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Micropolar curved rods. 2-D, high order, Timoshenko's and Euler-Bernoulli models

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Abstract: New models for micropolar plane curved rods have been developed. 2-D theory is developed from general 2-D equations of linear micropolar elasticity using a special curvilinear system of coordinates related to the middle line of the rod and special hypothesis based on assumptions that take into account the fact that the rod is thin. High order theory is based on the expansion of the equations of the theory of elasticity into Fourier series in terms of Legendre polynomials. First stress and strain tensors, vectors of displacements and rotation and body forces have been expanded into Fourier series in terms of Legendre polynomials with respect to a thickness coordinate. Thereby all equations of elasticity including Hooke's law have been transformed to the corresponding equations for Fourier coefficients. Then in the same way as in the theory of elasticity, system of differential equations in term of displacements and boundary conditions for Fourier coefficients have been obtained. The Timoshenko's and Euler-Bernoulli theories are based on the classical hypothesis and 2-D equations of linear micropolar elasticity in a special curvilinear system. The obtained equations can be used to calculate stress-strain and to model thin walled structures in macro, micro and nano scale when taking into account micropolar couple stress and rotation effects.

Keywords: Curved rod; micropolar; Legendre polynomial; Timoshenko theory; Euler-Bernoulli theory

1 Introduction

Classical linear theory of elasticity is the most popular and usable in engineering and scientific applications. It is based on assumptions that internal interactions between neighboring elements of an elastic continuum occur only by means of the symmetric force-based stress tensor, deformations are determined by symmetric tensor of deformation and motion of material particles are described by position vector. However the classical theory of elasticity fails to produce acceptable results for cases with large stress gradients (*e.g.* in the vicinity of holes and cracks), materials with significant microstructure contribution (*e.g.* composites, polymers, soil, and bone) and at micro- and nano-scale (*e.g.* micro- and nano-sized components such as thin films, beams, and plates, which are commonly used in MEMS and NEMS devices) [1, 2].

In order to improve the results of the classical theory of elasticity various theories of generalized continua have been developed [3–5]. The mechanics of generalized continua has long history of development. Since the publication of landmark book of Cosserat brothers [6] in the literature there are various known generalizations of the classical or Cauchy continuum which are summarized in many books [7–12] and papers [13–17]. Especially we have to mention remarkable books of Eringen [18] and Nowaski [19, 20] and fundamental treatise of Truesdell and Toupin [21]. For information about the experimental study and elastic constants in micropolar elasticity one can refer to [16, 22–24]. In the micropolar continuum the force-based stress tensor and deformations tensor are asymmetric and also are introduced the micropolar couple stress and twist tensors. Material particles can rotate independently from the surrounding medium and therefore its motion is described by position and rotation vectors. So every particle contains six degrees of freedom, three translational motions which are assigned to the macroelement and three rotational ones which are referred to the microstructure.

The rapid technological developments in engineering during the last few years have facilitated the applications of the micro- and nano-scale structures, in particular micro- and nano-scale beams and rods. They have been the subject of many investigations mainly because of their numerous applications in science and engineering. They are present in atomic force microscopes, biosensors, micro actuators, micro probes, micro switches, vibration shock sensors, and electrostatically excited micro and nano actuators. The size-dependent deformation behavior of beams

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at micro- and nano scale has been verified experimentally for different materials [1, 2]. This size-dependent behavior cannot be predicted by the classical continuum theories. Hence, new higher-order continuum theories have been developed to account for the size effect in the small scale structures.

The first micropolar models of shells and rods have been developed by Cosserat brothers [6], but only after publication of the paper [25], where by using the direct approach the Cosserat model has been extended to construction of the nonlinear mechanics of rods and shells, *i.e.* to 1-D and 2-D media, the generalized models of shells, plates and roods are extensively discussed in the literature. Among the many articles on generalized theories of plates and shell that we mention here [4, 5, 16, 26], for more information see the extensive review paper [14].

As it was mentioned before, micropolar beams and curved rods have attracted attention in conjunction with micro- and nano-scale analysis and the NEMS and MEMS applications. The model of micropolar beams based on the classical Euler-Bernoulli hypothesis has been considered in [27, 28] and on Timoshenko's hypothesis in [29–32]. Various models of micropolar cured rods have been considered in [3, 11, 25, 33–36].

There are several approaches to the development of the theories of thin-walled structures. One consists in improvement of the classical physical hypothesis and the development of theories that are more accurate. In the beams theory there is a well-known model that takes into account transversal deformations developed by Timoshenko. Another approach consists in the expansion of the stress-strain field components into polynomials series in terms of thickness. It was proposed by Cauchy and Poisson and at that a time was not popular. Significant extensions and developments of that method have been done by Kil'chevskii [37]. In [38–41] Legendre's polynomials have been used for the expansion of the equations of elasticity as well as the reduction of the 3-D problem to the 2-D one. Such an approach has significant advantages since Legendre's polynomials are orthogonal and as result the developed equations are simple.

In our previous publications [42–51] the approach based on the use of Legendre's polynomials series expansion has been applied to the development of high order models of shells, plates and rods. Thermoelastic contact problems of plates and shells when mechanical and thermal conditions are changed during deformation have been considered in [42, 43]. Then the proposed approach and methodology were further developed and extended to thermoelasticity of the laminated composite materials with the possibility of delamination along with mechanical and thermal contact in the temperature field in [44], the pencil-thin nuclear fuel rods modeling in [45], modeling of MEMS and NEMS in [49, 50] and the functionally graded shells in [46, 51]. Analysis and comparison with classical theory of elastic and thermoelastic plates and shells has been done in [47, 48].

In this paper 2-D high order, Timoshenko's and Euler-Bernoulli models for micropolar plane curved rods have been developed. In the 2-D model a special curvilinear system of coordinates related to the middle line of the rod and special hypothesis based on assumptions that take into account the fact that the rod is thin have been used. High order model is based on the expansion of equations of the theory of elasticity into Fourier series in terms of Legendre polynomials. The Timoshenko's and Euler-Bernoulli models are based on the classical hypothesis and 2-D equations of linear micropolar elasticity in special curvilinear system. Obtained equations can be used to calculate stress-strain and to model thin walled structures in macro-, micro- and nano-scales by taking into account micropolar couple stress and rotation effects. The proposed models can be efficient in MEMS and NEMS modeling as well as in computer simulation.

