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Energy methods for curved composite beams with partial shear interaction

DOI 10.1515/cls-2015-0020

Received July 3, 2015; accepted August 12, 2015

Abstract: This paper presents a derivation of the Rayleigh-Betti reciprocity relation for layered curved composite beams with interlayer slip. The principle of minimum of potential energy is also formulated for two-layer curved composite beams and its applications are illustrated by numerical examples. The solution of the presented problems are obtained by the Ritz method. The applications of the Rayleigh-Betti reciprocity relation proven are illustrated by some examples.

Keywords: Curved beam; two-layer; interlayer slip; potential energy; reciprocity

1 Introduction

Layered beams made of different linearly elastic materials have a wide application in many civil engineering structures, where both high strength-to-weight and stiffness-to-weight ratio are required. The beam components made of different materials are connected by the shear connectors to work together. It may be the connection between the beam components is perfect only in normal direction and the connection is not perfect in tangential direction. A lot of works deal with numerical and analytical solution of straight layered composite beam with imperfect shear connections [1–8]. In papers [1–4] variational principles for layered straight beams with imperfect shear connections are formulated and used to get approximate solutions. We note that there are a lot of works on the static problems of layered curved composite beams with perfect shear connections, such as [9–15]. One part of papers mentioned above use analytical methods [9–11] and other part

of papers give the solution of static problems of curved layered composite beams with numerical methods which are based on different FEM formulations [12–14]. In paper by Assma [15] the developed analytical method is checked by FEM solution (ANSYS 14.5) and it is tested by experimentally. There are several published papers in connection with the problem of layered curved composite beams with perfect shear connection. However, there exist relative few works on curved layered composite beams including the effect of partial shear interaction. Here, we deal with imperfect shear connections. For out-of plane deformation and loads the time dependent creep and shrinkage behavior of horizontally curved steel-concrete composite beams with partial shear interaction are analysed by Liu et al. [16]. Erkmen et al. [17] developed a total Lagrangian finite element formulation for elastic analysis of steel-concrete curved composite beams. A three-dimensional finite element model is used to simulate composite steel-concrete curved beams subjected to combined flexure and torsion [18]. Tan and Uy gave a detailed description of the torsion induced vertical slip [18]. Two-layer curved composite beam with imperfect shear connection for in-plane deformation caused by concentrated radial load is considered in paper Ecsedi and Lengyel [19]. Here, the solution of the equilibrium problem of two-layer curved composite beam with weak shear connection is given by energy method. Paper [19] used a direct analytical method which is based on the integration of equilibrium equations. The equilibrium equations are formulated in terms of radial displacement, cross-sectional rotations and slip function in paper by Ecsedi and Lengyel [19].

In this paper a Rayleigh-Betti type reciprocity relation is derived for two-layer curved composite beam with imperfect shear connection. The principle of minimum of potential energy is also formulated and its applications are illustrated by numerical examples.

2 Governing equations

The curved two-layer composite beam and its cross-section are shown in Fig. 1. The cross-section of curved beam com-

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ponent B_i is A_i ($i = 1, 2$). The common boundary surface of B_1 and B_2 is $\partial B_{12} = \partial A_{12} \times (0, \alpha)$ as illustrated in Fig. 1. The designations of this paper follow which were used

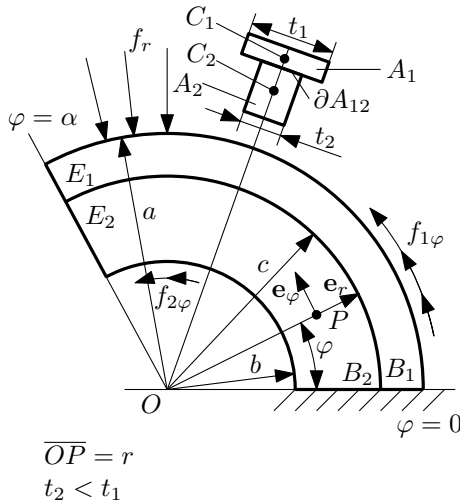


Figure 1: Two-layer curved beam.

in [19]. The connection of beam components B_1 and B_2 is perfect on their common boundary surface ∂B_{12} in radial direction but in the circumferential direction may be jump in the displacements, which is called interlayer slip. The formulation of the problem is given in the cylindrical coordinate system $Or\varphi z$. The in-plane deformation of two-layer curved beam is described by the next displacement field [9, 19]

$$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\varphi + w\mathbf{e}_z, \tag{1}$$

$$u = u(\varphi), w = 0, \tag{2}$$

$$v = r\phi_i(\varphi) + \frac{du}{d\varphi}, (r, \varphi, z) \in B_i, (i = 1, 2).$$

Following the derivations given in paper by Ecsedi-Lengyel [19] we get for the slip

$$s(\varphi) = c(\phi_1(\varphi) - \phi_2(\varphi)) \tag{3}$$

and for the interlayer shear force

$$T(\varphi) = kc^2t_2(\phi_1(\varphi) - \phi_2(\varphi)), \tag{4}$$

where k is the slip modulus. The virtual work of the distributed forces $f_r, f_{1\varphi}$ and $f_{2\varphi}$ on a small beam element (Fig. 2) can be computed as

$$d\tilde{W} = \left[f_r \tilde{u} + f_\varphi \frac{d\tilde{u}}{d\varphi} + m_1 \tilde{\phi}_1 + m_2 \tilde{\phi}_2 \right] d\varphi, \tag{5}$$

where

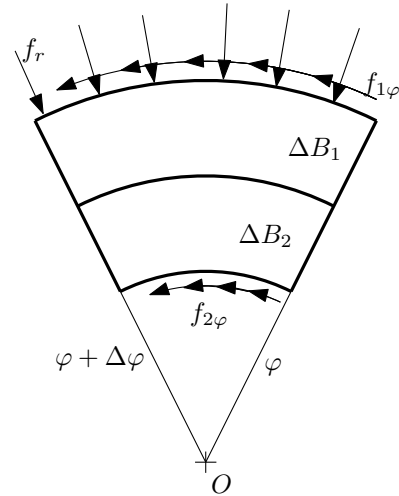


Figure 2: Virtual work of applied forces acting on a small beam element.

