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Bending analysis of laminated composite and sandwich beams according to refined trigonometric beam theory

Abstract: A trigonometric beam theory (TBT) is developed for the bending analysis of laminated composite and sandwich beams considering the effect of transverse shear deformation. The axial displacement field uses trigonometric function in terms of thickness coordinate to include the effect of transverse shear deformation. The transverse displacement is considered as a sum of two partial displacements, the displacement due to bending and the displacement due to transverse shearing. Governing equations and boundary conditions are obtained by using the principle of virtual work. To demonstrate the validity of present theory it is applied to the bending analysis of laminated composite and sandwich beams. The numerical results of displacements and stresses obtained by using present theory are presented and compared with those of other trigonometric theories available in literature along with elasticity solution wherever possible.

Keywords: bending; shear deformation; shear correction factor; shear stress; trigonometric; laminated; sandwich

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1 Introduction

Structural components made up of fibrous composite materials are increasingly being used in various engineering applications due to their attractive properties in strength, stiffness, and lightness. Common examples of fibrous composite materials are graphite-epoxy, Kevlar-epoxy, Carbon-epoxy, boron-aluminum, etc. Since the fibrous composite materials have a high extensional modulus to shear modulus ratio, shear deformation in beams made of such materials tend to be significant.

Beams are one of the fundamental structural or machine components. Composite beams are lightweight structures that can be found in many diverse applications including civil, aerospace, submarine, medical equipment and automotive industries. Buildings, steel framed structures and bridges are examples of beam applications in civil engineering. In these applications, beams exist as structural elements or components supporting the whole structure. In addition, the whole structure can be modeled at a preliminary level as a beam. For example, a high rise building can be modeled as a cantilever beam, or a bridge modeled as a simply supported beam. In addition, the whole wing of a plane is often modeled as a beam for some preliminary analysis. Innumerable other examples in these and other industries of beams exist.

The Euler-Bernoulli beam theory which is known as classical beam theory (CBT) usually predicts accurate bending behaviour of thin beams but it is inaccurate while predicting bending behaviour of thick beams where shear deformation is more significant. The Timoshenko beam theory [1] which is known as the first order shear deformation beam theory (FSDT) remedies this deficiency by permitting shear deformation to occur. Since the FSDT violates the zero shear stress conditions on the top and bottom surfaces of the beam, a shear correction factor is required to appropriately represent the strain energy of shear deformation.

To remove the discrepancies in the CBT and FSDT and to predict the realistic bending behaviour of laminated composite beams, higher order shear deformation theories are required. These theories are developed by assuming higher order variation of axial displacement. Many displacement based equivalent single layer shear deformation theories are available in the literature for the bending, buckling and free vibration analysis of laminated com-

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posite beams such as: Levinson [2, 3], Reddy [4], Krishna Murty [5], Ghugal and Shimpi [6], Soldatos [7], Karama et al. [8] and many more. All these theories are represented by single unified shear deformation theories by Sayyad [9] and Sayyad et al. [10]. Vo and Thai [11] presented two displacement based refined shear deformation theories to predict the static behaviour of laminated composite beams. Chakrabarti et al. [12] have developed a new finite element model based on zigzag theory for the bending analysis of laminated composite and sandwich beams. Aguiar et al. [13] presented mixed and displacement based finite element models for static analysis of composite beams of different cross-sections using equivalent single layer theories. Onate et al. [14] presented a new simple linear two-noded beam element adequate for the analysis of composite laminated and sandwich beams based on the Timoshenko beam theory and the refined zigzag kinematics proposed by Tessler et al. [15]. Carrera and Giunta [16] proposed several axiomatic refined theories based on a unified formulation for the linear static analysis of isotropic beams. Carrera et al. [17] presented hierarchical beam elements on the basis of the Carrera Unified Formulation for the linear static analysis of beams made of isotropic materials. A new higher-order shear deformation theory for the analysis of laminated composite beams based on global-local superposition technique is developed by Wanji and Zhen [18]. The theory satisfies the free surface conditions and the geometric and stress continuity conditions at interfaces. Kapuria et al. [19] have developed a new zigzag theory for the bending buckling and vibration analysis of laminated composite and sandwich beams and obtained solution is compared with 2D elasticity solution for various loading cases. Catapano et al. [20] developed several hierarchical theories based on Carrera's Unified Formulation for the static analysis of laminated composite beams. Icardi [21] developed three dimensional zigzag theory for the thick laminated composite and sandwich beams with general lay-up. Subramanian [22] carried out finite element analysis of symmetric laminated composite beams using a two noded C^1 continuity element having 8 degrees of freedom per node. Reddy [23] reformulated various beam theories using the nonlocal differential constitutive relations of Eringen for the bending, vibration and buckling analysis of beams. Goyal and Kapania [24] developed a shear-deformable beam element for the analysis of laminated composites to study the static and dynamic response of un-symmetrically laminated composite beams. Wanji et al. [25] developed a modified couple stress theory for composite laminated beam using first order shear deformation. Shi and Voyiadjis [26] developed a new sixth-order beam theory for the analysis of shear flexible beams. A two dimensional General Higher-order Equivalent Single Layer (GHESL) approach, based on the Carrera Unified Formulation (CUF) is proposed by Tornabene et al. [27] for the static analysis of doubly-curved laminated composite shells and panels. Pagani et al. [28, 29] formulated refined beam theories based on CUF for free vibration analysis of laminated composite beams and plates. Viola et al. [30, 31] presented a third order shear deformation theory for the static analysis of laminated composite and functionally graded truncated conical shells and panels. Natarajan et al. [32] applied sinusoidal shear deformation theory for the bending and free vibration analysis of cross-ply laminated composite plates using CUF and isogeometric analysis. Mohazzab and Dozio [33] carried out free vibration analysis of laminated composite cylindrical and spherical panels according to equivalent single layer and layerwise theories of variable order based on spectral collocation method.