2 2-D micropolar elasticity in orthogonal coordinates

In this study we are developing an approach based on the expansion of the equations of micropolar elasticity into Fourier series in terms of Legendre polynomials and apply it to create high order, Timoshenko's and Euler-Bernoulli curved rod theories. Therefore, we start our consideration with a 2-D formulation of the problem. We consider a curved elastic rod in a 2-D Euclidian space, which occupies the domain $V = \Omega \times [-h, h]$ with a smooth boundary ∂V. Here $2h$ is thickness, $\Omega = [-L, L]$ is the middle line of the rod and $2L$ is its length. The boundary of the rod ∂V can be presented in the form $\partial V = S \cup \Omega^{+} \cup \Omega^{-}$, where Ω^{+} and Ω^{-} are the upper and lower sides and S denotes lateral sides.

In micropolar elasticity it is assumed that the body consists of interconnected particles in the form of small rigid bodies. The position of each particle during the deformation is determined by the displacements $\mathbf{u}(\mathbf{x}, t)$and rotation $\boldsymbol{\omega}(\mathbf{x}, t)$ vectors as functions of coordinated and time. The internal forces (the interaction between adjacent elements) in a micropolar continuum are defined in terms of a classical force stress tensor $\sigma_{ij}(\mathbf{x}, t)$ and a micropolar couple stress tensor $\mu_{ij}(\mathbf{x}, t)$.The displacement field and the micropolar deformation are fully described by the displacements $\mathbf{u}(\mathbf{x}, t)$ and rotation $\boldsymbol{\omega}(\mathbf{x}, t)$ vectors and the asymmetric strain $\varepsilon_{ij}(\mathbf{x}, t)$ and twist $\chi_{ij}(\mathbf{x}, t)$ tensors.

We consider here a plane thin curved rod and assume all functions that define stress-strain state are independent of coordinate x_3 and correspond to the so-called plane stress state. As it was shown in [13, 31] the above mentioned functions have the forms

$$\boldsymbol{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \boldsymbol{\mu} = \begin{vmatrix} 0 & 0 & \mu_{13} \\ 0 & 0 & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{vmatrix}, \tag{1}$$

$$\boldsymbol{\varepsilon} = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}, \quad \boldsymbol{\chi} = \begin{vmatrix} 0 & 0 & \chi_{13} \\ 0 & 0 & \chi_{23} \\ 0 & 0 & 0 \end{vmatrix},$$

$$\mathbf{u} = \begin{vmatrix} u_1 \\ u_2 \\ 0 \end{vmatrix}, \quad \boldsymbol{\omega} = \begin{vmatrix} 0 \\ 0 \\ \omega_3 \end{vmatrix}$$

These quantities are not independent, they are related by the equations of linear micropolar elasticity [13, 29]. Taking into account that the theory of micropolar curved rods will be studied here, the curvilinear orthogonal system of coordinates will be used for that.

In the orthogonal system of coordinates $\mathbf{x} = (x_1, x_2)$, the position of an arbitrary point is defined by the radius vector $\mathbf{R}(\mathbf{x}) = \mathbf{e}_\alpha x_\alpha$. Here unit orthogonal basic vectors and their derivatives with respect to the coordinates are equal to

$$\mathbf{e}_\alpha = \frac{1}{H_\alpha}\frac{\partial \mathbf{R}}{\partial x_\alpha}, \quad \frac{\partial \mathbf{e}_\alpha}{\partial x_\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{e}_\gamma, \quad \alpha, \beta = 1, 2 \tag{2}$$

where H_α are Lamé coefficients and $\Gamma^\gamma_{\alpha\beta}$ are Christoffel symbols. They are calculated by the equations

$$H_\alpha = \left|\frac{\partial \mathbf{R}}{\partial x_\alpha}\right| = \sqrt{\frac{\partial \mathbf{R}}{\partial x_\alpha}\cdot\frac{\partial \mathbf{R}}{\partial x_\alpha}} \tag{3}$$

$$\Gamma^\gamma_{\alpha\beta} = -\frac{1}{H_\alpha}\frac{\partial H_\alpha}{\partial x_\beta}\delta_{\alpha\gamma} + \frac{1}{2H_\alpha H_\gamma}\left(\delta_{\beta\gamma}\frac{\partial H_\beta H_\gamma}{\partial x_\alpha} + \delta_{\alpha\gamma}\frac{\partial H_\alpha H_\gamma}{\partial x_\beta} - \delta_{\alpha\beta}\frac{\partial H_\alpha H_\beta}{\partial x_\gamma}\right) \tag{4}$$

Here $\delta_{\alpha\beta}$ is asymmetric tensor, called Kronecker's delta.

From last equation it follows that for indices $\alpha \neq \beta \neq \gamma$, $\alpha = \beta = \gamma$ and $\alpha = \beta \neq \gamma$ the Christoffel symbols are $\Gamma^\gamma_{\alpha\beta} = 0$. The nonzero Christoffel symbols are

$$\Gamma^\gamma_{\alpha\alpha} = -\frac{1}{H_\gamma}\frac{\partial H_\alpha}{\partial x_\gamma}, \quad \Gamma^\gamma_{\alpha\gamma} = \frac{1}{H_\alpha}\frac{\partial H_\gamma}{\partial x_\alpha} \text{ for } \alpha \neq \gamma \tag{5}$$

More specifically nonzero Christoffel symbols are

$$\Gamma^1_{22} = -\frac{1}{H_1}\frac{\partial H_2}{\partial x_1}, \quad \Gamma^2_{11} = -\frac{1}{H_2}\frac{\partial H_1}{\partial x_2}, \tag{6}$$

$$\Gamma^1_{21} = \frac{1}{H_2}\frac{\partial H_1}{\partial x_2}, \quad \Gamma^2_{12} = \frac{1}{H_1}\frac{\partial H_2}{\partial x_1}$$

The equations of motion have the form

$$\frac{1}{H_\beta}\frac{\partial \sigma_{\beta\alpha}}{\partial x_\beta} + \frac{\sigma_{\beta\alpha}}{H_\gamma}\Gamma^\gamma_{\beta\gamma} + \frac{\sigma_{\beta\gamma}}{H_\beta}\Gamma^\alpha_{\gamma\beta} + b_\alpha = \rho\frac{\partial^2 u_\alpha}{\partial t^2}, \tag{7}$$
$$\frac{1}{H_\beta}\frac{\partial \mu_{\beta 3}}{\partial x_\beta} + \in_{\alpha\beta 3}\sigma_{\alpha\beta} + m_3 = \rho\frac{\partial^2 \omega_3}{\partial t^2}$$