$$\tilde{\mathbf{u}} = \tilde{u}\mathbf{e}_r + \left(r\tilde{\phi}_i + \frac{d\tilde{u}}{d\varphi} \right) \mathbf{e}_\varphi, (r, \varphi, z) \in B_i, (i = 1, 2) \tag{6}$$

is the virtual displacement field and (Fig. 2)

$$f_\varphi = f_{1\varphi} + f_{2\varphi}, m_1 = af_{1\varphi}, m_2 = bf_{2\varphi}. \tag{7}$$

Equations of equilibrium can be formulated as [9, 19]

$$\frac{dN}{d\varphi} + S + f_\varphi = 0, \tag{8}$$

$$-N + \frac{dS}{d\varphi} + f_r = 0, \tag{9}$$

$$\frac{dM_1}{d\varphi} + m_1 - Q = 0, \tag{10}$$

$$\frac{dM_2}{d\varphi} + m_2 + Q = 0. \tag{11}$$

In Eqs. (8-11)

$$N = \int_A \sigma_\varphi dA, S = \int_A \tau_{r\varphi} dA, \tag{12}$$

$$M_1 = \int_{A_1} r\sigma_\varphi dA, M_2 = \int_{A_2} r\sigma_\varphi dA, \tag{13}$$

$$Q = K(\phi_1 - \phi_2), K = kc^3t_2, (t_2 < t_1). \tag{14}$$

The internal forces are illustrated in Fig. 3, where [19]

$$N_i = \int_{A_i} \sigma_\varphi dA, S_i = \int_{A_i} \tau_{r\varphi} dA, (i = 1, 2) \tag{15}$$

and we have

$$N = N_1 + N_2, S = S_1 + S_2. \tag{16}$$

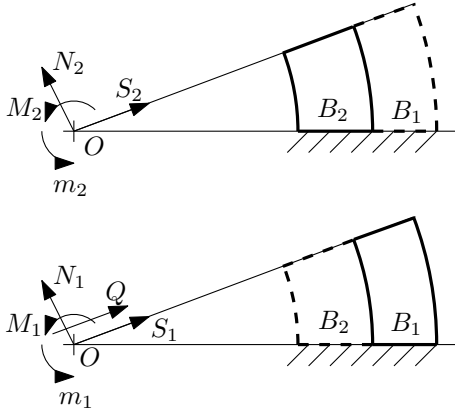


Figure 3: Illustration of internal forces and couples.

3 Rayleigh-Betti type reciprocity relation

Let us consider two equilibrium states of a two-layer curved beam with imperfect shear connection. These equilibrium states are denoted by upper one comma and upper two comma, respectively. Starting from Eqs. (8), (9) we can write

$$\left(\frac{dN'}{d\varphi} + S' + f'_\varphi \right) \frac{du''}{d\varphi} + \left(-N' + \frac{dS'}{d\varphi} + f'_r \right) u'' = 0. \quad (17)$$

Integration of Eq. (17) gives

$$\left\{ N' \frac{du''}{d\varphi} + S' u'' \right\}_0^\alpha + \int_0^\alpha \left(f'_\varphi \frac{du''}{d\varphi} + f'_r u'' \right) d\varphi - \int_0^\alpha N' \left(\frac{d^2 u''}{d\varphi^2} + u'' \right) d\varphi = 0, \quad (18)$$

where the next designation

$$\{F(\varphi)\}_0^\alpha = F(\alpha) - F(0) \quad (19)$$

is introduced. From Eqs. (10), (11) it follows that

$$\int_0^\alpha \left(\frac{dM'_1}{d\varphi} \phi''_1 + m'_1 \phi''_1 - Q' \phi''_1 + \frac{dM'_2}{d\varphi} \phi''_2 + m'_2 \phi''_2 + Q' \phi''_2 \right) d\varphi = \{M'_1 \phi''_1 + M'_2 \phi''_2\}_0^\alpha + \int_0^\alpha (m'_1 \phi''_1 + m'_2 \phi''_2) d\varphi - \int_0^\alpha \left[M_1 \frac{d\phi''_1}{d\varphi} + M_2 \frac{d\phi''_2}{d\varphi} + Q' (\phi''_1 - \phi''_2) \right] d\varphi = 0. \quad (20)$$

Let

$$W_{12} = \left\{ N' \frac{du''}{d\varphi} + S' u'' + M'_1 \phi''_1 + M'_2 \phi''_2 \right\}_0^\alpha + \int_0^\alpha \left(f'_r u'' + f'_\varphi \frac{du''}{d\varphi} + m'_1 \phi''_1 + m'_2 \phi''_2 \right) d\varphi \quad (21)$$

be. The mechanical meaning of W_{12} is obvious, the work done by the applied forces and reactions of the first equilibrium state on the displacement field caused by the forces applied in second equilibrium state. We define the mixed strain energy U_{12} for the states 1 and 2 as

$$U_{12} = \int_0^\alpha \left[N' \left(\frac{d^2 u''}{d\varphi^2} + u'' \right) + M'_1 \frac{d\phi''_1}{d\varphi} + M'_2 \frac{d\phi''_2}{d\varphi} + Q' (\phi''_1 - \phi''_2) \right] d\varphi. \quad (22)$$

The combination of Eq. (18) with Eq. (20) gives

$$W_{12} = U_{12}. \quad (23)$$

In paper by Ecsedi and Lengyel [19] it was proven that

$$N = \frac{AE_0}{R} \left(\frac{d^2 u}{d\varphi^2} + u \right) + A_1 E_1 \frac{d\phi_1}{d\varphi} + A_2 E_2 \frac{d\phi_2}{d\varphi}, \quad (24)$$

$$M_i = A_i E_i \left(\frac{d^2 u}{d\varphi^2} + u \right) + r_i A_i E_i \frac{d\phi_i}{d\varphi}, \quad (i = 1, 2), \quad (25)$$

$$M = M_1 + M_2 =$$

$$AE_0 \left(\frac{d^2 u}{d\varphi^2} + u \right) + r_1 A_1 E_1 \frac{d\phi_1}{d\varphi} + r_2 A_2 E_2 \frac{d\phi_2}{d\varphi}, \quad (26)$$

where (Fig. 1)

$$E_0 = \frac{A_1 E_1 + A_2 E_2}{A}, \quad A = A_1 + A_2, \quad (27)$$

$$r_i = \frac{1}{A_i} \int_{A_i} r dA, \quad r_i = \overline{OC}_i, \quad (i = 1, 2), \quad (28)$$

$$\frac{AE_0}{R} = E_1 \int_{A_1} \frac{dA}{r} + E_2 \int_{A_2} \frac{dA}{r}. \quad (29)$$

