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Topics

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Fundamental Non-convexity and Externalities: A Differentiable Approach

Abstract: It is well known that externalities cause fundamental non-convexity problems in the production set. We demonstrate that the differentiable approach is a proper tool. Existence of equilibrium obtains without requiring aggregate convexity in consumption or production. Our model allows general externalities in consumption and production and also price dependency.

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1 Introduction

Externalities prevail in the real world, yet they are difficult to deal with in general equilibrium models. Baumol (1972) first points out that the aggregate production possibility set of the polluter's activity and the pollutee's activity may present itself as a nonconvex set when the external damages are strong. For example, when a laundry (pollutee) and a steel mill (polluter) locate side by side, the production frontier becomes L-shaped with only the production of either of the two commodities possible. Even though individual production and consumption sets are convex, externalities create non-convexity in the aggregate and thus present a problem for the conventional convex analysis approach to finite economies. Moreover, the price hyperplane needs to separate production sets and consumption sets independently, yet with

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externalities, these sets are not independent. Starrett (1972) points out another type of fundamental non-convexity. When a positive price for the pollution right is determined in the Arrovian externalities market, the pollutee may want to sell an infinite amount of rights. Boyd and Conley (1997) argue that this type of non-convexity can be resolved by specifying an endowment bound for pollution rights. On the other hand, the Baumol type of non-convexity still persists.

Our paper presents a differentiable approach to externalities where convexity in the aggregate production or consumption is not required. Externalities may influence production and consumption in arbitrary ways. As long as consumers' preferences and firms' production sets are convex in own activities, i.e. in demand or net output, for fixed levels of externalities, a competitive equilibrium exists under standard assumptions. Our existence result is not only generic (which is common with the differentiable approach) but holds for all parameters. Our approach treats equilibrium of an economy as the intersection of manifolds, in line with Debreu (1970), Dierker (1975), Mas-Colell (1985), Balasko (1988), and Geanakoplos and Shafer (1990). A nonempty intersection obtains if these manifolds are transversal, and the fixed point mapping need not to be convex valued. We examine the first-order conditions of consumers' optimization problems without solving for excess demands. This is an approach used by Polemarchakis and Siconolfi (1997), Cass, Siconolfi, and Villanacci (2001), Villanacci et al. (2002), and Villanacci and Zenginobuz (2005).

The following authors address issues of externalities in competitive equilibrium. del Mercato (2006) and Bonnisseau and del Mercato (2010) study externalities when consumers have consumption constraints. Kung (2008) presents a public goods model with externalities in consumption but not in production. Noguchi and Zame (2006) use a continuous model of a distribution of consumptions over indivisible goods, where convexity is not required. Cornet and Topuzu (2005) study a two-period temporary equilibrium model as a reduced Walrasian economy with price dependency externalities. Balder (2004) demonstrates that an equilibrium exists if the externalities enter into preferences of each individual in the same way (which seems to exclude local externalities, externalities that diminish with distance, and externalities that have directional effects). Greenberg, Shitovitz, and Wieczorek's (1979) approach of abstract economy allows price dependency and consumption externalities (with aggregate production but no individual firms). In contrast to the literature, our model allows for production sets and individual preferences that are not convex in externalities and general externalities that firms and consumers experience in unrestricted ways. Hammond, Kaneko, and Wooders (1989) investigate widespread externalities in the economy with a coalition formation approach. We extend this differentiable approach to include production and externalities. Section 2 introduces the model and main results. Section 3 concludes.

2 The production economy

There are N private goods, I consumers, and J firms. The prices of private goods are denoted by $p \in S^N$ where $S^N = \left\{p \in \Re^N_{++} | \sum_{n=1}^N p_n = 1\right\}$ is the interior of the $(N-1)$-dimensional simplex.[1] Let $x_i \in \Re^N_{++}$ denote the consumption bundle of consumer i, and $y_j \in \Re^N$ denote net output of firm j. The activities of all consumers and firms enter into the utility functions of every consumer and the production technology of every firm. Each of consumer i and firm j is influenced by a profile of externalities including equilibrium prices. Let $T_i = ((x_h)_{h=1,h\neq i}^I, (y_j)_{j=1}^J)$ for consumer i and $T_j = ((x_i)_{i=1}^I, (y_h)_{h=1,h\neq j}^J)$ for firm j. All external activities are recorded as positive amounts. This model keeps track of the amount of the original activities such as the consumption of cigarettes, instead of the external by-products of these activities such as the amount of second-hand smoke. This framework includes public goods as a special case, for example, $\hat{x}_n$ is a public good if $x_{in} = \hat{x}_n$ for all i.