2 Trigonometric theories available in the literature

The theory which uses trigonometric functions in terms of thickness coordinates to include effect of transverse shear deformation is designated as trigonometric theory. The kinematics of the trigonometric theory is much richer than those of the other higher order shear deformation theories, because if the trigonometric term is expanded in power series, the kinematics of higher order theories are implicitly taken into account to good deal of extent. These theories are classified as equivalent single layer (ESL) theories, layerwise (LW) theories or zig-zag (ZZ) theories.

2.1 Equivalent single layer trigonometric theories

In the equivalent single layer displacement based theories one single expansion for each displacement component is used through the entire thickness of the laminate. Ghugal and Shimpi [6] developed an equivalent single layer trigonometric theory without considering effect of transverse normal strain for isotropic beams which is further extended by Arya et al. [34] for laminated composite beams. The displacement field of the theory is

$$u(x, z) = u_0(x) - z\frac{dw_0}{dx} + \frac{h}{\pi}\sin\frac{\pi z}{h}\phi(x), \quad w(x) = w_0(x)$$

where u_0, w_0 and ϕ are the unknown functions. Sayyad and Ghugal [35] have developed an equivalent single layer trigonometric shear deformation theory considering effects of transverse shear and normal strain and applied it for the bending analysis of laminated composite beams subjected to various static loadings. The displacement field of the theory is

$$u(x,\ z) = u_0(x) - z\frac{dw_0}{dx} + \frac{h}{\pi}\sin\frac{\pi z}{h}\phi(x),$$
$$w(x,z) = w_0(x) + \frac{h}{\pi}\cos\frac{\pi z}{h}\xi(x)$$

where u_0, w_0, ϕ and ξ are the unknown functions.

2.2 Layerwise trigonometric theory

In layerwise theories each layer is treated as a separate beam and discrete layer transverse shear effects are introduced into the assumed displacement field. Shimpi and Ghugal [36] have developed layerwise trigonometric shear deformation theory for the bending analysis of antisymmetric laminated beams and obtained solution by using Navier solution technique which is further extended by Ghugal and Shinde [37] for the flexural analysis of antisymmetric laminated beams with various boundary conditions by using general solution technique. The displacement field of the theory is

$$u^1(x,z) = -(z-\alpha h)\frac{dw_0}{dx} + h\left[C_1 + C_2\sin\left(\frac{\pi}{2}\frac{z/h-\alpha}{0.5+\alpha}\right)\right]\phi(x)$$

$$u^2(x,z) = -(z-\alpha h)\frac{dw_0}{dx} + h\left[C_3 + \sin\left(\frac{\pi}{2}\frac{z/h-\alpha}{0.5+\alpha}\right)\right]\phi(x),$$

$$w(x) = w_0(x)$$

Here u^1 and u^2 are the axial displacement components in the x direction, superscripts 1 and 2 refer to layer 1 and layer 2; $w(x)$ is the transverse displacement in the z–direction and C_1, C_2, C_3 and α are the constants. The function $\phi(x)$ is a rotation function or the warping function of the cross-section of the beam. The limitation of this theory is that, it is applicable for two layered antisymmetric laminated composite beams only.

2.3 Zig-zag trigonometric theory

In the zig-zag theories, the zig-zag form of inplane displacement distribution through the thickness is assumed. Arya et al. [34] developed zig-zag trigonometric theory for the bending analysis of symmetric laminated composite beams and compared numerical results with equivalent single layer trigonometric shear deformation theory. The displacement field of the theory is

$$u^k(x,z) = u_0 - z\frac{dw_0}{dx} + \left(A^k + zB^k + \sin\frac{\pi z}{h}\right)\eta_x,$$
$$w(x) = w_0(x)$$

where u^k is axial displacement in k^{th} layer, $w(x)$ is transverse displacement in z-direction and η_x is higher order term.

3 Present trigonometric beam theory (TBT)

Consider a beam of rectangular cross section (b x h) and length 'L' as shown in Fig. 1. The beam consists of 'N' number of layers perfectly bounded together. The beam is made up of linearly elastic orthotropic material and z-direction is taken positive in downward direction. The beam under consideration occupies the region $0 \le x \le L$, $-b/2 \le y \le b/2$, $-h/2 \le z \le h/2$ in Cartesian coordinate system. The displacements in x and z directions are represented by u and w respectively.

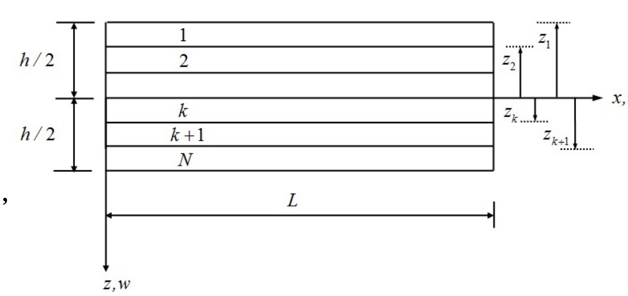

Figure 1: Geometry and coordinate system of laminated beam.