By the use of Eq. (14) and Eqs. (24), (25) we can reformulate the expression of U_{12} as

$$U_{12} = \int_0^\alpha \left[\frac{AE_0}{R} \left(\frac{d^2 u'}{d\varphi^2} + u' \right) \left(\frac{d^2 u''}{d\varphi^2} + u'' \right) + A_1 E_1 \left(\frac{d^2 u''}{d\varphi^2} + u'' \right) \frac{d\phi'_1}{d\varphi} + A_1 E_1 \left(\frac{d^2 u'}{d\varphi^2} + u' \right) \frac{d\phi''_1}{d\varphi} + A_2 E_2 \left(\frac{d^2 u''}{d\varphi^2} + u'' \right) \frac{d\phi'_2}{d\varphi} + A_2 E_2 \left(\frac{d^2 u'}{d\varphi^2} + u' \right) \frac{d\phi''_2}{d\varphi} + r_1 A_1 E_1 \frac{d\phi'_1}{d\varphi} \frac{d\phi''_1}{d\varphi} + r_2 A_2 E_2 \frac{d\phi'_2}{d\varphi} \frac{d\phi''_2}{d\varphi} \right] d\varphi$$

$$+K (\phi'_1 - \phi'_2) (\phi''_1 - \phi''_2)] d\varphi. \tag{30}$$

It is evident

$$U_{12} = U_{21} \text{ and } U_{21} = W_{21}, \tag{31}$$

where

$$W_{21} = \left\{ N'' \frac{du'}{d\varphi} + S''u' + M''_1\phi'_1 + M''_2\phi'_2 \right\}_0^\alpha + \int_0^\alpha \left(f''_r u' + f''_\varphi \frac{du'}{d\varphi} + m''_1\phi'_1 + m''_2\phi'_2 \right) d\varphi. \tag{32}$$

Comparison of Eq. (23) with Eq. (31) yields the next Rayleigh-Betti type reciprocity relation

$$W_{12} = W_{21}. \tag{33}$$

If the two equilibrium states are the same, that is

$$u = u' = u'', \quad \phi_1 = \phi'_1 = \phi''_1, \quad \phi_2 = \phi'_2 = \phi''_2, \dots, \tag{34}$$

then we have according to Clapeyron's theorem

$$U = W, \tag{35}$$

where U is the strain energy of the two-layer composite beam with imperfect shear connection

$$U = \frac{1}{2} \int_0^\alpha \left[\frac{AE_0}{R} \left(\frac{d^2u}{d\varphi^2} + u \right)^2 + 2A_1E_1 \left(\frac{d^2u}{d\varphi^2} + u \right) \frac{d\phi_1}{d\varphi} + 2A_2E_2 \left(\frac{d^2u}{d\varphi^2} + u \right) \frac{d\phi_2}{d\varphi} + r_1A_1E_1 \left(\frac{d\phi_1}{d\varphi} \right)^2 + r_2A_2E_2 \left(\frac{d\phi_2}{d\varphi} \right)^2 + K(\phi_1 - \phi_2)^2 \right] d\varphi \tag{36}$$

and W is the work of the applied forces which can be written in the next form

$$W = \frac{1}{2} \left[\left\{ N \frac{du}{d\varphi} + Su + M_1\phi_1 + M_2\phi_2 \right\}_0^\alpha + \int_0^\alpha \left(f_r u + f_\varphi \frac{du}{d\varphi} + m_1\phi_1 + m_2\phi_2 \right) d\varphi \right]. \tag{37}$$

4 Principle of minimum of potential energy

Let $\tilde{u} = \tilde{u}(\varphi)$, $\tilde{\phi}_1 = \tilde{\phi}_1(\varphi)$ and $\tilde{\phi}_2 = \tilde{\phi}_2(\varphi)$ be such functions which satisfy the geometric boundary conditions.

The geometric boundary conditions refer to deflection and cross-sectional rotation [19]. For this field we define the potential energy as [20, 21]

$$\Pi_L(\tilde{u}, \tilde{\phi}_1, \tilde{\phi}_2) = U(\tilde{u}, \tilde{\phi}_1, \tilde{\phi}_2) - \{ \text{virtual work of the prescribed forces on } \tilde{u}, \tilde{\phi}_1 \text{ and } \tilde{\phi}_2 \}. \tag{38}$$

It can be proven that according to the minimum property of potential energy [20, 21]

$$\Pi_L(u, \phi_1, \phi_2) \leq \Pi_L(\tilde{u}, \tilde{\phi}_1, \tilde{\phi}_2), \tag{39}$$

where $u = u(\varphi)$, $\phi_1 = \phi_1(\varphi)$ and $\phi_2 = \phi_2(\varphi)$ are the solution of the considered equilibrium problem of layered composite beam with weak shear connection. Next, it will be proven from the principle of minimum of potential energy the equations of equilibrium and force boundary conditions are obtained. Let $u = u(\varphi)$, $\phi_1 = \phi_1(\varphi)$, $\phi_2 = \phi_2(\varphi)$ be the solution of the considered equilibrium problem. The kinematically admissible radial displacement and cross-sectional rotations can be represented as

$$\tilde{u} = u + \delta u, \quad \tilde{\phi}_1 = \phi_1 + \delta\phi_1, \quad \tilde{\phi}_2 = \phi_2 + \delta\phi_2. \tag{40}$$

