Supplemental Material (not for publication)

Persistent vs. Permanent Income Shocks in the Buffer-Stock Model

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A Additional Figures and Tables

Figure A.1: Wealth and Consumption Age Profiles (estimating ρ , $\rho = 2$, $\phi = 0.03$).

Notes: Figure A.1 shows age profiles of 500.000 simulated households.

Figure A.2: MPC and MPCP Age Profiles (estimating ρ , $\rho = 2$, $\phi = 0.03$).

Notes: Figure A.2 shows age profiles of 500.000 simulated households.

Figure A.3: Wealth and Consumption Age Profiles (estimating ϕ , $\rho = 2$, $\phi = 0.03$).

Notes: Figure A.3 shows age profiles of 500.000 simulated households.

Figure A.4: MPC and MPCP Age Profiles (estimating ρ , $\rho = 2$, $\phi = 0.03$).

Notes: Figure A.4 shows age profiles of 500,000 simulated households.

Table A.1: Robustness. Estimating ρ using medians ($\rho = 2$, $\phi = 0.03$).

	BIAS($\hat{\rho}$)		DIFF(MPC)		DIFF(MPCP)	
	$\alpha = 0.99$	$\alpha = 0.97$	$\alpha = 0.99$	$\alpha = 0.97$	$\alpha = 0.99$	$\alpha = 0.97$
baseline	-4.2	-5.0	-0.1	-0.5	0.5	0.8
k = 26	-17.7	-32.3	-0.4	-1.4	0.5	0.6
k = 35	-11.7	-21.3	-0.3	-1.0	0.5	0.7
k = 55	4.3	16.6	0.1	-0.1	0.4	1.0

Notes: See notes to table 4.1 and 5.1.

Table A.2: Robustness. Estimating ϕ using medians ($\rho = 2$, $\phi = 0.03$).

	BIAS($\hat{\phi}$)		DIFF(MPC)		DIFF(MPCP)	
	$\alpha = 0.99$	$\alpha = 0.97$	$\alpha = 0.99$	$\alpha = 0.97$	$\alpha = 0.99$	$\alpha = 0.97$
baseline	2.5	2.8	-0.0	-0.5	0.4	0.8
k = 26	11.6	23.5	-0.2	-0.9	0.4	0.5
k = 35	7.4	14.4	-0.1	-0.7	0.4	0.6
k = 55	-2.6	-9.9	0.1	-0.2	0.4	1.1

Notes: See notes to table 4.1 and 5.1.

B Model: Details and Solution Algorithm

B.1 Generalized income process

In the model we solve, we use a somewhat more general formulation of the income process allowing for *both* persistent and permanent shocks simultaneously:

$$t \leq T_R : Y_t = \Gamma_t L_t P_t \xi_t \quad (\text{B.1})$$

$$P_t = G_t P_{t-1} \psi_t \quad (\text{B.2})$$

$$L_t = L_{t-1}^\alpha \eta_t \quad (\text{B.3})$$

$$\log \psi_t \sim \mathcal{N}(-0.5 \cdot \sigma_\psi^2, \sigma_\psi^2) \quad (\text{B.4})$$

$$\log \eta_t \sim \mathcal{N}(-0.5 \cdot \sigma_\eta^2, \sigma_\eta^2) \quad (\text{B.5})$$

$$\Gamma_t = \begin{cases} e^{-\frac{1}{2} \left(\frac{1-\alpha^{2t}}{1-\alpha^2} - \frac{1-\alpha^t}{1-\alpha} \right) \sigma_\eta^2} & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \end{cases} \quad (\text{B.6})$$

$$t > T_R : Y_t = \kappa \cdot \Gamma_{T_R} L_{T_R} P_{T_R} \quad (\text{B.7})$$

$$A_t \geq -\lambda_t L_t P_t \quad (\text{B.8})$$

B.2 The Normalization Factor Γ_t

Note the useful result that if

$$X = e^Z, \quad Z \sim \mathcal{N}(\mu, \sigma^2) \quad (\text{B.9})$$

then

$$\mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2} \quad (\text{B.10})$$

and

$$X^{\alpha^t} = (e^Z)^{\alpha^t} = e^{\alpha^t Z} = e^K, \quad K \sim \left(\mathcal{N}(\alpha^t \mu, \alpha^{2t} \sigma^2) \right) \quad (\text{B.11})$$

Given our assumption of

$$\begin{aligned} \eta_t &\sim e^{\mathcal{N}(\mu_\eta, \sigma_\eta^2)} \\ \mu_\eta &\equiv -\frac{1}{2} \cdot \sigma_\eta^2 \end{aligned} \quad (\text{B.12})$$

we thus have

$$\begin{aligned}
\mathbb{E}_0 [Y_t] &= \mathbb{E}_0 [\Gamma_t L_t P_t] \leftrightarrow \\
\Gamma_t^{-1} &= \mathbb{E}_0 [L_t] \cdot \mathbb{E}_0 [P_t] \\
&= \mathbb{E}_0 [\eta_1^{\alpha^{t-1}} \cdot \eta_2^{\alpha^{t-2}} \cdot \eta_3^{\alpha^{t-3}} \cdot \dots \cdot \eta_t] L_0 P_0 \\
&= e^{\mu_\eta + \frac{1}{2}\sigma_\eta^2} \cdot e^{\alpha\mu_\eta + \frac{1}{2}\alpha^2\sigma_\eta^2} \cdot e^{\alpha^2\mu_\eta + \frac{1}{2}\alpha^4\sigma_\eta^2} \cdot \dots \cdot e^{\alpha^{t-1}\mu_\eta + \frac{1}{2}\alpha^{2(t-1)}\sigma_\eta^2} \cdot L_0 P_0 \\
&= e^{(1+\alpha+\alpha^2+\dots+\alpha^{t-1})\mu_\eta} e^{\frac{1}{2}(1+\alpha+\alpha^2+\dots+\alpha^{2(t-1)})\alpha^2\sigma_\eta^2} \\
&= e^{\frac{1-\alpha^t}{1-\alpha}\mu_\eta + \frac{1}{2}\frac{1-\alpha^{2t}}{1-\alpha^2}\sigma_\eta^2} \leftrightarrow \\
\Gamma_t &= e^{-\frac{1}{2}\left(\frac{1-\alpha^{2t}}{1-\alpha^2} - \frac{1-\alpha^t}{1-\alpha}\right)\sigma_\eta^2}
\end{aligned} \tag{B.13}$$

