Contributions

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Directed Search with Endogenous Capacity

Abstract: We study a large economy where capacity-constrained sellers post prices to attract buyers. Capacity is costly and each seller chooses how many units to stock, which allows us to study the relationship between search frictions and capacity. We show that no equilibrium exists with linear costs; however, sufficiently convex costs are enough for existence. There may be several equilibria, but we show that the equilibrium payoffs to the agents are unique as well as constrained efficient. Finally, we numerically analyze how the magnitude of search frictions is affected by the capacity choices of the firms.

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1 Introduction

We study a market with a large number of buyers and capacity-constrained sellers. The sellers post prices and the buyers contact them in an uncoordinated fashion. This leads to local excess demand and excess supply. As a result there are sellers and buyers who would like to trade at a given price, but are not able to because they do not meet.¹ The interplay between the sellers' capacities and the buyers' uncoordinated actions, and their relation to the magnitude of the search frictions is the focus of this article.

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¹ Early contributions in this modeling tradition, often referred to as directed search, include Montgomery (1991), Peters (1991), Moen (1997), Shimer (1996) and Burdett, Shi, and Wright (2001), and Julien, Kennes and King (2000).

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It is clear that the magnitude of frictions depends on the capacities of the sellers, i.e. on how many units they stock. The details of this relationship are surprisingly little studied.² We analyze the issue by endogenizing the sellers' choice of capacity. In equilibrium the capacity is determined by the degree of competition, which is captured by the number of sellers as well as by their pricing decisions, and the cost of building capacity. As there is always the possibility of stockout high capacity is attractive to the buyers, and the high-capacity sellers are able to post high prices. Building capacity, however, is costly and the trade-off between the costs and benefits of high capacity plays a key role in the determination of equilibrium. In our model the magnitude of frictions is an equilibrium phenomenon and not imposed by the unit supply assumption.

In more detail, we study a three-stage game where, in the first stage, the sellers choose their capacity, or how many items to stock. In the second stage, they post prices having observed the capacities of the other sellers. In the third stage, the buyers observe the prices as well as the capacities, and contact the sellers based on this information; then the trades are consummated. The setting is of perfect information, and the only difference to the typical settings in the literature comes from allowing the sellers to choose the number of units they stock. The costs of capacity are sunk at the time the sellers learn how many buyers visit them.

We focus on a symmetric equilibrium with free entry. The equilibrium allocation is equivalent to the planner's solution, where the planner chooses the number of sellers and their capacities. As the planner's problem is easier to solve than the equilibrium this result simplifies our analysis.

To show the existence of equilibrium we have to assume that the cost function is sufficiently convex. Indeed, with linear costs we show that no equilibrium exists as there is always a profitable deviation to higher capacity. The reason is that the buyers find high capacity attractive because it reduces the probability of stockout. The increased number of buyers more than compensates for the increased costs of the deviating seller. Finally, we show that posted prices and auctions are payoff equivalent regardless of capacity.

Our main contribution is to provide a tractable setup for understanding trade in directed search markets with endogenous capacity. The distinguishing feature of our work is that capacity choice is made before meetings take place, and consequently the costs are sunk. In many settings this leads to a hold-up problem. The equilibrium is, however, constrained efficient because the buyers observe both the suggested terms of trade, or prices, and the capacities. Investment in capacity is rewarded in the form of more buyers, and the hold-up problem is avoided.

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² We discuss the existing literature addressing firm size and frictions in detail in the next section.

We also demonstrate, with numerical examples, how the level of frictions depends on capacity. We contrast the standard setting where each seller has one unit for sale with the setting where they have many goods in their inventory. In our first example the cost function is the primitive which determines the equilibrium capacity as well as the entry in the market. The equilibrium outcomes are then compared in terms of welfare and overall number of trades. In our second example we isolate the effects of the frictions by exogenously changing the sellers' capacities and the number of sellers, while retaining the overall number of the goods unchanged.

The structure of the paper is as follows. In Section 2 we introduce the model. In Section 3 we analyze the pricing subgame. The focus in Section 3.1 is on the case of symmetric seller capacities, and in Section 3.2 it is assumed that there are sellers of different capacities. In Section 3.3 we discuss the expected utilities of the market participants and note that price posting and auctions are payoff equivalent also with multi-unit capacities. In Section 4.1 we define the free entry equilibrium and show that there is no equilibrium if sellers have a linear cost function. Section 4.2 deals with the planner's problem and in it we show that the solution to the planner's problem coincides with the equilibrium allocation. This result gives us an easy way to calculate the equilibrium outcomes through the planner's problem. In Section 5 analogous results are derived without free entry. In Section 6 we focus on how the welfare in the economy is affected by the cost function and the capacity of the firms. In contrast to the other sections the analysis here is mainly numerical. Section 7 concludes the paper.

In the next section we review the relevant literature. We pay particular attention to the works that allow higher than unit capacity in a search setting.

1.1 Related literature

Unlike the literature on directed search, the literature addressing the trade-off between frictions and firm size or capacity is not very large. We briefly discuss the most relevant contributions.

Burdett, Shi, and Wright (2001) study the effects of capacity choice in a labor market when the firms may have one or two vacancies. They find that firm size affects matching frictions and thus both the equilibrium wages and unemployment levels. This finding is used to explain observed shifts in the Beveridge curve. Their result is indicative but doesn't allow the firms to choose their sizes.

Lester (2010) also studies a labor market setting and endogenizes capacity by allowing the firms to choose between posting one or two vacancies. To get rid of trivial equilibria he assumes that firms face increasing per unit costs; producing

one unit is costless, while producing two units involves a strictly positive production cost. Lester's model is similar to ours both in that capacity is set before search and in that there is curvature in the production costs. The first feature is important in general as sunk capacity costs at the time of pricing makes the firm vulnerable to the hold-up problem. The main difference between our paper and Lester (2010) is that we allow the firms to choose any level of capacity, which allows us to quantify the relationship between frictions and firm size. This is of importance since there are several interesting trade-offs. First, frictions are, of course, at minimum when there is only one large firm. But when the costs of capacity are convex an economy with just one firm is not usually an equilibrium outcome. Further, even without convex costs such a firm would be in a monopoly position, and earn strictly positive profits. With free entry there cannot be positive profits, and a frictionless economy cannot be an equilibrium outcome, either.

Tan (2012) explains why large firms pay higher wages than small firms by assuming that firms have different optimal sizes and that they incur a cost for operating below their capacity. In her model small firms are of size 1 and large firms of size 2. She characterizes the equilibrium in which there are firms of both sizes.

Watanabe (2010) describes a model where sellers can choose to be farmers that produce one good or merchants/middlemen who can store several goods but need to buy them from the farmers. The middlemen are able to restock their goods between periods and buy the leftover goods from farmers at price zero between the periods. The reason for the zero price and lack of competition between merchants at the restocking market is that the discount factor of all agents is zero and thus they do not value future payoffs. Given the capacities of merchants he determines the equilibrium steady state measure of them. Watanabe endogenizes the choice to become a merchant but he does not endogenize the capacity that merchants hold.

In Watanabe (2013) the assumption of the zero discount factor is relaxed. The cost of this relaxation is that the sellers are no longer allowed the choice between remaining as sellers (with capacity at unity) or middlemen; it is assumed that there are always enough sellers for the restocking price to be the sellers' continuation value. The measure of middlemen is then determined for any capacity by imposing a zero profit condition on the middlemen. The middlemen are found to have a positive effect on overall utility. The difference to our model is that we treat capacity as a choice variable.

Geromichalos (2012) analyzes firms that compete by posting production plans in a market closely related to the one analyzed in the current paper, but with one major difference, namely, the timing of production. In his model production is decided and costs are born after demand is realized while in our model costs are born before the seller-specific demand uncertainties are

resolved. The difference is the same as between continuous production (or shops without inventory) and batch production (or shops that can sell a good only if it is in their inventory). Geromichalos shows that the equilibrium production plan of the firms must be efficient given the ex post realizations of demand. This result allows him to solve for the announced maximum capacity without need to take into account how it affects the expected demands facing each firm. (In our model capacity is paid for in advance and this solution method doesn't work). Geromichalos then characterizes the optimal production plans in small and in large markets both with and without free entry.

Hawkins (2013) allows firms to post any number of vacancies, possibly with different wages. Workers are homogeneous and each worker applies to a vacancy. After the vacancy-specific matches are realized firms go on to actually create (and pay for) the vacancies. Then they employ the cheapest workers first. A competitive search equilibrium is shown to exist, and it is characterized when firms announce at most two vacancies. The focus of the paper is on the efficiency properties of the equilibrium; if firms cannot commit to the number of workers to be hired then the equilibrium need not be constrained efficient.

2 The model

We analyze a large market consisting of ex ante homogeneous sellers and homogeneous buyers. The measure of buyers is unity and there is a large measure of potential sellers of whom θ^{-1} are active in the market. Each active seller invests in capacity and posts a binding price. The good is indivisible and the cost of holding k units of capacity is $c(k)$. Each buyer wants to acquire a single unit of the good and can visit only one seller. Buyers value the good at one, sellers at zero.

We simplify the interaction of the agents by assuming a static game in three stages. In stage 1, each active seller acquires a capacity $k \in \mathbb{N}$ at the cost of $c(k)$. In stage 2, the sellers post binding prices. In stage 3, the buyers choose which seller to visit. This is a game of perfect information, as the actions of the previous stages are perfectly observed by the players.³

We capture the frictions by focusing on symmetric equilibrium strategies for the buyers. We further assume that the strategies of both the sellers and the

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³ Allowing the sellers to set both price and capacity in the same stage would not change the symmetric equilibrium outcome although it is possible that new equilibria would arise. The real-world motivation for the separate stages is that pricing decisions are easy to revise whereas capacity/production decisions are not. Therefore sellers could always revise their prices after observing the capacities. We restrict to three stages out of convenience.

buyers are anonymous in that sellers with the same capacity and the same price are treated identically by the buyers and all buyers are treated identically by the sellers.

A seller i chooses his capacity k_i and posts a price p_i , that depends on his capacity and the distribution H of capacities chosen by sellers. The pricing strategy p_i is a map $p_i : \Psi \times \mathbb{Z}_+ \rightarrow [0, 1]$, where $H \in \Psi$, the set of distributions, and $k_i \in \mathbb{Z}_+$.