Here $\in_{ijk} = -(\mathbf{i}_j \times \mathbf{i}_i)\cdot\mathbf{i}_k$ is skew symmetric Levi-Civita tensor, b_α and m_3 are body forces and body momentum, respectively. Taking into account (5) and (6) the equations of motion can be presented as

$$\frac{1}{H_1H_2}\left(\frac{\partial(H_2\sigma_{11})}{\partial x_1} + \frac{\partial(H_1\sigma_{12})}{\partial x_2}\right) + \frac{\sigma_{12}}{H_1H_2}\frac{\partial H_1}{\partial x_2} - \frac{\sigma_{22}}{H_1H_2}\frac{\partial H_2}{\partial x_1} + b_1 = \rho\frac{\partial^2 u_1}{\partial t^2} \tag{8}$$
$$\frac{1}{H_1H_2}\left(\frac{\partial(H_2\sigma_{21})}{\partial x_1} + \frac{\partial(H_1\sigma_{22})}{\partial x_2}\right) + \frac{\sigma_{21}}{H_2H_1}\frac{\partial H_2}{\partial x_1} - \frac{\sigma_{11}}{H_2H_1}\frac{\partial H_1}{\partial x_2} + b_2 = \rho\frac{\partial^2 u_2}{\partial t^2}\, \frac{1}{H_1}\frac{\partial \mu_{13}}{\partial x_1} + \frac{1}{H_2}\frac{\partial \mu_{23}}{\partial x_2} + \sigma_{12} - \sigma_{21} + m_3 = j\frac{\partial^2 \omega_3}{\partial t^2}$$

where ρ is a density of material and j is rotational inertia.

Strain and torsion tensors are related with vectors of displacements and rotation by the following kinematic relations

$$\varepsilon_{\alpha\beta} = \frac{1}{H_\alpha}\frac{\partial u_\beta}{\partial x_\alpha} - \Gamma^\gamma_{\alpha\beta}u_\gamma + \in_{3\alpha\beta}\omega_3, \tag{9}$$
$$\chi_{\alpha 3} = \frac{1}{H_\alpha}\frac{\partial \omega_3}{\partial x_\alpha} - \Gamma^3_{\alpha 3}\omega_3,$$

Taking into account (5) and (6) they can be presented in the form

$$\varepsilon_{11} = \frac{1}{H_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{H_1H_2}\frac{\partial H_1}{\partial x_2}, \tag{10}$$
$$\varepsilon_{22} = \frac{1}{H_2}\frac{\partial u_2}{\partial x_2} + \frac{u_1}{H_1H_2}\frac{\partial H_2}{\partial x_1}$$
$$\varepsilon_{12} = \frac{1}{H_1}\frac{\partial u_2}{\partial x_1} - \frac{u_1}{H_1H_2}\frac{\partial H_1}{\partial x_2} - \omega_3,$$
$$\varepsilon_{21} = \frac{1}{H_2}\frac{\partial u_1}{\partial x_2} - \frac{u_2}{H_1H_2}\frac{\partial H_2}{\partial x_1} + \omega_3,$$
$$\chi_{12} = \frac{1}{H_1}\frac{\partial \omega_3}{\partial x_1}, \quad \chi_{21} = \frac{1}{H_2}\frac{\partial \omega_3}{\partial x_2}$$

In this paper constitutive relations (generalized Hooke 's law) for micropolar liners elasticity in the form introduced by Nowacki [19, 20] have been used. They are presented by equations

$$\sigma_{\alpha\beta} = \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + (\mu + \alpha)\varepsilon_{\alpha\beta} + (\mu - \alpha)\varepsilon_{\beta\alpha}, \tag{11}$$
$$\mu_{\alpha 3} = (\gamma + \varepsilon)\chi_{\alpha 3}$$

and in component form

$$\begin{aligned}\sigma_{11} &= \lambda(\varepsilon_{11}+\varepsilon_{22})+2\mu\varepsilon_{11},\\ \sigma_{22} &= \lambda(\varepsilon_{11}+\varepsilon_{22})+2\mu\varepsilon_{22},\\ \sigma_{12} &= (\mu+\alpha)\varepsilon_{12}+(\mu-\alpha)\varepsilon_{21},\\ \sigma_{21} &= (\mu+\alpha)\varepsilon_{21}+(\mu-\alpha)\varepsilon_{12}\\ \mu_{13} &= (\gamma+\varepsilon)\chi_{13},\quad \mu_{23}=(\gamma+\varepsilon)\chi_{23}\end{aligned} \tag{12}$$

where $\lambda = \bar{\lambda}\frac{2\mu}{\bar{\lambda}+2\mu}$, $\bar{\lambda}$ and μ are Lamé constants of classical elasticity, α, γ and ε additional elastic constants.

We have to point out that other notations and elastic constants can be used. For more information about different approaches, notations and their comparison one can refer to [16].

By substituting kinematic relations (10) into generalized Hooke 's law (11) and the obtained result into the equations of motion (8) the differential equations of motion in form of displacements and rotation can be obtained. In vector form they can be represented as the following

$$\begin{aligned}&(\mu+\alpha)\Delta\mathbf{u}+(\lambda+\mu-\alpha)\nabla(\nabla\cdot\mathbf{u})\\ &\quad+2\alpha\nabla\times\mathbf{e}_3\omega_3+\mathbf{b}=\rho\ddot{\mathbf{u}}\\ &(\gamma+\varepsilon)\Delta\omega_3-4\alpha\omega_3+2\alpha(\nabla\times\mathbf{u})_3+M_3=j\ddot{\omega}_3\end{aligned} \tag{13}$$

By taking into account the expressions for differential operators in the orthogonal system of coordinates (A.1)–(A.7) differential equations of motion in the form of displacements and rotation can be represented in the form

$$\begin{aligned}&(\mu+\alpha)\frac{1}{H_1H_2}\left(\frac{\partial}{\partial x_1}\left(\frac{H_2}{H_1}\frac{\partial u_1}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{H_1}{H_2}\frac{\partial u_1}{\partial x_2}\right)\right)\\ &+(\lambda+\mu-\alpha)\frac{1}{H_1}\frac{\partial}{\partial x_1}\left(\frac{1}{H_1H_2}\frac{\partial(H_2u_1)}{\partial x_1}+\frac{1}{H_1H_2}\frac{\partial(H_1u_2)}{\partial x_2}\right)\\ &+2\alpha\frac{1}{H_2}\frac{\partial\omega_3}{\partial x_2}+b_1=\rho\frac{\partial^2u_1}{\partial t^2}\\ &(\mu+\alpha)\frac{1}{H_1H_2}\left(\frac{\partial}{\partial x_1}\left(\frac{H_2}{H_1}\frac{\partial u_2}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{H_1}{H_2}\frac{\partial u_2}{\partial x_2}\right)\right)\\ &+(\lambda+\mu-\alpha)\frac{1}{H_2}\frac{\partial}{\partial x_2}\left(\frac{1}{H_1H_2}\frac{\partial(H_2u_1)}{\partial x_1}+\frac{1}{H_1H_2}\frac{\partial(H_1u_2)}{\partial x_2}\right)\\ &-2\alpha\frac{1}{H_1}\frac{\partial\omega_3}{\partial x_1}+b_1=\rho\frac{\partial^2u_2}{\partial t^2}\\ &(\gamma+\varepsilon)_3\frac{1}{H_1H_2}\left(\frac{\partial}{\partial x_1}\left(\frac{H_2}{H_1}\frac{\partial\omega_3}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\frac{H_2}{H_1}\frac{\partial\omega_3}{\partial x_1}\right)\right)\\ &+2\alpha\frac{1}{H_1H_2}\left(\frac{\partial(H_2u_2)}{\partial x_1}-\frac{\partial(H_1u_1)}{\partial x_2}\right)-4\alpha\omega_3+m_3\\ &=j\frac{\partial^2\omega_3}{\partial t^2}\end{aligned} \tag{14}$$

