider the textbook neo-Keynesian model with staggered prices and wages in discrete time. We prove analytically that the Taylor principle holds in this case. When both contracts exhibit sluggish adjustment to market conditions, the policy maker faces a trade-off between stabilizing three welfare relevant variables: output, price inflation and wage inflation. We consider a monetary policy rule designed accordingly: the central banker can react to both inflations and the output gap. In addition to generalizing the Taylor principle we show that the frontier of determinacy embeds the frontier derived with staggered prices only, generalizes the frontier of determinacy in the limit case of continuous time and is symmetric in price and wage inflations.

Keywords: dynamic stochastic general equilibrium model; monetary policy rule; sun spot equilibria; taylor principle.

JEL: C62; C68; E52; E58.

1 Introduction

In (Taylor 1993), John Taylor advocates the use of monetary policy rules where the central banker reacts to both price inflation and output as a benchmark to be used judgementslly.

In the simplest neo-Keynesian model with staggered prices only, this Taylor rule is key in ruling out sun-spot fluctuations. The Taylor principle associated to such rules states that the central banker should overreact to inflation to ensure the uniqueness of the solution under rational expectations.

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Extensions to models with both staggered prices and wages have emphasized the welfare optimization problem while assuming, backed on numerical simulations, that the Taylor principle still holds in these cases (Erceg, Henderson, and Levin 2000; Galí 2008).

Flaschel, Franke, and Proaño (2008) derive analytically the frontier of determinacy in the same model but on the limit case of continuous time using a simpler strategy. Although the authors expect their demonstration in continuous time to be informative of the discrete time case, based on the precept that such a property should not depend on the time length of the period, they acknowledge in their concluding remarks that their intuition is “*not unchallenged*”. In this paper we prove them right: the Taylor principle does hold in the discrete time version of the model and is a generalization of their result.

We consider the same model as (Galí 2008, chapter 6) or (Erceg, Henderson, and Levin 2000) and a monetary policy rule in line with Erceg et al.’s results: the central banker can react to both inflations and the output gap. With straightforward notations, the Taylor rule takes the following form:

$$i_t = \Phi_p \pi_t^p + \Phi_w \pi_t^w + \Phi_y y_t \quad (1)$$

We show that the necessary and sufficient condition to rule out sun-spot equilibria is symmetric in inflations:

$$\Phi_p + \Phi_w + \frac{1-\beta}{\tilde{\kappa}} \Phi_y > 1 \quad (2)$$

with $\tilde{\kappa}$ a coefficient depending symmetrically on both slopes of the prices and wages Phillips curves and β the discount factor. This frontier, taking the form of a condition on $\Phi_p + \Phi_w$ eased as Φ_y increases, is in line with numerical investigations (Galí 2008, chapter 6).

Though the symmetry may not appear straightforward, it is deeply rooted in the model. In the simple model with staggered prices only, the Phillips curve implies that stabilizing price inflation is equivalent to stabilizing the output gap, a result (Blanchard and Galí 2007) present as a *divine coincidence* because it allows the central banker to enforce the social optimum. Blanchard and Galí (2007) note a weaker form of the *divine coincidence* in Erceg et al.’s model: combining the two Phillips curves yields that stabilizing the output gap is equivalent to stabilizing a weighted average of price and wage inflation (with the weight on each inflation being the slope of the other Phillips curve). Similar symmetry arises when studying the optimal monetary policy (see the functional form of the welfare criterion derived by Galí and Erceg et al.) as well as in our demonstration.

Our result generalizes that of Flaschel, Franke, and Proaño (2008) in the limit case of continuous time.

It is also a direct generalization of the frontier derived in the simpler case of staggered prices only (Woodford 2001). Moreover it has an identical interpretation in the long run: the central banker should react more than one for one to permanent changes in inflation (Woodford 2011, chapter 4).

In the remainder of this paper, Section 2 recalls the model, Section 3 outlines the proof detailed in Appendix A and Section 4 concludes.

2 A monetary model with sticky wages and prices

We study the model exposed in (Galí 2008, chapter 6) and (Erceg, Henderson, and Levin 2000). This model extends the standard neo-Keynesian model for monetary policy analysis which consist of an *IS* curve relating the output gap to the expected real interest rate, a Phillips curve relating inflation, expected inflation and output gap and a monetary policy rule describing how the interest rate is set by the central banker. The present extension of the model considers wage rigidities under the form of Calvo contracts. It follows from this rigidity that real wages may deviate from their flexible equivalent due to exogenous disturbances.

