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A *n*th-order shear deformation theory for composite laminates in cylindrical bending

Abstract: The present study investigates whether an *n*th-order shear deformation theory is applicable for the composite laminates in cylindrical bending. The theory satisfies the traction free conditions at top and bottom surfaces of the plate and does not require problem dependent shear correction factor which is normally associated with the first order shear deformation theory. The well-known classical plate theory at ($n = 1$) and higher order shear deformation theory of Reddy at ($n = 3$) are the perticular cases of the present theory. The governing equations of equilibrium and boundary conditions are obtained using the principle of virtual work. A simply supported laminated composite plate infinitely long in y-direction is considered for the detail numerical study. A closed form solution for simply supported boundary conditions is obtained using Navier's technique. The displacements and stresses are obtained for different aspect ratios and modular ratios.

Keywords: *n*th-order shear deformation theory; shear correction factor; traction free conditions; laminated composites; cylindrical bending

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1 Introduction

Composite materials are those formed by combining two or more materials on a macroscopic scale such that they have better engineering properties such as stiffness, strength, weight reduction, corrosion resistance, thermal properties, fatigue life, and wear resistance. The structural elements (beams, plates and shells) made up of such materials are required accurate structural analysis to predict the correct bending behaviour. Therefore, over the years, researchers have developed many computational models for the one dimensional and two dimensional bending of laminated composite plates such as: Mindlin [1], Reddy [2], Touratier [3], Soldatos [4], Karama et al. [5], Akavci [6] and many more. When one dimension of the plate is infinitely long as compared to other two dimensions, it is called as a cylindrical bending problem (plane strain problem). Pagano [7] has presented the exact elasticity solution for cylindrical bending of laminated composite plates. Reddy [8] has solved several problems on cylindrical bending of composite laminated plates using classical plate theory (CPT), first order shear deformation theory (FSDT) and higher order shear deformation theory (HSDT). Soldatos [4] presented cylindrical bending of orthotropic plates using trigonometric and hyperbolic functions. Soldatos and Watson [9] have developed a new stress analysis method for the cylindrical bending of cross-ply laminated plates which is further applied to angle-ply laminated plates by Shu and Soldatos [10]. A semi-exact method based on the assumption of inextensibility of the plate through the thickness is developed by Jalali and Taheri [11] to examine the bending response of cross-ply laminated plates under cylindrical bending. Perel and Palazotta [12] developed a new plate theory for the cylindrical bending of sandwich plate simplifying the assumptions regarding the distribution of transverse strain components in the thickness direction. Khdeir [13] presented free and forced vibration analysis of angle-ply laminated composite plates based on CPT and FSDT for arbitrary boundary conditions and loading conditions using the state space concept. Park and Lee [14] presented a new exponential theory for the cylindrical bending of laminated composite plates. Chen and Lee [15] developed an elasticity method to study the bending and the free vibration response of simply-supported angle-ply laminated cylindrical panels in the cylindrical bending using method of state-space. Lu et al. [16] obtained elasticity solutions for free vibration of angle-ply laminates subjected to cylindrical bending using semi-analytical approach. Carrera [17] studied the non-linear response of antisymmetrically laminated composite plates under cylindrical bending.

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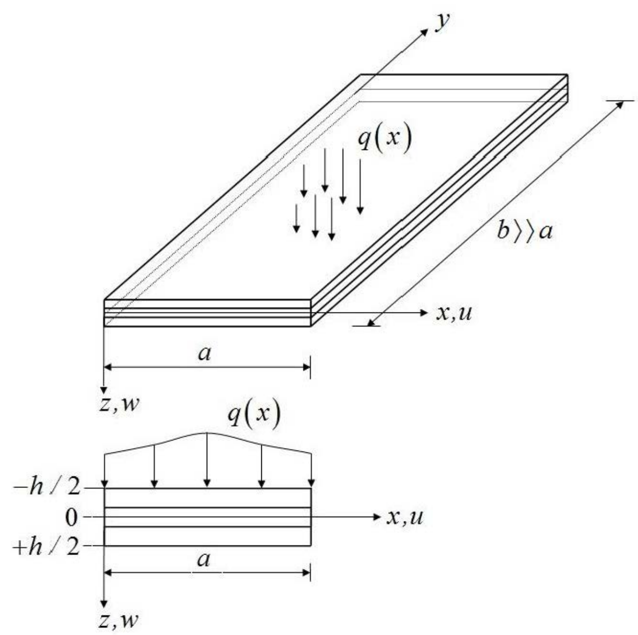

Figure 1: Plate geometry and coordinate system

Ghugal and Sayyad [18] have developed a trigonometric shear deformation theory taking into account transverse shear deformation effect as well as the transverse normal strain effect for the cylindrical bending of orthotropic plate. Recently Sayyad et al. [19] developed a new trigonometric shear deformation theory for the flexural analysis of laminated composite and sandwich beams whereas Natarajan et al. [20] studied the bidirectional bending and free vibration analysis of cross-ply laminated composite plates using trigonometric shear deformation theory. Sun and Harik [21] extend the analytical strip method to stiffened antisymmetric cross-ply or angle-ply laminated composite plates with bending-extension coupling. Carpentieri et al. [22] developed an accurate one dimensional theory for the dynamic analysis of laminated composite curved beams. Ferreira et al. [23] proposed layerwise theory for the bidirectional bending analysis of sandwich plates using generalized differential quadrature method. A number of refined beam theories are applied by Carrera et al. [24] for the free vibration analysis of laminated composite beams. Tornabene [25] applied the generalized differential quadrature for the dynamic behaviour of laminated composite doubly-curved shells of revolution. Tornabene et al. [26] proposed a two dimensional general higher-order equivalent single layer approach, based on the Carrera Unified Formulation for the static analysis of doubly-curved laminated composite shells and panels.

Xiang et al. [27] was first to developed an *n*th order shear deformation for the free vibration analysis of isotropic plates. Further, it is extended for several problems of laminated composite and functionally graded plates by Xiang et al. [28], Xiang and Kang [29] and Xiang et al. [30]. However, the theory is applied to bidirectional problems of plates only. This study investigates whether the *n*th order shear deformation proposed by Xiang et al. [27] is applicable for one dimensional (cylindrical bending) problems of laminated composite plates. In the view of this, Sayyad et al. [31] have applied this theory for the cylindrical bending of homogenous plate made up of orthotropic material. Therefore, in this paper the theory is extended for the cylindrical bending of laminated composite plates. The Governing equations and boundary conditions are obtained using an analytical form of the principle of virtual work. A simply supported laminated composite plate is considered for the numerical study. The present results are compared with those obtained by HSDT of Reddy [2] FSDT of Mindlin [1] and exact elasticity solution provided by Pagano [7].