The complete system of linear micropolar equation of elasticity is considered in detail here. In the next sections it will be simplified and applied for the development of approximate theories of the curved rods.

3 2-D micropolar elasticity in coordinates related to the middle line

For convenience we introduce curvilinear coordinates related to the middle line of the rod. In this case coordinate x_1 is associated with the main curvature k_1 of the middle line of the rod and coordinate x_2 is perpendicular to it. The position vector $\mathbf{R}(\mathbf{x})$ of any point in domain V, occupied by material points of the rod may be presented as

$$\mathbf{R}(\mathbf{x}) = \mathbf{r}(x_1)+x_2\mathbf{n}(x_1) \tag{15}$$

where $\mathbf{r}(x_1)$ is the position vector of the points located on the middle line of the rod, and $\mathbf{n}(\mathbf{x}_\alpha)$ is a unit vector normal to the middle line of the rod.

In this case the 2-D equations of micropolar elasticity can be simplified with taking into account that Lamé coefficients and their derivatives have the simpler form

$$\begin{aligned}&H_1(x_1,x_2)=A_1(x_1)(1+k_1x_2),\quad H_2=1,\\ &\frac{\partial H_1}{\partial x_1}=\frac{\partial A_1}{\partial x_1}(1+k_1x_2),\quad \frac{\partial H_1}{\partial x_2}=k_1A_1,\quad \frac{\partial H_2}{\partial x_\alpha}=0\end{aligned} \tag{16}$$

where $A_1(x_1) = \frac{\partial \mathbf{r}(x_1)}{\partial x_1}$ is the coefficient of the first quadratic form of the middle line.

Taking into account that we consider relatively thin rods, the following assumptions can be applied

$$1+k_1x_2\approx 1\rightarrow H_1\approx A_1,\quad \frac{1}{H_2}\frac{\partial H_1}{\partial x_2}=k_1A_1, \tag{17}$$

Therefore Christoffel symbols (6) become

$$\Gamma^1_{22}=0,\quad \Gamma^2_{11}=-k_1A_1,\quad \Gamma^1_{21}=k_1A_1,\quad \Gamma^2_{12}=0 \tag{18}$$

After substitution of the simplified Lamé coefficients (16) and the Christoffel symbols (18) into equations of motion (7) and (8) they are simplified and take the form

$$\begin{aligned}&\frac{1}{A_1}\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{21}}{\partial x_2}+k_1(\sigma_{21}+\sigma_{12})+b_1=\rho\frac{\partial^2u_1}{\partial t^2}\\ &\frac{1}{A_1}\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+k_1(\sigma_{22}-\sigma_{11})+b_2=\rho\frac{\partial^2u_2}{\partial t^2}\\ &\frac{1}{A_1}\frac{\partial\mu_{13}}{\partial x_1}+\frac{\partial\mu_{23}}{\partial x_2}+(\sigma_{12}-\sigma_{21})+m_3=j\frac{\partial^2\omega_3}{\partial t^2}\end{aligned} \tag{19}$$

In the same way kinematic relations simplify and have the form

$$\begin{aligned}&\varepsilon_{11}=\frac{1}{A_1}\frac{\partial u_1}{\partial x_1}+k_1u_2,\quad \varepsilon_{22}=\frac{\partial u_2}{\partial x_2},\\ &\varepsilon_{12}=\frac{1}{A_1}\frac{\partial u_2}{\partial x_1}-k_1u_1-\omega_3,\quad \varepsilon_{21}=\frac{\partial u_1}{\partial x_2}-k_1u_2+\omega_3,\end{aligned} \tag{20}$$

$$\chi_{13} = \frac{1}{A_1}\frac{\partial \omega_3}{\partial x_1}, \quad \chi_{23} = \frac{\partial \omega_3}{\partial x_2}$$

By substituting these kinematic relations into the constitutive relations (12) we obtain

$$\begin{aligned}
\sigma_{11} &= \lambda\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2 + \frac{\partial u_2}{\partial x_2}\right) \\
&\quad + 2\mu\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2\right), \\
\sigma_{22} &= \lambda\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2 + \frac{\partial u_2}{\partial x_2}\right) + 2\mu\frac{\partial u_2}{\partial x_2}, \\
\sigma_{12} &= (\mu+\alpha)\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \omega_3\right) \\
&\quad + (\mu-\alpha)\left(\frac{\partial u_1}{\partial x_2} - k_1 u_2 + \omega_3\right), \\
\sigma_{21} &= (\mu+\alpha)\left(\frac{\partial u_1}{\partial x_2} - k_1 u_2 + \omega_3\right) \\
&\quad + (\mu-\alpha)\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \omega_3\right) \\
\mu_{13} &= (\gamma+\varepsilon)\frac{1}{A_1}\frac{\partial \omega_3}{\partial x_1}, \quad \mu_{23} = (\gamma+\varepsilon)\frac{\partial \omega_3}{\partial x_2}
\end{aligned} \tag{21}$$

Finally the differential equations of motion in the form of displacements and rotation (14) can be represented in the simpler form.

$$\begin{aligned}
&(\mu+\alpha)\left(\frac{1}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2} + k_1\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2^2}\right) \\
&+ (\lambda+\mu-\alpha)\frac{1}{A_1}\frac{\partial}{\partial x_1}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 A_1 u_2 + A_1\frac{\partial u_2}{\partial x_2}\right) \\
&+ 2\alpha\frac{\partial \omega_3}{\partial x_2} + b_1 = \rho\frac{\partial^2 u_1}{\partial t^2} \\
&(\mu+\alpha)\left(\frac{1}{A_1^2}\frac{\partial^2 u_2}{\partial x_1^2} + k_1\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2^2}\right) \\
&+ (\lambda+\mu-\alpha)\frac{\partial}{\partial x_2}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 A_1 u_2 + A_1\frac{\partial u_2}{\partial x_2}\right) \\
&- 2\alpha\frac{1}{A_1}\frac{\partial \omega_3}{\partial x_1} + b_2 = \rho\frac{\partial^2 u_2}{\partial t^2} \\
&(\gamma+\varepsilon)_3\left(\frac{1}{A_1^2}\frac{\partial^2 \omega_3}{\partial x_1^2} + k_1\frac{\partial \omega_3}{\partial x_2} + \frac{\partial \omega_3}{\partial x_2^2}\right) \\
&+ 2\alpha\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \frac{\partial u_1}{\partial x_2}\right) - 4\alpha\omega_3 + m_3 = j\frac{\partial^2 \omega_3}{\partial t^2}
\end{aligned} \tag{22}$$

In this section differential operators (A.8)–(A.13) have been used frequently to simplify of the equations of micropolar elasticity. These equations will be used for development of the approximate theories of the curved rods.