The model takes the following linear form:

$$\pi_t^p = \beta E(\pi_{t+1}^p | t) + \kappa_p y_t + \lambda_p \omega_t \quad (3)$$

$$\pi_t^w = \beta E(\pi_{t+1}^w | t) + \kappa_w y_t - \lambda_w \omega_t \quad (4)$$

$$\omega_{t-1} = \omega_t - \pi_t^w + \pi_t^p + \Delta \omega_t^n \quad (5)$$

$$y_t = E(y_{t+1} | t) - \frac{1}{\sigma} (i_t - E(\pi_{t+1}^p | t) - r_t^n) \quad (6)$$

$$i_t = \Phi_p \pi_t^p + \Phi_w \pi_t^w + \Phi_y y_t + v_t \quad (7)$$

with π^p price inflation, π^w wage inflation, ω the real wage gap, y output and i the nominal interest rate. $\Delta \omega^n$ is an exogenous wage gap shock, r^n the natural real interest rate and v a monetary policy shock.

The complete derivation of the model is exposed in full details in (Galí 2008, chapter 6) with the same notations.

Denoting $x_t = [y_t, \pi_t^p, \pi_t^w, \omega_{t-1}]^T$, the endogenous variables, and $z_t = [r_t^n - v_t, \Delta \omega_t^n]^T$, the exogenous variables, the equations (3) to (7) can be written in the form:

$$x_t = A^{-1}(E(x_{t+1}|t) + Bz_t) \quad (8)$$

In equation (8), the matrix of interest A is:

$$A = \begin{bmatrix} 1 + \frac{\kappa_p + \Phi_y}{\sigma\beta} & \frac{\beta\Phi_p - 1 - \lambda_p}{\sigma\beta} & \frac{\beta\Phi_w + \lambda_p}{\sigma\beta} & \frac{\lambda_p}{\sigma\beta} \\ \frac{-\kappa_p}{\beta} & \frac{1 + \lambda_p}{\beta} & \frac{-\lambda_p}{\beta} & \frac{-\lambda_p}{\beta} \\ \frac{-\kappa_w}{\beta} & \frac{-\lambda_w}{\beta} & \frac{1 + \lambda_w}{\beta} & \frac{\lambda_w}{\beta} \\ 0 & -1 & 1 & 1 \end{bmatrix} \quad (9)$$

There are three forward looking variables in this model: $[y_t, \pi_t^p, \pi_t^w]$. According to (Blanchard and Kahn 1980), the system (8) has a unique solution if and only if the matrix A defined by (9) has three eigenvalues strictly larger than one in modulus and one eigenvalue smaller than one in modulus. In addition, this solution has no unit root if this last eigenvalue is strictly smaller than one in modulus.

3 Outline of the proof

Defining the frontier of indeterminacy is based on the study of the roots of the characteristic polynomial of matrix A , a fourth degree polynomial. Though it is not complex mathematics, it is rather cumbersome.¹

In Appendix A.1 we develop the proof in the case $\Phi_y = 0$. We use the intuition that in this case the frontier of determinacy is $\Phi_p + \Phi_w = 1$ and decompose the polynomial as a fourth degree polynomial corresponding to this case plus deviations from this case in both directions (Φ_p, Φ_w) . In the limit case $\Phi_p + \Phi_w = 1$, the model is at the limit of solution determinacy: 1 is a root of its characteristic polynomial; its real roots are non-negative and at most one real root is in $]0, 1[$; its complex roots have a modulus strictly greater than one. The deviation from this limit case ensures the uniqueness of the model's solution if and only if the deviation from $\Phi_p + \Phi_w = 1$ is positive: the root 1 moves in the direction ensuring the uniqueness of the model's solution (depending on the existence of another root smaller than one), the other roots are kept outside or inside the unit disk depending on their initial position.

¹ We are particularly grateful to Yvon Maday and other mathematicians at Laboratoire Jacques-Louis Lions for proofreading and comments.

In Appendix A.2 we show that the case $\Phi_y \neq 0$ can be treated identically to the case $\Phi_y = 0$. We consider the frontier of indeterminacy under the form $\Phi_p + \Phi_w = 1 - \theta$ and show that setting $\theta = \Phi_y \frac{(1-\beta)(\lambda_p + \lambda_w)}{\kappa_w \lambda_p + \kappa_p \lambda_w}$ allows a decomposition of the characteristic polynomial which has the same properties as in the case $\Phi_y = 0$. We can conclude that equation (10) generalizes the frontier of indeterminacy.