The present equivalent single layer trigonometric beam theory has the following noteworthy features

1. The transverse displacement (w) is considered as a sum of two partial displacements, the displacement due to bending w_b and the displacement due to transverse shearing w_s.
2. The bending component of axial displacement is similar to that given by the CBT
3. The shear component of axial displacement gives rise to the higher-order variation of shear strain.
4. The theory satisfies the zero transverse shear stress conditions on the top and bottom surfaces of the beam and thus obviates the need of shear correction factor.

The displacement field of the present theory is:

$$u(x,z) = u_0(x) - z\frac{dw_b}{dx} - \left[z - \frac{h}{\pi}\sin\frac{\pi z}{h}\right]\frac{dw_s}{dx},$$
$$w(x) = w_b(x) + w_s(x) \tag{1}$$

where u_0 is the axial displacement of the center line of the beam, w_b and w_s are the bending and shear components of transverse displacement along the center line of the beam, respectively. Normal and shear strains corresponding to assumed displacement field of the present theory are obtained within the framework of linear theory of elasticity. The non-zero strain components at any point in the beam are

$$\varepsilon_x = \varepsilon_x^0 + z\,k_x^b + f(z)\,k_x^s \quad \text{and} \quad \gamma_{zx}^k = k_{zx}^0\, g(z) \tag{2}$$

where

$$\varepsilon_x^0 = \frac{du_0}{dx},\quad k_x^b = -\frac{d^2 w_b}{dx^2},\quad k_x^s = -\frac{d^2 w_s}{dx^2},\quad k_{zx}^0 = \frac{dw_s}{dx},$$
$$f(z) = \left[z - \frac{h}{\pi}\sin\frac{\pi z}{h}\right],\quad g(z) = \cos\left(\frac{\pi z}{h}\right) \tag{3}$$

where ε_x^0 is the axial strain, $k_x^b$and k_x^s are the bending curvatures. The state of stress in k^{th} layer is given by

$$\sigma_x^{(k)} = Q_{11}^{(k)}\,\varepsilon_x \quad \text{and} \quad \tau_{zx}^{(k)} = Q_{55}^{(k)}\,\gamma_{zx} \tag{4}$$

where k is the layer number, $Q_{11}^{(k)}$ is the Young's modulus in the axial direction of fibre, $Q_{55}^{(k)}$ is the shear modulus. The force and the moment resultants are defined in the following form:

$$N = \int_{-h/2}^{h/2} \sigma_x^{(k)}\, dz,\quad M = \int_{-h/2}^{h/2} \sigma_x^{(k)}\, z\, dz,$$
$$P = \int_{-h/2}^{h/2} \sigma_x^{(k)} f(z)\, dz,\quad Q = \int_{-h/2}^{h/2} \tau_{zx}^{(k)}\, g(z)\, dz \tag{5}$$

where N and Q are the force resultants, M and P are the moments resultants. The principle of virtual work is used to obtain the governing equations and boundary conditions associated with the present theory. The analytical form of the principle of virtual work is:

$$b\int_0^L\int_{-h/2}^{h/2}\left(\sigma_x^{(k)}\delta\varepsilon_x + \tau_{zx}^{(k)}\delta\gamma_{zx}\right) dz\,dx - \int_0^L q\,(\delta w_b + \delta w_s)\,dx = 0 \tag{6}$$

Using the equations (2) – (5), the principle of virtual work can be rewritten as:

$$b\int_0^L\int_{-h/2}^{h/2}\sigma_x^{(k)}\left(\frac{d\delta u_0}{dx} - z\frac{d^2\delta w_b}{dx^2} - f(z)\frac{d^2\delta w_s}{dx^2}\right) dz\,dx$$
$$+b\int_0^L\int_{-h/2}^{h/2}\tau_{zx}^{(k)} g(z)\frac{d\delta w_s}{dx}\,dz\,dx - \int_0^L q\,(\delta w_b + \delta w_s)\,dx = 0 \tag{7}$$

In terms of the force and the moment resultants it can be written as follows:

$$\int_0^L\left(N\frac{d\delta u_0}{dx} - M\frac{d^2\delta w_b}{dx^2} - P\frac{d^2\delta w_s}{dx^2} + Q\frac{d\delta w_s}{dx}\right) dx$$
$$-\int_0^L q\,(\delta w_b + \delta w_s)\,dx = 0 \tag{8}$$

where δ is the variational operator. Integrating equation (8) by parts and collecting the coefficients of $\delta u_0, \delta w_b$ and δw_s, one can obtain the governing equations and boundary conditions of the beam associated with the present theory using fundamental lemma of calculus of variations. The variationally consistent governing equations of the present theory in terms of force and moment resultants are as follows:

$$\delta u_0:\ \frac{dN}{dx} = 0$$
$$\delta w_b:\ \frac{d^2M}{dx^2} + q = 0 \tag{9}$$
$$\delta w_s:\ \frac{d^2P}{dx^2} + \frac{dQ}{dx} + q = 0$$

The following sets of boundary conditions at the ends $x = 0$ and $x = L$ of the beam are obtained after the integration of equation (8)

$$\begin{array}{llll} \text{Either} & N = 0 & or & u_0 = 0 \\ \text{Either} & dM/dx = 0 & or & w_b = 0 \\ \text{Either} & M = 0 & or & dw_b/dx = 0 \\ \text{Either} & dP/dx + Q = 0 & or & w_s = 0 \\ \text{Either} & P = 0 & or & dw_s/dx = 0 \end{array} \tag{10}$$