Here, δu , $\delta\phi_1$ and $\delta\phi_2$ satisfy homogeneous kinematic boundary conditions where u , ϕ_1 or ϕ_2 are prescribed as a boundary condition. Assuming that all the boundary conditions are force boundary conditions, this means that $N(0)$, $N(\alpha)$, $S(0)$, $S(\alpha)$, $M_1(0)$, $M_1(\alpha)$ and $M_2(0)$, $M_2(\alpha)$ are prescribed. In this case

$$\begin{aligned} \Pi(\tilde{u}, \tilde{\phi}_1, \tilde{\phi}_2) &= \frac{1}{2} \int_0^\alpha \left[\frac{AE_0}{R} \left(\frac{d^2\tilde{u}}{d\varphi^2} + \tilde{u} \right)^2 + 2A_1E_1 \left(\frac{d^2\tilde{u}}{d\varphi^2} + \tilde{u} \right) \frac{d\tilde{\phi}_1}{d\varphi} + 2A_2E_2 \left(\frac{d^2\tilde{u}}{d\varphi^2} + \tilde{u} \right) \frac{d\tilde{\phi}_2}{d\varphi} + r_1A_1E_1 \left(\frac{d\tilde{\phi}_1}{d\varphi} \right)^2 + r_2A_2E_2 \left(\frac{d\tilde{\phi}_2}{d\varphi} \right)^2 + K(\tilde{\phi}_1 - \tilde{\phi}_2)^2 \right] d\varphi - \int_0^\alpha \left(f_r \tilde{u} + f_\varphi \frac{d\tilde{u}}{d\varphi} + m_1\tilde{\phi}_1 + m_2\tilde{\phi}_2 \right) d\varphi - \left\{ \bar{N} \frac{d\tilde{u}}{d\varphi} + \bar{S}\tilde{u} + \bar{M}_1\tilde{\phi}_1 + \bar{M}_2\tilde{\phi}_2 \right\}_0^\alpha, \tag{41} \end{aligned}$$

where the quantities with overbar are given. By a lengthy, but elementary computation which includes the application of the integration of parts for

$$\Delta\Pi_L = \Pi_L(u + \delta u, \phi_1 + \delta\phi_1, \phi_2 + \delta\phi_2) - \Pi_L(u, \phi_1, \phi_2) \tag{42}$$

the next result can be derived

$$\Delta\Pi_L = \delta\Pi + U(\delta u, \delta\phi_1, \delta\phi_2), \tag{43}$$

where

$$\begin{aligned} \delta \Pi = & \int_0^\alpha \left[\left(N + \frac{d^2 N}{d\varphi^2} - f_r + \frac{df_\varphi}{d\varphi} \right) \delta u - \right. \\ & - \left(\frac{dM_1}{d\varphi} + m_1 - Q(\phi_1 - \phi_2) \right) \delta \phi_1 - \\ & - \left(\frac{dM_2}{d\varphi} + m_2 + Q(\phi_1 - \phi_2) \right) \delta \phi_2 \Big] d\varphi + \\ & + \left\{ (N - \bar{N}) \frac{d}{d\varphi} \delta u - \left(\frac{dN}{d\varphi} + f_\varphi + \bar{S} \right) \delta u + \right. \\ & + (M_1 - \bar{M}_1) \delta \phi_1 + (M_2 - \bar{M}_2) \delta \phi_2 \Big\}_0^\alpha. \end{aligned} \quad (44)$$

For admissible variation of u , ϕ_1 , ϕ_2 , which are δu , $\delta \phi_1$ and $\delta \phi_2$ are arbitrary except where the kinematic boundary conditions are specified. Since with arbitrary admissible variation of u , ϕ_1 , ϕ_2

$$\Delta \Pi_L \geq 0 \quad (45)$$

according to the principle of minimum of potential energy. From Eq. (43) and inequality relation (45) and

$$U(\delta u, \delta \phi_1, \delta \phi_2) \geq 0 \quad (46)$$

we obtain

$$\delta \Pi = 0. \quad (47)$$

A detailed form of Eq. (47), which can be derived by means of the fundamental lemma of calculus of variation [22] gives the equations of equilibrium

$$\frac{d^2 N}{d\varphi^2} + N - f_r + \frac{df_\varphi}{d\varphi} = 0, \quad 0 < \varphi < \alpha, \quad (48)$$

$$\frac{dM_1}{d\varphi} + m_1 - Q(\phi_1 - \phi_2) = 0, \quad 0 < \varphi < \alpha, \quad (49)$$

$$\frac{dM_2}{d\varphi} + m_2 + Q(\phi_1 - \phi_2) = 0, \quad 0 < \varphi < \alpha, \quad (50)$$

and boundary conditions

$$N - \bar{N} = 0 \text{ for } \varphi = 0 \text{ and } \varphi = \alpha, \quad (51)$$

$$\frac{dN}{d\varphi} + f_\varphi + \bar{S} = 0 \text{ for } \varphi = 0 \text{ and } \varphi = \alpha, \quad (52)$$

$$M_1 - \bar{M}_1 = 0 \text{ for } \varphi = 0 \text{ and } \varphi = \alpha, \quad (53)$$

$$M_2 - \bar{M}_2 = 0 \text{ for } \varphi = 0 \text{ and } \varphi = \alpha. \quad (54)$$

We note, Eq. (48) is obtained from Eqs. (8), (9) with the elimination of $S = S(\varphi)$ and the validity of boundary condition (52) follows from Eq. (8). In Eqs. (48-50) N , M_1 , M_2 , Q are given by Eqs. (14), (24), (25) in terms of u , ϕ_1 and ϕ_2 .

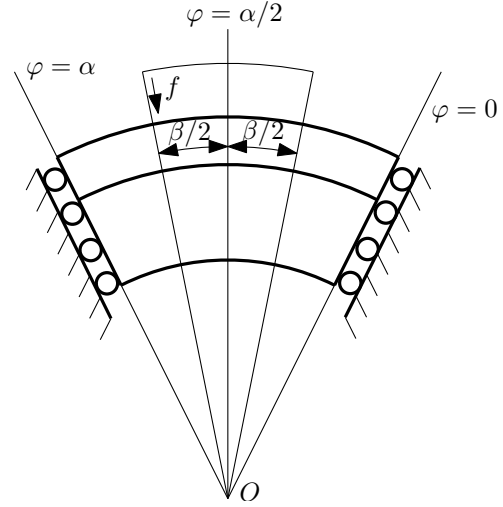


Figure 4: Uniformly loaded curved composite beam.