4 Conclusion

In the textbook neo-Keynesian model with both staggered prices and wages, we show that any monetary policy rule satisfying

$$\Phi_p + \Phi_w + \Phi_y \frac{(1-\beta)(\lambda_p + \lambda_w)}{\kappa_w \lambda_p + \kappa_p \lambda_w} > 1 \quad (10)$$

rules out sunspot fluctuations.² The admissibility of a policy rule symmetrically depends on wage inflation and price inflation: when the central bank does not respond to changes in output, the condition on the monetary policy parameters comes down to $\Phi_p + \Phi_w > 1$ in line with Galí's numerical investigations. Also in line with Galí's numerical investigations, when the central bank reacts to changes in output, doing so relaxes the constraint above, proportionally to Φ_y with a factor $\frac{(1-\beta)(\lambda_p + \lambda_w)}{\kappa_w \lambda_p + \kappa_p \lambda_w}$. This coefficient crucially and symmetrically depends on the Phillips curves of prices and wages: more impatient agents (smaller β) or flatter Phillips curves (smaller λ or κ), facilitate the task of the central banker to prevent sun spot fluctuations.

In continuous time, Flaschel, Franke, and Proaño (2008) derive an identical result; determinacy holds under the following condition:

$$\Phi_p + \Phi_w + \Phi_y \frac{\rho(\lambda_p + \lambda_w)}{\kappa_w \lambda_p + \kappa_p \lambda_w} > 1 \quad (11)$$

with ρ related to the discount factor as follows $1/\beta(t) = 1 + \rho t$. In (11) ρ is the limit of $(1-\beta(t))/t$ when the time step t tends to zero, while the time step is one in (10).

² Using Dynare (Adjemian et al. 2011), it is possible to verify numerically frontier (10); code available upon request.

Hence the combination of these results proves, as expected by Flaschel et al., that the determinacy condition of this model can be taken to the limit of continuous time.

In a model with staggered prices only, the frontier of determinacy is (12) (Woodford 2001), of which our result (10) is a direct generalization.

$$\Phi_p + \frac{1-\beta}{\kappa_p} \Phi_y > 1 \quad (12)$$

With both rigidities, a permanent shift in price inflation ($\tilde{\pi}$) implies an identical permanent shift in wage inflation [equation (5)]. The Phillips curves [equations

(3) and (4)] imply a proportional shift in output gap $\tilde{y} = \frac{(1-\beta)(\lambda_p + \lambda_w)}{\kappa_w \lambda_p + \kappa_p \lambda_w} \tilde{\pi}$. In turn,

the Taylor rule (7) implies that the reaction of the central banker is to raise

the nominal interest rate by $\tilde{i} = \left[\Phi_p + \Phi_w + \Phi_y \frac{(1-\beta)(\lambda_p + \lambda_w)}{\kappa_w \lambda_p + \kappa_p \lambda_w} \right] \tilde{\pi}$. Thus, as in the

standard neo-Keynesian model without wage rigidities (Woodford 2011, chapter 4), our frontier of indeterminacy can also be interpreted as a strict Taylor principle applied to permanent shocks: the *central banker should react more than one for one to permanent changes in inflation*.

Finally, if wage inflation and price inflation play similar roles for the design of the optimal monetary policy (Erceg, Henderson, and Levin 2000; Galí 2008), we show that they also play symmetric roles for eliminating sun-spot fluctuations. This extended conclusion remains “*at odds with the practice of most central banks, which seem to attach little weight to wage inflation as a target variable*” (Galí 2008).

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The characteristic polynomial of the matrix defined in (9) can be expressed as follows, with implicit dependency on X :

$$P_{\gamma, \xi}(t) = at^4 - bt^3 + c_{\gamma, \xi}t^2 - d_{\gamma, \xi}t + e_{\gamma, \xi}. \quad (16)$$

with the coefficients:

$$\begin{aligned} a &= \sigma\beta^2 \\ b &= \beta[\kappa_p + \sigma(2 + 2\beta + \lambda_p + \lambda_w)] \\ c_{\gamma, \xi} &= \kappa_p[1 + \beta + \lambda_w + \beta\xi] + \kappa_w[\lambda_p + \beta\gamma] + \sigma[1 + 4\beta + \beta^2 + (\lambda_p + \lambda_w)(1 + \beta)] + \beta\phi_p\kappa_p \\ &\quad + \beta(1 - \phi_p)\kappa_w \\ d_{\gamma, \xi} &= \kappa_p[1 + \lambda_w(1 + \gamma + \xi) + (1 + \beta)\xi] + \kappa_w[\lambda_p(1 + \gamma + \xi) + (1 + \beta)\gamma] + \sigma[2(1 + \beta) + \lambda_p + \lambda_w] \\ &\quad + (1 + \beta)\kappa_p\phi_p + (1 + \beta)\kappa_w(1 - \phi_p) \\ e_{\gamma, \xi} &= \sigma + \kappa_p\xi + \kappa_w\gamma + \kappa_p\phi_p + \kappa_w(1 - \phi_p) \end{aligned}$$