2 Mathematical Formulation of Cylindrical Bending Problem

Let us consider a laminated composite plate constructed of an arbitrary number of layers of linearly elastic orthotropic material. The layers are perfectly bounded together. Each ply obeys the hook's law of the plane strain problem. The transverse normal strain ε_z is negligible. The geometry and coordinate system of the plate is shown in Fig. 1. It is assumed that the plate is of an infinite extent in the y direction while it is simply supported at its edges $x = 0$ and x = a. The plate has constant thickness 'h'. The downward z-direction is assumed as positive, therefore, the thickness coordinate of the upper surface is $-h/2$ and lower surface is $+h/2$. A transverse load $q(x)$ is applied at the upper surface of the plate, i.e. $z = -h/2$. The displacements in the x and z directions are denoted by u and w respectively.

2.1 Displacement function

Due to the symmetries involved in this cylindrical bending problem (plane strain problem), the displacement function of the *n*th-order shear deformation theory [31] is as follows:

$$u(x,z) = u_0 + z\phi(x) - C_1 z^n \left(\phi + \frac{dw_0}{dx}\right)$$
$$v(x,z) = 0 \tag{1}$$
$$w(x) = w_0(x)$$

where w_0 is the transverse displacement of a point on mid plane (i.e. $z = 0$) and ϕ is the shear slope associated with cross sectional warping. Due to the symmetries involved in both the geometrical and loading characteristics, both the displacement components ϕ and w_0 are independent of the y coordinate. The CPT and HSDT of Reddy [2] are the modified form of the present theory at $n = 1$ and $n = 3$ respectively.

2.2 Strain components

Upon applying the kinematic relations of plane strain elasticity problem to the displacement function given by equation (1), one can obtain the following non-zero linear strain components associated with the present theory:

$$\varepsilon_x = \frac{du}{dx} = \frac{du_0}{dx} + z\frac{d\phi}{dx} - C_1 z^n \left(\frac{d\phi}{dx} + \frac{d^2 w_0}{dx^2}\right) \tag{2}$$

$$\gamma_{zx} = \frac{du}{dz} + \frac{dw}{dx} = \left[1 - C_2 z^{n-1}\right]\left(\phi + \frac{dw_0}{dx}\right) \tag{3}$$

where ε_x is the normal strain and γ_{zx} is the shear strain.

2.3 Stress components

The stress-strain relationship for the k^{th} layer of the plate can be written as:

$$\sigma_x^k = Q_{11}^k \varepsilon_x^k \quad \text{and} \quad \tau_{zx}^k = Q_{55}^k \gamma_{zx}^k \tag{4}$$

The substitution of strains from equations (2) and (3) into the equation (4) gives the following values of stresses.

$$\sigma_x^k = Q_{11}^k \left[\frac{du_0}{dx} + z\frac{d\phi}{dx} - C_1 z^n \left(\frac{d\phi}{dx} + \frac{d^2 w_0}{dx^2}\right)\right] \tag{5}$$

$$\tau_{zx}^k = Q_{55}^k \left[1 - C_2 z^{n-1}\right]\left(\phi + \frac{dw_0}{dx}\right) \tag{6}$$

where

$$C_1 = \frac{1}{n}\left(\frac{2}{h}\right)^{n-1}, \quad C_2 = \left(\frac{2}{h}\right)^{n-1},$$
$$Q_{11}^k = \frac{E_1^k}{1-\mu_{12}\mu_{21}} \quad \text{and} \quad Q_{55}^k = G_{31}^k \tag{7}$$

where E_1, G_{31}, μ_{12}, μ_{21} are the elastic constants of plate material. Here subscripts 1, 2, 3 correspond to x, y, z directions of Cartesian coordinate system respectively. Equation (6) implies that the transverse shear stress vanish on the surface $z = \pm h/2$.

2.4 Stress resultants

The in-plane force (N_x) and moment (M_x, P_x) resultants associated with the present theory are defined as follows:

$$N_x = \sum_{k=1}^{N} \int_{-h/2}^{h_k} \sigma_x dz =$$
$$= A_{11}\frac{du_0}{dx} + (B_{11} - C_1 C_{11})\frac{d\phi}{dx} - C_1 C_{11}\frac{d^2 w_0}{dx^2} \tag{8}$$

$$M_x = \sum_{k=1}^{N} \int_{-h/2}^{h_k} \sigma_x z dz =$$
$$= B_{11}\frac{du_0}{dx} + (D_{11} - C_1 E_{11})\frac{d\phi}{dx} - C_1 E_{11}\frac{d^2 w_0}{dx^2} \tag{9}$$

$$P_x = \sum_{k=1}^{N} \int_{-h/2}^{h_k} \sigma_x z^n dz =$$
$$= C_{11}\frac{du_0}{dx} + (E_{11} - C_1 F_{11})\frac{d\phi}{dx} - C_1 F_{11}\frac{d^2 w_0}{dx^2} \tag{10}$$

The shear (R_x, Q_x) resultants associated with the present theory are defined as follows:

$$R_x = \sum_{k=1}^{N} \int_{-h/2}^{h_k} \tau_{xz} z^{n-1} dz =$$
$$= (E_{55} - C_2 F_{55})\left(\phi + \frac{dw_0}{dx}\right) \tag{11}$$

$$Q_x = \sum_{k=1}^{N} \int_{-h/2}^{h_k} \tau_{xz}\, dz = (A_{55} - C_2 E_{55})\left(\phi + \frac{dw_0}{dx}\right) \tag{12}$$

3 Governing Equations and Boundary Conditions

The governing equations and boundary conditions associated with the assumed displacement function are obtained using the principle of virtual work. The analytical form of the principle of virtual work is

$$\int_{-h/2}^{h/2}\int_0^a \left[\sigma_x \delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx}\right] dxdz - \int_0^a q(x)\,\delta w_0 dx = 0 \tag{13}$$