The buyers choose which seller to visit to maximize their utility. When there are different capacity–price pairs the buyers adjust their behavior so that they are indifferent between visiting the different types of sellers and expect the market utility M from them all. This adjustment of behavior leads to different ratios of buyers to sellers, i.e. expected queue lengths,⁴ that the sellers face. A seller with k goods and price p faces queue length $q_{k,p}(\theta, M)$ that depends on the overall market tightness θ and the market utility M . In the remainder of the paper we simplify the notation to q_k or q when there is no risk of confusion. When the queue length is q the number of buyers that visit a seller is a discrete random variable that has the Poisson distribution with parameter q . The motivation for the Poisson distribution is as follows. The buyers randomize over the identical sellers. When the number of buyers and sellers is finite the number of buyers that a seller meets follows the binomial distribution. Holding the ratio of buyers to sellers constant and letting their numbers tend to infinity the binomial distribution converges to the Poisson distribution. The probability that a seller is visited by exactly j buyers is then $P[x = j] = e^{-q} \frac{q^j}{j!}$.

2.1 Sellers' profit

When all the sellers have the same capacity k and price p the queue length they face is equal to the market tightness, $q = \theta$. If a seller is visited by $i \leq k$ buyers he sells i units, otherwise he sells all k units. The price is p in both cases. The seller's expected payoff is given by

$$\begin{aligned} \pi^s &= p \left[\sum_{i=0}^k e^{-\theta} \frac{\theta^i}{i!} i + k \left(\sum_{i=k+1}^{\infty} e^{-\theta} \frac{\theta^i}{i!} \right) \right] \\ &= p \left[\theta \sum_{i=0}^{k-1} e^{-\theta} \frac{\theta^i}{i!} + k \left(1 - \sum_{i=0}^k e^{-\theta} \frac{\theta^i}{i!} \right) \right] \\ &= p[\theta F_{\theta}(k-1) + k(1 - F_{\theta}(k))], \end{aligned} \tag{1}$$

⁴ In the remainder of the paper we refer to expected queue length as queue length when the meaning is clear.

where $F_\theta(k) = \sum_{i=0}^k e^{-\theta} \frac{\theta^i}{i!}$ is the Poisson cumulative distribution function giving the probability that at most k buyers visit a seller.

2.2 Buyers' utility

A buyer visiting a seller expects the following utility:

$$\begin{aligned} u^b &= (1-p) \left[\sum_{i=0}^{k-1} e^{-\theta} \frac{\theta^i}{i!} + \sum_{i=k}^{\infty} e^{-\theta} \frac{\theta^i}{i!} \frac{k}{i+1} \right] \\ &= (1-p) \left[\sum_{i=0}^{k-1} e^{-\theta} \frac{\theta^i}{i!} + \frac{k}{\theta} \sum_{i=k}^{\infty} e^{-\theta} \frac{\theta^{i+1}}{(i+1)!} \right] \quad [2] \\ &= (1-p) \left[F_\theta(k-1) + \frac{k}{\theta} (1 - F_\theta(k)) \right]. \end{aligned}$$

If a buyer trades, his utility is $(1-p)$. With probability $F_\theta(k-1)$ fewer than k other buyers visit the same seller and he trades with certainty. With probability $\sum_{i=k}^{\infty} e^{-\theta} \frac{\theta^i}{i!}$ at least k other buyers visit the seller. The buyer then trades with probability $\frac{k}{i+1}$, where i is the number of other buyers at same location. The overall probability of trade for the buyer is simplified to $F_\theta(k-1) + \frac{k}{\theta} (1 - F_\theta(k))$.

We point out that even though the directed search setting allows the analysis of the agents' strategic behavior, choice of price or capacity, this is somewhat limited by our large market assumption. The main reason for this is to keep the model tractable as the analysis of markets with a finite number of agents is quite involved. We also believe that the functioning of the large markets is mostly very close to that of small markets. Still, there are differences, and for example our result that there is no equilibrium with linear costs clearly does not hold in the small market where reasonable deviations in capacity are constrained by the number of the buyers.

3 The pricing subgame

3.1 Identical capacities

We begin by analyzing price formation when all the sellers have the same capacity k . To find a symmetric equilibrium in price first assume that the sellers

post price $p(k)$. The queue length that they face is then just the overall market tightness θ . Consider the problem of a single seller who deviates to some price $\tilde{p}(k)$. The expected queue length q that the deviator faces is determined by the indifference condition of the buyers, who in equilibrium expect to receive the market utility M from any seller.⁵ Thus q is given by

$$(1 - p(k)) \left[F_\theta(k-1) + \frac{k}{\theta} (1 - F(k)) \right] = M = (1 - \tilde{p}(k)) \left[F_q(k-1) + \frac{k}{q} (1 - F_q(k)) \right],$$

where the LHS is the expected utility of a buyer visiting a non-deviating seller and the RHS is the expected utility of a buyer visiting a deviator. The deviating seller maximizes his expected profit given the above constraint

$$\max_{\tilde{p}} \tilde{p}(k) [qF_q(k-1) + k(1 - F_q(k))] \text{ s.t. } (1 - \tilde{p}(k)) \left[F_q(k-1) + \frac{k}{q} (1 - F_q(k)) \right] = M.$$

By solving the constraint for

$$\tilde{p}(k) = 1 - \frac{M}{\left[F_q(k-1) + \frac{k}{q} (1 - F_q(k)) \right]},$$

we can rewrite the objective as a function of the queue length q . The deviator thus maximizes

$$\max_q qF_q(k-1) + k(1 - F_q(k)) - Mq.$$

The first-order condition⁶

$$F_q(k-1) - M = 0$$

⁵ The market utility approach goes back to McAfee (1993), Peters (1997), Moen (1997) and Shimer (1996). The idea is that buyers respond to sellers' deviations so that they receive the same utility by going to the deviators as going to the non-deviators, i.e. the market utility.

An alternative approach where the idea is to mimic a finite economy is taken in Godenhielm (2012), where we, following Kultti (2003a), derive the queue lengths off the equilibrium path by analyzing deviations from price p by a small measure ε of sellers that post some other $\tilde{p}$ and then let ε tend to zero.

Galenianos and Kircher (2009) introduce a fraction of ε of noise sellers that post every price in $[0, 1]$ and let ε tend to zero. This way one can define the queue lengths and thus expected profits for any price in the support.

Burdet, Shi and Wright (2001) solve for subgame perfect equilibria in a finite model and let the number of buyers and sellers tend to infinity keeping their ratio constant.

All these approaches yield the same result.

⁶ It is surprisingly simple as $\frac{\partial [qF_q(k-1) + k(1 - F_q(k)) - Mq]}{\partial q} = F_q(k-1) - q \frac{e^{-q} q^{k-1}}{(k-1)!} + k \frac{e^{-q} q^k}{k!} - M = F_q(k-1) - M$.

allows us to express the deviator's optimal price as

$$\tilde{p}(k) = \frac{k(1 - F_q(k))}{k(1 - F_q(k)) + qF_q(k-1)}.$$

For $p(k)$ to be the symmetric equilibrium price we need $\tilde{p}(k) = p(k)$, which leads to $q = \theta$ and⁷

$$p(k) = \frac{k(1 - F_\theta(k))}{k(1 - F_\theta(k)) + \theta F_\theta(k-1)}. \quad [3]$$

We show in the Appendix that the second-order condition holds. The denominator in eq. [3] is equal to the expected number of trades for a seller. The numerator is equal to the probability that more than k buyers visit the seller.

The equilibrium price [3] is decreasing in capacity k , as long as the overall market tightness θ is kept constant. The price is thus lower in an economy where θ^{-1} sellers stock $k+1$ goods than in an economy where they stock k goods. This result, the derivation of which is relegated to the Appendix, intuitively reflects that competition for buyers increases when all sellers have more goods. An interesting observation is that when k tends to infinity, i.e. when the capacity restriction is lifted, the firms engage in Bertrand competition and the price tends to zero. For more on the connection between directed search and Bertrand competition, see Geromichalos (2014).

3.2 Different capacities

When there are sellers of several different capacities the equilibrium prices are derived the same way as before. The buyers have to be indifferent between the sellers and expect the market utility M from any seller. Buyers visiting sellers with capacity k thus expect

$$(1 - p(k)) \left[F_{q_k}(k-1) + \frac{k}{q_k} (1 - F_{q_k}(k)) \right] = M. \quad [4]$$

By repeating the steps of the identical capacities case we solve for the first-order condition of a seller with capacity k that deviates to some price $\tilde{p}(k)$ as

$$F_\beta(k-1) = M, \quad [5]$$

⁷ Note that when all sellers have unit capacity, i.e. $k=1$, then the equilibrium price is $p(1) = \frac{1-e^{-\theta}-\theta e^{-\theta}}{1-e^{-\theta}}$, just as in the standard directed search model.

where β is the queue length of the deviator. The equilibrium price for sellers with capacity k is then

$$p(k) = \frac{k(1 - F_{q_k}(k))}{k(1 - F_{q_k}(k)) + q_k F_{q_k}(k-1)}. \quad [6]$$

The queue length $q_k = \frac{\omega_k}{s_k}$, where s_k is the measure of sellers with capacity k and ω_k is the measure of buyers visiting these sellers. The queue lengths are determined by the market utility condition [4], i.e. by the buyers, who have to be indifferent between visiting the sellers of different capacities, and can be simplified to $F_{q_k}(k-1) = M$ for each capacity k . In addition, the overall measure of buyers in the economy sums up to unity, or $\sum \omega_k = 1$ and the overall measure of sellers sums up to θ^{-1} , or $\sum s_k = \theta^{-1}$, so that the overall market tightness remains θ .

Proposition 1 *The equilibrium price of a seller with capacity k is given by eq. [6].*

Proof. The necessary conditions are as above. We show in the Appendix that there are no profitable deviations from eq. [6]. ■

It immediately follows from Proposition 1 that there is ex ante price dispersion when the sellers are of different size.⁸

3.3 Posted prices and auctions

The expected profit $\pi(k)$ of a seller with capacity k is determined by the following expression:

$$\pi(k) = p(k) [q_k F_{q_k}(k-1) + k(1 - F_{q_k}(k))] - c(k) = k(1 - F_{q_k}(k)) - c(k), \quad [7]$$

where $[q_k F_{q_k}(k-1) + k(1 - F_{q_k}(k))]$ is the expected number of trades, c is the cost function and $p(k)$ is given by eq. [6].

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8 As an example assume that proportion s of the sellers have capacity l and proportion $1 - s$ of the sellers have capacity m . The equilibrium prices are then $p(l) = \frac{l(1 - F_{q_l}(l))}{l(1 - F_{q_l}(l)) + q_l F_{q_l}(l-1)}$ and $p(m) = \frac{m(1 - F_{q_m}(m))}{m(1 - F_{q_m}(m)) + q_m F_{q_m}(m-1)}$. As there are only two capacities the queue lengths can be given as $q_m = \frac{1-\omega}{1-s}\theta$ and $q_l = \frac{\omega}{s}\theta$, where s is the proportion of sellers with capacity l and ω is the proportion of buyers visiting these sellers. The queue lengths are determined by the buyers' indifference condition $F_{q_m}(m-1) = F_{q_l}(l-1)$.