The production technology of firm j is represented by a C^2 transformation function $f_j(y_j, T_j, p) : \Re^{IN}_{++} \times \Re^{JN} \times S^N \to \Re$, which follows standard assumptions: f_j is *differentiably strictly decreasing* in y_j, i.e. $D_{y_j} f_j \ll 0$. And f_j is *differentiably strictly quasiconcave* in y_j, i.e. if $D_{y_j} f_j v = 0$, then $v D^2_{y_j} f_j v < 0$ for all $v \in \Re^N \backslash \{0\}$.

Firm j chooses a production plan y_j to maximize profit taking prices p and externalities T_j as given:

$$\begin{aligned} &max_{y_j \in \Re^N} \; p y_j \\ &\text{s.t. } f_j(y_j, T_j, p) = 0 \end{aligned}. \qquad [1]$$

With $v_j \in \Re$ being the multiplier, the first-order conditions are

$$\begin{aligned} p - v_j D_{y_j} f_j(y_j, T_j, p) &= 0, \\ f_j(y_j, T_j, p) &= 0. \end{aligned} \qquad [2]$$

1 We can safely exclude zero prices because of assumed strict quasiconcavity.

Each consumer i is endowed with private goods $e_i \in \Re^N_{++}$ and a share $s_{ij} \in [0,1]$ of firm j. Preferences of consumer i are represented by a C^2 utility function $u_i(x_i, T_i, p) : \Re^{IN}_{++} \times \Re^{JN} \times S^N \rightarrow \Re$, which follows standard assumptions: u_i is *differentiably strictly increasing* in x_i, i.e. $D_{x_i} u_i \gg 0$. And u_i is *differentiably strictly quasiconcave* in x_i, i.e. if $D_{x_i} u_i v = 0$ then $v D^2_{x_i} u_i v < 0$ for all $v \in \Re^N \backslash \{0\}$. Moreover, u_i satisfies the *boundary condition*[2]: for all T_i such that for any bundle $x_i' \in \Re^N_{++}$, the upper contour set $\{x_i \in \Re^N_{++} | u_i(x_i, T_i, p) \geq u_i(x_i', T_i, p)\}$ is closed in $\Re^N_{++}$.

Consumer i chooses a consumption bundle to maximize utility taking prices p and externalities T_i as given:

$$\begin{aligned} & max_{xx_i \in \Re^N_{++}} u_i(x_i, T_i, p) \\ & \text{s.t. } p(x_i - e_i) - \sum_{j=1}^{J} s_{ij} p y_j = 0. \end{aligned} \tag{3}$$

With $\lambda_i \in \Re$ being the multiplier, the first-order conditions are

$$\begin{aligned} & D_{x_i} u_i(x_i; T_i, p) - \lambda_i p = 0, \\ & p(x_i - e_i) - \sum_{j=1}^{J} s_{ij} p y_j = 0, \end{aligned} \tag{4}$$

The markets clear with

$$\sum_{i=1}^{I} (x_i - e_i) - \sum_{j=1}^{J} y_j = 0. \tag{5}$$

The *equilibrium* of an economy is a list $((x_i)_{i=1}^I, (y_j)_{j=1}^J, p)$ of consumption bundles $(x_i)_{i=1}^I \in \Re^{IN}_{++}$, production plans $(y_j)_{j=1}^J \in \Re^{JN}$, and price vector $p \in S^N$ such that consumers maximize utility solving eq. [3], firms maximize profits solving eq. [1], and markets clear (eq. [5]). Because of the assumed differentiability and strict quasiconcavity, the first-order conditions of consumers (eq. [4]) and of firms (eq. [2]) are also sufficient. We redefine the equilibrium with the first-order conditions and market clearing condition, eqs [2], [4], and [5], without solving for demand and supply functions. The equilibrium variables are expanded to include the multipliers, $(\lambda_i)_{i=1}^I \in \Re^I$ and $(v_j)_{j=1}^J \in \Re^J$, of consumers and firms maximization problems.