4 Converting to the 1-D formulation of the problem

In order to reduce the 2-D problem for micropolar elastic curved rods to a 1-D one, we expand the functions that describe the stress-strain state of the rod into the Legendre polynomials series along the coordinate x_2 and represent them in the form

$$\begin{aligned}
u_\alpha(x_1,x_2) &= \sum_{k=0}^{\infty} u_\alpha^k(x_1)\,P_k(\varpi), \\
\omega_3(x_1,x_2) &= \sum_{k=0}^{\infty} \omega_3^k(x_1)\,P_k(\varpi), \\
\sigma_{\alpha\beta}(x_1,x_2) &= \sum_{k=0}^{\infty} \sigma_{\alpha\beta}^k(x_1)\,P_k(\varpi), \\
\mu_{\alpha 3}(x_1,x_2) &= \sum_{k=0}^{\infty} \mu_{\alpha 3}^k(x_1)\,P_k(\varpi), \\
\varepsilon_{\alpha\beta}(x_1,x_2) &= \sum_{k=0}^{\infty} \varepsilon_{\alpha\beta}^k(x_1)\,P_k(\varpi), \\
\chi_{\alpha 3}(x_1,x_2) &= \sum_{k=0}^{\infty} \chi_{\alpha 3}^k(x_1)\,P_k(\varpi),
\end{aligned} \tag{23}$$

where $\varpi = x_2/h \in [-1,1]$ is a normalized variable and coefficients of expansion have the form

$$\begin{aligned}
u_\alpha^k(x_1) &= \frac{2k+1}{2h}\int_{-h}^{h} u_\alpha(x_1,x_2)\,P_k(\varpi)dx_2, \\
\omega_3^k(x_1) &= \frac{2k+1}{2h}\int_{-h}^{h} \omega_3(x_1,x_2)\,P_k(\varpi)dx_2 \\
\sigma_{\alpha\beta}^k(x_1) &= \frac{2k+1}{2h}\int_{-h}^{h} \sigma_{\alpha\beta}(x_1,x_2)\,P_k(\varpi)dx_2, \\
\mu_{\alpha 3}^k(x_1) &= \frac{2k+1}{2h}\int_{-h}^{h} \mu_{\alpha 3}(x_1,x_2)\,P_k(\varpi)dx_2 \\
\varepsilon_{\alpha\beta}^k(x_1) &= \frac{2k+1}{2h}\int_{-h}^{h} \varepsilon_{\alpha\beta}(x_1,x_2)\,P_k(\varpi)dx_2, \\
\chi_{\alpha 3}^k(x_1) &= \frac{2k+1}{2h}\int_{-h}^{h} \chi_{\alpha 3}(x_1,x_2)\,P_k(\varpi)dx_2\ ,
\end{aligned} \tag{24}$$

Generally, all of the functions that are considered here also depend on time t, but to reduce typing the variable of time has been omitted

For derivatives of the considered functions with respect to x_1 the following relations take place

$$\begin{aligned}
\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial u_\alpha(x_1,x_2)}{\partial x_1}P_k(\varpi)\,dx_2 &= \frac{\partial u_\alpha^k(x_1)}{\partial x_1}, \\
\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial \omega_3(x_1,x_2)}{\partial x_1}P_k(\varpi)\,dx_2 &= \frac{\partial \omega_3^k(x_1)}{\partial x_1},
\end{aligned} \tag{25}$$

$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial\sigma_{\alpha\beta}(x_1,x_2)}{\partial x_1}P_k(\varpi)\,dx_2=\frac{\partial\sigma^k_{\alpha\beta}(x_1)}{\partial x_1},$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial\mu_{\alpha3}(x_1,x_2)}{\partial x_1}P_k(\varpi)\,dx_2=\frac{\partial\mu^k_{\alpha3}(x_1)}{\partial x_1}$$

Derivatives of the displacements and rotation with respect to x_2 following [39, 40] can be represented in the form

$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial u_\alpha(x_1,x_2)}{\partial x_2}P_k(\varpi)\,dx_2=\underline{u^k_\alpha}(x_1),\qquad(26)$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial\omega_3(x_1,x_2)}{\partial x_2}P_k(\varpi)\,dx_2=\underline{\omega^k_3}(x_1)$$

where

$$\underline{u^k_\alpha}(x_1)=\frac{2k+1}{h}\left(u^{k+1}_\alpha(x_1)+u^{k+3}_\alpha(x_1)+\dots\right),\qquad(27)$$
$$\underline{\omega^k_3}(x_1)=\frac{2k+1}{h}\left(\omega^{k+1}_3(x_1)+\omega^{k+3}_3(x_1)+\dots\right),$$

In the equations (26) evident representations have been used for derivative of function

$$\frac{\partial u_\alpha(x_1,x_2)}{\partial x_2}=\frac{1}{h}\sum_{k=0}^{\infty}u^k_\alpha(x_1)\frac{\partial P_k(\varpi)}{\partial x_2},\qquad(28)$$
$$\frac{\partial\omega_3(x_1,x_2)}{\partial x_2}=\frac{1}{h}\sum_{k=0}^{\infty}\omega^k_3(x_1)\frac{\partial P_k(\varpi)}{\partial x_2}$$

and the following relation between Legendre polynomials and their derivatives [52]

$$\frac{\partial P_k(\varpi)}{\partial\varpi}=\left((2k-1)P_{k-1}(\varpi)+(2k-5)P_{k-3}(\varpi)+\dots\right)\qquad(29)$$

Derivatives of the classical force stress tensor and a micropolar couple stress tensor with respect to x_2 can be transformed using integration by parts and formulas (29). Finally they can be represented in the form