By substituting strains from equations (2) and (3) into the equation (13), one can obtain

$$\int_{-h/2}^{h/2}\int_0^a \sigma_x \left[\frac{d\delta u_0}{dx} + z\frac{d\delta\phi}{dx} - C_1 z^n \left(\frac{d\delta\phi}{dx} + \frac{d^2\delta w_0}{dx^2}\right)\right] dxdz$$
$$+ \int_{-h/2}^{h/2}\int_0^a \tau_{zx}\left[1 - C_2 z^{n-1}\right]\left(\delta\phi + \frac{d\delta w_0}{dx}\right) dxdz \tag{14}$$
$$- \int_0^a q(x)\,\delta w_0 dx = 0$$

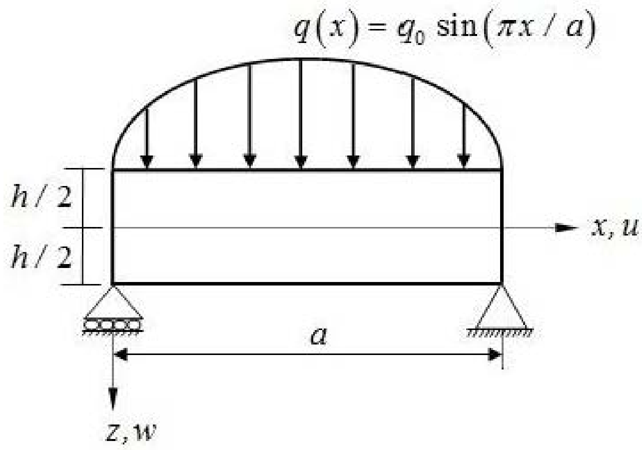

Figure 2: Simply supported plate in xz plane subjected to sinusoidally distributed load

After carrying out integrations with respect to z, the equation (14) leads to the following form:

$$\int_0^a \left[N_x \frac{d\delta u_0}{dx} + M_x \frac{d\delta\phi}{dx} - C_1 P_x \left(\frac{d\delta\phi}{dx} + \frac{d^2\delta w_0}{dx^2}\right)\right] dx + \int_0^a [Q_x - C_2 R_x]\left(\delta\phi + \frac{d\delta w_0}{dx}\right) dx - \int_0^a q(x)\,\delta w_0 dx = 0 \tag{15}$$

Integrating the equation (15) by parts and setting the coefficients of δu_0, δw_0 and $\delta\phi$ zero, the governing equations and boundary conditions are obtained. The governing equations of the present theory are simplified as follows:

$$\delta u_0 : \quad \frac{dN_x}{dx} = 0 \tag{16}$$

$$\delta w_0 : \quad C_2 \frac{dR_x}{dx} - \frac{dQ_x}{dx} - C_1 \frac{d^2 P_x}{dx^2} = q \tag{17}$$

$$\delta\phi : \quad \frac{dM_x}{dx} - C_1 \frac{dP_x}{dx} - Q_x + C_2 R_x = 0 \tag{18}$$

To determine the form of boundry conditions, one can consider the boundry integrals in equation (15). The boundary conditions at the edges $x = 0$ and $x = a$ are as follows:

$$\text{Either } N_x = 0 \quad \text{or} \quad u_0 = 0 \tag{19}$$

$$\text{Either } C_2 R_x - Q_x + C_1 \frac{dP_x}{dx} = 0 \quad \text{or} \quad w_0 = 0 \tag{20}$$

$$\text{Either } P_x = 0 \quad \text{or} \quad \frac{dw_0}{dx} = 0 \tag{21}$$

$$\text{Either } M_x - C_1 P_x = 0 \quad \text{or} \quad \phi = 0 \tag{22}$$

In conjection with stress resultants from equations (8) – (12), the governing equations in terms of displacement variables (u_0, ϕ and w_0) are rewritten as:

$$\delta u_0 : \quad -A_{11}\frac{d^2u_0}{dx^2} - (B_{11} - C_1 C_{11})\frac{d^2\phi}{dx^2} + C_1 C_{11}\frac{d^3 w_0}{dx^3} = 0 \tag{23}$$

$$\delta w_0 : \quad -C_1 C_{11}\frac{d^3u_0}{dx^3} + C_1^2 F_{11}\frac{d^4 w_0}{dx^4} - (A_{55} - 2C_2E_{55} + C_2^2 F_{55})\frac{d^2 w_0}{dx^2} - \left(A_{55} - 2C_2 E_{55} + C_2^2 F_{55}\right)\frac{d\phi}{dx} + \left(C_1^2 F_{11} - C_1 E_{11}\right)\frac{d^3\phi}{dx^3} = q \tag{24}$$

$$\delta\phi : \quad -(B_{11} - C_1 C_{11})\frac{d^2u_0}{dx^2} + (C_1 E_{11} - C_1^2 F_{11})\frac{d^3 w_0}{dx^3} + (A_{55} - 2C_2E_{55} + C_2^2 F_{55})\frac{dw_0}{dx} - \left(D_{11} - 2C_1 E_{11} + C_1^2 F_{11}\right)\frac{d^2\phi}{dx^2} + \left(A_{55} - 2C_2 E_{55} + C_2^2 F_{55}\right)\phi = 0 \tag{25}$$

The stiffness coefficients ($A_{11}, B_{11}, C_{11}, F_{11}, D_{11}, E_{11}, A_{55}, E_{55}, F_{55}$) appeared in the governing equations are as follows:

$$\begin{aligned} (A_{11}, A_{55}) &= (Q_{11}, Q_{55}) \int_{-h/2}^{h/2} dz; \\ B_{11} &= Q_{11} \int_{-h/2}^{h/2} z dz; \\ C_{11} = Q_{11} \int_{-h/2}^{h/2} z^n dz; &\quad F_{11} = Q_{11} \int_{-h/2}^{h/2} z^{2n} dz; \\ D_{11} = Q_{11} \int_{-h/2}^{h/2} z^2 dz; &\quad E_{11} = Q_{11} \int_{-h/2}^{h/2} z^{n+1} dz; \\ E_{55} = Q_{55} \int_{-h/2}^{h/2} z^{n-1} dz; &\quad F_{55} = Q_{55} \int_{-h/2}^{h/2} z^{2n-2} dz \end{aligned} \tag{26}$$

This completes the development of governing equations and boundary conditions of the present theory for cylindrical bending problem.