Using equations (5), the governing equations (9) can be written in terms of displacement variables as:

$$\delta u_0:\ -A_{11}\frac{d^2u_0}{dx^2} + B_{11}\frac{d^3w_b}{dx^3} + C_{11}\frac{d^3w_s}{dx^3} = 0$$
$$\delta w_b:\ -B_{11}\frac{d^3u_0}{dx^3} + D_{11}\frac{d^4w_b}{dx^4} + E_{11}\frac{d^4w_s}{dx^4} = q$$
$$\delta w_s:\ -C_{11}\frac{d^3u_0}{dx^3} + E_{11}\frac{d^4w_b}{dx^4} + F_{11}\frac{d^4w_s}{dx^4} - G_{55}\frac{d^2w_s}{dx^2} = q \tag{11}$$

where the extensional, coupling, bending and transverse shear rigidities are defined as follows:

$$(A_{11},\ B_{11},\ D_{11}) = Q_{11}^{(k)} \int_{-h/2}^{h/2} \left(1,\ z,\ z^2\right) dz,$$
$$(C_{11},\ E_{11},\ F_{11}) = Q_{11}^{(k)} \int_{-h/2}^{h/2} f(z)(1,\ z,\ f(z))\ dz, \quad (12)$$
$$G_{55} = Q_{55}^{(k)} \int_{-h/2}^{h/2} [g(z)]^2\ dz$$

4 Navier Solutions

In this section a closed form solution of simply supported laminated composite and sandwich beams with length 'L' and overall thickness 'h' is obtained. The Navier solution procedure is used to determine the analytical solutions for the simply-supported boundary conditions. The following simply-supported boundary conditions at $x = 0$, $x = L$ are considered

$$N = w_b = w_s = M = P = 0 \quad (13)$$

For this purpose, the displacement functions are expressed as product of undetermined coefficients and known trigonometric functions to satisfy the governing equations and the boundary conditions. The following displacement fields are assumed to be of the form:

$$u_0 = \sum_{m=1,3,5}^{\infty} u_m \cos\alpha x,\ w_b = \sum_{m=1,3,5}^{\infty} w_{bm} \sin\alpha x,$$
$$w_s = \sum_{m=1,3,5}^{\infty} w_{sm} \sin\alpha x \quad (14)$$

where u_m, w_{bm}, w_{sm} are the unknown Fourier coefficients to be determined for each m value, and $\alpha = m\pi/L$. The beam is subjected to distributed transverse load, $q(x)$ on the top surface i.e. $z = -h/2$. The applied transverse load $q(x)$ is expanded in single trigonometric Fourier series as

$$q(x) = \sum_{m=1,3,5}^{\infty} q_m \sin\alpha x \quad (15)$$

where q_m are the Fourier coefficients, and are given for distributed load as follows:

$$q_m = q_0 \text{ for sinusoidally distributed load } (m = 1)$$
$$q_m = \frac{4q_0}{m\pi} \text{ for uniformly distributed load } (m = 1, 3, 5\ldots.) \quad (16)$$

where q_0 denotes the intensity of the load at the center of the beam. Substituting equations (14) – (16) into the equation (11) one can obtain

$$\begin{bmatrix} A_{11}\alpha^2 & -B_{11}\alpha^3 & -C_{11}\alpha^3 \\ & D_{11}\alpha^4 & E_{11}\alpha^4 \\ \text{Symmetric} & & \left(F_{11}\alpha^4 + G_{55}\alpha^2\right) \end{bmatrix} \begin{Bmatrix} u_m \\ w_{bm} \\ w_{sm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ q_m \\ q_m \end{Bmatrix} \quad (17)$$

Solving equation (17), the unknowns u_m, w_{bm}, w_{sm} can be readily determined. Having obtained values of these coefficients one can then calculate all the displacement and stress components within the beam using equations (1) – (4).

4.1 Estimation of transverse shear stresses

Kant and Manjunatha [38] have presented the detail procedure of estimation of transverse shear stresses in composite laminates. The evaluation of transverse shear stresses from the constitutive relations leads to discontinuity at the interface of two adjacent layers of a laminate and thus violates the equilibrium conditions. The three-dimensional analysis becomes very complex due to the thickness variation of constitutive layers and continuity requirements of transverse stresses across the interfaces. Thus, elasticity equilibrium equation neglecting the body force is used to derive expression for the transverse stress in the k^{th} lamina of laminated composite beams.

$$\tau_{zx}^{(k)} = -\sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \frac{d\sigma_x^{(k)}}{dx}\ dz \quad (18)$$

From equation (18) the transverse stress (τ_{zx}) can be evaluated through integration with respect to the laminate thickness coordinate (z). The in-plane stress (σ_x) obtained by using equation (4) is substituted in equation (18). The constants of integrations can be determined by substituting the boundary conditions. It is expected that this procedure will produce an accurate transverse shear stresses.