To study the eigenvalues of this polynomial in deviation from the limit case $\phi_p + \phi_w = 1$, we use the simplifying notations $P_0 = P_{0,0}$, $c = c_{0,0}$, $d = d_{0,0}$ and $e = e_{0,0}$ and the following decomposition:

$$P_{\gamma, \xi}(t) = P_0(t) + \gamma\kappa_w Q(t) + \xi\kappa_p S(t) \quad \text{where} \quad (17)$$

$$Q(t) = \beta t^2 - \left[\lambda_w \frac{\kappa_p}{\kappa_w} + (1 + \beta + \lambda_p) \right] t + 1 \quad (18)$$

$$S(t) = \beta t^2 - \left[\lambda_p \frac{\kappa_w}{\kappa_p} + (1 + \beta + \lambda_w) \right] t + 1 \quad (19)$$

It is noteworthy that the deviation in direction $\Phi_p(\xi\kappa_p S(t))$ is exactly symmetric to the deviation in direction $\Phi_w(\gamma\kappa_w Q(t))$.

Property A.1: For every vector of parameters $X \in D_X$, $(a, b) \in (\mathbb{R}_+^*)^2$ and $\forall \gamma \geq -\phi_p$, $\forall \xi \geq \phi_p - 1$, $(c_{\gamma, \xi}, d_{\gamma, \xi}, e_{\gamma, \xi}) \in (\mathbb{R}_+^*)^3$. This implies that $\forall t \leq 0$, $P_{\gamma, \xi}(t) > 0$.

Proof: From their definition and the definition (15) of D_X , coefficients a to $e_{\gamma, \xi}$ are all positive. The sign of the polynomial $P_{\gamma, \xi}(t)$ derives from (16). ■

Property A.2: The limit case polynomial P_0 verifies the following properties:

(i) 1 is a root of P_0 which can be written $P_0(t)=(t-1)R_0(t)$ with

$$R_0(t)=at^3-(b-a)t^2+(a-b+c)t+a-b+c-d$$

(ii) P_0 has no negative root,

(iii) P_0 has at most two complex roots, these roots are conjugate and their modulus is strictly larger than one

(iv) if P_0 has a multiple root in 1, it is a double one and the other roots are real and larger than one

(v) P_0 has at most one root in $]0, 1[$

Proof:

(i) From the definition of P_0 and its coefficients, one can check that $a-b+c-d+e=P_0(1)=0$ ♦

(ii) This property derives directly from Property A.1 ♦

(iii) P_0 is a fourth degree polynomial with real coefficients and one real root (1). It follows that it has at most two complex roots which are conjugate. These roots are also the roots of polynomial R_0 which we denote z and $\bar{z}$ and let r denote the remaining real root. We can write:

$$\begin{aligned} R_0(t) &= at^3 - (b-a)t^2 + (a-b+c)t + a-b+c-d \\ &= a(t-r)(t-z)(t-\bar{z}) = a(t-r)(t^2 + |z|^2) \\ &= at^3 - art^2 + a|z|^2 t - r|z|^2 \end{aligned}$$

$$\text{property A.6} \Rightarrow r = \frac{b-a}{a} > 3 \text{ \& } |z|^2 = \frac{a-b+c}{a} > 3$$

Thus $z, \bar{z}$ are outside the unit disk. ♦

(iv) We showed above that if P_0 has complex roots, the two remaining real roots are 1 and $r > 3$. So 1 can not be a double root of P_0 in this case. Property A.8 implies that if $P'_0(1)=0$ then $P''_0(1)>0$, thus 1 can not be a triple root of P_0 . If 1 is a double root of P_0 , the two other roots, necessarily real and positive, can not be both smaller than one as the product of the four is larger than one. If we assume that one of them is smaller than one, a table of variation of P_0 shows that it implies that $P''(1)<0$, which contradicts Property A.8. ♦

(v) We showed above that if P_0 has complex roots, the two remaining real roots are 1 and $r > 3$.

If P_0 has four real roots, one of them is 1 and their product is $\frac{e}{a} > 1$. Thus at least one of its roots is strictly larger than 1.