The expected utility of a buyer visiting a seller with capacity k is given by eq. [4]. Substituting in the optimal price $p(k)$ we get

$$(1 - p(k)) \left[F_{q_k}(k-1) + \frac{k}{q_k} (1 - F_{q_k}(k)) \right] = F_{q_k}(k-1), \quad [8]$$

where $F_{q_k}(k-1) + \frac{k}{q_k} (1 - F_{q_k}(k))$ is the probability that the buyer trades. As before, the queue lengths q_k are determined by the market utility condition(s) [5].

Within the directed search literature there are several papers that analyze sellers that compete by offering a larger set of mechanisms than just prices. Kultti (1999) analyzes an economy with homogenous buyers (with unit demand), and sellers with a single good, and finds that auctions and price posting yield the same expected payoffs in large markets. Key to the result is that the buyers observe (and base their bids on) how many other buyers meet with the same seller. The optimal behavior in, for example, first price auctions or second price auctions with no reserve price⁹ is to bid zero as long as no other buyer shows up and to bid one otherwise. Translated to the current model, a buyer bids zero as long as there are at most as many goods as buyers at his auction, and bids one otherwise. The expected utility of a buyer visiting an auction with k goods is then equal to $F_{q_k}(k-1)$, which is the probability that at most $k-1$ other buyers visit the auction. When there are auctions of differing sizes the queue lengths q_k are again determined the market utility condition, which for auctions is $F_{q_k}(k-1) = M$. The buyers' expected utility is thus the same under both trading mechanisms. The bidding behavior of the buyers coupled with the market utility condition implies that the expected profit of sellers (auctioneers) with k goods is $k(1 - F_{q_k}(k)) - c(k)$. Thus, also the sellers expected profits are the same as under price posting. This implies that the revenue equivalence between posted prices and auctions generalizes to the multi-unit case.

Observation *Auctions and posted prices are payoff equivalent.*

The equivalence result holds for large markets where the buyers have linear utility functions. It does not hold for small markets. In small markets sellers prefer auctions to posted prices (see, e.g. Julien, Kennes, and King 2001; Kultti 2003b, or Coles and Eeckhout 2003). Of these papers Coles and Eeckhout (2003) analyze a large set of mechanisms and note that auctions are just an example of

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⁹ Eeckhout and Kircher (2010) show that when sellers compete in second price auctions with reserve prices their optimal strategy is to set the reserve price at zero.

a mechanism where the price depends on the realized ex post demand. Selcuk (2012) shows that if the buyers have concave utility functions, then price posting (fixed prices) is the only constrained efficient mechanism. The reason is that the buyers are willing to pay a premium for fixed prices that act as an insurance against price uncertainty.

4 Free entry

4.1 Free entry equilibrium

We define equilibrium in a standard way (see, e.g. Lester 2010). Then we show existence and describe an algorithm that finds equilibrium. After this we discuss efficiency properties of equilibrium. The free entry equilibrium consists of the following parts.

a) Buyers' optimal choice.

The buyers maximize their expected utility. Given the distribution of different capacity–price pairs $K(k_i, p_i)$, they adjust their behavior in equilibrium so that they are indifferent between visiting different sellers and receive the market utility M from each. This adjustment of behavior leads to different expected queue lengths $q_{k_i, p_i}(\theta, M)$ for the different sellers, so that the expected utility of all buyers is M and $\int q_{k_i, p_i}(\theta, M) dK(k_i, p_i) = \theta$.

b) A seller's optimal price $p^*(k)$ given his capacity k is, following Section 2.1, equal to

$$\arg \max_{p(k)} p(k) \left[q_{k, p(k)}(\theta, M) F_{q_{k, p(k)}(\theta, M)}(k-1) + (1 - F_{q_{k, p(k)}(\theta, M)}(k))k \right].$$

c) A seller's optimal choice of capacity k^* is equal to

$$\begin{aligned} & \arg \max_k \pi_k(M) \\ &= \arg \max_k \left\{ p^*(k) \left[q_{k, p^*(k)}(\theta, M) \cdot F_{q_{k, p^*(k)}(\theta, M)}(k-1) + k(1 - F_{q_{k, p^*(k)}(\theta, M)}(k)) \right] - c(k) \right\} \\ &= \arg \max_k \left\{ k \left(1 - F_{q_{k, p^*(k)}(\theta, M)}(k) \right) - c(k) \right\}. \end{aligned}$$

d) Free entry of sellers

The free entry condition implies that the measure of active sellers θ^{-1} is such that the expected profit of sellers, $\pi_k(\theta, M)$, is zero.

Definition 2 Let $\pi^*(M) = \max_k \{\pi_k(M)\}$. A free entry equilibrium is a distribution $K^*(k_i, p_i)$ of capacities and prices across firms, a market utility M and queue lengths $q_{k_i, p_i}(\theta, M)$ such that (i) $\pi_{k_i}(p_i, M) = \pi_{k_i}^*(M)$ for all (k_i, p_i) such that $dK^*(k_i, p_i) > 0$; (ii) $\pi_{k_i}(p_i, M) \leq \pi_{k_i}^*(M)$ for all (k_i, p_i) such that $dK^*(k_i, p_i) = 0$; (iii) M and $q_{k_i, p_i}(\theta, M)$ constitute a symmetric equilibrium of the third stage subgame where each buyer contacts a seller. The queue length $q_{k_i, p_i}(\theta, M)$ is given by the market utility condition whenever $p < 1 - M$ and is set at zero whenever $p \geq 1 - M$ and (iv) the expected profit of all sellers is zero $\pi_k(\theta, M) = 0$.

The existence of equilibrium requires restrictions on the cost function. We begin by showing that there is no equilibrium¹⁰ with a linear costs.

Proposition 3 There is no equilibrium with constant unit cost.

Proof. Assume any distribution of seller capacities and consider sellers with capacity k . A deviation to $l > k$ by a single seller is profitable if

$$\pi(k) = k(1 - F_{q_k}(k)) - ak < l(1 - F_{q_l}(l)) - al = \pi(l), \quad [9]$$

where, as usual, the queue length q_l that the deviator faces is determined by the buyers' indifference condition $F_{q_k}(k - 1) = M = F_{q_l}(l - 1)$. Let $\pi(k) \geq 0$ and $0 < k < l$. A deviation leads to higher profits if it leads to higher per unit profits. This happens whenever $(1 - F_{q_k}(k)) - a < (1 - F_{q_l}(l)) - a$, which simplifies to

$$F_{q_k}(k) > F_{q_l}(l). \quad [10]$$

The buyers' indifference condition

$$F_{q_k}(k - 1) = F_{q_l}(l - 1), \quad [11]$$

can be written with the help of upper incomplete gamma functions as

$$Q(k, q_k) = Q(l, q_l),$$

where

$$Q(k, q_k) = \frac{\Gamma(k, q_k)}{\Gamma(k)} = \frac{1}{(k-1)!} \int_{q_k}^{\infty} y^{k-1} e^{-y} dy = \sum_{i=0}^{k-1} e^{-q_k} \frac{q_k^i}{i!} = F_{q_k}(k-1), \quad [12]$$

and

¹⁰ For the trivial case of a unit cost of $c \geq 1$ it is obvious that producing $k = 0$ is an equilibrium.

$$Q(l, q_l) = \frac{\Gamma(l, q_l)}{\Gamma(l)} = \frac{1}{(l-1)!} \int_{q_l}^{\infty} y^{l-1} e^{-y} dy = \sum_{i=0}^{l-1} e^{-q_l} \frac{q_l^i}{i!} = F_{q_l}(l-1).$$

We denote the probability $Q(k, q_k)$ by t , where $t \in (0, 1)$. The indifference condition of the buyers can then be written as

$$Q(k, q_k) = t = Q(l, q_l). \quad [13]$$

Now the inverted-regularized incomplete gamma function $Q^{-1}(l, t)$ is the solution in q_l to $Q(l, q_l) = t$. Next define

$$C_1(l, t) \equiv Q(l+1, Q^{-1}(l, t)), \quad [14]$$

where the inverse $Q^{-1}(l, t)$ is just the value of q_l satisfying eq. [13]. Furman and Zitikis (2008) show that the function $C_1(l, t)$ is decreasing in l . To see that this implies eq. [10] first note that for $l = k$ we get $Q^{-1}(l, t) = q_k$. Thus

$$C_1(k, t) = Q(k+1, q_k) = F_{q_k}(k). \quad [15]$$

Similarly $l = k+1$ gives us $Q^{-1}(k+1, t) = q_l$ and we have

$$C_1(k+1, t) = Q(k+2, q_{k+1}) = F_{q_{k+1}}(k+1). \quad [16]$$

Then by $C_1(l, t)$ being decreasing in l it follows that $F_{q_k}(k) > F_{q_k}(k+1)$ implying that eq. [10] holds. Thus the marginal revenue of a seller offering the buyers the market utility is increasing in capacity. This means that given any candidate equilibrium there always exists a profitable deviation to a larger capacity. Therefore no equilibrium with constant marginal cost exists. ■

This result is in line with the intuition that search frictions are eased in submarkets where firms offer a higher capacity. Proposition 3 would be evident if the production decisions of the firms were made after the realization of the firm specific demands as it would then be clear that the sellers would want to satisfy any demand. Here, we show that deviating to a higher capacity results in a higher per unit expected revenue for any seller. The key to this result is that the deviator has to offer the market utility to the buyers.

For intuition we utilize the equivalence result of Section 3.3 and consider the case of competing auctions. From a single buyer's perspective the market utility condition is now the equality between the probability that at most $k-1$ (other) buyers visit a k -good auction and the probability that k (other) buyers visit a $k+1$ -good auction. As shown above, the probability that at most k other buyers visit the k -good auction is then higher than the probability that at most $k+1$ buyers visit the $k+1$ -good auction. Consequently, the probability that the k -good auction receives a revenue of k is lower than the probability that the

$k + 1$ -good auction receives a revenue of $k + 1$. As the alternative for each type of auctions is to receive a revenue of zero (see Section 3.3), the result implies that the expected revenue per unit, and hence the marginal revenue, always increases with a deviation to a higher capacity. Constant marginal costs are therefore not enough to keep the firms “small,” even when they have to bear the costs of production *ex ante*.