4 Numerical Examples

A laminated composite plate subjected to sinusoidally distributed transverse load on the upper surface ($z = -h/2$) is considered for the detail numerical study. The transverse loading independent of the y co-ordinate acting on the plate is shown in Fig. 2. The plate satisfies following boundary conditions at simply supported edges ($x = 0$, $x = a$):

$$N_x = 0, \quad w = 0, \quad P_x = 0, \quad M_x - C_1 P_x = 0 \tag{27}$$

Table 1: Comparison of normalized displacements and stress for two layered (0°/90°) un-symmterically laminated composite plates in cylindrical bending (Material 1).

h/a	Theory	Model	$\bar{u}$ $(0, h/2)$	w $(a/2, 0)$	$\bar{\sigma}_x$ $(a/2, 0)$	$\bar{\tau}_{zx}^{CR}$ $(0, 0)$	$\bar{\tau}_{zx}^{EE}$ $(0, 0)$
0.25	Present	$n = 1$ (CPT)	1.4176	2.6188	27.9048	– – –	2.9468
	Present	$n = 3$ (HSDT[a])	1.7073	4.4445	33.6062	2.4769	2.9794
	Present	$n = 5$	1.6521	4.5763	32.5202	2.2348	2.9387
	Present	$n = 7$	1.6083	4.5947	31.6576	2.1329	2.9293
	Present	$n = 9$	1.5769	4.5888	31.0394	2.0729	2.9278
	Mindlin [1]	FSDT	1.4212	4.7966	27.9049	1.8189	2.9352
	Pagano [7]	Exact	1.5288	4.7080	30.0190	2.7212	—
0.1	Present	$n = 1$ (CPT)	22.1505	2.6188	174.4054	—	7.3670
	Present	$n = 3$ (HSDT[a])	22.8869	2.9159	180.2035	6.2973	7.3560
	Present	$n = 5$	22.7405	2.9341	179.0512	5.6238	7.3395
	Present	$n = 7$	22.6286	2.9360	178.1700	5.3492	7.3357
	Present	$n = 9$	22.5493	2.9346	177.5455	5.1913	7.3351
	Mindlin [1]	FSDT	22.2060	2.9728	174.4052	4.5473	7.3670
	Pagano [7]	Exact	22.4760	2.9611	176.5300	7.2678	—
0.01	Present	$n = 1$ (CPT)	22150.47	2.6188	17440.54	—	73.674
	Present	$n = 3$ (HSDT[a])	22158.21	2.6219	17445.90	63.171	73.380
	Present	$n = 5$	22158.93	2.6223	17442.80	56.304	73.371
	Present	$n = 7$	22155.79	2.6217	17442.60	53.523	73.373
	Present	$n = 9$	22153.23	2.6219	17443.70	55.859	73.379
	Mindlin [1]	FSDT	22151.56	2.6224	17441.40	45.475	73.674

[a]HSDT of Reddy [2]

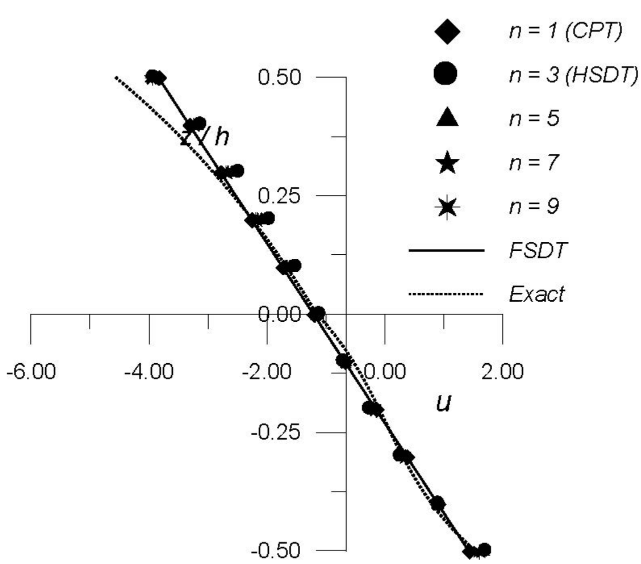

Figure 3: Through thickness distribution of in-plane displacement ($\bar{u}$) for simply supported two layered (0°/90°) un-symmetrically laminated composite plates with $a/h = 4$ and $E_1/E_2 = 25$

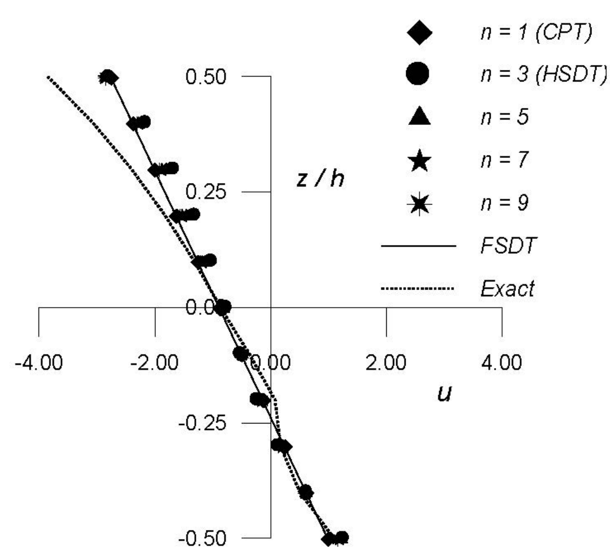

Figure 4: Through thickness distribution of in-plane displacement ($\bar{u}$) for simply supported two layered ($0^0/90^0$) un-symmetrically laminated composite plates with $a/h = 4$ and $E_1/E_2 = 40$

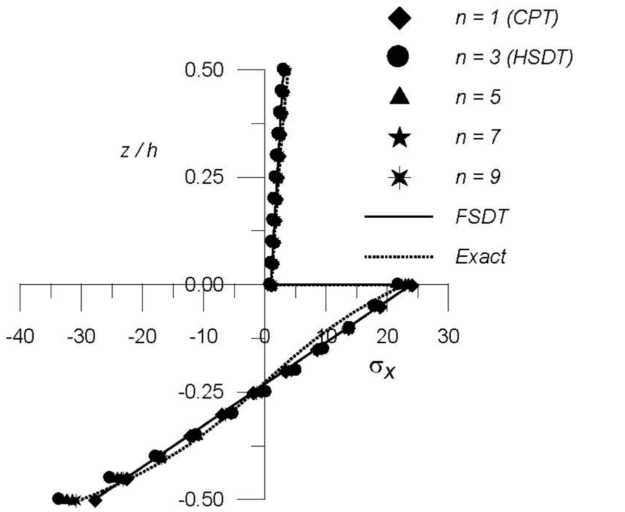

Figure 5: Through thickness distribution of bending stress ($\bar{\sigma}_x$) for simply supported two layered (0°/90°) un-symmetrically laminated composite plates with a/h = 4 and E_1/E_2 = 25