4.2 Numerical Results

In order to show the accuracy and reliability of the present trigonometric beam theory (TBT) for the analysis of the laminated composite and sandwich beams, following three types of laminated beams are analyzed: a) A simply supported two layered ($0^0/90^0$) laminated composite beam subjected to sinusoidally distributed load *b*) A simply supported three layered (0°/90°/0°) laminated composite beam subjected to sinusoidally distributed load *c*) A simply supported five layered (0°/90°/*Core*/90°/0°) sandwich beam subjected to uniformly distributed load. The laminated composite and sandwich beams have following material properties: for 0° layer: $Q_{11} = 25$, $Q_{55} = 0.5$, for 90° layer: $Q_{11} = 1.0$, $Q_{55} = 0.2$, for *core*: $Q_{11} = 4.0$, $Q_{55} = 0.06$. For the simplicity, the numerical results are presented in the following non-dimensional forms:

$$\bar{u}\left(0,-\frac{h}{2}\right)=\frac{buE_3}{q_0 h},\ \bar{w}\left(\frac{L}{2},0\right)=\frac{100wh^3E_3}{q_0L^4},$$
$$\bar{\sigma}_x\left(0,-\frac{h}{2}\right)=\frac{b\sigma_x}{q_0},\ \bar{\tau}_{zx}(0,0)=\frac{b\tau_{zx}}{q_0},\ E_3=1.0 \quad (19)$$

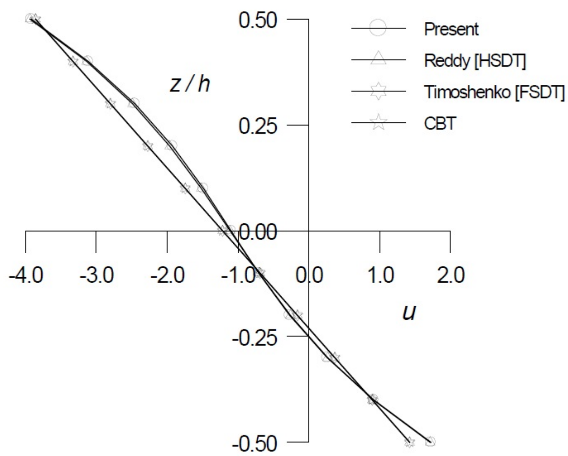

Figure 2: Through thickness distribution of axial displacement ($\bar{u}$) for two layered (0°/90°) laminated composite beam subjected to sinusoidally distributed load at $L/h = 4$

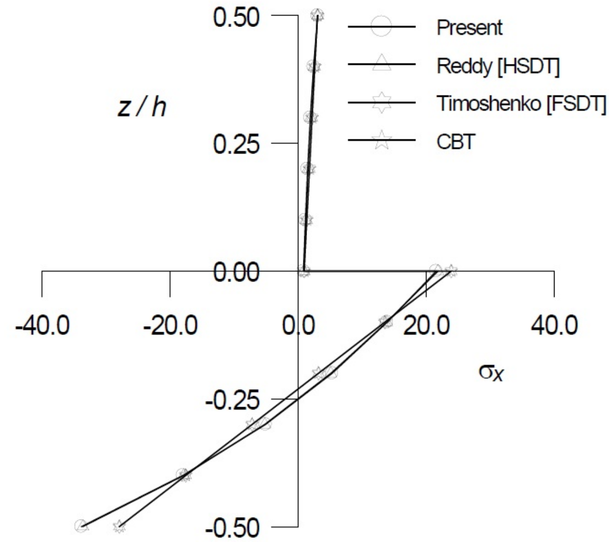

Figure 3: Through thickness distribution of bending stress ($\bar{\sigma}_x$) for two layered (0°/90°) laminated composite beam subjected to sinusoidally distributed load at $L/h = 4$

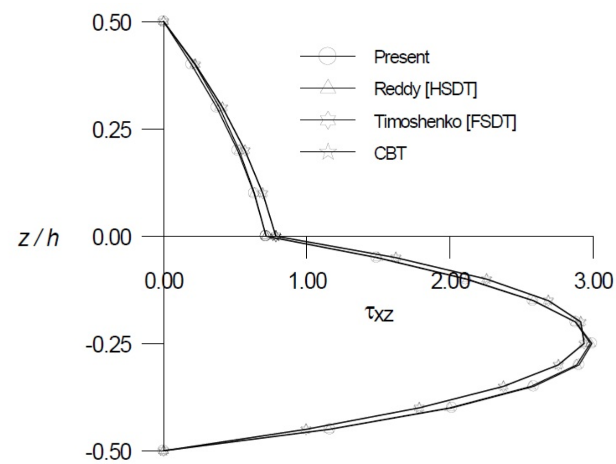

Figure 4: Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}^{EE}$) for two layered (0°/90°) laminated composite beam subjected to sinusoidally distributed load at $L/h = 4$

4.3 Discussion of Results

The displacements and stresses are obtained for simply supported laminated composite and sandwich beams. The results of equivalent single layer trigonometric shear deformation theories (TSDTs) of Arya *et al.* [34], Sayyad and Ghugal [35], layerwise trigonometric shear deformation theory (LTSDT) of Shimpi and Ghugal [36] and exact elasticity solution [39] available in the literature are used as a basis for comparison wherever possible. The numerical results are also generated by using higher order shear deformation theory (HSDT) of Reddy [4], first order shear deformation theory (FSDT) of Timoshenko [1] and classical beam theory (CBT) for the graphical comparison of present results. The transverse shear stress τ_{zx} is obtained by constitutive relations ($\bar{\tau}_{zx}^{CR}$) and equilibrium equations of theory of elasticity ($\bar{\tau}_{zx}^{EE}$).