2 This guarantees an interior solution with positive demands for all goods (Mas-Colell 1985).

Definition 1. An *equilibrium* of the benchmark economy (e, s) is a list $((x_i)_{i=1}^{I}, (\lambda_i)_{i=1}^{I}, (y_j)_{j=1}^{J}, (v_j)_{j=1}^{J}, p) \in \Xi$, where $\Xi = \Re_{++}^{IN} \times \Re^{I} \times \Re^{JN} \times \Re^{J} \times S^{N}$, that satisfies the following C^1 equations:

Utility maximization:

$$D_{x_i} u_i(x_i, T_i, p) - \lambda_i p = 0, \forall i,$$
$$p(x_i - e_i) - \sum_{j=1}^{J} s_{ij} p y_j = 0, \forall i,$$

Profit maximization:

$$p - v_j D_{y_j} f_j(y_j, T_i, p) = 0, \forall j,$$
$$f_j(y_j, T_i, p) = 0, \forall j,$$

Market clearing:

$$\sum_{i=1}^{I} (x_i - e_i) - \sum_{j=1}^{J} y_j = 0.$$

Perturbing the economy: Our existence proof relies on the transversality of the system of equations defining the equilibrium. This requires the Jacobian matrix to have full rank at equilibrium. Parameters in the benchmark model may not provide enough rank or may be difficult to check. We equipped the economy with an augmented parameter space (similar perturbation methods are used in Allen 1981; Mas-Colell and Nachbar 1991, and Berliant and Kung 2006). The real world has many parameters that are omitted from a model (for example, the classic Arrow–Debreu model has only endowment as parameters). We add parameters in the utility functions and technology functions to provide enough variance in consumers and firms. These extra parameters perturb the system orthogonally so that its Jacobian matrix has full rank and thus provide enough independent directions for the system to be transversal. This technique can disentangle the interdependency generated by externalities. While perturbing around the parameters, all externalities variables stay fixed.

Take ε small enough so that it does not alter the properties of u_i and f_j assumed above. We perturb the utility function with $\alpha_i \in \Re_{+}^{N}$.

$$u_i(x_i, T_i, p) + \varepsilon \alpha_i x_i.$$

Firm specific parameters $\beta_j \in \Re^N$ and $\gamma_j \in \Re$ (let $\gamma = (\gamma_j)_{j=1}^J$) perturb around transformation function f_j.

$$f_j(y_j, T_j, p) + \varepsilon\left(\beta_j y_j + \gamma_j\right).$$

Let $s_{-1} = (s_{i1})_{i=2}^I$; it is the profile of all consumers' shares of firm 1 except for $i = 1$. We augment the parameter space into $\theta = ((\alpha_i)_{i=1}^I, s_{-1}, (\beta_j)_{j=1}^J, \gamma, e_1) \in \Theta = \Re_+^{IN} \times [0,1]^{I-1} \times \Re^{JN} \times \Re^J \times \Re_{++}^N$. A benchmark economy is parameterized at $(0, s_{-1}, 0, 0, e_1)$.[3] Notice that Θ serves as an example parameter space. Our theorem does not rely on these particular parameters. Any parameter space containing Θ also works.

Definition 2. An *equilibrium of the economy θ in the augmented parameter space* Θ is a list $((x_i)_{i=1}^I, (\lambda_i)_{i=1}^I, (y_j)_{j=1}^J, (v_j)_{j=1}^J, p) \in \Xi$, that satisfies the following conditions:

$$\begin{aligned}
D_{x_i} u_i(x_i, T_i, p) + \varepsilon\alpha_i - \lambda_i p &= 0, \forall i, \\
p(x_i - e_i) - \sum_{j=1}^{J} s_{ij} p y_j &= 0, \forall i \neq 1, \\
p - v_j\left(D_{y_j} f_j(y_j, T_j, p) + \varepsilon\beta_j\right) &= 0, \forall j, \\
f_j(y_j, T_j, p) + \varepsilon\left(\beta_j y_j + \gamma_j\right) &= 0, \forall j, \\
\sum_{i=1}^{I}(x_i - e_i) - \sum_{j=1}^{J} y_j &= 0.
\end{aligned} \tag{6}$$

Consumer 1's budget constraint is satisfied automatically by Walras' Law. Denote the left-hand side of system [6] as a C^1 map ϕ where $\phi : \Xi \times \Theta \to \Re^{I(N+1)+J(N+1)+N-1}$. Let $\chi \in \Xi$ denote an element of Ξ.