A few words of caution regarding the interpretation of the result are appropriate here. Note that the non-existence result regarding the symmetric equilibrium is due to there being an infinite number of buyers in the market. In finite markets there is no incentive for a seller to deviate and offer a capacity larger than the number of buyers in the market. When the number of buyers in the market is infinite this threshold is never reached.

In order to guarantee the existence of equilibrium with an infinite number of buyers and sellers we need to make further assumptions on the cost function. One straightforward way would be to require that the marginal cost would increase faster than $\frac{d(l(1-C_1(l, t^M)))}{dl}$, the marginal revenue of a seller offering the market utility t^M . However, as the exact shape of the $C_1(l, t^M)$ function is hard pin down and the market utility t^M is endogenous this approach is not practical. Instead, we assume that the cost function is convex enough to discourage “too high” capacities, thereby making the set of relevant capacities finite.

Assumption A The cost function $c(k)$ satisfies a) $c(0) = 0$, b) $c'(x) > 0$, c) $c''(x) > 0$ and d) there exists some $\hat{k}$ such that $\frac{c(x)}{x} > 1$ for $x \geq \hat{k}$.

As the buyers value a good at unity any seller with a capacity above $\hat{k}$ would make negative profits even if he sold all his goods. Thus, the set of relevant capacities is finite. It is clear that, e.g. strongly convex cost functions satisfy Assumption A.¹¹

With the definition of equilibrium at hand we next derive the planner's solution. Then we show that any equilibrium allocation coincides with a solution to the planner's problem.

4.2 Planner's problem

The measure of welfare is the number of trades in the economy multiplied by the value of each trade minus the capacity costs. The measure of trades $m(\theta, k)$ in

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¹¹ The authors thank a referee for pointing out that only (d) of assumption A is necessary for existence.

the economy when all θ^{-1} sellers have k goods, is stated as the probability that a buyer makes a trade times the measure of buyers in the economy. This is

$$m(\theta, k) = F_\theta(k-1) + \frac{k}{\theta}(1 - F_\theta(k)) = [\theta F_\theta(k-1) + k(1 - F_\theta(k))]\theta^{-1}. \quad [17]$$

The costs of capacity $c(k)$ are born by the θ^{-1} sellers. Thus welfare is given by

$$S(\theta, k) = m(\theta, k) - \theta^{-1}c(k) = [\theta F_\theta(k-1) + k(1 - F_\theta(k))]\theta^{-1} - c(k)\theta^{-1}. \quad [18]$$

The planner chooses the capacities as well as the measure of sellers offering each capacity and allocates the buyers over the sellers (see, e.g. Shimer 2005). In other words, the planner chooses the queue length for every capacity k , the only constraint being that the measure of buyers sums to unity. As the planner is free to put weight on several capacities the planner's problem is

$$\max_{q_k, s_k} \sum_{k=1}^{\hat{k}} S(q_k, s_k, k) \equiv \max_{q_k, s_k} \sum_{k=1}^{\hat{k}} m(q_k, k) - c(k)s_k,$$

where q_k is the queue length at capacity k , and s_k is the corresponding measure of sellers. Now it is clear that if there are sellers of different capacities in the planner's solution, say capacities, $\bar{k}$ and $\bar{l}$, then the planner chooses the queue lengths $q_{\bar{k}}$ and $q_{\bar{l}}$ so that $S(q_{\bar{k}}, s_{\bar{k}}, \bar{k}) = S(q_{\bar{l}}, s_{\bar{l}}, \bar{l})$, as she could otherwise increase the social welfare by increasing both the measure of sellers and the measure of buyers visiting the sellers generating the higher welfare. Therefore we can without loss of generality restrict the planner to choose identical capacities for all active sellers. As any solution to the planner's problem leads to the same expected welfare all solutions are found by this simplified problem.

The planner's problem is then

$$\max_{k, \theta} S(\theta, k). \quad [19]$$

The first-order condition (FOC) with respect to the overall market tightness θ is simply¹²

12 The intermediate steps are

$$\begin{aligned} \frac{\partial S(\theta, k)}{\partial \theta} &= \frac{\partial [\theta F_\theta(k-1) + k(1 - F_\theta(k)) - c(k)]\theta^{-1}}{\partial \theta} \\ &= -\frac{e^{-\theta}\theta^{k-1}}{(k-1)!} + \frac{k}{\theta} \frac{e^{-\theta}\theta^k}{k!} + \frac{-k(1 - F_\theta(k)) + c(k)}{\theta^2} \\ &= \frac{-k(1 - F_\theta(k)) + c(k)}{\theta^2}. \end{aligned}$$

$$k(1 - F_{\theta}(k)) - c(k) = 0. \quad [20]$$

Note that for different values of k the θ s making the condition true are different. Thus, to avoid confusion, we henceforth let θ_k denote the value of θ satisfying the first-order condition [20] when the sellers have k goods.

Utilizing the first-order condition, the planner's problem simplifies to

$$\begin{aligned} \max_k S(\theta_k, k) &= \max_k [\theta_k F_{\theta_k}(k-1) + k(1 - F_{\theta_k}(k))] \theta_k^{-1} - c(k) \theta_k^{-1} \\ &= \max_k [F_{\theta_k}(k-1)], \end{aligned} \quad [21]$$

where θ_k is the solution of θ in [20]. With a cost function that satisfies Assumption A it is clear that only $k \in \{0, \dots, \hat{k}\}$ are relevant in [21]. Thus, the planner's problem is to pick a capacity coupled with a measure of sellers from a finite set of such pairs and hence the planner's solution exists.

Proposition 4 *A solution to the planner's problem exists.*

Proof. The proof is above. ■

The solution to the planner's problem is straightforward to find. It only requires to pick the (k, θ_k) pair that maximizes welfare. It is in principle possible that there are many such pairs implying that there would be sellers of different capacities. All the solutions still generate the same welfare. We proceed by showing that there is a one-to-one mapping between any planner's solution and free entry equilibria.

Proposition 5 *Any solution to the planner's problem constitutes a free entry equilibrium.*

Proof. Assume that the planner chooses capacity k^* and queue length, θ_{k^*} , i.e. k^* is the solution to

$$\arg \max_k [F_{\theta_k}(k-1)], \quad [22]$$

where, for each capacity k , the queue length θ_k satisfies

$$k(1 - F_{\theta_k}(k)) - c(k) = 0. \quad [23]$$

We show by contradiction that there cannot exist a profitable deviation from k^* for a seller.

Let there exist a profitable deviation from k^* to quantity $k^* + d$. Then

$$\overbrace{(k^*)(1 - F_{\theta_{k^*}}(k^*)) - c(k^*)}^{=0} < (k^* + d)(1 - F_{\theta_{k^*+d}}(k^* + d)) - c(k^* + d). \quad [24]$$

The RHS of eq. [24] is then higher than the profit under the free entry condition for capacity $k^* + d$, i.e.

$$\overbrace{(k^* + d)(1 - F_{\theta_{k^*+d}}(k^* + d)) - c(k^* + d)}^{=0} < (k^* + d)(1 - F_{\theta_{k^*+d}}(k^* + d)) - c(k^* + d).$$

But as

$$\partial \frac{F_q(k)}{\partial q} = -\frac{q^k e^{-q}}{k!} < 0,$$

this means that

$$q_{k+d} > \theta_{k+d}.$$

In words: the queue length of a seller deviating from capacity k to $k + d$ is higher than the queue length (or market tightness) when the planner chooses for all sellers to offer capacity $k + d$. Next we define the buyer utilities $t_{k^*} = F_{\theta_{k^*}}(k^* - 1)$ and $t_{k^*+d} = F_{\theta_{k^*+d}}(k^* + d - 1)$. The queue lengths can be expressed as inverted-regularized incomplete gamma functions $q_{k^*+d} = Q^{-1}(k^* + d, t_{k^*})$ and $\theta_{k^*+d} = Q^{-1}(k^* + d, t_{k^*+d})$. The equation then states that the queue length of a seller deviating from q to $q + d$ satisfies the buyers' indifference condition as both give an expected utility of t_{k^*} . The second equation states that the queue length when the planner chooses $k + d$ is such that welfare (i.e. buyer utility) is t_{k^*+d} . As

$$\frac{\partial Q^{-1}(k + d, t)}{\partial t} < 0$$

$q_{k^*,d} > \theta_{k^*+d}$ implies that

$$F_{\theta_{k^*}}(k^* - 1) < F_{\theta_{k^*+d}}(k^* + d - 1)$$

contradicting the claim that k^* is the solution to

$$\max_k [F_{\theta_k}(k - 1)].$$

■

The first-order condition of the planner's problem with respect to θ is identical to the zero profit condition of the sellers in the free entry equilibrium and welfare is identical to the buyers' expected utility. This finding is next used to show that

there is a one-to-one correspondence between any planner's solution and free entry equilibrium.

Proposition 6 *Any free entry equilibrium is constrained efficient.*

Proof. The proof can be found in the Appendix. ■

Constrained efficiency is, almost, a defining feature in directed search models with fixed capacity, and is therefore not a surprising result. It is, however, not, a priori, obvious that it must hold when sellers are free to choose their capacities, especially as there might exist multiple equilibria. The significance of Propositions 5 and 6 is that they allow us to find any equilibrium by solving the planner's problem, which is much simpler than directly solving for the equilibria.

We further illustrate the connection between equilibrium and the planner's solution with the help of figures. In Figure 1 we analyze three scenarios, or three (k, θ_k) -pairs that satisfy the zero profit condition of the sellers.¹³ The measure of buyers is fixed at unity and the measure of sellers is determined by the zero profit condition. The focus is on a single seller, his capacity is given on the x-axis and his expected profit is given on the y-axis. The queue length that our seller faces given his choice of capacity is as usual determined by the buyers'

Figure 1: A seller's profit given three different utility levels of the buyers.

¹³ The cost function in the example is $c(k) = 0.2k^{1.15}$.

indifference condition. To highlight this we have, for each of the scenarios, drawn a curve along which the utility of the buyers is fixed. The curves can therefore be seen as isoutility curves. The expected profit of our seller varies according to the capacity he chooses and can for each capacity be read off the curve corresponding to the scenario in question. As can be seen, the seller has a profitable deviation (upward along the red curve) from the scenario where all sellers have capacity 10 as well as from the scenario where all have capacity 23 (to the left along the green curve). There is no profitable deviation from the scenario where all sellers have capacity 16 (see the blue dotted curve) as this is the equilibrium capacity.