5 Examples

5.1 A curved composite beam with uniformly distributed radial load

Both ends of curved two-layer composite beam with flexible shear connection are radially guided and loaded by uniformly distributed radial forces as shown in Fig. 4. The applied radial load is expressed as

$$f_r(\varphi) = -f \left[H \left(\varphi - \frac{\alpha}{2} + \frac{\beta}{2} \right) - H \left(\varphi - \frac{\alpha}{2} - \frac{\beta}{2} \right) \right], \quad (55)$$

where

$$0 < \beta \leq \alpha \quad (56)$$

and $H = H(\varphi)$ is the Heaviside function. In this problem the boundary conditions are as follows [19]

$$\phi_1(0) = \phi_1(\alpha) = \phi_2(0) = \phi_2(\alpha) = 0, \quad (57)$$

$$S(0) = S(\alpha) = 0. \quad (58)$$

The minimum of the potential energy is obtained by the application of Ritz method. Assumed form of the solution is

$$u(\varphi) = u_0 + \sum_{p=1}^{\infty} u_p \cos \frac{p\pi}{\alpha} \varphi, \quad (59)$$

$$\phi_1(\varphi) = \sum_{p=1}^{\infty} \phi_{1p} \sin \frac{p\pi}{\alpha} \varphi, \quad \phi_2(\varphi) = \sum_{p=1}^{\infty} \phi_{2p} \sin \frac{p\pi}{\alpha} \varphi. \quad (60)$$

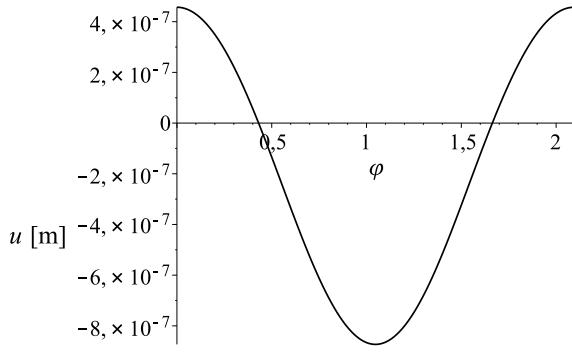
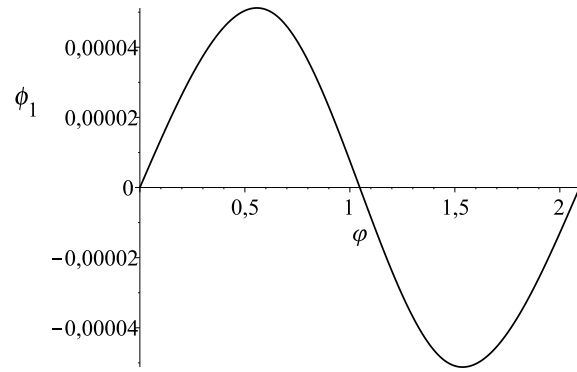


Figure 5: The graph of the radial displacement.

Figure 6: The graph of $\phi_1 = \phi_1(\varphi)$.

With arbitrary ϕ_{1p} ($i = 1, 2; p = 1, 2, \dots$) the cross-sectional rotations given by Eq. (60) satisfy the geometric boundary conditions formulated in Eq. (57). Substitution of Eqs. (59), (60) into the expression of potential energy we obtain that

$$\begin{aligned} \Pi_L(u_0, u_p, \phi_{1p}, \phi_{2p}; p = 1 \dots) = & \frac{1}{2} \frac{AE_0}{R} \alpha u_0^2 + \\ & + \frac{\alpha}{4} \sum_{p=1}^{\infty} \left\{ \frac{AE_0}{R} \left(1 - \left(\frac{p\pi}{\alpha}\right)^2\right)^2 u_p^2 + \right. \\ & + 2A_1 E_1 \left(1 - \left(\frac{p\pi}{\alpha}\right)^2\right) \frac{p\pi}{\alpha} u_p \phi_{1p} + \\ & + 2A_2 E_2 \left(1 - \left(\frac{p\pi}{\alpha}\right)^2\right) \frac{p\pi}{\alpha} u_p \phi_{2p} + \\ & + r_1 A_1 E_1 \left(\frac{p\pi}{\alpha}\right)^2 \phi_{1p}^2 + r_2 A_2 E_2 \left(\frac{p\pi}{\alpha}\right)^2 \phi_{2p}^2 + \\ & \left. + K(\phi_{1p} - \phi_{2p})^2 \right\} + f u_0 \beta + \\ & + \sum_{p=1}^{\infty} f u_p \frac{\alpha}{p\pi} \left[\sin \frac{p\pi}{2\alpha} (\alpha + \beta) - \sin \frac{p\pi}{2\alpha} (\alpha - \beta) \right]. \end{aligned} \quad (61)$$

The necessary condition of minimum of Π_L as a function of $u_0, u_p, \phi_{1p}, \phi_{2p}$ ($p = 1, 2, \dots$) can be formulated as

$$\frac{\partial \Pi_L}{\partial u_0} = 0, \quad \frac{\partial \Pi_L}{\partial u_p} = 0, \quad \frac{\partial \Pi_L}{\partial \phi_{1p}} = 0, \quad \frac{\partial \Pi_L}{\partial \phi_{2p}} = 0, \quad (p = 1, 2, \dots). \quad (62)$$

From Eq. (62) it follows that

$$u_0 = -f \frac{R}{AE_0} \frac{\beta}{\alpha} \quad (63)$$

and $u_p, \phi_{1p}, \phi_{2p}$ are the solution of the following system of linear equations

$$\mathbf{M}_p \mathbf{x}_p = \mathbf{b}_p, \quad \mathbf{M}_p = [m_{p ij}], \quad \mathbf{x}_p = [u_p, \phi_{1p}, \phi_{2p}]^T, \quad (64)$$

$$\mathbf{b}_p = [b_p, 0, 0]^T, \quad (65)$$

where

$$m_{p11} = \frac{AE_0}{R} \left(1 - \left(\frac{p\pi}{\alpha}\right)^2\right)^2, \quad (66)$$

$$m_{p12} = A_1 E_1 \frac{p\pi}{\alpha} \left(1 - \left(\frac{p\pi}{\alpha}\right)^2\right),$$

$$m_{p13} = A_2 E_2 \frac{p\pi}{\alpha} \left(1 - \left(\frac{p\pi}{\alpha}\right)^2\right),$$

$$m_{p21} = m_{p12}, \quad m_{p22} = r_1 A_1 E_1 \left(\frac{p\pi}{\alpha}\right)^2 + K, \quad (67)$$

$$m_{p23} = -K,$$

$$m_{p31} = m_{p13}, \quad m_{p32} = m_{p23}, \quad (68)$$

$$m_{p33} = r_2 A_2 E_2 \left(\frac{p\pi}{\alpha}\right)^2 + K,$$

$$b_p = -\frac{2f}{p\pi} \left[\sin \frac{p\pi}{2\alpha} (\alpha + \beta) - \sin \frac{p\pi}{2\alpha} (\alpha - \beta) \right]. \quad (69)$$