Theorem 1. *Equilibrium exists for every economy $\theta \in \Theta$.*

Proof. The main object of this proof is a homotopy map that transforms a seed system diffeomorphically into our economy. The seed system is purely a mathematical construction that has a unique solution. The property of odd number solutions, being of degree 1, carries through the homotopy into the economy, if (i) the preimage of zero is closed in the homotopy map (Lemma 1) and is bounded (Lemma 2), and (ii) the Jacobian matrices of the homotopy and its two boundaries have full rank at value zero. We show that generically in almost all

3 The full parameters for our economy should be (θ, s_1, e_{-1}). Notice that our results that hold for θ will also hold for (θ, s_1, e_{-1}), since it contains Θ as a subspace.

parameters, these matrices do have full rank at zero (Lemma 3). Finally, using continuity, we conclude that for every parameter, not just generically, an equilibrium exists.

A simplified seed system without externalities is defined as follows. Let $\hat{u}(x_i) = \sum_{i=1}^{I} \ln x_{in}/N$; consumers have preferences $\hat{u}(x_i) + \varepsilon\alpha_i$. Firm 1 has linear production technology $\beta_1 y_1 + \gamma_1 = 0$. Take a differentiably strictly decreasing and quasiconcave function $\hat{f}(y_j)$, the transformation functions for other firms $j{\neq}1$ are $\hat{f}(y_j) + \varepsilon\left(\beta_j y_j + \gamma_j\right)$. Thus, the following C^1 map η,

$$\eta(\chi,\theta) = \begin{pmatrix} D_{x_i}\hat{u}(x_i) + \varepsilon\alpha_i - \lambda_i p, \forall i \\ p(x_i - e_i), \forall i{\neq}1 \\ p - v_1\beta_1 \\ p - v_j\left(D_{y_j}\hat{f}_j(y_j) + \varepsilon\beta_j\right), \forall j{\neq}1 \\ \beta_1 y_1 + \gamma_1 \\ \hat{f}(y_j) + \varepsilon\left(\beta_j y_j + \gamma_j\right), \forall j{\neq}1 \\ \sum_{i=1}^{I}(x_i - e_i) - \sum_{j=1}^{J} y_j \end{pmatrix}$$

defines the solution of the seed system at $\eta(\chi,\theta) = 0$. Notice that there may not be a corresponding economy to this system. There is a unique solution χ^* to $\eta(\chi,\theta) = 0$: Let $v_1^* = 1/\sum_{h=1}^{N}\beta_{1h}$, then we can solve prices as $p^* = v_1^*\beta_1$. Due to strict quasiconcavity of the transformation functions and utility functions, prices p^* then uniquely determine the production plan y_j^*, multiplier v_j^* of firm $j{\neq}1$, consumer i's bundle x_i^*, and multiplier λ_i^* Finally, $y_1^* = \sum_{i=1}^{I}\left(x_i^* - e_i\right) - \sum_{j=1}^{J} y_j^*$.

This seed system $\eta(\chi,\theta)$ will be deformed diffeomorphically into $\phi(\chi,\theta)$ via a homotopy while its topological properties are preserved. Define a homotopy $\Phi : \Xi \times [0,1] \times \Theta \to \Re^{I(N+1)+J(N+1)+N-1}$ where $\Phi(\chi,0,\theta) = \eta(\chi,\theta)$ and $\Phi(\chi,1,\theta) = \varphi(\chi,\theta)$.

$$\Phi(\chi,\rho,\theta) = \begin{pmatrix} \rho D_{x_i}u_i(x_i,T_i,p) + (1-\rho)D_{x_i}\hat{u}(x_i) + \varepsilon\alpha_i - \lambda_i p, \forall i \\ p(x_i - e_i) - \rho\sum_{j=1}^{J} s_{ij}py_j, \forall i{\neq}1 \\ p - \rho v_1\left(D_{y_1}f_1(y_1,T_1,p) + \varepsilon\beta_1\right) - (1-\rho)v_1\beta_1 \\ p - v_j\left(\rho D_{y_j}f_j(y_j,T_j,p) + (1-\rho)D_{y_j}\hat{f}_j(y_j) + \varepsilon\beta_j\right), \forall j{\neq}1 \\ \rho f_1(y_1,T_1,p) + (\rho\varepsilon + 1 - \rho)(\beta_1 y_1 + \gamma_1) \\ \rho f_j(y_j,T_j,p) + (1-\rho)\hat{f}(y_j) + \varepsilon\left(\beta_j y_j + \gamma_j\right), \forall j{\neq}1 \\ \sum_{i=1}^{I}(x_i - e_i) - \sum_{j=1}^{J} y_j \end{pmatrix}.$$

The following shows that the preimage of zero in the homotopy is closed.