In Figure 2 we study the planner's problem. The planner maximizes welfare by choosing the measure of sellers and their capacities so that eqs [22] and [23] are satisfied. All allocations satisfying eq. [23] imply zero expected profits to the sellers, therefore the planner's choice corresponds to choosing a point, or actually a (k, θ_k) -pair, on the x -axis. Of the (k, θ_k) -pairs satisfying eq. [23] she then chooses the one(s) corresponding to the highest level of utility for the buyers. As the utility of the buyers increases further down in the figure we move the planner chooses the lowest lying curve that has a common point with the x -axis. In the figure we focus on capacity price pairs $(10, \theta_{10})$, $(16, \theta_{16})$ and $(23, \theta_{23})$ and note that the planner chooses $(16, \theta_{16})$ as this choice corresponds with the highest buyer utility/welfare. The isoutility curve associated with this

Figure 2: A seller's profit given three different utility levels of the buyers.

point cannot cross the x -axis as the planner could otherwise increase welfare by either decreasing the measure of sellers or by choosing another capacity. It follows that no single seller has a profitable deviation from the planner's solution which therefore corresponds to the equilibrium of the decentralized economy.

4.2.1 Uniqueness

We have shown that the free entry equilibrium exists and that any equilibrium is a planner's solution and vice versa. The solution to the planner's problem is to pick the welfare maximizing pair (k, θ_k) satisfying eq. [20] from a finite set. The expected utilities associated with equilibrium or the planner's solution are uniquely pinned down. The solution need not be unique. Indeed, numerical analysis suggests that the equilibrium capacity is either unique or consists of two consecutive capacities.¹⁴ In this case there are two pure equilibria and a continuum of equilibria where both capacities are offered and the sellers and the buyers are distributed so that queue lengths are as in the pure equilibria. The difficulty in establishing that there are no other kind of equilibria (or solutions to the planner's problem), e.g. such that two non-consecutive capacities are offered, is that both the expected marginal revenue of a seller offering buyers the market utility, and the marginal cost function, are positive and upward sloping.¹⁵ Should there exist additional equilibria, then it is clear from eqs [20], [21] and Propositions 5 and 6 that they are payoff equivalent for both the sellers, who earn zero expected profits, and the buyers, who earn the market utility. With this result in hand we can in Section 6 numerically analyze the effects that varying the convexity of the cost function has on the equilibrium capacities as well as the market utility and measure of trades.

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14 For an example of a non-unique solution where $\bar{k}$ different symmetric solutions are possible consider a cost function

$$\begin{cases} c(k) = k(1 - C_1(k, t)), & \text{for } k \leq \bar{k}, \text{ and} \\ c(k) > k & \text{otherwise,} \end{cases}$$

where $C_1(k, t)$ is defined on page 14.

15 See the proof of Proposition 3.

5 Equilibrium without free entry

In this section we fix the measure of sellers. However, each seller can make the choice of producing zero goods, or leave the market, in case production would result in negative expected profits. Therefore, there is, in a sense, free exit.

One can think of this section as describing a market with barriers to entry. Alternatively, it describes the stages after entry, in a game where overall demand is uncertain before the sellers decide on production.

The equilibrium definition from the last section carries over with the obvious change that the measure of sellers θ^{-1} is exogenous. When $\theta^{-1} \geq \theta_{k^*}^{-1}$, i.e. when there are more sellers than in the free entry equilibrium, the solution from the last section is available. Then exactly $\theta_{k^*}^{-1}$ of the sellers produce k^* and the rest produce zero. The sellers are indifferent between the two options as their expected profit in both cases is zero.

When $\theta^{-1} < \theta_{k^*}^{-1}$, i.e. when there are fewer sellers than in the free entry equilibrium, it is harder to find equilibrium. As in the last section, we show that also in this case there is a solution to the planner's problem and that any solution corresponds to an equilibrium of the decentralized economy and vice versa.

In order to solve the game for an arbitrary θ^{-1} we first show that a solution to the planner's problem exists when the cost function is given by assumption A.

Proposition 7 *A solution to the planner's problem exists.*

The planner chooses the proportion of sellers producing each quality and the proportion of buyers visiting these sellers. Assumption A implies that there is a capacity $\bar{k}$ beyond which the average costs per unit are above unity, the value of a match. The planner never chooses such a capacity for any seller. The set of capacities that she considers is therefore finite. Her problem of maximizing welfare by distributing the sellers and buyers over a finite set of capacities is therefore well defined and has a solution.

Next we compare the planner's solution(s) to the equilibrium(s) in the directed search game. We argue that no single agent has, in a directed search game, a profitable deviation from any capacity endowed with a positive measure of agents in the planner's solution. Therefore any solution by the planner constitutes an equilibrium. To be more specific, we show that the first-order conditions of the planner's problem are identical to the equilibrium indifference conditions of the buyers and sellers in the directed search equilibrium.

We assume, for simplicity, that no capacities except k and l have a positive mass of sellers and buyers in the planner's solution.¹⁶ The planner maximizes welfare by determining the proportion of sellers and buyers associated with each capacity

$$\text{Max}_{s,\omega} \left[\left[F_{q_k}(k-1) + \frac{k}{q_k} (1 - F_{q_k}(k)) \right] (1 - \omega) - (1-s)\theta^{-1}c(k) \right. \\ \left. + \left[F_{q_l}(l-1) + \frac{l}{q_l} (1 - F_{q_l}(l)) \right] \omega - s\theta^{-1}c(l) \right],$$

where s is the proportion of sellers producing l and ω is the proportion of buyers visiting these sellers and the queue lengths are $q_k = \frac{1-\omega}{1-s}\theta$ and $q_l = \frac{\omega}{s}\theta$. We get the following first-order condition with respect to s

$$-\frac{k}{q_k^2} (1 - F_{q_k}(k)) \left[\frac{(1-\omega)\theta}{(1-s)^2} \right] (1 - \omega) + \theta^{-1}c(k) \\ - \frac{l}{q_l^2} (1 - F_{q_l}(l)) \left[-\frac{\omega\theta}{s^2} \right] \omega - \theta^{-1}c(l) = 0$$

or

$$-\frac{k}{\theta} (1 - F_{q_k}(k)) + \theta^{-1}c(k) + \frac{l}{\theta} (1 - F_{q_l}(l)) - \theta^{-1}c(l) = 0.$$

The first-order condition with respect to ω is

$$-F_{q_k}(k-1) - \frac{k}{q_k} (1 - F_{q_k}(k)) + \frac{k}{q_k^2} (1 - F_{q_k}(k)) \frac{(1-\omega)\theta}{1-s} \\ + F_{q_l}(l-1) + \frac{l}{q_l} (1 - F_{q_l}(l)) + \frac{l}{q_l^2} (1 - F_{q_l}(l)) \frac{\omega}{s} \theta,$$

which simplifies to

$$-F_{q_k}(k-1) + F_{q_l}(l-1) = 0.$$

Thus $F_{q_k}(k-1) = F_{q_l}(l-1)$ and $k(1 - F_{q_k}(k)) - c(k) = l(1 - F_{q_l}(l)) - c(l)$. But these are the indifference conditions of the buyers and the sellers in the directed search game. Thus, no seller has an incentive to deviate from k to l or vice versa.

Before dealing with other deviations we note that the queue length at l is undefined for $s = 0$, i.e. when the planner chooses not to put any weight on l . To compare the planner's solution to the directed search equilibrium we need to define the queue lengths at capacities on which the planner has not put any weight.

¹⁶ It is clear that this generalizes to any number of capacities.

Assumption B Consider the planner's problem with capacities $K \subset \{0, 1, 2, \dots, \hat{k}\}$, each $k \in K$ with strictly positive mass of sellers. To emphasize the dependence of the queue lengths on the distribution of sellers on the capacities denote $q_k = q(s_k, s_{K \setminus \{k\}}, k)$ where s_k is the proportion of sellers with capacity k and $s_{K \setminus \{k\}}$ is the vector of proportions of sellers with capacities in $K \setminus \{k\}$. If a measure zero of sellers are assigned capacity $m \notin K$ then the corresponding queue length is given by $q_m = \lim_{s_m \rightarrow 0} q(s_m, s_K, m)$.

Note that Assumption B pertains to measure zero of agents and has no effect on welfare. It imposes continuity on the queue length $q_l = q(s, 1 - s, l)$ at $s = 0$ as $q(0, 1, l) = \lim_{s \rightarrow 0} q(s, 1 - s, l)$. The queue length q_l at $s = 0$ is then determined by $F_\theta(k - 1) = F_{q_l}(l - 1)$, which is the indifference condition of the buyers.¹⁷ Next we note that the planner chooses $s = 0$ only if $k(1 - F_\theta(k)) - c(k) \geq l(1 - F_{q_l}(l)) - c(l)$, or when no seller has a profitable deviation to l .

To show that there are no other profitable deviations, from the planner's solution by any seller, consider some capacity m , which is not in the support of the planner's solution. As the planner chooses not to put any weight on this capacity, the FOCs of a problem over capacities k, l and m give us $k(1 - F_{q_k}(k)) - c(k) \geq m(1 - F_{q_m}(m)) - c(m)$ or $l(1 - F_{q_l}(l)) - c(l) \geq m(1 - F_{q_m}(m)) - c(m)$ whenever $F_{q_k}(k - 1) = F_{q_l}(l - 1) = F_{q_m}(m - 1)$. Because a single seller's deviation leaves all queue lengths unaffected the above implies that no seller has a profitable deviation to capacity m . This point is demonstrated in more detail as separate proof in the Appendix.

Proposition 8 Any solution to the planner's problem satisfying Assumption B constitutes an equilibrium.

Proof. The proof can be found above. ■

Proposition 9 Any equilibrium is constrained efficient.

The proof of Proposition 9 is in the Appendix. In it we show that if there are several equilibria, then they must be payoff equivalent to the buyers and hence to the sellers as well. Therefore, the planner is indifferent between all equilibria. Proposition 8 states that any solution to the planner's problem is an equilibrium.

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¹⁷ If the planner chooses just capacities and the buyers choose whom to visit the solution to the planner's problem remains unchanged. In this case imposing the buyers' indifference condition at $s = 0$ is an even clearer choice, and in accordance with the market utility approach.

This, together with the payoff equivalence of all equilibria, implies that they are constrained efficient.

6 Capacity and the cost of frictions

The purpose of this section is to quantify the level of frictions when capacity choice is endogenous. To this end we use specific cost functions and compare the welfare levels and number of trades under equilibrium capacities to the benchmark case (with unit capacities) as well as to a frictionless market.

We begin by numerically analyzing the effects that changing the convexity of the cost function has on equilibrium capacity. More specifically we consider cost functions of the form $c(k) = ak^b$, where $0 < a < 1$ and $b > 1$. It is immediate that the equilibrium capacity is 1 for b high enough. In Figure 3 we plot the equilibrium capacities as functions of b (on the x -axis) for fixed values of a between 0.02 and 0.98. Numerical analysis indicates that the equilibrium capacities grow without limit when b approaches 1. For $b = 1$ there is no equilibrium as the sellers then always have a profitable deviation to a higher capacity as we learned from Proposition 3. The lower the value of the fixed parameter a in a curve the further up and to the right it is. Each curve is decreasing in b .