The next data are used in Example 5.1: $\alpha = \frac{2\pi}{3}$, $a = 0.04$ [m], $b = 0.02$ [m], $c = 0.03$ [m], $t_1 = t_2 = t = 0.03$ [m], $E_1 = 10^{11}$ [Pa], $E_2 = 8 \times 10^{10}$ [Pa], $k = 80 \times 10^8$ [Pa/m³], $f = 5000$ [N/m], $\beta = \frac{\pi}{4}$. Figs. 5, 6 and 7 show the graphs of deflection and cross-sectional rotation functions. The normal force as a function of φ is shown in Fig. 8. The graphs of shear force function $S = S(\varphi)$ and bending moment function $M = M(\varphi)$ are illustrated in Figs. 9 and 10.

5.2 A curved composite beam subjected to uniformly distributed radial load on its total length

In this example we consider the case $\alpha = \beta$ which is shown in Fig. 11. Here, the same data are used as in Example 5.1 except β ($\beta = \frac{2\pi}{3}$). In this case we have from Eq. (68)

$$b_p = 0, \quad (p = 1, 2, \dots). \quad (70)$$

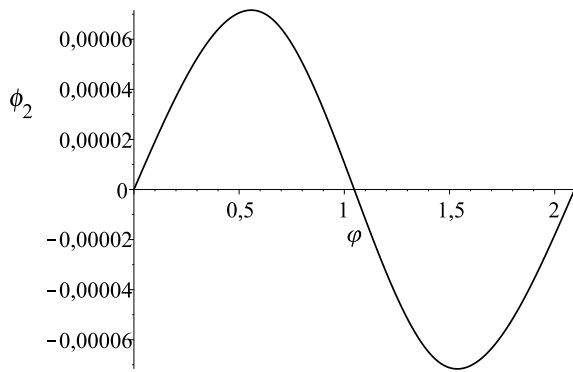


Figure 7: The graph of $\phi_2 = \phi_2(\varphi)$.

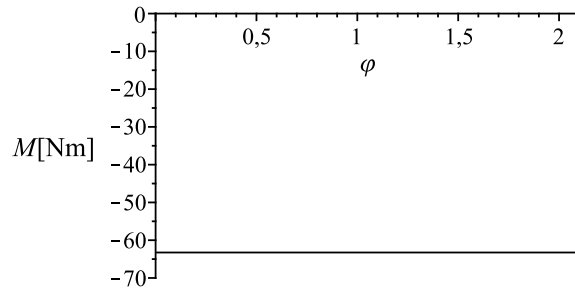


Figure 10: The graph of $M = M(\varphi)$.

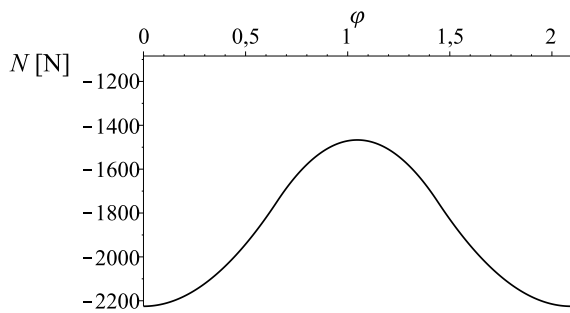


Figure 8: The plot of the normal force function.

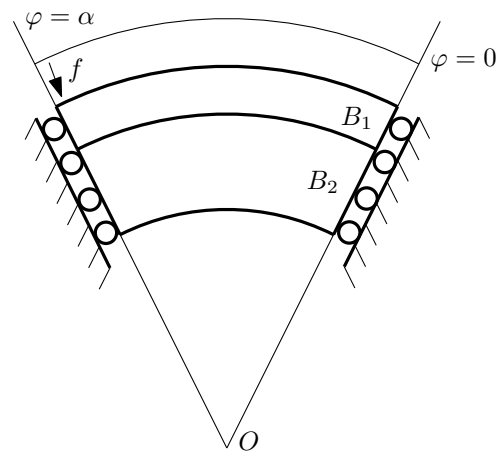


Figure 11: The case of $\beta = \alpha$.

The solution of this problem is as follows

$$u = -f \frac{R}{AE_0}, \quad \phi_1(\varphi) = 0, \quad \phi_2(\varphi) = 0, \quad (71)$$

$$N = -f, \quad S = 0, \quad M = -Rf. \quad (72)$$

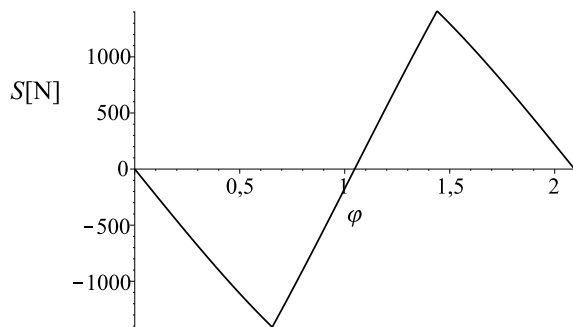


Figure 9: The plot of the shear force function.