Property A.7 also implies if P_0 has four real roots, that $3b^2 > 8ac$ and P_0'' has two real roots, denoted t_- and t_+ . As $P_0''(t) = 12at^2 - 6bt + 2c$, we obtain that:

$$t_{\pm} = \frac{b}{4a} \pm \left(\left(\frac{b}{4a} \right)^2 - \frac{c}{6a} \right)^{\frac{1}{2}}.$$

We proceed by contradiction and assume that P_0 has two roots $\in]0, 1[$. A table of variations of P_0 and its derivatives show that this assumption implies $P_0'(1) < 0$ and $t_- < 1$.

$$\begin{aligned} 0 < \frac{b}{4a} - 1 < \left(\left(\frac{b}{4a} \right)^2 - \frac{c}{6a} \right)^{\frac{1}{2}} &\Leftrightarrow 1 - \frac{b}{2a} < -\frac{c}{6a} \\ &\Leftrightarrow 6a - 3b + c < 0 \Leftrightarrow P_0''(1) < 0. \end{aligned}$$

This is in contradiction with Property A.8, both the first and second derivatives of P_0 can not be negative in 1. Thus P_0 has at most one root in $]0, 1[$. ■

Property A.3: *There exist a neighborhood of 1 where both $Q(t) < 0$ and $S(t) < 0$*

Proof: From their definition, it follows that both Q and S are strictly negative in 1, so there is a neighborhood of 1 where each of them is negative. On the intersection of these neighborhoods they are both negative. ■

Property A.4: $\forall \xi > 0, \gamma \geq 0$ or $\forall \gamma > 0, \xi \geq 0$ (that is on any point in the dashed domain on Figure 1), $P_{\gamma, \xi}$ has only one root within the unit disk. This root is in $]0, 1[$ and if it has complex roots they remain outside the unit disk.

Proof: We consider the dynamic of the roots of $P_{\gamma, \xi}$ starting from the limit case P_0 . Due to Property A.1, the real roots of $P_{\gamma, \xi}$ can not be negative. Property A.2 implies that if one is a single root of P_0 , there is at most one root of P_0 in $]0, 1[$, 1 is another of its roots and the remaining real roots, if any, are strictly greater than one. The other possible configuration is, 1 is a double root of P_0 , the other roots are real and strictly greater than one. Applying Lemma A.1 to this configuration shows that:

- if P_0 has a root strictly smaller than one, then $\forall \xi > 0, \gamma \geq 0$ or $\forall \gamma > 0, \xi \geq 0$ the root 1 will shift to the right,
- if P_0 has no root strictly smaller than one, then $\forall \xi > 0, \gamma \geq 0$ or $\forall \gamma > 0, \xi \geq 0$ the root 1 will shift to the left,
- if 1 is a double root of P_0 , then $\forall \xi > 0, \gamma \geq 0$ or $\forall \gamma > 0, \xi \geq 0$, the root 1 will split on both sides of the unit disk.

Still in application of Lemma A.1, 1 is a repulsive point for the dynamic of the roots of $P_{\gamma,\xi}$: with γ and ξ increasing, if a root of $P_{\gamma,\xi}$ enters the neighborhood of 1 were both Q and S are negative, this root will be pushed back to the left (resp. right) if it is smaller (resp. larger) than one.

Thus $\forall \xi > 0, \gamma \geq 0$ or $\forall \gamma > 0, \xi \geq 0$, $P_{\gamma,\xi}$ has one real root in $]0, 1[$ and its other real roots are larger than 1.

If $P_{\gamma,\xi}$ has complex roots, it can have only two complex conjugate roots, since $P_{\gamma,\xi}(0) > 0$, $\lim_{t \rightarrow +\infty} P_{\gamma,\xi}(t) = +\infty$ and $P_{\gamma,\xi}(1) < 0$. Let $r^\pm$ denote the real roots of $P_{\gamma,\xi}$ and $z, \bar{z}$ its complex roots. *A fortiori* $r^- < 1 < r^+$.

$$P_{\gamma,\xi}(t) = a(t - r^+)(t - r^-)(t^2 + |z|^2) \quad (20)$$

$$= a(t^4 - (r^+ + r^-)t^3 + (|z|^2 + r^+r^-)t^2 - (r^+ + r^-)|z|^2 t + r^+r^-|z|^2) \quad (21)$$

By identification we get

$$\begin{aligned} r^+ + r^- &= \frac{b}{a} |z|^2 + r^+r^- = \frac{c_{\gamma,\xi}}{a} (r^+ + r^-) |z|^2 = \frac{d_{\gamma,\xi}}{a} r^+r^- |z|^2 = \frac{e_{\gamma,\xi}}{a} \\ r^- < 1 &\Rightarrow r^+ < \frac{b-a}{a} \Rightarrow |z|^2 + r^+r^- = \frac{c_{\gamma,\xi}}{a} < \frac{b-a}{a} + |z|^2 \\ &\Rightarrow \frac{c_{\gamma,\xi} - b + a}{a} < |z|^2 \\ \text{Property A.6} &\Rightarrow 3 < \frac{c_{\gamma,\xi} - b + a}{a} < |z|^2 \end{aligned}$$

Hence, if $P_{\gamma,\xi}$ has complex roots, their modulus is strictly larger than one. ■

Property A.5: $\forall \xi < 0$ or $\forall \gamma < 0$ (that is on any point in the dotted domain on Figure 1), $P_{\gamma,\xi}$ has two or zero real roots and two or zero complex roots within the unit disk.