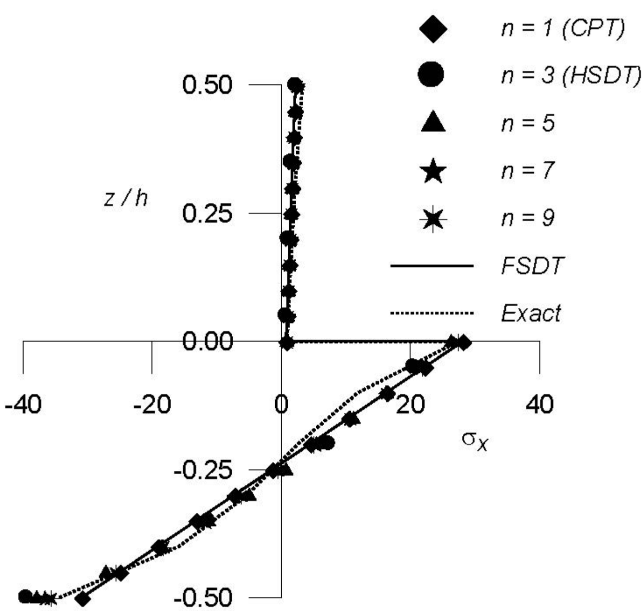

Figure 6: Through thickness distribution of bending stress ($\bar{\sigma}_x$) for simply supported two layered (0°/90°) un-symmetrically laminated composite plates with a/h = 4 and E_1/E_2 = 40

Navier's solution procedure is adopted to determine unknown displacement variables (u_0, w_0 and ϕ). The following is the solution form assumed for displacement variables u_0, w_0 and ϕ satisfying the simply supported boundary conditions exactly.

$$u_0(x) = u_1 \cos\frac{\pi x}{a}, \quad w_0(x) = w_1 \sin\frac{\pi x}{a} \quad \text{and} \quad \phi(x) = \phi_1 \cos\frac{\pi x}{a} \tag{28}$$

where u_1, w_1 and ϕ_1 are the unknown coefficients to be determine. Substitution of equation (28) into governing

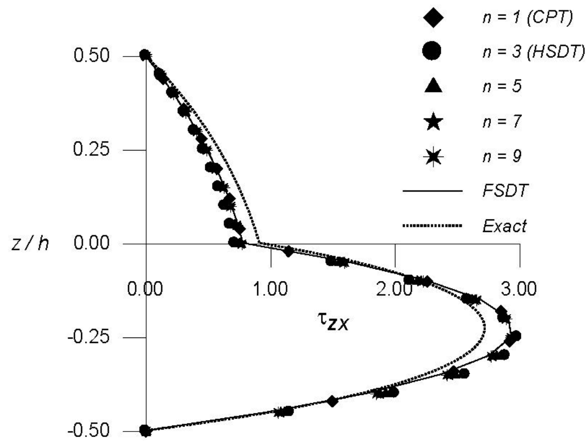

Figure 7: Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}$) for simply supported two layered (0°/90°) un-symmetrically laminated composite plates with a/h = 4 and E_1/E_2 = 25

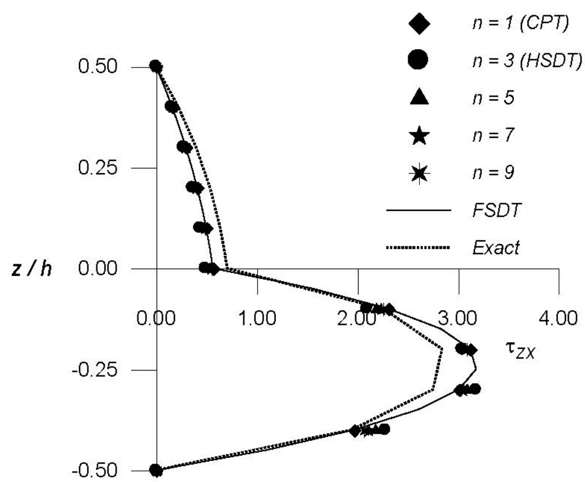

Figure 8: Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}$) for simply supported two layered (0°/90°) un-symmetrically laminated composite plates with a/h = 4 and E_1/E_2 = 40

equations (23) – (25) leads to the following matrix form:

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ \phi_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ q_0 \\ 0 \end{Bmatrix} \tag{29}$$

Table 2: Comparison of normalized displacements and stress for two layered (0°/90°) un-symmterically laminated composite plates in cylindrical bending (Material 2).

h/a	Theory	Model	$\bar{u}$ $(0, h/2)$	w $(a/2, 0)$	$\bar{\sigma}_x$ $(a/2, 0)$	$\bar{\tau}_{zx}^{CR}$ $(0, 0)$	$\bar{\tau}_{zx}^{EE}$ $(0, 0)$
0.25	Present	$n = 1$ (CPT)	2.7694	1.8669	30.9674	—	3.1883
	Present	$n = 3$ (HSDT[a])	2.8024	3.6206	39.5038	2.4163	3.2233
	Present	$n = 5$	2.8708	3.7880	38.0765	2.2095	3.1729
	Present	$n = 7$	2.8807	3.8217	36.8172	2.1194	3.1636
	Present	$n = 9$	2.8763	3.8233	35.8845	2.0646	3.1642
	Mindlin [1]	FSDT	2.7694	4.0381	30.9674	1.8189	3.1883
	Pagano [7]	Exact	2.7354	2.6569	34.4062	2.8426	—
0.1	Present	$n = 1$ (CPT)	43.2726	1.8669	193.546	—	7.9708
	Present	$n = 3$ (HSDT[a])	43.3569	2.1545	202.295	6.1913	7.9711
	Present	$n = 5$	43.5286	2.1773	200.726	5.5787	7.9506
	Present	$n = 7$	43.5521	2.1812	199.424	5.3242	7.9468
	Present	$n = 9$	43.5406	2.1808	198.476	5.1753	7.9471
	Mindlin [1]	FSDT	43.2726	2.2143	193.546	4.5473	7.9707
0.01	Present	$n = 1$ (CPT)	43272.63	1.8669	19354.65	—	79.707
	Present	$n = 3$ (HSDT[a])	43273.41	1.8698	19363.41	62.205	79.542
	Present	$n = 5$	43272.58	1.8699	19360.67	55.890	79.535
	Present	$n = 7$	43273.98	1.8700	19359.88	53.291	79.537
	Present	$n = 9$	43276.01	1.8703	19360.17	55.699	79.540
	Mindlin [1]	FSDT	43272.59	1.8704	19354.63	45.475	79.707