$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial\sigma_{2\alpha}(x_1,x_2)}{\partial x_2}P_k(\varpi)\,dx_2\qquad(30)$$
$$=\frac{2k+1}{h}\left[\sigma^+_{\alpha2}(x_1)-(-1)^k\sigma^-_{\alpha2}(x_1)\right]-\underline{\sigma^k_{2\alpha}}(x_1),$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial\mu_{23}(x_1,x_2)}{\partial x_2}P_k(\varpi)\,dx_2$$
$$=\frac{2k+1}{h}\left[\mu^+_{\alpha3}(x_1)-(-1)^k\mu^-_{\alpha3}(x_1)\right]-\underline{\mu^k_{23}}(x_1)$$

where

$$\underline{\sigma^k_{2\alpha}}(x_1)=\frac{2k+1}{h}\left(\sigma^{k-1}_{2\alpha}(x_1)+\sigma^{k-3}_{2\alpha}(x_1)+\dots\right)\qquad(31)$$
$$\underline{\mu^k_{23}}(x_1)=\frac{2k+1}{h}\left(\mu^{k-1}_{23}(x_1)+\mu^{k-3}_{23}(x_1)+\dots\right)$$

Now, when multiplying equations of motion (19) by $P_k(\varpi)$ and integrating with respect to x_2 from $-h$ to h and taking into account (25) and (30) we obtain 1-D equations of motion in the form

$$\frac{1}{A_1}\frac{\partial\sigma^k_{11}}{\partial x_1}+\left(\sigma^k_{21}+\sigma^k_{12}\right)k_1-\underline{\sigma^k_{21}}+\tilde b^k_1=\rho\frac{\partial^2u^k_1}{\partial t^2},\qquad(32)$$
$$\frac{1}{A_1}\frac{\partial\sigma^k_{12}}{\partial x_1}+\left(\sigma^k_{22}-\sigma^k_{11}\right)k_1-\underline{\sigma^k_{22}}+\tilde b^k_2=\rho\frac{\partial^2u^k_2}{\partial t^2},$$
$$\frac{1}{A_1}\frac{\partial\mu^k_{13}}{\partial x_1}+\left(\sigma^k_{12}-\sigma^k_{21}\right)-\underline{\mu^k_{23}}+\tilde m^k_3=j\frac{\partial^2\omega^k_3}{\partial t^2}$$

where

$$\tilde b^k_\alpha(x_1)=b^k_\alpha(x_1)\qquad(33)$$
$$+\frac{2k+1}{h}\left(\sigma^+_{\alpha2}(x_1)-(-1)^k\sigma^-_{\alpha2}(x_1)\right)$$
$$\tilde m^k_3(x_1)=m^k_3(x_1)+\frac{2k+1}{h}\left(\mu^+_{23}(x_1)-(-1)^k\mu^-_{23}(x_1)\right)$$

In the same way by considering (25) and (26) the 2-D Cauchy relations can be found in the form

$$\varepsilon^k_{11}=\frac{1}{A_1}\frac{\partial u^k_1}{\partial x_1}+k_1u^k_2,\quad\varepsilon^k_{12}=\frac{1}{A_1}\frac{\partial u^k_2}{\partial x_1}-k_1u^k_1-\omega^k_3,\qquad(34)$$
$$\varepsilon_{21}=\underline{u^k_1}-k_1u^k_2+\omega^k_3\quad\varepsilon^k_{22}=\underline{u^k_2},$$
$$\chi^k_{13}=\frac{1}{A_1}\frac{\partial\omega^k_3}{\partial x_1},\quad\chi^k_{23}=\underline{\omega^k_3}$$

Constitutive relations (generalized Hooke's law) in index notations (11) become

$$\sigma^k_{\alpha\beta}=\lambda\varepsilon^k_{\gamma\gamma}\delta_{\alpha\beta}+(\mu+\alpha)\varepsilon^k_{\alpha\beta}+(\mu-\alpha)\varepsilon^k_{\beta\alpha},\qquad(35)$$
$$\mu^k_{\alpha3}=(\gamma+\varepsilon)\chi^k_{\alpha3}$$

and in component form (12) become

$$\sigma^k_{11}=\lambda\left(\varepsilon^k_{11}+\varepsilon^k_{22}\right)+2\mu\varepsilon^k_{11},\qquad(36)$$
$$\sigma^k_{11}=\lambda\left(\varepsilon^k_{11}+\varepsilon^k_{22}\right)+2\mu\varepsilon^k_{22},$$
$$\sigma^k_{12}=(\mu+\alpha)\varepsilon^k_{12}+(\mu-\alpha)\varepsilon^k_{21},$$
$$\sigma^k_{21}=(\mu+\alpha)\varepsilon^k_{21}+(\mu-\alpha)\varepsilon^k_{12}$$
$$\mu^k_{13}=(\gamma+\varepsilon)\chi^k_{13},\quad\mu^k_{23}=(\gamma+\varepsilon)\chi^k_{23}$$

Substitution of the kinematic relations (34) into the generalized Hooke's law (36) gives us the following equations

$$\sigma^k_{11}=(\lambda+2\mu)\left(\frac{1}{A_1}\frac{\partial u^k_1}{\partial x_1}+k_1u^k_2\right)+\lambda\underline{u^k_2},\qquad(37)$$
$$\sigma^k_{22}=(\lambda+2\mu)\underline{u^k_2}+\lambda\left(\frac{1}{A_1}\frac{\partial u^k_1}{\partial x_1}+k_1u^k_2\right),$$

$$\sigma_{12}^k = (\mu+\alpha)\left(\frac{1}{A_1}\frac{\partial u_2^k}{\partial x_1} - k_1 u_1^k - \omega_3^k\right) + (\mu-\alpha)\left(\underline{u_1^k} - k_1 u_2^k + \omega_3^k\right),$$

$$\sigma_{21}^k = (\mu+\alpha)\left(\underline{u_1^k} - k_1 u_2^k + \omega_3^k\right) + (\mu-\alpha)\left(\frac{1}{A_1}\frac{\partial u_2^k}{\partial x_1} - k_1 u_1^k - \omega_3^k\right),$$

$$\mu_{13}^k = (\gamma+\varepsilon)\frac{1}{A_1}\frac{\partial \omega_3^k}{\partial x_1}, \quad \mu_{23}^k = (\gamma+\varepsilon)\underline{\omega_3^k}.$$