Proof: Symmetrically to the previous one, this property derives from the application of Lemma A.1 with $\xi < 0$ or $\gamma < 0$ in the neighborhood of one.

- if P_0 has a root strictly smaller than one, then with $\xi < 0$ or $\gamma < 0$ the root 1 shifts to the left,
- if P_0 has no root strictly smaller than one, then with $\xi < 0$ or $\gamma < 0$ the root 1 shifts to the right,
- if 1 is a double root of P_0 , then with $\xi < 0$ or $\gamma < 0$ the root 1 becomes two complex conjugate roots.

From this configuration, 1 is a repulsive point for the dynamic of the roots of $P_{\gamma,\xi}$ which implies that this configuration remains with γ or ξ decreasing.

As for the complex roots, they are conjugate thus either both inside or outside the unit disk. ■

Theorem A.1: For every $X \in D_X$, if $\Phi_y = 0$, for all $(\Phi_p, \Phi_w) \in (\mathbb{R}_+^*)^2$, model (3) to (7) has a unique and stable solution **if and only if** $\Phi_p + \Phi_w > 1$

Proof: Properties A.4 and A.5 imply that for all admissible values of γ and ξ (namely such that $\Phi_p > 0$ and $\Phi_w > 0$), $P_{\gamma, \xi}$ has only one root smaller than one in modulus and three roots larger than one in modulus **if and only if** $\gamma + \xi > 0$, i.e. $\Phi_p + \Phi_w > 1$. The Blanchard and Kahn (1980) condition implies that this configuration of the roots of $P_{\gamma, \xi}$ is equivalent to the model's solution determinacy. ■

A.2 The case for $\Phi_y \neq 0$

A priori the frontier of indeterminacy in this case can be written $\phi_p + \phi_w = 1 - \theta$, with θ positive, decreasing with Φ_y and to be determined (Galí 2008). We introduce the following parametrization:

$$\Phi_p = \phi_p + \xi \quad \Phi_w = 1 - \phi_p - \theta + \gamma \quad \Phi_y = \phi_y + \zeta \quad (22)$$

$$0 < \phi_p < 1 \quad 0 < \theta < 1 - \phi_p \quad (23)$$

$$\xi \text{ s.t. } \Phi_p > 0, \quad \gamma \text{ s.t. } \Phi_w > 0, \quad \zeta \text{ s.t. } \Phi_y > 0 \quad (24)$$

The characteristic polynomial P of A is now defined by:

$$P_{\gamma, \xi, \zeta}(t) = at^4 - b_{\zeta}t^3 + c_{\gamma, \xi, \zeta}t^2 - d_{\gamma, \xi, \zeta}t + e_{\gamma, \xi, \zeta}.$$

It is equivalent to (16) with augmented coefficients:

$$a = \sigma\beta^2$$

$$b_{\zeta} = \beta[\kappa_p + \Phi_y\beta + \sigma(2 + 2\beta + \lambda_p + \lambda_w)]$$

$$c_{\gamma, \xi, \zeta} = \kappa_p[1 + \beta + \lambda_w + \beta\xi] + \kappa_w[\lambda_p + \beta\gamma] + \sigma[1 + 4\beta + \beta^2 + (\lambda_p + \lambda_w)(1 + \beta)] \\ + \beta\phi_p\kappa_p + \beta(1 - \theta - \phi_p)\kappa_w + \Phi_y\beta[\lambda_p + \lambda_w + 2 + \beta]$$

$$d_{\gamma, \xi, \zeta} = \kappa_p[1 + \lambda_w(1 + \gamma + \xi) + (1 + \beta)\xi] + \kappa_w[\lambda_p(1 + \gamma + \xi) + (1 + \beta)\gamma] \\ + \sigma[2(1 + \beta) + \lambda_p + \lambda_w] + (1 + \beta)\kappa_p\phi_p + (1 + \beta)\kappa_w(1 - \theta - \phi_p) \\ + \Phi_y[1 + \lambda_w + \lambda_p + 2\beta] - (\kappa_p\lambda_w + \kappa_w\lambda_p)\theta$$

$$e_{\gamma, \xi, \zeta} = \sigma + \kappa_p\xi + \kappa_w\gamma + \kappa_p\phi_p + \kappa_w(1 - \theta - \phi_p) + \Phi_y.$$