[a]HSDT of Reddy [2]

where,

$$
\begin{gathered}
K_{11} = A_{11}\frac{\pi^2}{a^2}, \quad K_{12} = -C_1 C_{11}\frac{\pi^3}{a^3}, \\
K_{13} = -\left(C_1 C_{11} - B_{11}\right)\frac{\pi^2}{a^2}, \\
K_{22} = C_1^2 F_{11}\frac{\pi^4}{a^4} + \left(A_{55} - 2C_2 E_{55} + C_2^2 F_{55}\right)\frac{\pi^2}{a^2}, \\
K_{23} = \left(C_1^2 F_{11} - C_1 E_{11}\right)\frac{\pi^3}{a^3} + \\
+ \left(A_{55} - 2C_2 E_{55} + C_2^2 F_{55}\right)\frac{\pi}{a}, \\
K_{33} = \left(D_{11} - 2C_1 E_{11} + C_1^2 F_{11}\right)\frac{\pi^2}{a^2} + \\
+ \left(A_{55} - 2C_2 E_{55} + C_2^2 F_{55}\right), \\
K_{21} = K_{12}, \quad K_{31} = K_{13}, \quad K_{32} = K_{23},
\end{gathered} \tag{30}
$$

Solution of equation (29) gives the values of unknown coefficients $u_1 w_1$ and ϕ_1. Having obtained values of these coefficients one can then calculate all the displacements from equation (1) and stress components within the plate from equations (6) and (7). Following examples are considered for the detail numerical study.

Example 1: cylindrical bending of simply supported two layered (0°/90°) un-symmetrically laminated composite plate

Example 2: cylindrical bending of simply supported three layered (0°/90°/0°) symmetrically laminated composite plate

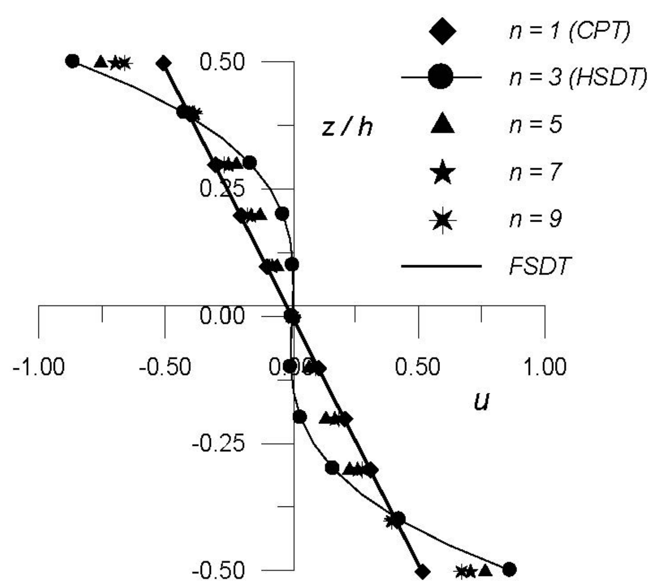

Figure 9: Through thickness distribution of in-plane displacement ($\bar{u}$) for simply supported three layered (0°/90°/0°) symmetrically laminated composite plates with $a/h = 4$ and $E_1/E_2 = 25$

Table 3: Comparison of normalized displacements and stress for two layered (0°/90°/0°) symmterically laminated composite plates in cylindrical bending (Material 1).

h/a	Theory	Model	$\bar{u}$ $(0, h/2)$	w $(a/2, 0)$	$\bar{\sigma}_x$ $(a/2, 0)$	$\bar{\tau}_{zx}^{CR}$ $(0, 0)$	$\bar{\tau}_{zx}^{EE}$ $(0, 0)$
0.25	Present	$n = 1$ (CPT)	0.5124	0.5097	10.0854	—	1.7690
	Present	$n = 3$ (HSDT[a])	0.8640	2.6985	17.0063	2.4650	1.5568
	Present	$n = 5$	0.7601	2.5294	14.9618	2.2009	1.6590
	Present	$n = 7$	0.7023	2.4308	13.8242	2.0289	1.7031
	Present	$n = 9$	0.6661	2.3686	13.1115	1.9291	1.9117
	Mindlin [1]	FSDT	0.5124	2.4094	10.0854	1.5915	1.7690
	Pagano [7]	Exact	0.9500	2.8870	17.9500	1.4300	—
0.1	Present	$n = 1$ (CPT)	8.0057	0.5097	63.0339	—	4.4226
	Present	$n = 3$ (HSDT[a])	8.9197	0.8738	70.2305	6.4081	4.3342
	Present	$n = 5$	8.6336	0.8373	67.9780	5.5788	4.3780
	Present	$n = 7$	8.4835	0.8190	66.7963	5.1044	4.3961
	Present	$n = 9$	8.3914	0.8081	66.0703	4.8391	4.9231
	Mindlin [1]	FSDT	8.0057	0.8136	63.0339	3.9789	4.4226
	Pagano [7]	Exact	9.1850	0.8900	71.5000	4.2500	—
0.01	Present	$n = 1$ (CPT)	8005.66	0.5097	6303.39	—	44.226
	Present	$n = 3$ (HSDT[a])	8014.92	0.5133	6310.64	64.565	44.217
	Present	$n = 5$	8011.74	0.5129	6308.16	55.934	44.220
	Present	$n = 7$	8010.45	0.5128	6307.27	51.105	44.224
	Present	$n = 9$	8009.46	0.5126	6306.63	53.351	49.502
	Mindlin [1]	FSDT	8005.65	0.5127	6303.38	39.788	44.226

[a]HSDT of Reddy [2]

Table 4: Comparison of normalized displacements and stress for two layered (0°/90°/0°) symmterically laminated composite plates in cylindrical bending (Material 2).