In the first example, an efficiency of present theory is checked for a simply supported two layered (0°/90°) laminated composite beam. The layers are of equal thickness. The beam is subjected to sinusoidally distributed load. The aspect ratios (L/h) are considered as 4, 10 and 100. The non-dimensional displacements and stresses at critical points are presented in Table 1. The present results are compared with other trigonometric theories available

Table 1: Comparison of axial displacement ($\bar{u}$), transverse displacement ($\bar{w}$), bending stress ($\bar{\sigma}_x$) and transverse shear stress ($\bar{\tau}_{zx}$) for two layered (0°/90°) laminated composite beam subjected sinusoidally distributed load.

L/h	Theory	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	τ_{xz}^{CR}	τ_{xz}^{EE}
4	Present (TBT)	1.7189	4.3989	33.8001	2.5162	2.9001
	TSDT (ESL) [34]	1.7220	4.4026	33.8121	2.5187	2.9885
	TSDT [35]	1.7183	4.3955	33.8252	2.5185	2.9885
	LTSDT [36]	1.5574	4.7437	30.5650	2.9895	2.6784
	HSDT [4]	1.7108	4.4511	33.5921	2.4769	2.8831
	FSDT [1]	1.4212	4.7966	27.9049	1.8189	2.9118
	CBT	1.4212	2.6254	27.9049	—-	2.9468
	Exact [39]	1.5288	4.7080	30.0190	2.7212	—
10	Present (TBT)	22.9419	2.9150	180.4211	6.4191	7.3604
	TSDT (ESL) [34]	22.9736	2.9156	180.4339	6.4260	7.3814
	TSDT [35]	22.9353	2.9113	180.5434	6.4254	7.3845
	LTSDT [36]	22.563	2.9744	177.1400	7.6948	7.2609
	HSDT [4]	22.9424	2.9225	180.1890	6.2973	7.2637
	FSDT [1]	22.2060	2.9728	174.4052	4.5473	7.2794
	CBT	22.2060	2.6254	174.4052	—-	7.3670
	Exact [39]	22.4760	2.9611	176.530	7.2678	—
100	Present (TBT)	22213.67	2.6283	17446.57	64.430	73.3831
	HSDT [4]	22213.37	2.6284	17446.33	63.174	72.7923
	FSDT [1]	22205.98	2.6283	17440.54	45.473	72.7938
	CBT	22205.97	2.6254	17440.53	—	73.6702
	Exact [39]	22275.00	2.6366	17495.00	73.963	—

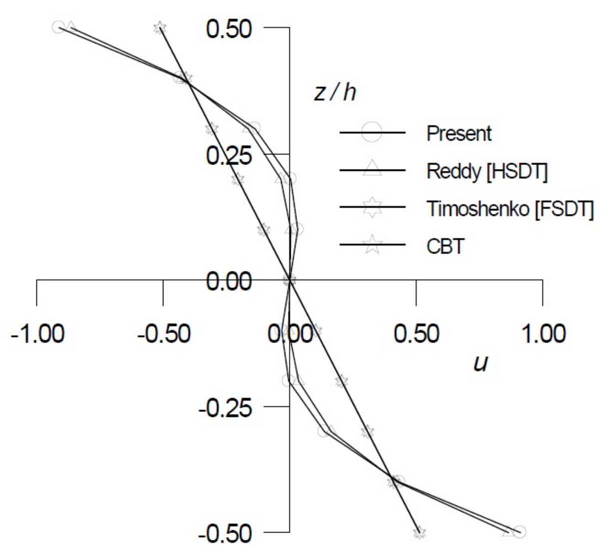

Figure 5: Through thickness distribution of axial displacement ($\bar{u}$) for three layered (0°/90°/0°) laminated composite beam subjected to sinusoidally distributed load at $L/h = 4$

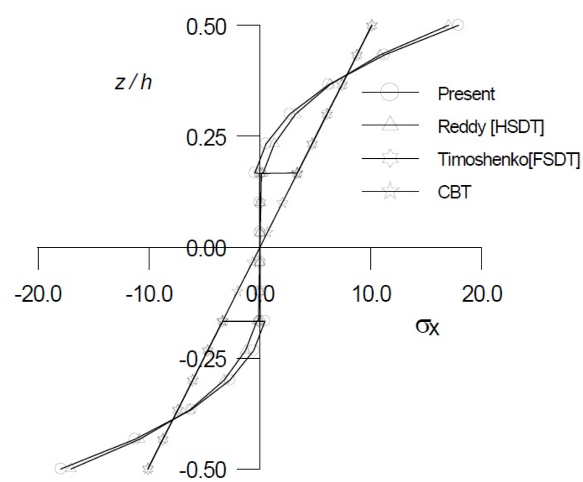

Figure 6: Through thickness distribution of bending stress ($\bar{\sigma}_x$) for three layered (0°/90°/0°) laminated composite beam subjected to sinusoidally distributed load at $L/h = 4$

in the literature. The present results are also compared graphically with those generated by HSDT, FSDT and CBT.

From the examination of Table 1 it is observed that the displacements and stresses obtained using present theory are in excellent agreement with those obtained by TSDT of Sayyad and Ghugal [35] which considered the effect of

Table 2: Comparison of axial displacement ($\bar{u}$), transverse displacement ($\bar{w}$), bending stress ($\bar{\sigma}_x$) and transverse shear stress ($\bar{\tau}_{zx}$) for three layered (0°/90°/0°) laminated composite beam subjected sinusoidally distributed load.