5.3 Checking the previous examples

In this example we check the exactness of solution of Examples 5.1 and 5.2 by the application of Rayleigh-Betti reciprocity relation. The first equilibrium state of the curved composite beam with deformable shear connection is shown in Fig. 4 and the second equilibrium state is illustrated in Fig. 11. In this example

$$W_{12} = - \int_0^\alpha f_r(\varphi) f \frac{R}{AE_0} d\varphi, \quad (73)$$

where $f_r = f_r(\varphi)$ is given by Eq. (55), and

$$W_{21} = \int_0^\alpha (-f) u(\varphi) d\varphi, \tag{74}$$

where $u = u(\varphi)$ is given by Eq. (59). A simple computation gives

$$\begin{aligned} W_{12} &= 0.00204454608 \text{ [Nm]}, \\ W_{21} &= 0.00204454606 \text{ [Nm]}. \end{aligned} \tag{75}$$

5.4 A curved composite beam with concentrated radial load

This example deals with the case of concentrated radial load applied at $\varphi = \frac{\alpha}{2}$ as shown in Fig. 12. The Ritz type solution is based on the assumed forms of radial displacement and cross-sectional rotations given by Eqs. (59) and (60). The virtual work of the concentrated radial force on the radial displacement (59) can be computed as

$$W_F = (-F) \left[u_0 + \sum_{p=1}^\infty u_p \cos \frac{p\pi}{2} \right]. \tag{76}$$

From the principle of minimum of potential energy by the use of expression of W_F we obtain that

$$u_0 = -F \frac{R}{\alpha E_0 A}, \quad b_p = -\frac{2F}{\alpha} \cos \frac{p\pi}{2}, \quad (p = 1, 2, \dots). \tag{77}$$

Let $F = 5000$ [N] be. The same geometrical and material properties are used to solve Eq. (64) with the new value of b_p ($p = 1, 2, \dots$). The results of the computations are shown in Figs. 13-18. The radial displacement and cross-sectional rotations as functions of φ are shown in Figs. 13, 14 and 15. The graphs of normal force, shear force and bending moment are illustrated in Figs. 16, 17 and 18. Here we note, that the shear force function has a jump at $\varphi = \frac{\alpha}{2}$ (Fig. 17). The small oscillation of $S = S(\varphi)$ at $\varphi = \frac{\alpha}{2}$ follows from its representation by „truncated” Fourier series.

5.5 Checking the results of the curved beam with concentrated radial load

By the use of Rayleigh-Betti type reciprocity relation we check the accuracy of the solution obtained for concentrated radial load. The first equilibrium state of the curved composite beam is shown in Fig. 12 and the second equilibrium state is illustrated in Fig. 11. For these equilibrium states we have

$$W_{12} = (-F)(-f) \frac{R}{AE_0}, \tag{78}$$

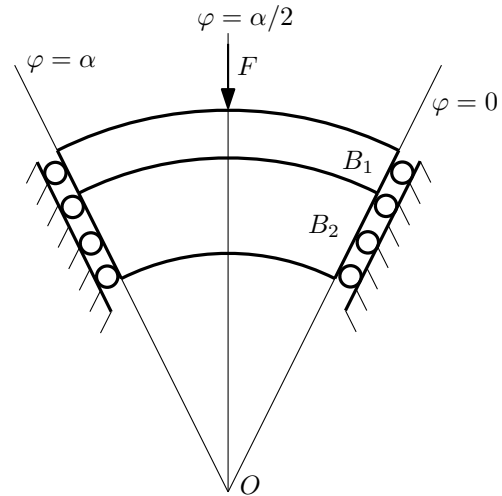


Figure 12: The case of concentrated radial load.

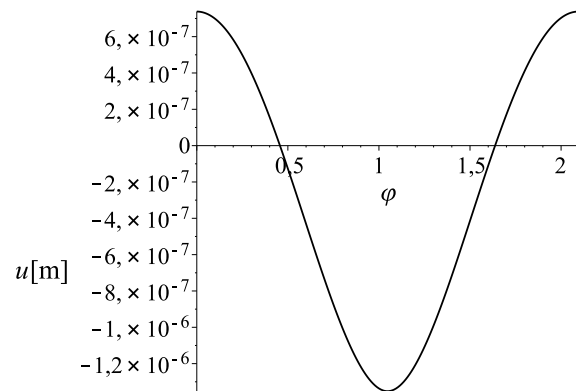


Figure 13: The plot of the radial displacement.

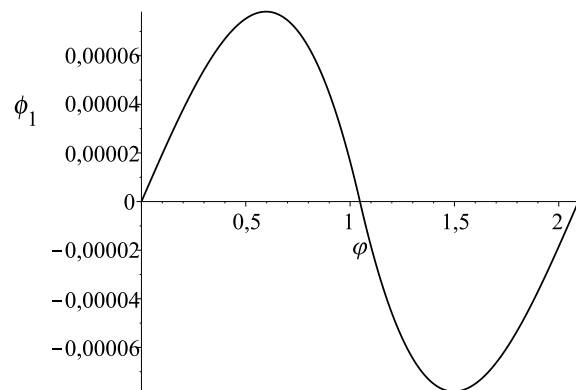


Figure 14: The plot of $\phi_1 = \phi_1(\varphi)$.

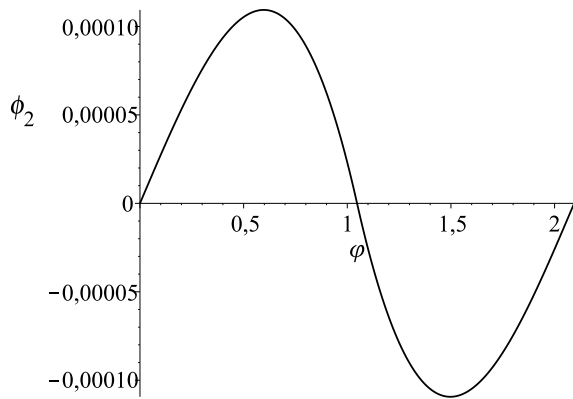


Figure 15: The plot of $\phi_2 = \phi_2(\varphi)$.

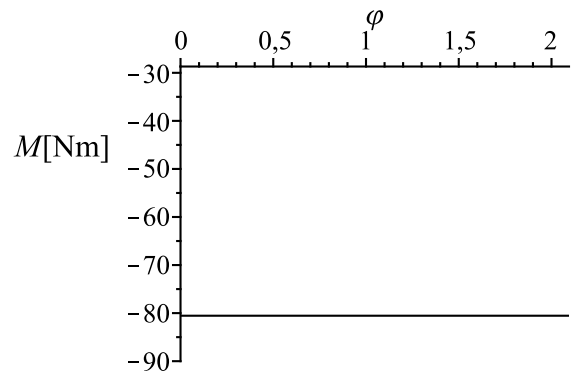


Figure 18: The graph of $M = M(\varphi)$.