Then the differential equations of motion for micropolar elasticity in the form of displacements and rotations (22) are transformed into its 1-D form

$$(\lambda+2\mu)\frac{1}{A_1^2}\frac{\partial^2 u_1^k}{\partial x_1^2} + \frac{(\lambda+4\mu)k_1}{A_1}\frac{\partial u_2^k}{\partial x_1} + \frac{\lambda}{A_1}\underline{\frac{\partial u_2^k}{\partial x_1}} \tag{38}$$
$$- 2k_1^2\left(u_1^k + u_2^k - \underline{u_1^k}\right) - 4\mu k_1 \omega_3^k - \underline{\sigma_{21}^k} + f_1^k = \rho\frac{\partial^2 u_1^k}{\partial t^2}$$
$$\frac{(\mu+\alpha)}{A_1^2}\frac{\partial^2 u_2^k}{\partial x_1^2} - \frac{(3\mu+\alpha)k_1}{A_1}\frac{\partial u_1^k}{\partial x_1} - \frac{(\mu-\alpha)k_1}{A_1}\frac{\partial u_2^k}{\partial x_1}$$
$$+ \frac{(\mu-\alpha)k_1}{A_1}\underline{\frac{\partial u_1^k}{\partial x_1}} - \frac{2\alpha}{A_1}\frac{\partial \omega_3^k}{\partial x_1} - 2\mu k_1^2 u_2^k + 2\mu k_1 \underline{u_2^k} - \underline{\sigma_{22}^k} + f_2^k$$
$$= \rho\frac{\partial^2 u_2^k}{\partial t^2}$$
$$(\gamma+\varepsilon)\frac{1}{A_1^2}\frac{\partial^2 \omega_3^k}{\partial x_1^2} + \frac{2\alpha}{A_1}\frac{\partial u_2^k}{\partial x_1} - 2\alpha k_1\left(u_1^k + u_2^k\right) - 2\alpha k_1 \underline{u_1^k}$$
$$- 4\alpha\omega_3^k - \underline{\mu_{23}^k} + \tilde{m}_3^k = j\frac{\partial^2 \omega_3^k}{\partial t^2}$$

where

$$\underline{\sigma_{21}^k}(x_1) = A_1\frac{2k+1}{h}\left((\mu+\alpha)\left(\underline{u_1^{k-1}} - \frac{1}{A_1}\frac{\partial u_2^{k-1}}{\partial x_1}\right) - \mu k_1\left(u_1^{k-1} + u_2^{k-1}\right) + \alpha\left(k_1\left(u_1^{k-1} - u_2^{k-1}\right) + 2\omega_3^{k-1}\right) + (\mu+\alpha)\left(\underline{u_1^{k-3}} - \frac{1}{A_1}\frac{\partial u_2^{k-3}}{\partial x_1}\right) - \mu k_1\left(u_1^{k-3} + u_2^{k-3}\right) + \alpha\left(k_1\left(u_1^{k-3} - u_2^{k-3}\right) + 2\omega_3^{k-3}\right) + \ldots\right) \tag{39}$$

$$\underline{\sigma_{22}^k}(x_1) = A_1\frac{2k+1}{h}\left((\lambda+2\mu)\left(\underline{u_2^{k-1}} + \underline{u_2^{k-3}} + \ldots\right) + \lambda\left(\frac{1}{A_1}\frac{\partial u_1^{k-1}}{\partial x_1} + k_1 u_2^{k-1} + \frac{1}{A_1}\frac{\partial u_1^{k-3}}{\partial x_1} + k_1 u_2^{k-3} + \ldots\right)\right)$$

$$\underline{\mu_{23}^k} = A_1\frac{2k+1}{h}\left(\mu_{23}^{k-1}(x_1) + \mu_{23}^{k-3}(x_1) + \ldots\right) = A_1(\gamma+\varepsilon)\frac{2k+1}{h}\left(\underline{\omega_3^{k-1}} + \underline{\omega_3^{k-3}} + \ldots\right)$$

Now instead of the 2-D system of the differential equations in displacements (22) we have an infinite system of 1-D differential equations for coefficients of the Legendre's polynomial series expansion (38). In order to simplify the problem an approximate theory has to be developed and only a finite number of members have to be taken into account in the expansion (23) and in all of the above relations. For example if we consider n-order approximate shell theory, only $n+1$ members in the expansion (23) are taken into account

$$u_\alpha(x_1, x_2) = \sum_{k=0}^{n} u_\alpha^k(x_1) P_k(\varpi), \tag{40}$$
$$\omega_3(x_1, x_2) = \sum_{k=0}^{n} \omega_3^k(x_1) P_k(\varpi),$$
$$\sigma_{\alpha\beta}(x_1, x_2) = \sum_{k=0}^{n} \sigma_{\alpha\beta}^k(x_1) P_k(\varpi),$$
$$\mu_{\alpha 3}(x_1, x_2) = \sum_{k=0}^{n} \mu_{\alpha 3}^k(x_1) P_k(\varpi),$$

In this case we consider that $u_\alpha^k = 0$, $\omega_3^k = 0$, $\sigma_{\alpha\beta}^k = 0$, $\mu_{\alpha 3}^k = 0$, and $\theta^k = 0$ for $k < 0$ and for $k > n$.

Then the 1-D differential equations of micropolar elasticity in displacements and rotations (38) can be presented in the matrix form

$$\mathbf{L}_u \cdot \mathbf{u} + \mathbf{f} = \mathbf{M}_u \cdot \mathbf{u} \tag{41}$$

where

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^{00} & L_{12}^{00} & L_{13}^{00} & \cdots & L_{11}^{0n} & L_{12}^{0n} & L_{13}^{0n} \\ L_{21}^{00} & L_{22}^{00} & L_{23}^{00} & \cdots & L_{21}^{0n} & L_{22}^{0n} & L_{23}^{0n} \\ L_{31}^{00} & L_{32}^{00} & L_{33}^{00} & \cdots & L_{31}^{0n} & L_{32}^{0n} & L_{33}^{0n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ L_{11}^{n0} & L_{12}^{n0} & L_{13}^{n0} & \cdots & L_{11}^{nn} & L_{12}^{nn} & L_{13}^{nn} \\ L_{21}^{n0} & L_{22}^{n0} & L_{23}^{n0} & \cdots & L_{21}^{nn} & L_{22}^{nn} & L_{23}^{nn} \\ L_{31}^{n0} & L_{32}^{n0} & L_{33}^{n0} & \cdots & L_{31}^{nn} & L_{32}^{nn} & L_{33}^{nn} \end{vmatrix}, \tag{42}$$

$$\mathbf{u} = \begin{vmatrix} u_1^0 \\ u_2^0 \\ \omega_3^0 \\ \vdots \\ u_1^n \\ u_2^n \\ \omega_3^n \end{vmatrix}, \quad \tilde{\mathbf{b}} = \begin{vmatrix} \tilde{b}_1^0 \\ \tilde{b}_2^0 \\ \tilde{m}_3^0 \\ \vdots \\ \tilde{b}_1^n \\ \tilde{b}_2^n \\ \tilde{m}_3^n \end{vmatrix},$$

Matrix operator $\mathbf{M}_u$ has nonzero elements only on the main diagonal, which equal to $\rho\frac{\partial^2}{\partial t^2}$ and $j\frac{\partial^2}{\partial t^2}$, for the translational and rotational modes, respectively.