Lemma 1. $\Phi^{-1}(0) = \{(\chi,\rho,\theta) \in \Xi \times [0,1] \times \Theta | \Phi(\chi,\rho,\theta) = 0\}$ *is closed in* $\Xi \times [0,1] \times \Theta$.

Proof. See the Appendix.

Next we show that solutions to $\Phi(.,.,\theta) = 0$ can be bounded in the interior of a manifold, so that no sequence of solutions approaches the boundary. Let $B^N(r) = \{x \in \Re^N | x \leq r\}$ denote the N-dimensional ball with radius r.

Lemma 2. *For each $\theta \in \Theta$ there is a manifold*

$$E(\theta) = \left(B^{IN}(\bar{r}_\theta) \cap \Re^{IN}_{++}\right) \times B^{I+J(N+1)}(\bar{r}_\theta) \times S^N \subset \Xi \times [0,1]$$

such that the following holds true:
(i) *If $\Phi(\chi,\rho,\theta) = 0$, then $(\chi,\rho) \in E(\theta)$.*
(ii) *If there is a sequence $(\chi_k,\rho_k) \to (\chi',\rho')$ with $\Phi(\chi_k,\rho_k,\theta) = 0$, then $\chi' \notin \partial E(\theta)$.*

Proof. See the Appendix.

In the following, we can safely restrict the domain of $\Phi(.,.,\theta)$ to the manifold $E(\theta)$ and show that there is a solution to $\phi(.,\theta) = 0$ for almost all θ. We need the Jacobian matrices of maps Φ, ϕ, and η to have full rank at value zero; that is, 0 is a regular value for maps Φ, ϕ, and η.[4]

Lemma 3. 0 *is a regular value for $\Phi(.,.,\theta)$ except for θ in a closed set of measure zero in Θ.*

Proof. See the Appendix.

Since the above result holds for all $\rho \in [0,1]$, we have that $D_{(\chi,\theta)}\phi$ has full rank whenever $\phi(\chi,\theta) = \Phi(\chi,1,\theta) = 0$, and $D_{(\chi,\theta)}\eta$ has full rank whenever $\eta(\chi,\theta) = \Phi(\chi,0,\theta) = 0$. Immediately following Lemma 3, we have that 0 is a regular value for both $\phi(.,\theta)$ and $\eta(.,\theta)$ except for θ in a closed set of measure zero in Θ.

Lemma 4. *If* 0 *is a regular value for $\Phi(.,.,\theta)$, $\phi(.,\theta)$, and $\eta(.,\theta)$at $\theta \in \Theta$, then $\phi(.,\theta) = 0$ has a solution.*

Proof. See the Appendix.

4 For a C^r map $f: M \to N$ between manifolds, $y \in N$ is a *regular value* if $Df(x)$ has full rank for all $x \in f^{-1}(y)$.

Therefore, generic in θ, there is a solution to $\phi(.,\theta)=0$. Moreover, all critical value θ in Lemma 4 such that 0 is not a regular value are in a nowhere dense set of Θ. For a critical $\bar{\theta}\in\Theta$, we can find a sequence $\theta_k\to\bar{\theta}$ such that 0 is a regular value for those maps in Lemma 4 at each θ_k, and each θ_k has an associated equilibrium χ_k. Since Lemma 1 shows that $\Phi^{-1}(0)$ is closed, sequence $\chi_k\to\bar{\chi}$, and by continuity $\phi(\bar{\chi},\bar{\theta})=0$ and $\bar{\chi}$ is an equilibrium for $\bar{\theta}$. ■

3 Conclusion

This paper demonstrates that the differentiable approach is a proper tool for treating general externalities, because convexity in the aggregate is not required. Externalities are allowed to influence consumers and firms in arbitrary ways. Utility and production functions can be nonconvex in externalities, and externalities can be price dependent. We study equilibrium of an economy as the intersection of manifolds. An augmented parameter space is chosen for the system to be transversal. We use parameters in utility and production functions which are the primitives of the economy. By examining the first-order conditions of consumers and firms without solving for excess demands, we can bypass aggregate convexity and check the Jacobian matrix of the equilibrium system. The equilibrium exists by showing that the economy can be deformed through a homotopy into a simplified system with a unique solution. Therefore, as long as preferences and production are convex in own activities for fixed levels of externalities, existence of competitive equilibrium obtains under standard assumptions. Our existence is not only generic but holds for all parameters. This result does not rely on the chosen set of particular parameters; any parameter space containing the chosen parameter space will also yield existence and genericity.