We define the baseline polynomial by choosing $\{\theta, \phi_y\}$ such that $P_0(1)=0$, which implies that θ is proportional to ϕ_y :

$$\theta = \phi_y \frac{(1-\beta)(\lambda_p + \lambda_w)}{\kappa_w \lambda_p + \kappa_p \lambda_w}. \quad (25)$$

Our limit case, i.e. with $\xi=\gamma=\xi=0$, is:

$$\phi_p + \phi_w + \phi_y \frac{(1-\beta)(\lambda_p + \lambda_w)}{\kappa_w \lambda_p + \kappa_p \lambda_w} = 1 \quad (26)$$

Let now X denote the extended vector of parameters:

$$X = [\beta, \phi_p, \theta, \kappa_p, \lambda_p, \lambda_w, \sigma] \in]0, 1[^3 \times (\mathbb{R}_+^*)^5 = D_X. \quad (27)$$

We can decompose polynomial $P_{\gamma, \xi, \zeta}$ similarly to (17):

$$P_{\gamma, \xi}(t) = P_0(t) + \gamma \kappa_w Q(t) + \xi \kappa_p S(t) + \zeta T(t) \quad \text{where} \quad (28)$$

$$Q(t) = \beta t^2 - \left[\lambda_w \frac{\kappa_p}{\kappa_w} + (1 + \beta + \lambda_p) \right] t + 1 \quad (29)$$

$$S(t) = \beta t^2 - \left[\lambda_p \frac{\kappa_w}{\kappa_p} + (1 + \beta + \lambda_w) \right] t + 1 \quad (30)$$

$$T(t) = -\beta t^3 + \beta(\lambda_p + \lambda_w + 2 + \beta)t^2 - (1 + \lambda_p + \lambda_w + 2\beta)t + 1 \quad (31)$$

From the definition of $P_{\gamma, \xi, \zeta}$ and its coefficients, we can generalize Property A.1 to the case $\Phi_y \neq 0$. The limit case polynomial also verifies Property A.2 (the proof is identical). Since $T(1) < 0$, we can generalize Property A.3 as well. All the developments derived in the case $\Phi_y \neq 0$ apply and Theorem A.1 can be generalized:

Theorem A.2: For every $X \in D_X$, for all $(\Phi_p, \Phi_w, \Phi_y) \in (\mathbb{R}_+^*)^3$, model (3) to (7) has a

unique and stable solution **if and only if** $\Phi_p + \Phi_w + \Phi_y \frac{(1-\beta)(\lambda_p + \lambda_w)}{\kappa_w \lambda_p + \kappa_p \lambda_w} > 1$

A.3 Some useful properties

A.3.1 On the polynomial coefficients if $\Phi_y \neq 0$

Property A.6: *The following inequalities hold:*

$$\forall X \in D_X \quad \forall \gamma \geq 0 \quad \forall \xi \geq 0 \quad b > 4a \quad c_{\gamma, \xi} > 2a + b > 6a \quad e_{\gamma, \xi} > a$$

The same inequalities hold for the coefficients of $P_{\gamma, \xi, \zeta}$ when $\Phi_y \neq 0$.

Proof: These inequalities derive directly for the definitions of the coefficients. ■

Property A.7: $\forall X \in D_X, \forall \gamma \geq 0, \forall \xi \geq 0$ $P_{\gamma, \xi}$ has two conjugate complex roots if and only if $3b^2 < 8ac_{\gamma, \xi}$

The same property applies to $P_{\gamma, \xi, \zeta}$ when $\Phi_y \neq 0$.

Proof: The discriminant of the second order derivative $P''_{\gamma, \xi}$ is $\Delta(P''_{\gamma, \xi}) = 12[3b^2 - 8ac_{\gamma, \xi}]$; if it is negative, this polynomial is positive for every t . In this case, $P'_{\gamma, \xi}$ is strictly increasing and has only one real root, and $P_{\gamma, \xi}$ is strictly decreasing and then strictly increasing. As we already know that it has two real roots, the two others are complex. ■

Property A.8: *Whether $\Phi_y \neq 0$ or not, with the respective definitions of polynomial P_0 and the parameters domain D_X : for all $X \in D_X$, $P'_0(1) = 4a - 3b + 2c - d$ and $P''_0(1) = 12a - 6b + 2c$ cannot be both negative or null.*