To analyze the effects of capacity on frictions we pick three cost functions to focus upon.

Figure 3: Equilibrium capacities k plotted as a function of b for values of a between 0.02 and 0.98.

Example 10 Let the cost functions be a) $c(k) = 0.2k^{1.25}$, b) $c(k) = 0.2k^{1.15}$ and c) $c(k) = 0.2k^{1.05}$.

a) The planner's solution satisfies eqs [22] and [23], which we replicate below for the readers benefit. The planner solves

$$\arg \max_k [F_{\theta_k}(k-1)], \quad [22]$$

where, for each capacity k , the queue length θ_k is the solution to

$$k(1 - F_{\theta_k}(k)) - c(k) = 0. \quad [23]$$

The algorithm for finding the planner's solution and hence the free entry equilibrium is quite simple. It is just to solve eq. [23] for θ_k for each capacity for which $\frac{c(k)}{k} < 1$ and pick the (k, θ_k) -pair resulting in the highest welfare $F_{\theta_k}(k-1)$.¹⁸ With the help of Maple we find that the free entry equilibrium capacity in a) is $k = 6$ and the measure of active sellers is $\theta_6^{-1} = 0.182$. The expected utility of the buyers, i.e. welfare, is $F_{\theta_6}(6-1) \approx 0.53$ and the expected measure of trades is $F_{\theta_6}(5) + \frac{6}{\theta_6}(1 - F_{\theta_6}(6)) \approx 0.872$. As each buyer wants to acquire a single good the maximum number of trades in the economy is the minimum of the measure of buyers and the measure of goods in the economy, i.e. $\min\{1, k\theta_k^{-1}\}$. Here $k\theta_k^{-1} > 1$ and 87.2% of the possible trades take place.

In a standard model with capacities set $k = 1$ and the same cost function the free entry measure of sellers is $\theta_1^{-1} = 1.213$. The level of welfare is $F_{\theta_1}(k-1) \approx 0.44$ and the measure of trades is $F_{\theta_1}(k-1) + \frac{k}{\theta_k}(1 - F_{\theta_k}(k)) \approx 0.681$. The welfare is thus 20.8% higher and 28% more trades take place in the model with endogenous capacity choice compared to the model where k is set to 1.

Figure 4 shows welfare levels given the capacities of the sellers. The cost function is $c(k) = 0.2k^{1.25}$ and the free entry condition is assumed to be satisfied for each capacity. Welfare is measured on the vertical axis and the capacity of

¹⁸ When the cost function is not "convex enough" there might exist a large number of capacities for which $\frac{c(k)}{k} < 1$. Then we can use the following algorithm to find the planner's solution. Step 1: set $k = 1$ and solve eq. [22] so that θ_k satisfies eq. [23] for each k from $k = 1$ to $k = \bar{k}$, where $\bar{k}$ is the lowest integer for which $F_{\theta_{\bar{k}}}(\bar{k}-1) > F_{\theta_{\bar{k}+1}}(\bar{k})$. Now $(\bar{k}, \theta_{\bar{k}})$ is a potential solution to the planner's problem. Step 2: To check that $(\bar{k}, \theta_{\bar{k}})$ is a solution to the planner's problem it is enough to repeat step 1 for capacities from $k = \bar{k} + 1$ up to a capacity such that $\frac{c(k)}{k} > (1 - F_{\theta_{\bar{k}}}(\bar{k}-1))$. The reason is that the $C_1(k, t)$ function (defined on page 14) is decreasing and hence $1 - F_{\theta_{\bar{k}}}(\bar{k}-1) > 1 - F_{\theta_{\bar{k}+d}}(\bar{k}+d)$ for any d . Step 3: If no (k, θ_k) -pair leads to a higher welfare than $(\bar{k}, \theta_{\bar{k}})$ then the planner's solution is $(\bar{k}, \theta_{\bar{k}})$. Step 4: If there is a pair $(\hat{k}, \theta_{\hat{k}})$ that gives a higher welfare than $(\bar{k}, \theta_{\bar{k}})$ then repeat step 1 from $k = \hat{k}$ until $\frac{c(k)}{k} > (1 - F_{\theta_{\hat{k}}}(\hat{k}-1))$.

This algorithm comes to a stop before $\frac{c(k)}{k} > 1$.

Figure 4: Planner’s solution.

the sellers on the horizontal axis. The free entry equilibrium capacity $k = 6$ is shown as a blue circle.

- b) With cost function $c(k) = 0.2k^{1.15}$ the free entry equilibrium capacity is $k = 16$ and the measure of sellers $\theta_{16}^{-1} = 6.8237 \times 10^{-2}$. Welfare is 0.60 and 93% of all possible trades take place. The welfare with endogenous capacity choice is now 37.6% higher and 45.3% more trades take place in the model with endogenous capacity compared to the standard model.
- c) With cost function $0.2k^{1.05}$ the free entry equilibrium capacity is $k = 162$ and the measure of active sellers is 0.00647. Welfare is 0.716 which is almost 73% higher than in the standard model and 99.16% of all possible trades take place, which is an increase of more than 54.9% compared to the standard model.

We collect the results in Table 1, where we compare the equilibrium welfare of the examples to both the standard model where the sellers capacity is set at 1 and to a model without frictions. With the above cost function there would be an equal number of buyers and sellers in the model without frictions. All sellers would have one good and the welfare would be 0.8.¹⁹

Table 1 demonstrates that the less convex the cost function is the higher is the equilibrium capacity and the lower is the equilibrium measure of sellers. Welfare and the expected measure of trades increase with the (equilibrium) capacity of the firms. The increase in welfare is most pronounced in the last

19 In a Walrasian equilibrium the price would be 0.2.

Table 1: Convexity of cost functions and welfare.

Cost function	$0.2k^{1.25}$	$0.2k^{1.15}$	$0.2k^{1.05}$
FE capacity	6	16	162
FE measure of sellers	0.18	$6.8 \cdot 10^{-2}$	$6.5 \cdot 10^{-3}$
Aggregate measure of goods	1.08	1.09	1.05
FE welfare	0.53	0.60	0.72
FE measure of trades	0.87	0.93	0.99
FE price	0.39	0.35	0.28
Standard model ($k = 1$): welfare	0.44	0.44	0.44
Standard model: measure of sellers	1.21	1.21	1.21
Standard model: price	0.36	0.36	0.36
Standard model measure of trades	0.64	0.64	0.64
Walrasian price	0.20	0.20	0.20
Walrasian welfare	0.80	0.80	0.80

column where the cost function is the least convex and each firm stocks 162 units. Here the welfare is already remarkably high compared to the prediction of a standard directed search model and is actually closer to the welfare in a frictionless Walrasian market.

Figure 4 and Table 1 demonstrate how the inefficiencies due to frictions are alleviated when we allow firms to choose their capacity. The model allows for the prediction of the number of matches and the welfare, given the cost function and the size distribution of the firms in the economy. In addition, it allows us to quantify the effect that shocks or policies that affect the cost of holding capacity has on prices, welfare and the expected measure of matches.

The example above shows how endogenizing capacity affects welfare and the measure of trades. It doesn't, however, isolate the effect of firm size on frictions. To see why, first remember that the standard case with capacity $k = 1$ can be achieved with a cost function that is convex enough, for example, $c(k) = 0.2k^b$ with a sufficiently large b . Now changing b , e.g. to the values in Table 1, changes not only the frictions but also has a direct effect on welfare and the measure of active firms. (It can be thought of as changing the production technology in addition to changing the frictions.)

To isolate the effect of firm capacity on frictions we exogenously change the capacity of the firms as well as the measure of firms while keeping the aggregate measure of goods fixed at unity. Then we compare the outcomes in terms of expected measure of trades.

When all firms have capacity $k = 1$, and the total measure of goods is unity, 63.2% of all possible trades take place. With firm capacity $k = 2$, the percentage

of buyers that trade is 72.9% and for $k = 6$, it is 83.9%. For $k = 16$ and $k = 162$ the corresponding percentages are 90% and 96.8%. The results show that the capacity of the firms has a drastic effect on frictions. However, search frictions still play a role even when firms have quite high capacities. Even capacities as high as $k = 1,000$ or $k = 10,000$ leave a large amount of potential trade unrealized. We collect the results in Table 2.

Ignoring informational aspects, housing and marriage are prime examples of markets where search frictions are prevalent. In these markets each seller has a single good. In commodities' markets frictions are less of a problem, and stockouts are rather uncommon. Tables 1 and 2 demonstrate this intuition by providing a metric of how important frictions are, in terms of missed trading opportunities, given the seller capacities.

Table 2: Capacity and number of trade.

Seller capacity	1	2	6	16	162	1,000	10,000
Measure of sellers	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{16}$	$\frac{1}{162}$	$\frac{1}{1,000}$	$\frac{1}{10,000}$
Price	0.42	0.44	0.47	0.48	0.49	0.50	0.50
Trades (% of max.)	63.2%	72.9%	83.9%	90%	96.8%	98.7%	99.2%

7 Conclusion

In this paper we derive the equilibrium capacities, or sizes of the inventories, of sellers and the ensuing prices in dispersed markets. We show that there is a one-to-one relationship between equilibrium capacity and the planner's solution both with free entry and a fixed number of sellers. In addition, we show that no equilibrium exists when the cost of capacity is a linear. Equilibrium exists when the number of economically sensible capacities is bounded above, which is the case, for example, with a strongly convex cost function.

Our numerical exercises point out that the frictions are clearly less severe when the capacity is a choice variable than when capacity is imposed at the outset to unity. It looks that markets with severe frictions cannot be achieved in this kind of directed search model without introducing heterogeneity or informational asymmetries.

Our model is static and it is clear that many more issues can be addressed in a dynamic setting. For instance, if the goods are not perishable then after one period of trading there will be a non-degenerate distribution of goods around and it is well possible that the sellers do not stock the inventories up to identical capacities.

Of particular interest would be to consider a fully dynamic model of a labor market. One could then analyze employment levels, wages and the expected duration of unemployment spells as a function of the number of vacancies per firm. A feature of this model is that even if all firms have an identical number of vacancies in the beginning of a period they are different after the location-specific matches are realized. Thus, the firms have different numbers of vacancies in the next period, which leads to a non-degenerate size distribution. The different sized firms then post different wages in order to attract unemployed workers. A version of this model with firm capacity set to 2 can be found in Godenhielm (2012).