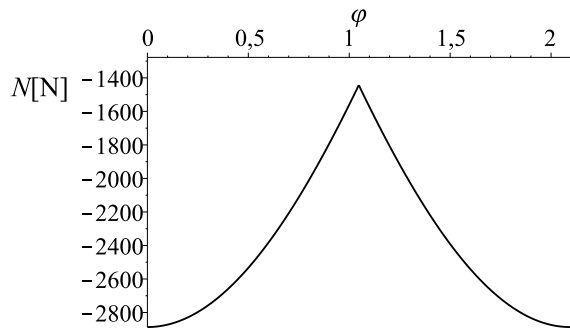


Figure 16: The graph of normal force function.

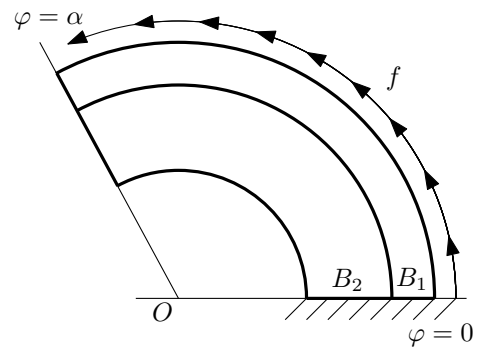


Figure 19: Two-layer curved composite beam uniformly loaded by tangential forces.

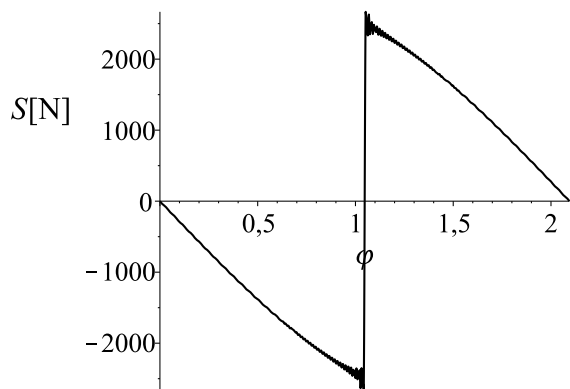


Figure 17: The graph of the shear force function.

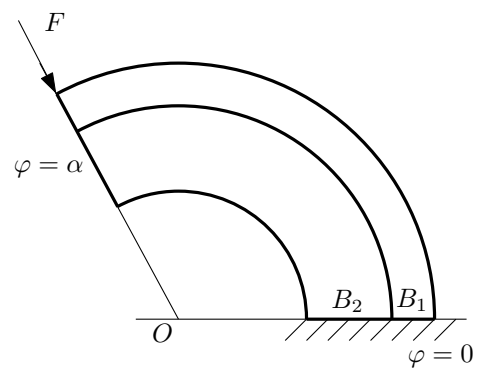


Figure 20: Two-layer composite beam with concentrated radial load.

$$W_{21} = (-f) \int_0^{\alpha} u(\varphi) d\varphi, \quad (79)$$

where $u = u(\varphi)$ is obtained in Example 5.4. A simple computation gives

$$\begin{aligned} W_{12} &= 0.002603196319 \text{ [Nm]}, \\ W_{21} &= 0.002603196318 \text{ [Nm]} \end{aligned} \quad (80)$$

according to the proven Rayleigh-Betti type reciprocity relation.

5.6 A curved composite beam uniformly loaded by tangential forces

Fig. 19 shows a two-layer composite beam with deformable shear connection loaded by uniform tangential load on its outer cylindrical boundary. The geometrical and material properties of the considered curved beam is the same as in Example 5.1. In paper [19] the solution of two-layer composite beam with weak shear connection for radial concentrated load applied at its one of the end cross-section was derived (Fig. 20). The first equilibrium state of the composite curved beam is shown in Fig. 19 and the second equilibrium state of the same curved composite beam is given in Fig. 20. Our aim is to obtain the deflection of the end cross-section of curved beam loaded by uniformly distributed tangential forces (Fig. 19). According to the Rayleigh-Betti theorem we can write that

$$\begin{aligned} W_{12} &= f \int_0^{\alpha} v''(a, \varphi) d\varphi = \\ &= f \left[a \int_0^{\alpha} \phi_1''(\varphi) d\varphi + u''(\alpha) \right], \end{aligned} \quad (81)$$

$$W_{21} = -Fu'(\alpha). \quad (82)$$

From Eqs. (81) and (82) it follows that

$$\begin{aligned} u'(\alpha) &= -\frac{f}{F} \int_0^{\alpha} v''(a, \varphi) d\varphi = \\ &= -\frac{f}{F} \left[a \int_0^{\alpha} \phi_1''(\varphi) d\varphi + u''(\alpha) \right]. \end{aligned} \quad (83)$$

Let

$$f = 1500 \text{ [N]}, \quad F = 1000 \text{ [N]} \quad (84)$$

be in Eq. (83). Other data are given in Example 5.1. By these data using the solution presented for $\phi_1'' = \phi_1''(\varphi)$ and $u'' = u''(\varphi)$ in [19] we get

$$u'(\alpha) = -0.0000188033149 \text{ [m]}. \quad (85)$$

6 Conclusions

In this paper the in-plane deformation of two-layer composite beam with flexible shear connection is analysed by energy methods. A detailed derivation of the Rayleigh-Betti type reciprocity relation is presented and its applications are illustrated by several numerical examples. A formulation of the principle of minimum of potential energy is also given. By the combination of principle of minimum of potential energy with the Ritz method the solution of two-layer composite beam with radially guided ends under the action of radial loads are given. The validity of the solution obtained by Ritz method is checked by the application of the proven reciprocity relation. The numerical results derived for two-layer curved composite beams with deformable shear connection can be used as benchmark solutions to check the accuracy of solutions obtained by different approximate methods such as FEM, finite differences etc.

Acknowledgement: This research (partially) carried out in the framework of the Center of Excellence of Innovative Engineering Design and Technologies at the University of Miskolc.

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