Appendix: Proofs

Proof of Lemma 1. Take a sequence $(\chi_k,\rho_k,\theta_k)\to(\bar{\chi},\bar{\rho},\bar{\theta})$ such that $(\chi_k,\rho_k,\theta_k)\in\Phi^{-1}(0)$ for every k. By continuity, $\Phi(\bar{\chi},\bar{\rho},\bar{\theta})=0$. Hence we are left to check that all $\bar{x}_i$ are interior. Since utility functions are differentiably strictly increasing, we can see that the left-hand side of consumer i's first-order condition $\rho D_{x_i}u_i(x_i;T_i,p)+(1-\rho)D_{x_i}\hat{u}(x_i)+\varepsilon\alpha_i=\lambda_i p$ is always strictly positive for small ε, and hence $\bar{p}\gg 0$. We show $\bar{x}_i\notin\Re^N_+\backslash\Re^N_{++}$ for all i in the following. Suppose there is $\bar{x}_{\bar{i}\bar{n}}=0$ for some $\bar{i}$ and some $\bar{n}$. Denote the left-hand side of consumer i's first-order condition as

$$v_i(x_i, \rho, \theta) = \rho u_i(x_i; T_i, p,) + (1-\rho)\hat{u}(x_i) + \varepsilon \alpha_i x_i, \text{ and}$$

$$D_{x_{in}} v_i(x_i, \rho, \theta) = \rho D_{x_{in}} u_i(x_i; T_i, p) + (1-\rho) D_{x_{in}} \hat{u}(x_i) + \varepsilon \alpha_{in}.$$

Notice that the first-order condition says, for all $n \neq \bar{n}$,

$$D_{x_{\bar{i}n}} v_{\bar{i}}\left(\bar{x}_{\bar{i}}, \bar{\rho}, \bar{\theta}\right) = D_{x_{\bar{i}n}} v_{\bar{i}}\left(\bar{x}_{\bar{i}}, \bar{\rho}, \bar{\theta}\right) \frac{\bar{p}_{\bar{n}}}{\bar{p}_n}.$$

Since the price ratio $\bar{p}_{\bar{n}}/\bar{p}_n$ is bounded and away from 0, we can find two points χ' and χ'' in the neighborhood of $\bar{\chi}$. So that χ' is interior by adding ε to $\bar{x}_{\bar{i}\bar{n}}$ and χ'' is on the boundary and $v_i\left(x''_i, \bar{\rho}, \bar{\theta}\right) \geq v_i\left(x'_i, \bar{\rho}, \bar{\theta}\right)$: By continuity, we can find $n' \in \{1, \ldots, N\} \backslash \bar{n}$, a small ε, and two points $\chi' = \left((x'_i)_{i=1}^I, (\bar{\lambda}_i)_{i=1}^I, \left(\bar{y}_j\right)_{j=1}^J, (\bar{v}_j)_{j=1}^J, \bar{p} \right)$ and $\chi'' = \left((x''_i)_{i=1}^I, (\bar{\lambda}_i)_{i=1}^I, \left(\bar{y}_j\right)_{j=1}^J, (\bar{v}_j)_{j=1}^J, \bar{p} \right)$, where for all $i \neq \bar{i}$, $x'_i = x''_i = \bar{x}_i$, and for $\bar{i}$ we have (i) $x'_{\bar{i}n} = x''_{\bar{i}n} = \bar{x}_{\bar{i}n}$ for all $n \neq \bar{n}, n'$, (ii) $x'_{\bar{i}\bar{n}} = \varepsilon$ and $x'_{\bar{i}}$ is interior, and (iii) $x''_{\bar{i}\bar{n}} = \bar{x}_{\bar{i}\bar{n}} = 0$, $x''_{\bar{i}n'}(>\bar{x}_{\bar{i}n'})$ is picked to outweigh the positive amount $x'_{\bar{i}\bar{n}} = \varepsilon$. This violates the boundary condition assumed for utility function since $x''_{\bar{i}n} \in \Re^N_+ \backslash \Re^N_{++}$. ■