Appendix

A.1 Omitted proofs

This is a proof of the claim $p(k) = \frac{k(1-F_\theta(k))}{k(1-F_\theta(k))+\theta F_\theta(k-1)}$ **is decreasing in capacity** k .

Proof.

$$\begin{aligned}
 p(k) > p(k+1) &\Leftrightarrow \frac{1}{p(k+1)} > \frac{1}{p(k)} \\
 &\Leftrightarrow 1 + \frac{\theta F_\theta(k)}{(k+1)(1-F_\theta(k+1))} > 1 + \frac{\theta F_\theta(k-1)}{k(1-F_\theta(k))} \\
 &\Leftrightarrow F_\theta(k)k(1-F_\theta(k)) > F_\theta(k-1)(k+1)(1-F_\theta(k+1)) \\
 &\Leftrightarrow e^{-\theta} \frac{\theta^k}{k!} k(1-F_\theta(k)) > F_\theta(k-1)(k+1)(1-F_\theta(k+1)) \quad [25] \\
 &\quad - F_\theta(k-1)k(1-F_\theta(k)) \\
 &\Leftrightarrow e^{-\theta} \frac{\theta^k}{k!} k(1-F_\theta(k)) > F_\theta(k-1)(1-F_\theta(k)) \\
 &\quad - F_\theta(k-1)(k+1)e^{-\theta} \frac{\theta^{k+1}}{(k+1)!}
 \end{aligned}$$

To see that eq. [25] always holds we first recall that the mean of the Poisson cdf $F_\theta(k)$ is θ and the mode is $\lceil \theta \rceil - 1$ and $\lfloor \theta \rfloor$. In addition the Poisson pmf $e^{-\theta} \frac{\theta^k}{k!}$ is increasing before the mode and decreasing after the mode. Therefore when $k+1 \leq \theta$, the claim holds as $e^{-\theta} \frac{\theta^k}{k!} k > F_\theta(k-1)$ and hence $e^{-\theta} \frac{\theta^k}{k!} k(1-F_\theta(k)) > F_\theta(k-1)(1-F_\theta(k))$. When $k+1 > \theta$, the claim holds as $(1-F_\theta(k)) < (k+1)e^{-\theta} \frac{\theta^{k+1}}{(k+1)!}$, with the result that the RHS of eq. [25] is then negative while the LHS is always positive. ■

This is a proof of the claim $p(k) = \frac{k(1 - F_\theta(k))}{k(1 - F_\theta(k)) + \theta F_\theta(k-1)}$ **approaches zero when k grows without limit.**

Proof. It is immediate that $\theta F_\theta(k-1)$ approaches θ when k grows without limit. It remains to show that $k(1 - F_\theta(k))$ also converges to zero. The Chernoff bound for the tail probability of a Poisson distribution X with parameter θ when $k+1 > \theta$ is given by

$$\Pr(X \geq k+1) = 1 - F_\theta(k) \leq \frac{e^{-\theta}(e\theta)^{k+1}}{(k+1)^{k+1}}$$

and consequently

$$k(1 - F_\theta(k)) \leq k \frac{e^{-\theta}(e\theta)^{k+1}}{(k+1)^{k+1}} < (k+1) \frac{e^{-\theta}(e\theta)^{k+1}}{(k+1)^{k+1}} = \frac{e^{-\theta}(e\theta)^{k+1}}{(k+1)^k}$$

which certainly approaches zero when k grows without bound. ■

Proposition 1 *The equilibrium price of a seller with capacity k is given by eq. [6].*

$$p(k) = \frac{k(1 - F_{q_k}(k))}{k(1 - F_{q_k}(k)) + \theta F_{q_k}(k-1)},$$

where the queue lengths q_k are determined by the market utility condition

$$F_{q_k}(k-1) = M.$$

The overall measure of buyers sums up to unity and the overall measure of sellers sums up to θ^{-1} so that the overall market tightness remains θ .

Proof. We have shown the necessary conditions in the text. Next we show that there are no profitable deviations from eq. [6].

Price $p(k)$, as in eq. [6], gives expected revenue

$$p(k) [q_k F_{q_k}(k-1) + k(1 - F_{q_k}(k))] = k(1 - F_{q_k}(k)), \quad [26]$$

deviating to some $\bar{p}(k)$ gives

$$\bar{p}(k) [\beta F_\beta(k-1) + k(1 - F_\beta(k))],$$

where β is derived from the buyers' indifference condition

$$(1 - p(k)) \left[F_{q_k}(k-1) + \frac{k}{q_k} (1 - F_{q_k}(k)) \right] = (1 - \bar{p}(k)) \left[F_\beta(k-1) + \frac{k}{\beta} (1 - F_\beta(k)) \right].$$

Rewriting we get

$$\begin{aligned} & \frac{q_k F_{q_k}(k-1)}{k(1-F_{q_k}(k)) + q_k F_{q_k}(k-1)} \left[F_{q_k}(k-1) + \frac{k}{q_k} (1-F_{q_k}(k)) \right] \\ &= (1-\bar{p}(k)) \left[F_\beta(k-1) + \frac{k}{\beta} (1-F_\beta(k)) \right] \\ \Leftrightarrow & \\ & F_{q_k}(k-1) = (1-\bar{p}(k)) \left[F_\beta(k-1) + \frac{k}{\beta} (1-F_\beta(k)) \right]. \end{aligned}$$

Thus,

$$\bar{p}(k) = \frac{F_\beta(k-1) + \frac{k}{\beta} (1-F_\beta(k)) - F_{q_k}(k-1)}{F_\beta(k-1) + \frac{k}{\beta} (1-F_\beta(k))}.$$

A deviator expects to get

$$\begin{aligned} & \frac{F_\beta(k-1) + \frac{k}{\beta} (1-F_\beta(k)) - F_{q_k}(k-1)}{F_\beta(k-1) + \frac{k}{\beta} (1-F_\beta(k))} [\beta F_\beta(k-1) + k(1-F_\beta(k))] \\ &= \beta F_\beta(k-1) + k(1-F_\beta(k)) - \beta F_{q_k}(k-1). \end{aligned}$$

For $p(k)$ to be an equilibrium we need

$$k(1-F_{q_k}(k)) \geq \beta F_\beta(k-1) + k(1-F_\beta(k)) - \beta F_{q_k}(k-1). \quad [27]$$

Differentiating the RHS with respect to β

$$\begin{aligned} & \frac{d}{d\beta} (\beta F_\beta(k-1) + k(1-F_\beta(k)) - \beta F_{q_k}(k-1)) \\ &= \frac{d}{d\beta} \left(\sum_1^k e^{-\beta} \frac{\beta^i}{i!} i + \sum_{k+1}^\infty e^{-\beta} \frac{\beta^i}{i!} i - \sum_1^k e^{-q_i} \frac{q_i^{i-1}}{i!} i \beta \right) \\ &= -e^{-\beta} \sum_1^k \frac{\beta^i}{i!} i - e^{-\beta} \sum_{k+1}^\infty \frac{\beta^i}{i!} i + e^{-\beta} \sum_1^k i \frac{\beta^{i-1}}{i!} i + e^{-\beta} \sum_{k+1}^\infty i \frac{\beta^{i-1}}{i!} i - e^{-q_i} \sum_1^k \frac{q_i^{i-1}}{i!} i \\ &= F_\beta(k-1) - F_{q_i}(k-1). \end{aligned} \quad [28]$$

This is zero when $\beta = q_k$. The second-order condition

$$\begin{aligned} & \frac{d}{d\beta} (F_\beta(k-1) - F_{q_k}(k-1)) = -e^{-\beta} \sum_0^{k-1} \frac{\beta^i}{i!} + e^{-\beta} \sum_0^{k-1} \frac{\beta^{i-1}}{(i-1)!} \\ &= -e^{-\beta} \frac{\beta^{k-1}}{(k-1)!} < 0 \end{aligned} \quad [29]$$

is negative meaning that $\beta = q_k$ is a global maximum. Thus there is no profitable deviation from

$$p(k) = \frac{k(1 - F_{q_k}(k))}{k(1 - F_{q_k}(k)) + q_k F_{q_k}(k - 1)}$$

and $p(k)$ is the equilibrium price. ■

Proposition 6 Any free entry equilibrium is constrained efficient.

Proof. If a capacity k with corresponding queue length θ_k is a free entry equilibrium then by definition there cannot exist a profitable deviation from it and by free entry the expected profit of the sellers is zero. Now assume that (k, θ_k) is the equilibrium but that it is not the planner's solution. Then there exists a welfare level $F_{\theta_{k+d}}(k + d - 1)$, associated with some capacity $k + d$, that is higher than $F_{\theta_k}(k - 1)$, the welfare level of the free entry equilibrium. The queue length θ_{k+d} is determined by eq. [20]. Now as $\partial \frac{F_q(l)}{\partial q} = -\frac{q^l e^{-q}}{l!} < 0$, it is possible for the planner to increase the queue length from θ_{k+d} to θ'_{k+d} and thus decrease welfare to $F_{\theta'_{k+d}}(k + d - 1) = F_{\theta_k}(k - 1)$. With these queue lengths the buyers are indifferent between sellers offering the two capacities. However, with queue length θ'_{k+d} the sellers make positive profits as $(k + d)(1 - F_{\theta'_{k+d}}(k + d)) - c(k + d) > 0$. But this implies that (k, θ_k) cannot be an equilibrium in the first place as the sellers have a profitable deviation from it. ■

It is a Proof that the planner always has a solution where all sellers have identical capacities

Proof. Assume that there are two capacities k and l . The planner chooses s_k and s_l and ω

$$\text{Max}_{s_k, s_l, \omega} \left[\begin{aligned} & \left[F_{q_k}(k - 1) + \frac{k}{q_k} (1 - F_{q_k}(k)) \right] \omega - s_k c(k) \\ & + \left[F_{q_l}(l - 1) + \frac{l}{q_l} (1 - F_{q_l}(l)) \right] (1 - \omega) - s_l c(l) \end{aligned} \right] \quad [30]$$

where $q_k = \frac{\omega}{s_k}$ and $q_l = \frac{1 - \omega}{s_l}$. We get the following first-order conditions with respect to s_k and s_l

$$k(1 - F_{q_k}(k)) - c(k) = 0, \text{ and}$$

$$l(1 - F_{q_l}(l)) - c(l) = 0$$

The first-order condition with respect to ω is

$$\begin{aligned}
 & F_{q_k}(k-1) + \frac{k}{q_k}(1 - F_{q_k}(k)) - \frac{k}{q_k^2}(1 - F_{q_k}(k)) \frac{\omega}{s_k} \\
 & - F_{q_l}(l-1) - \frac{l}{q_l}(1 - F_{q_l}(l)) + \frac{l}{q_l^2}(1 - F_{q_l}(l)) \frac{(1-\omega)}{s_l} = 0 \\
 \Leftrightarrow & \\
 & F_{q_k}(k-1) + \frac{k}{\omega}(1 - F_{q_k}(k))s_k - \frac{k}{\omega}(1 - F_{q_k}(k)) \\
 & - F_{q_l}(l-1) - \frac{l}{1-\omega}(1 - F_{q_l}(l))s_l + \frac{l}{(1-\omega)}(1 - F_{q_l}(l)) = 0 \\
 \Leftrightarrow & \\
 & F_{q_k}(k-1) - F_{q_l}(l-1) = 0
 \end{aligned}$$