h/a	Theory	Model	$\bar{u}$ $(0, h/2)$	w $(a/2, 0)$	$\bar{\sigma}_x$ $(a/2, 0)$	$\bar{\tau}_{zx}^{CR}$ $(0, 0)$	$\bar{\tau}_{zx}^{EE}$ $(0, 0)$
0.25	Present	$n = 1$ (CPT)	0.3207	0.3190	10.0912	—	1.7668
	Present	$n = 3$ (HSDT[a])	0.6627	2.4496	20.8518	2.3996	1.4396
	Present	$n = 5$	0.5659	2.3191	17.8069	2.1796	1.5939
	Present	$n = 7$	0.5097	2.2315	16.0390	2.0198	1.6625
	Present	$n = 9$	0.4740	2.1734	14.9157	1.9245	1.8718
	Mindlin [1]	FSDT	0.3207	2.2188	10.0912	1.5915	1.7668
0.1	Present	$n = 1$ (CPT)	5.0111	0.3190	63.0702	—	4.4169
	Present	$n = 3$ (HSDT[a])	5.9202	0.6815	74.5113	6.3784	4.2778
	Present	$n = 5$	5.6377	0.6461	70.9569	5.5697	4.3462
	Present	$n = 7$	5.4885	0.6281	69.0781	5.1005	4.3748
	Present	$n = 9$	5.3965	0.6173	67.9208	4.8372	4.9095
	Mindlin [1]	FSDT	5.0111	0.6230	63.0702	3.9789	4.4169
0.01	Present	$n = 1$ (CPT)	5011.13	0.3190	6307.02	—	44.169
	Present	$n = 3$ (HSDT[a])	5020.34	0.3227	6318.62	64.554	44.155
	Present	$n = 5$	5017.37	0.3223	6314.88	55.930	44.161
	Present	$n = 7$	5015.95	0.3221	6313.09	51.102	44.165
	Present	$n = 9$	5015.29	0.3223	6312.26	53.351	49.529
	Mindlin [1]	FSDT	5011.13	0.3221	6307.02	39.788	44.169

[a]HSDT of Reddy [2]

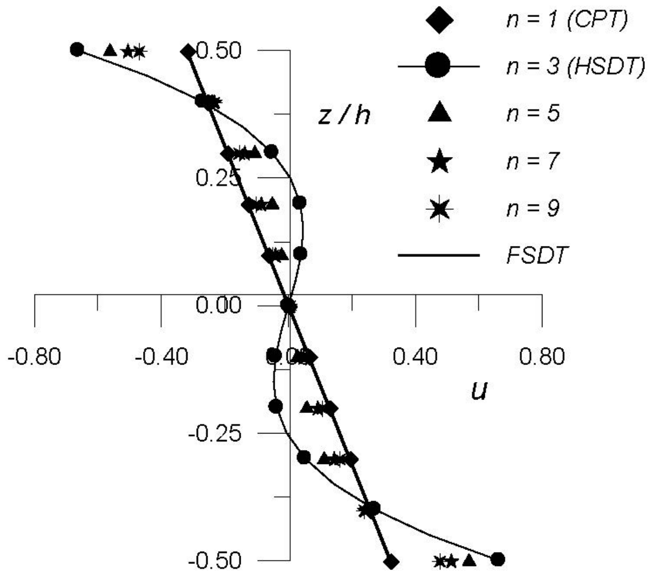

Figure 10: Through thickness distribution of in-plane displacement ($\bar{u}$) for simply supported three layered (0°/90°/0°) symmetrically laminated composite plates with a/h = 4 and E_1/E_2 = 40

5 Results and Discussions

The displacements and stresses of simply supported laminated composite plates are obtained at n = 1, 3, 5, 7 and 9. The classical plate theory (CPT) and higher order shear deformation theory (HSDT) of Reddy [2] are the special cases of present theory at n = 1 and n = 3 respectively. To verify the correctness of present theory, the obtained results are compared with exact elasticity solution given by Pagano [7] wherever possible. Also numerical results are generated by using first order shear deformation theory (FSDT) of Mindlin [1] for the comparison purpose. The results are presented in the following normalized form.

$$\bar{u} = \frac{u(0,z)E_3}{qh}; \quad \bar{w} = \frac{w(a/2,z)100h^3E_3}{qa^4}; \quad \bar{\sigma}_x = \frac{\sigma_x(a/2,z)h^2}{qa^2}; \quad \bar{\tau}_{zx} = \frac{\tau_{zx}(0,z)h}{qa} \tag{31}$$

The following material properties for high modulus graphite-epoxy laminated plates given by Pagano [7] are used to obtain the normalized displacements and stresses.

$$\textbf{Material 1}: \ E_1/E_3 = 25, \quad G_{31}/E_3 = 0.5, \quad G_{23}/E_3 = 0.2 \quad \text{and} \quad \mu_{13} = 0.25 \tag{32}$$

$$\textbf{Material 2}: \ E_1/E_3 = 40, \quad G_{31}/E_3 = 0.5, \quad G_{23}/E_3 = 0.2 \quad \text{and} \quad \mu_{13} = 0.25 \tag{33}$$

where the E_1/E_2 is the modular/stiffness ratio.

$$\text{For } 0^\circ \text{ layer: } Q_{11} = \frac{E_1}{1-\mu_{12}\mu_{21}} \quad \text{and} \quad Q_{55} = G_{31} \tag{34}$$

$$\text{For } 90^\circ \text{ layer: } Q_{11} = \frac{E_3}{1-\mu_{13}\mu_{31}} \quad \text{and} \quad Q_{55} = G_{23} \tag{35}$$

Example 1: In this example, displacements and stresses of a simply supported two layered (0°/90°) un-symmetrically laminated composite plate subjected to sinusoidally distributed load are obtained. The overall thickness of the plate is equally distributed among the layers. The transverse shear stresses are obtained using the constitutive relation ($\bar{\tau}_{zx}^{CR}$) and equilibrium equations ($\bar{\tau}_{zx}^{EE}$) of the theory of elasticity. The stress continuity condition is satisfied when transverse shear stresses are obtained using equilibrium equations of the theory of elasticity. In Table 1 results are presented for the plate made up of material 1 whereas in Table 2 results are presented for the plate made up of material 2. From Table 1 it can be observed that, the CPT