B.3 Equalizing Income Variances

The variance of the *permanent* income component in period k is given by

$$\begin{aligned}
\mathbb{V}[\log P_k] &= \mathbb{V}[\log P_0] + \sum_{j=1}^k \mathbb{V}[\log \psi_j] \\
&= \mathbb{V}[\log P_0] + k \cdot \sigma_\psi^2
\end{aligned} \tag{B.14}$$

The variance of the *persistent* income component in period k is given by

$$\begin{aligned}
\mathbb{V}[\log L_k] &= \alpha^k \cdot \mathbb{V}[\log L_0] + \sum_{j=1}^k \alpha^{2(k-j)} \cdot \mathbb{V}[\log \eta_k] \\
&= \alpha^k \cdot \mathbb{V}[\log L_0] + \sum_{j=1}^k \alpha^{2(k-j)} \cdot \mathbb{V}[\log \eta_k] \\
&= \alpha^k \cdot \mathbb{V}[\log L_0] + \left(\alpha^{-2(k-1)} + \dots + \alpha^4 + \alpha^2 + \alpha\right) \cdot \sigma_\eta^2 \\
&= \alpha^k \cdot \mathbb{V}[\log L_0] + \frac{1 - \alpha^{2k}}{1 - \alpha^2} \cdot \sigma_\eta^2
\end{aligned} \tag{B.15}$$

To ensure that the variances are identical in period k , we set

$$\begin{aligned}
\frac{1 - \alpha^{2k}}{1 - \alpha^2} \cdot \sigma_\eta^2 &= k \cdot \sigma_\psi^2 \\
\sigma_\eta^2 &= k \frac{1 - \alpha^2}{1 - \alpha^{2k}} \sigma_\psi^2
\end{aligned}$$

and we assume $\mathbb{V}[\log P_0] = \mathbb{V}[\log L_0] = 0$ in our simulations.

B.4 Normalization

Relying on the homogeneity in P_t , the model can be written in ratio form ($a_t = \frac{A_t}{P_t}$, $m_t = \frac{M_t}{P_t}$ etc.) as

$$\begin{aligned}
v_t(L_t, m_t) &= \max_{c_t \geq 0} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[(G_{t+1} \psi_{t+1})^{1-\rho} v_{t+1}(L_{t+1}, m_{t+1}) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \frac{R}{G_{t+1} \psi_{t+1}} a_t + \Lambda_{t+1} \xi_{t+1} L_{t+1} \\
L_{t+1} &= L_t^\alpha \eta_{t+1} \\
a_t &\geq \lambda_t
\end{aligned} \tag{B.16}$$

Denoting the optimal consumption choice by $c_t^*(L_t, m_t)$, we have that the first order condition imply

$$c_t^*(L_t, m_t)^{-\rho} \geq \beta R \cdot \mathbb{E}_t \left[\left(G_{t+1} \psi_{t+1} c_{t+1}^*(L_{t+1}, m_{t+1}) \right)^{-\rho} \right] \tag{B.17}$$

which holds with equality if

$$a_t^*(L_t, m_t) \equiv m_t - c_t^*(L_t, m_t) > \lambda_t$$

and otherwise

$$c_t = m_t + \lambda_t$$

B.5 Accuracy of the Numerical Solution

The model is solved using the endogenous-grid-point method (EGM) proposed by [Carroll \(2006\)](#). The grids of a_t (200 nodes) and L_t (301 nodes) are chosen exogenously, and the expectation is taken using 8 Gauss-Hermite nodes for ξ_{t+1} , η_{t+1} , and ψ_{t+1} and by relying on standard bi-linear interpolation of the next period consumption function.

The left panel of figure [B.1](#) shows the age profile of the average Euler error based on simulated data from the numerically solved model. Particularly, we simulate $N = 500,000$ individuals for $T = 34$ time periods and calculate the average Euler error for a given age as

$$\mathcal{E}_t = \frac{1}{N} \sum_{i=1}^N \beta R (C_{it}/C_{it-1})^{-\rho}. \tag{B.18}$$

Binding credit constraints would generally imply an average Euler error below one; to mitigate this, the right panel of figure [B.1](#) shows the average Euler error conditioning on lagged wealth being above some threshold such that the households are not credit constrained. As seen, the average Euler errors fluctuates around and are very close to

one, and, importantly, do not differ significantly across the two considered levels of α .

Figure B.1: Average Euler Error Age Profiles

Notes: Figure B.1 shows average Euler errors (see equation B.18) for 500,000 simulated households.

References

CARROLL, C. D. (2006): “The method of endogenous gridpoints for solving dynamic stochastic optimization problems,” *Economics Letters*, 91(3), 312–320.