Proof of Lemma 2. (i) The following defines the maximum amount of the n-good that can be produced (using all other goods as inputs) by firms in the economy (ρ, θ).

$$\tilde{y}_n(\rho, \theta) = \max_{y_j \in \Re^N} \sum_{j=1}^J y_{jn}$$

$$\text{s.t.} \begin{cases} \rho f_1(y_1, T_1, p) + (\rho\varepsilon + 1 - \rho)(\beta_1 y_1 + \gamma_1) = 0, \\ \rho f_j\left(y_j, T_j, p\right) + (1-\rho)\hat{f}_j\left(y_j\right) + \varepsilon\left(\beta_j y_j + \gamma_j\right) = 0, \forall j \neq 1, \\ \sum_{i=1}^I e_{in'} + \sum_{j=1}^J y_{jn'} \geq 0, \forall n' \neq n. \end{cases}$$

It has a unique solution by strict quasiconcavity. Next, let

$$\tilde{x}(\theta) = \max_{n \in \{1, \ldots N\}, \rho \in [0.1]} \left[\tilde{y}_n(\rho, \theta) + \sum_{i=1}^I e_{in} \right] + 1.$$

This is more than the maximum amount of any n-good potentially available to consumers in economy (ρ, θ) for all ρ. Thus, each x_i is bounded by $B^N(\tilde{x}(\theta)) \cap \Re^N_{++}$, and y_j is bounded by $B^N(\tilde{x}(\theta))$.

Since the values of all x_i and y_j are bounded, the multipliers λ_i and v_j are bounded by the first-order conditions in $\Phi(\chi, \rho, \theta) = 0$. Denote their bounds by $\tilde{\lambda}_i(\rho, \theta)$ and $\tilde{v}_j(\rho, \theta)$. Take the maximum value of all these bounds:

$$\bar{r}_\theta = \max\left[\tilde{x}(\theta), \max_{i=1,\ldots,I,\, j=1,\ldots,J,\, \rho\in[0.1]} \left[\tilde{\lambda}_i(\rho,\theta), \tilde{v}_j(\rho,\theta)\right]\right].$$

We have the manifold $E(\theta)$.

(ii) At the limit (χ', ρ'), we have $\Phi(\chi', \rho', \theta) = 0$. The boundary problem $\chi' \in \partial E(\theta)$ only happens when there is zero consumption in x_i' or a zero price in p'. These are ruled out by Lemma 1. ■

Proof of Lemma 3. We need $D_{(\chi,\rho,\theta)}\Phi$ to have full rank whenever $\Phi(\chi, \rho, \theta) = 0$. And $D_\theta \Phi =$

$$\begin{bmatrix}
\varepsilon I_N & 0 & & & \cdots & & & & & 0 \\
0 & \ddots & 0 & \cdots & & & & & & \\
 & 0 & -\rho p y_1 I_{N-1} & 0 & & & & & & \\
 & \cdots & 0 & -(\varepsilon\rho + 1 - \rho) v_1 I_N & 0 & \cdots & & & & \\
 & & \cdots & 0 & \varepsilon v_j I_N & 0 & \cdots & & & \\
\vdots & & & 0 & & \ddots & & & & \vdots \\
 & \cdots & 0 & (\rho\varepsilon + 1 - \rho) y_1 & 0 & 0 & \varepsilon\rho + 1 - \rho & 0 & \cdots & \\
 & & \cdots & 0 & \varepsilon y_j & 0 & 0 & \varepsilon & 0 & \\
 & & & \cdots & 0 & \ddots & & 0 & \ddots & 0 \\
0 & \cdots & & & \cdots & & & & 0 & -I_N
\end{bmatrix}$$

$\underbrace{\qquad}_{\alpha_i}$ $\underbrace{\qquad}_{s_{-1}}$ $\underbrace{\qquad}_{\beta_1}$ $\underbrace{\qquad}_{\beta_{j,j\neq 1}}$ $\underbrace{\qquad}_{\gamma_1}$ $\underbrace{\qquad}_{\gamma_{j,j\neq 1}}$ $\underbrace{\quad}_{e_1}$

always has full rank, since all the diagonal elements are non-zero and all off-diagonal elements can be eliminated by row operations. Therefore, $D_{(\chi,\rho,\theta)}\Phi$ always has full rank.