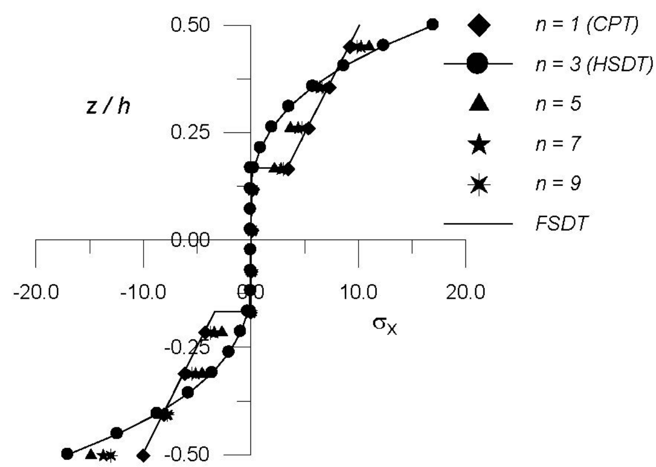

Figure 11: Through thickness distribution of bending stress ($\bar{\sigma}_x$) for simply supported three layered (0°/90°/0°) symmetrically laminated composite plates with a/h = 4 and E_1/E_2 = 25

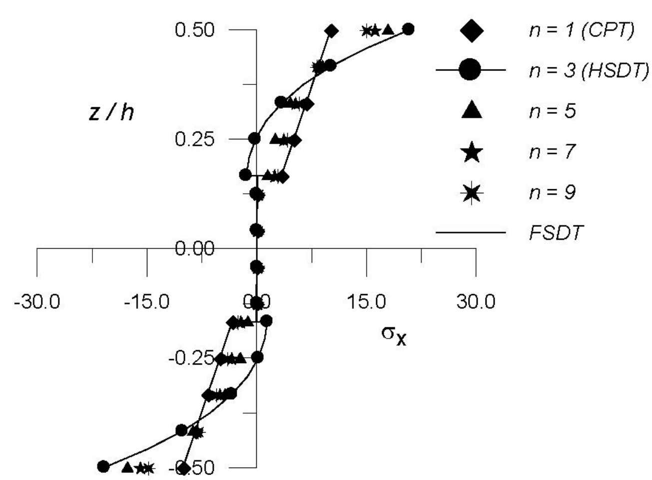

Figure 12: Through thickness distribution of bending stress ($\bar{\sigma}_x$) for simply supported three layered (0°/90°/0°) symmetrically laminated composite plates with a/h = 4 and E_1/E_2 = 40

(n = 1) and FSDT underestimates the values of displacements and bending stresses for all aspect ratios. For n = 3, 5 and 7, displacements and bending stresses predicted by present theory are in excellent agreement with respect to the exact elasticity solution. The thorough thickness variation of in-plane displacement and bending stresses are shown in Figs. 3 through 6. The present theory predicts excellent values of transverse shear stress when obtained using equilibrium equations. The distributions of transverse shear stress across the thickness of the plate are plotted in Figs. 7 and 8.

Example 2: This example presents displacements and stresses of a simply supported three layered (0°/90°/0°) symmetrically laminated composite plate. In this problem also the overall thickness of the plate is equally distributed among the layers. The comparison of displacements and stresses are shown in Tables 3 and 4. For symmetrically laminated composite plates, present theory at n = 3 (HSDT) is more efficient than the other models (n = 5, 7 and 9). The CPT and FSDT show less accurate results.

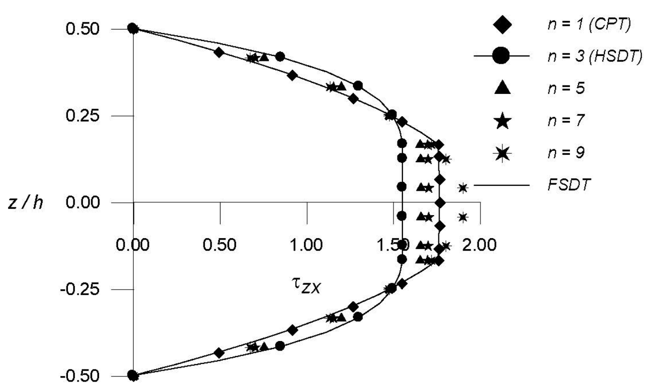

Figure 13: Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}$) for simply supported three layered (0°/90°/0°) symmetrically laminated composite plates with a/h = 4 and E_1/E_2 = 25

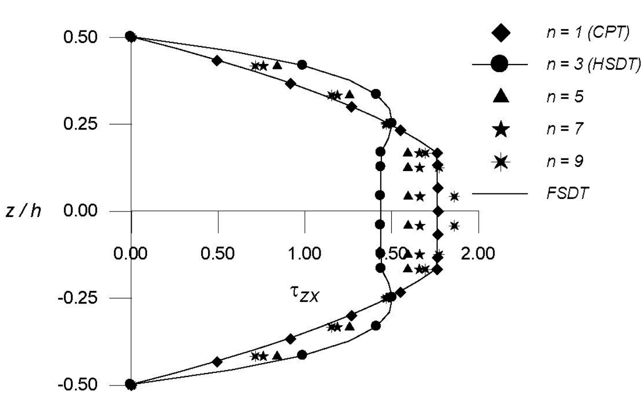

Figure 14: Through thickness distribution of transverse shear stress ($\bar{\tau}_{zx}$) for simply supported three layered (0°/90°/0°) symmetrically laminated composite plates with a/h = 4 and E_1/E_2 = 40

The distribution of in-plane displacements and stresses are shown in Figs. 9 through 14.

6 Conclusions

In this paper, a nth-order shear deformation theory developed by Xiang et al. [18] is successfully extended for the cylindrical bending of laminated composite plates. The theory accounts for a parabolic variation of the transverse shear stress across the thickness and satisfies the traction free boundary conditions on the top and bottom surfaces of the plate without using problem dependent shear correction factors. The variationally consistent governing equations and boundary conditions are obtained using the principle of virtual work. The numerical results presented in the study prove that, the classical plate theory and higher order shear deformation theory of Reddy are the special cases of the present theory. The present theory gives excellent results of displacements for higher values of n (3, 5, 7 and 9). In case of un-symmetrically laminated composite plates, present theory predicts excellent values of stresses at all values of n except n = 1. In case of symmetrically laminated composite plates, the excellent results are obtained at n = 3.

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