Proof: If $\Phi_y \neq 0$:

$$\begin{aligned} P'_0(1) &= 4a - 3b + 2c - d \\ &= \kappa_p \lambda_w + \kappa_w \lambda_p - (1 - \beta)((1 - \phi_p)(\kappa_w - \kappa_p) - \sigma(\lambda_p + \lambda_w)) \end{aligned}$$

$$\begin{aligned} \frac{1}{2} P''_0(1) &= 6a - 3b + c \\ &= \kappa_p [1 - \beta + \lambda_w] + \kappa_w \lambda_p + \sigma[(1 - \beta)^2 + (\lambda_p + \lambda_w)(1 - \beta)] \\ &\quad + \beta((1 - \phi_p)(\kappa_w - \kappa_p) - \sigma(\lambda_p + \lambda_w)) \end{aligned}$$

It is a necessary condition for $P'_0(1)$ to be negative or zero that:

$$(1 - \phi_p)(\kappa_w - \kappa_p) - \sigma(\lambda_p + \lambda_w) > 0 \quad (32)$$

However, if this is the case, then

$$\frac{1}{2}P''_0(1) > \kappa_p[1-\beta+\lambda_w] + \kappa_w\lambda_p + \sigma[(1-\beta)^2 + (\lambda_p + \lambda_w)(1-\beta)] > 0 \quad (33)$$

If $\Phi_y \neq 0$:

$$P'_0(1) = 4a - 3b + 2c - d \\ = \kappa_p\lambda_w + \kappa_w\lambda_p - (1-\beta)((1-\phi_p)(\kappa_w - \kappa_p) + \phi_y(1-\beta) - \sigma(\lambda_p + \lambda_w) - \theta\kappa_w)$$

$$\frac{1}{2}P''_0(1) = 6a - 3b + c \\ = \kappa_p[1-\beta+\lambda_w] + \kappa_w\lambda_p + \sigma[(1-\beta)^2 + (\lambda_p + \lambda_w)(1-\beta)] + \phi_y\beta(\lambda_p + \lambda_w) \\ + \beta((1-\phi_p)(\kappa_w - \kappa_p) + \phi_y(1-\beta) - \sigma(\lambda_p + \lambda_w) - \theta\kappa_w)$$

It is a necessary condition for $P'_0(1)$ to be negative or zero that:

$$(1-\phi_p)(\kappa_w - \kappa_p) + \phi_y(1-\beta) - \sigma(\lambda_p + \lambda_w) - \theta\kappa_w > 0 \quad (34)$$

However, if this is the case, then

$$\frac{1}{2}P''_0(1) > \kappa_p[1-\beta+\lambda_w] + \kappa_w\lambda_p + \sigma[(1-\beta)^2 + (\lambda_p + \lambda_w)(1-\beta)] + \phi_y\beta(\lambda_p + \lambda_w) > 0 \quad (35)$$

■

A.3.2 Real roots shift

Lemma A.1: Let P be a polynomial and λ a simple or double root of this polynomial.

If the root is simple ($P(\lambda)=0$ but $P'(\lambda)\neq 0$) and we add a real quantity q sufficiently small to the polynomial, it translates the root λ in the direction defined by the sign of $-P'(\lambda)q$, + defining a translation to the right, - to the left.

If the root is double ($P(\lambda)=0$, $P'(\lambda)=0$ but $P''(\lambda)\neq 0$) and we add a real quantity q sufficiently small to the polynomial, the root becomes two distinctive roots on both sides of λ if $-P'(\lambda)q > 0$ and become complex roots otherwise.

Proof: If $P(\lambda)=0$ but $P'(\lambda)\neq 0$, the property is a direct application of the implicit function theorem to $f(t, q)=P(t)+q$ in the neighborhood of $(t, q)=(\lambda, 0)$.

Let's assume that $P(\lambda)=0$, $P'(\lambda)=0$ and $P''(\lambda)>0$. For regularity, we consider a neighborhood V of λ where $P''>0$. In V , a table of variation of P and its derivatives shows that P is strictly positive except in λ . If we add a positive q , arbitrarily small, to polynomial P , $P+q$ becomes strictly positive on V . By a continuity argument the double root λ became two conjugate complex roots.

Also, there exists $\delta > 0$, arbitrarily small, and $\lambda^- < \lambda < \lambda^+$ such that $P(\lambda^-) = P(\lambda^+) = \delta$. If we subtract q , arbitrarily small, to polynomial P , $P(\lambda) - q$ is strictly negative while $P(\lambda^-) - q = P(\lambda^+) - q = \delta - q$ can be set strictly positive. Since P' is strictly monotonous on V , $P - q$ has two roots in V , one on each side of the original root λ .

All in all, if $P''(\lambda) > 0$, the double root become complex if we add a positive quantity, arbitrarily small, to P and two real roots on each side of the initial one if this quantity is negative. A symmetric argument completes the proof for the case if $P''(\lambda) < 0$.

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