L/h	Theory	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	τ_{xz}^{CR}	τ_{xz}^{EE}
4	Present (TBT)	0.9128	2.8612	17.9230	1.0685	1.5173
	TSDT (ESL) [34]	0.8913	2.7252	17.5007	1.1614	1.5294
	TSDT [35]	0.8899	2.7157	17.5575	1.1609	1.5284
	HSDT [4]	0.8653	2.7000	16.9898	1.1094	1.5570
	FSDT [1]	0.5136	2.4107	10.0854	0.6366	1.7690
	CBT	0.5136	0.5109	10.0854	—	1.7690
	Exact [39]	0.9150	2.8870	17.8800	1.4250	—
10	Present (TBT)	9.0218	0.8861	70.8572	2.6647	4.3810
	TSDT (ESL) [34]	9.0167	0.8828	70.8169	3.0475	4.3220
	TSDT [35]	8.9969	0.8815	70.8363	3.0474	4.3217
	HSDT [4]	8.9398	0.8751	70.2128	2.8839	4.3344
	FSDT [1]	8.0257	0.8149	63.0339	1.5915	4.4226
	CBT	8.0257	0.5109	63.0339	—	4.4226
	Exact [39]	9.1050	0.8800	71.3000	4.2000	—
100	Present (TBT)	8036.36	0.5147	6311.75	66.596	45.041
	HSDT [4]	8034.93	0.5146	6310.62	64.567	44.217
	FSDT [1]	8025.72	0.5135	6303.39	35.366	44.226
	CBT	8025.72	0.5109	6303.39	—	44.226
	Exact [39]	8040.00	0.5153	6315.00	44.150	—

Table 3: Comparison of axial displacement ($\bar{u}$), transverse displacement ($\bar{w}$), bending stress ($\bar{\sigma}_x$) and transverse shear stress ($\bar{\tau}_{zx}$) for five layered (0°/90°/*Core*/90°/0°) sandwich beam subjected uniformly distributed load.

L/h	Source	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	τ_{xz}^{EE}
10	Present (TBT)	30.120	4.4041	264.870	3.3563
	HSDT [4]	37.350	4.0479	276.226	5.1350
	FSDT [1]	35.650	3.2875	267.38	5.2070
	CBT	35.650	2.2282	267.38	5.2070
20	Present (TBT)	231.24	2.2780	895.5	8.9908
	HSDT [4]	288.60	2.6834	1078.44	10.374
	FSDT [1]	285.20	2.4930	1069.55	10.414
	CBT	285.20	2.2282	1069.55	10.414
100	Present (TBT)	27702.0	1.7537	20773.88	40.805
	HSDT [4]	35667.0	2.2464	26747.79	52.061
	FSDT [1]	35650.0	2.2387	26738.62	52.069
	CBT	35650.0	2.2282	26738.81	52.070

transverse normal strain. The present results are also in good agreement with HSDT of Reddy [4] and exact elasticity solution given by Pagano [39]. The LTSDT of Shimpi and Ghugal [36] shows more accurate results as compared to present theory because it is a layerwise theory and the present theory is an equivalent single layer theory. CBT underestimates the values of displacement and bending stress and overestimates the values of transverse shear stress due to neglect of transverse shear deformation. The through thickness distribution of axial displacement ($\bar{u}$), bending stress ($\bar{\sigma}_x$) and transverse shear stress *via* equations of equilibrium ($\bar{\tau}_{zx}^{EE}$) are plotted in Figs. 2 – 4. The observation of these figures reveals that the in-plane dis-

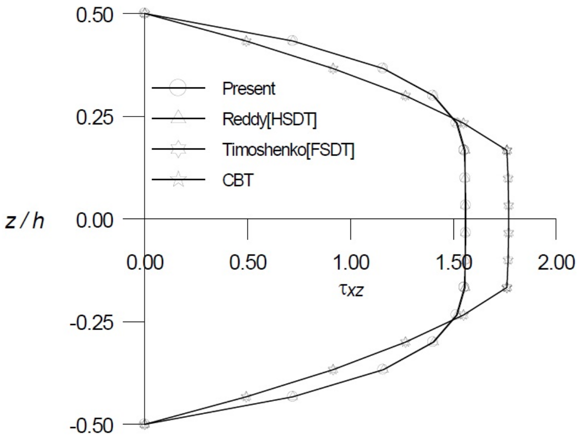

Figure 7: Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}^{EE}$) for three layered (0°/90°/0°) laminated composite beam subjected to sinusoidally distributed load at $L/h = 4$

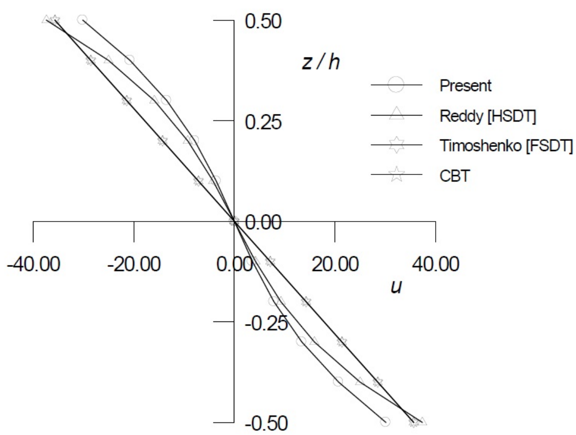

Figure 8: Through thickness distribution of axial displacement ($\bar{u}$) for five layered (0°/90°/*core*/90°/0°) sandwich beam subjected to uniformly distributed load at $L/h = 10$