Thus in an interior solution with capacities k and l and masses ω^* , s_k^* and s_l^*

$$\begin{aligned}
 & \left[F_{q_k^*}(k-1) + \frac{k}{q_k^*}(1 - F_{q_k^*}(k)) \right] \omega^* - s_k^* c(k) \\
 & + \left[F_{q_l^*}(l-1) + \frac{l}{q_l^*}(1 - F_{q_l^*}(l)) \right] (1 - \omega^*) - s_l^* c(l) \\
 & = F_{q_k^*}(k-1)\omega^* + F_{q_l^*}(l-1)(1 - \omega^*) \\
 & = F_{q_l^*}(l-1) = F_{q_k^*}(k-1).
 \end{aligned}$$

It is immediate that this result generalizes to any interior solution as the solutions are bound to be finite and $\sum \omega_i^* = 1$.

Proposition 8 *Any solution to the planner's problem satisfying Assumption B constitutes an equilibrium.*

We complete the argument in the end of the proof of proposition 8 with the following demonstration. For simplicity the focus is on a solution, where all sellers have capacity k .

Proof. Assume that all sellers have capacity k but this is not an equilibrium. Then a seller has a profitable deviation to some capacity m or

$$k(1 - F_\theta(k)) - c(k) < m(1 - F_{\bar{q}_m}(m)) - c(m), \quad [31]$$

where the queue length of the deviator is given by the buyers' indifference condition $F_\theta(k-1) = F_{\bar{q}_m}(m-1)$

Next we look at the objective function of a planner distributing sellers between capacities k and m . We assume the planner lets proportion s of the buyers have capacity m and that ω of the buyers are directed to them so that

$$F_{q_k}(k-1) = F_{q_m}(m-1), \quad [32]$$

where $q_k = \frac{1-\omega}{1-s}\theta$ and $q_m = \frac{\omega}{s}\theta$. We totally differentiate eq. [32] to get

$$d\omega \left[\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} \frac{\theta}{1-s} + \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} \frac{\theta}{s} \right] + ds \left[-\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} \frac{1-\omega}{(1-s)^2} \theta - \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} \frac{\omega}{s^2} \theta \right] = 0.$$

Thus

$$\begin{aligned} \frac{d\omega}{ds} &= \frac{\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} \frac{1-\omega}{(1-s)^2} + \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} \frac{\omega}{s^2}}{\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} \frac{1}{1-s} + \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} \frac{1}{s}} \\ &= \frac{\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} q_k \frac{1}{(1-s)} + \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} q_m \frac{1}{s}}{\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} \frac{1}{1-s} \theta + \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} \frac{1}{s} \theta} \cdot \frac{s}{s} \\ &= \frac{\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} q_k \frac{s}{(1-s)} + \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} q_m}{\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} \frac{s}{1-s} \theta + \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} \theta}. \end{aligned} \quad [33]$$

The planner's objective is

$$\text{Max}_s \left[\begin{aligned} & \left[F_{q_k}(k-1) + \frac{k}{q_k} (1 - F_{q_k}(k)) \right] (1-\omega) - (1-s)c(k) \\ & + \left[F_{q_m}(m-1) + \frac{l}{q_m} (1 - F_{q_l}(l)) \right] \omega - sc(l) \end{aligned} \right]. \quad [34]$$

Differentiating eq. [34] with respect to s we get

$$\begin{aligned} & -\frac{k}{\theta} (1 - F_{q_k}(k)) + \theta^{-1} c(k) + \frac{m}{\theta} (1 - F_{q_m}(m)) - \theta^{-1} c(m) \\ & + [-F_{q_k}(k-1) + F_{q_m}(m-1)] \frac{d\omega}{ds}. \end{aligned} \quad [35]$$

Next evaluate eq. [35], the planner's objective's derivate, at $s=0$. We denote q_m at $s=0$ as $q(0,1,m)$. Then, following Assumption B, $q(0,1,m) = \lim_{s \rightarrow 0} q(s,1-s,m)$. When s tends to zero so does ω and hence $q_k = \lim_{s \rightarrow 0} q(1-s,s,k) = \frac{1-\omega}{1-s}\theta = \theta$. Therefore eq. [32] at $s=0$ is $F_{\theta}(k-1) = F_{q(0,1,m)}(m-1)$, which implies that $q(0,1,m) = \bar{q}_m$ and hence eq. [32] is equivalent to the indifference condition of the buyers in eq. [31]. In addition, we note that when s is zero

$$\frac{d\omega}{ds}\bigg|_{s=0} = \lim_{s \rightarrow 0} \frac{\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} q_k \frac{s}{(1-s)} + \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} q_m}{\frac{e^{-q_k} q_k^{k-1}}{(k-1)!} \frac{s}{1-s} \theta + \frac{e^{-q_m} q_m^{m-1}}{(m-1)!} \theta} = \frac{\frac{e^{-q_m} q_m^{m-1}}{(m-1)!} q_m}{\frac{e^{-q_m} q_m^{m-1}}{(m-1)!} \theta} = \frac{\bar{q}_m}{\theta},$$

which is positive and finite. Hence eq. [32] implies that $[-F_{q_k}(k-1) + F_{q_m}(m-1)] \frac{d\omega}{ds} = 0$. Evaluated at $s = 0$ eq. [35] is therefore equivalent to

$$\frac{k}{\theta}(1 - F_{\theta}(k)) + \theta^{-1}c(k) + \frac{m}{\theta}(1 - F_{q_m}(m)) - \theta^{-1}c(m).$$

We assumed that eq. [31] holds, or that it is not an equilibrium for all sellers to have capacity k . Therefore eq. [35], evaluated at $s = 0$, is positive, which contradicts the claim that all sellers have capacity k in the planner's solution. ■

Proposition 9 *Any equilibrium is constrained efficient.*

We show that all equilibria are payoff equivalent. Because the equilibrium conditions and the planner's first-order conditions coincide (see Proposition 8, page 24), the planner is indifferent between any equilibria, and all equilibria are constrained efficient.

In the first part of the proof we show that there cannot exist more than one pure strategy equilibrium. In the second part we show that if there are many equilibria (mixed or pure) then they must be payoff equivalent.

Proof. Assume that there are two equilibria, A and B . It is clear that if both the sellers and the buyers are better off in, say A , then the other cannot be an equilibrium. Assume that the sellers are better off in A and the buyers are better off in B . We denote the buyers' utility by t .

Let $t = \bar{t}$ in A and let $t = \hat{t}$ in B , where $\bar{t} < \hat{t}$

Assume that all the sellers have capacity h in A and that all sellers have capacity k in B , where $h < k$. For any t the buyers' indifference condition gives the queue length α for sellers with capacity h and β for sellers with capacity k , or

$$F_{\alpha}(h-1) = t = F_{\beta}(k-1),$$

which can be written with the help of the upper incomplete gamma functions as

$$F_{\alpha}(h-1) = Q(h, \alpha) = t = F_{\beta}(k-1) = Q(k, \beta), \quad [36]$$

as described on page 13.

For A to be an equilibrium the following inequality has to be satisfied:

$$h(1 - F_\alpha(h)) - c(h) - k(1 - F_\beta(k)) + c(k) \geq 0, \quad [37]$$

at $t = \bar{t}$.²⁰

We rewrite eq. [37], as

$$h(1 - Q(h + 1, Q^{-1}(h, t))) - c(h) - h(1 - Q(k + 1, Q^{-1}(k, t))) + c(k) \geq 0, \quad [38]$$

where, as on page 14, $Q^{-1}(h, t)$ is the inverted-regularized incomplete gamma function, $Q^{-1}(h, t) = \alpha$ and $Q^{-1}(k, t) = \beta$.

Differentiating eq. [38] with respect to the buyers' utility t we get

$$\begin{aligned} & \left(h \frac{e^{-Q^{-1}(h, t)} Q^{-1}(h, t)^h}{h!} \right) \left(-e^{Q^{-1}(h, t)} Q^{-1}(h, t)^{-(h-1)} (h-1)! \right) \\ & + \left(k \frac{e^{-Q^{-1}(k, t)} Q^{-1}(k, t)^k}{k!} \right) e^{Q^{-1}(k, t)} Q^{-1}(k, t)^{-(k-1)} (k-1)! \\ & = -Q^{-1}(h, t) + Q^{-1}(k, t) = -\alpha + \beta, \end{aligned} \quad [39]$$

which is strictly positive, when $h < k$, as can be seen from eq. [36]. As eq. [38] is non-negative for $t = \bar{t}$ and its derivative with respect to t is positive for any t in the interval $[\bar{t}, \hat{t}]$, it must be positive also at $t = \hat{t}$. Hence B cannot be an equilibrium, as $h(1 - F_\alpha(h)) - c(h) - k(1 - F_\beta(k)) + c(k) \geq 0$, at $t = \hat{t}$.

Thus, there cannot exist two or more pure strategy equilibria. We next show that if there are many equilibria (e.g. in mixed strategies), then they must be payoff equivalent.

We again consider two potential equilibria, A and B , with the buyers' utility being higher in B .

If all capacities in the support of B are higher than the capacities in the support of A it is clear that the buyers are better off in B as there are then more goods per buyer in B . Then we can replicate steps [36]–[39] above and $-\alpha + \beta$ in eq. [39] is again positive, which implies that B is not an equilibrium.

In case not all capacities in the support of B are higher than the capacities in the support of A we denote the lowest capacity in the support of A by h and the highest capacity in B by k . Then we can, again, replicate the steps [36]–[39], with the same result.

²⁰ Note that when all the sellers have capacity h , then $\alpha = \theta$.

Thus, both A and B can be equilibria only if they lead to the same expected utilities to the buyers. But in this case eq. [38] has to be binding, and hence the sellers are also equally well off. Proposition 8 states that any solution to the planner's problem is an equilibrium. This, together with the payoff equivalence of all equilibria, implies that they are constrained efficient. ■

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