Applying the transversality theorem (see Guillemin and Pollack 1974, 68, and Mas-Colell 1985, 320), we have that 0 is generically a regular value for Φ.

Transversality theorem. *Suppose that* $\phi : X \times S \to \Re^m$ *is a* C^r *map where* X, S *are* C^r *boundaryless manifolds with* $r > \max\{0, \dim(X) - m\}$; *let* $\phi_s(x) = \phi(x, s)$, $\phi_s : X \to \Re^m$. *If* $y \in \Re^m$ *is a regular value for* ϕ, *then except for* s *in a set of measure zero in* S, y *is a regular value for* ϕ_s.

Therefore, 0 is a regular value for $\Phi(.,.,\theta)$ except for θ in a set of measure zero. (Notice that the transversality theorem holds for Θ, a parameter space with partial boundary, since the boundary has zero measure.) The set of critical θ such that 0 is not a regular value is actually closed. Suppose there is a sequence of $\theta_k \in \Theta$ with associated solutions $(\chi_k, \rho_k, \theta_k) \in \Phi^{-1}(0)$ such that $\theta_k \to \bar{\theta}$ and $D_{(\chi,\rho)}\Phi(\chi_k, \rho_k, \theta_k)$ has zero determinant (no full rank) for every k. By Lemma 1, there is a limit point $(\bar{\chi}, \bar{\rho}, \bar{\theta}) \in \Xi \times [0,1] \times \Theta$ such that $(\chi_k, \rho_k, \theta_k) \to (\bar{\chi}, \bar{\rho}, \bar{\theta})$. By continuity, $\Phi(\bar{\chi}, \bar{\rho}, \bar{\theta}) = 0$ and $D_{(\chi,\rho)}\Phi(\bar{\chi}, \bar{\rho}, \bar{\theta})$ does not have full rank; $\bar{\theta}$ is also critical. ■

Proof of Lemma 4. We apply the following version of the preimage theorem (Guillemin and Pollack 1974, 60, also Mas-Colell 1985, 38).

Theorem. *Let ϕ be a smooth map of a manifold X with boundary onto a boundaryless manifold Y, and suppose that both $\phi : X \to Y$ and $\partial\phi : \partial X \to Y$ are transversal with respect to a boundaryless submanifold Z in Y. Then the preimage $\phi^{-1}(Z)$ is a manifold with boundary $\partial\{\phi^{-1}(Z)\} = \phi^{-1}(Z) \cap \partial X$, and the codimension of $\phi^{-1}(Z)$ in X equals the codimension of Z in Y.*

We apply this theorem to $\Phi(.,.,\theta)$ with $E(\theta) \times [0,1]$ as X, $E(\theta) \times \{0\} \cup E(\theta) \times \{1\}$ as ∂X, $\Re^{I(N+1)+J(N+1)+N-1}$ as Y, and $\Phi(.,0,\theta) \cup \Phi(.,1,\theta)$ is the boundary $\partial\Phi(.,.,\theta)$. Note that map ϕ is transversal to a point z means that z is a regular value for ϕ. Therefore, we have $\Phi(.,.,\theta)$ and $\partial\Phi(.,.,\theta)$ both transversal to 0.

So, $\Phi^{-1}(0,\theta)$ is a one-dimensional C^1 manifold with boundary, whose boundary is on the boundary of the domain $E(\theta) \times \{0\} \cup E(\theta) \times \{1\}$. We know that there is already a unique boundary point $(\chi^*, 0) \in E(\theta) \times \{0\}$ where $\eta(\chi^*, \theta) = 0$. By the classification theorem of one-dimensional manifolds (Hirsch 1976, 32 and Guillemin and Pollack 1974, 64), this boundary point of $\eta(.,\theta) = 0$ is either part of a closed curve diffeomorphic to $[0,1]$ or a half-open curve diffeomorphic to $[0,1)$. Suppose it is a half-open curve. Then, its open end cannot approach the boundary $\partial E(\theta)$ by Lemma 4 (ii), and this open end cannot be in $E(\theta)$ since this violates continuity of Φ. Thus, $\Phi^{-1}(0,\theta)$ is a closed C^1 curve with another end point $(\chi^{**}, 1) \in E(\theta) \times \{1\}$ where $\phi(\chi^{**}, \theta) = 0$. ■

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