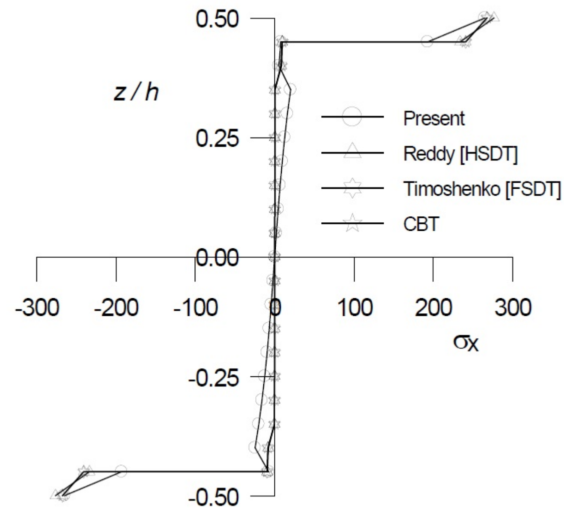

Figure 9: Through thickness distribution of bending stress ($\bar{\sigma}_x$) for five layered (0°/90°/*core*/90°/0°) sandwich beam subjected to uniformly distributed load at $L/h = 10$

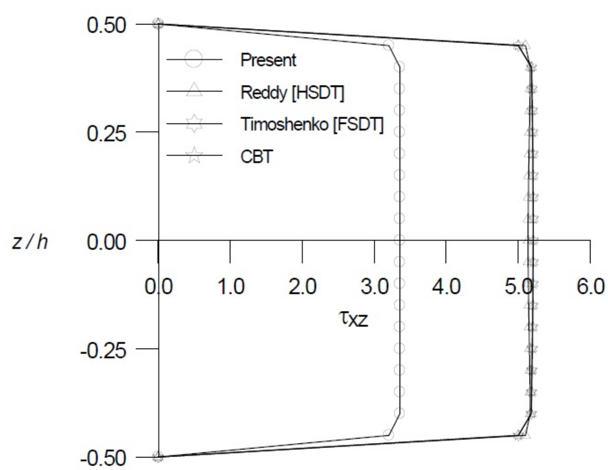

Figure 10: Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}^{EE}$) for five layered (0°/90°/*core*/90°/0°) sandwich beam subjected to uniformly distributed load at $L/h = 10$

placement is minimum in 0° layers (fibres are along the length) whereas stresses are maximum in 0° layers.

In second problem, present theory is applied for the bending analysis of simply supported three layered laminated composite beams. In this problem also total thickness is equally distributed among the layers. The displacements and stresses are obtained for aspect ratios 4, 10 and 100 when beam is subjected to sinusoidally distributed load. In Table 2, the results for the non-dimensional displacements and stresses are presented. In this Table results using LTSDT are not reported because it is applicable to two layered beam only. From Table 2 it is pointed out that the present theory is more accurate while predicting displacements and stresses of three layered laminated composite beams as compared to other trigonometric, HSDT, FSDT and CBT models. Figs. 5 – 7 show the through thickness distribution of axial displacement ($\bar{u}$), bending stress ($\bar{\sigma}_x$) and transverse shear stress *via* equations of equilibrium ($\bar{\tau}_{zx}^{EE}$). From these figures it is observed that the in-plane displacement and the in-plane normal stresses are maximum at top and bottom surfaces of the beam (i.e. $z = \pm h/2$) and those are zero at neutral axis (i.e. $z = 0$). The transverse shear stress is zero at top and bottom surfaces of the beam whereas maximum at neutral axis.

In third and last example, accuracy of present theory is checked for five layered (0°/90°/*Core*/90°/0°) sandwich beams. The top two and bottom two layers are called as face sheets whereas central portion is called as soft core. The thickness of each face sheet is '0.05h' whereas thickness of core is '0.8h' where h is the total thickness of the beam. A simply supported sandwich beam subjected to uniformly distributed load is considered. The results of non-dimensional displacements and stresses are presented in Table 3 for aspect ratios 10, 20 and 100. All these results are presented for the first time using Navier solution technique. The numerical results for this problem using other trigonometric theories are not available in the literature. Therefore, for the comparison purpose numerical results are also generated using HSDT of Reddy [4], FSDT of Timoshenko [1] and CBT. From Table 3 it is observed that for aspect ratio 100 results of all theories are more or less same. The through thickness distributions of axial displacement ($\bar{u}$), bending stress ($\bar{\sigma}_x$) and transverse shear stress *via* equations of equilibrium ($\bar{\tau}_{zx}^{EE}$) are plotted in Figs. 6 – 9. The transverse shear stress distribution satisfies the shear stress free boundary conditions at the top and bottom faces of the beam and also satisfies the continuity condition at the interface between the layers.

5 Conclusions

In this paper, a new trigonometric beam theory has been developed and presented for laminated composite and sandwich beams. The theory does not require shear correction factor. Variationally consistent governing equations and boundary conditions associated with the present theory are derived using principal of virtual work. Bending problems of laminated composite and sandwich beams are solved to prove the efficiency of present theory. The numerical results are compared with other trigonometric theories available in the literature. The numerical evaluations have proved that the present theory is in good agreement while predicting bending behaviour of laminated composite and sandwich beams.

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