The order of the system of differential equations depends on the assumption regarding thickness distribution of the stress-strain parameters of the shell. The higher the order of approximation, the better accuracy of the result obtained using the proposed theory. The complete system of linear micropolar equation of elasticity of any order can be obtained using the equations presented here. In the next section will be considered the first order approximation theory in detail.

5 First order approximation

In the case that only first two terms of the Legendre polynomials series is considered in the expansion (23) we have the first approximation micropolar rod theory. In this case the functions, which describe the stress-strain state of the rod, can be presented in the form

$$\begin{aligned} u_\alpha(x_1,x_2) &= u_\alpha^0(x_1)P_0(\omega) + u_\alpha^1(x_1)P_1(\omega), \\ \omega_3(x_1,x_2) &= \omega_3^0(x_1)P_0(\omega) + \omega_3^1(x_1)P_1(\omega) \\ \sigma_{\alpha\beta}(x_1,x_2) &= \sigma_{\alpha\beta}^0(x_1)P_0(\omega) + \sigma_{\alpha\beta}^1(x_1)P_1(\omega), \\ \mu_{\alpha 3}(x_1,x_2) &= \mu_{\alpha 3}^0(x_1)P_0(\omega) + \mu_{\alpha 3}^1(x_1)P_1(\omega) \\ \varepsilon_{\alpha\beta}(x_1,x_2) &= \varepsilon_{\alpha\beta}^0(x_1)P_0(\omega) + \varepsilon_{\alpha\beta}^1(x_1)P_1(\omega), \\ \chi_{\alpha 3}(x_1,x_2) &= \chi_{\alpha 3}^0(x_1)P_0(\omega) + \chi_{\alpha 3}^1(x_1)P_1(\omega) \end{aligned} \tag{43}$$

The equations of motion (32) now have the form

$$\begin{aligned} &\frac{1}{A_1}\frac{\partial\sigma_{11}^0}{\partial x_1} + \left(\sigma_{12}^0 + \sigma_{21}^0\right)k_1 + \tilde{b}_1^0 = \rho\frac{\partial^2 u_1^0}{\partial t^2}, \\ &\frac{1}{A_1}\frac{\partial\sigma_{12}^0}{\partial x_1} + \left(\sigma_{22}^0 - \sigma_{11}^0\right)k_1 + \tilde{b}_2^0 = \rho\frac{\partial^2 u_2^0}{\partial t^2}, \\ &\frac{1}{A_1}\frac{\partial\mu_{13}^0}{\partial x_1} + \left(\sigma_{12}^0 - \sigma_{21}^0\right) + \tilde{m}_3^0 = j\frac{\partial^2\omega_3^0}{\partial t^2} \\ &\frac{1}{A_1}\frac{\partial\sigma_{11}^1}{\partial x_1} + \left(\sigma_{12}^1 + \sigma_{21}^1\right)k_1 - \frac{3}{h}\sigma_{21}^0 + \tilde{b}_1^1 = \rho\frac{\partial^2 u_1^1}{\partial t^2}, \\ &\frac{1}{A_1}\frac{\partial\sigma_{12}^1}{\partial x_1} + \left(\sigma_{22}^1 - \sigma_{11}^1\right)k_1 - \frac{3}{h}\sigma_{22}^0 + \tilde{b}_2^1 = \rho\frac{\partial^2 u_2^1}{\partial t^2}, \\ &\frac{1}{A_1}\frac{\partial\mu_{13}^1}{\partial x_1} + \left(\sigma_{12}^1 - \sigma_{21}^1\right) - \frac{3}{h}\mu_{23}^0 + \tilde{m}_3^1 = j\frac{\partial^2\omega_3^1}{\partial t^2} \end{aligned} \tag{44}$$

Kinematic relations (34) have form

$$\begin{aligned} &\varepsilon_{11}^0 = \frac{1}{A_1}\frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0, \\ &\varepsilon_{12}^0 = \frac{1}{A_1}\frac{\partial u_2^0}{\partial x_1} - k_1 u_1^0 - \omega_3^0, \quad \varepsilon_{21} = \frac{u_1^1}{h} - k_1 u_2^0 + \omega_3^0 \\ &\varepsilon_{22}^0 = \frac{u_2^1}{h}, \quad \chi_{13}^0 = \frac{1}{A_1}\frac{\partial\omega_3^0}{\partial x_1}, \quad \chi_{23}^0 = \frac{\omega_3^1}{h} \\ &\varepsilon_{11}^1 = \frac{1}{A_1}\frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1, \\ &\varepsilon_{12}^1 = \frac{1}{A_1}\frac{\partial u_2^1}{\partial x_1} - k_1 u_1^1 - \omega_3^1, \quad \varepsilon_{21} = -k_1 u_2^1 + \omega_3^1 \\ &\varepsilon_{22}^1 = 0, \quad \chi_{13}^1 = \frac{1}{A_1}\frac{\partial\omega_3^1}{\partial x_1}, \quad \chi_{23}^1 = 0 \end{aligned} \tag{45}$$

The generalized Hooke's law for micropolar elasticity (36) has the form

$$\begin{aligned} &\sigma_{11}^0 = \lambda\left(\varepsilon_{11}^0 + \varepsilon_{22}^0\right) + 2\mu\varepsilon_{11}^0, \\ &\sigma_{22}^0 = \lambda\left(\varepsilon_{11}^0 + \varepsilon_{22}^0\right) + 2\mu\varepsilon_{22}^0, \\ &\sigma_{12}^0 = (\mu+\alpha)\varepsilon_{12}^0 + (\mu-\alpha)\varepsilon_{21}^0, \\ &\sigma_{21}^0 = (\mu+\alpha)\varepsilon_{21}^0 + (\mu-\alpha)\varepsilon_{12}^0 \\ &\mu_{13}^0 = (\gamma+\varepsilon)\chi_{13}^0, \quad \mu_{23}^0 = (\gamma+\varepsilon)\chi_{23}^0 \\ &\sigma_{11}^1 = \lambda\left(\varepsilon_{11}^1 + \varepsilon_{22}^1\right) + 2\mu\varepsilon_{11}^1, \\ &\sigma_{11}^1 = \lambda\left(\varepsilon_{11}^1 + \varepsilon_{22}^1\right) + 2\mu\varepsilon_{22}^1, \\ &\sigma_{12}^1 = (\mu+\alpha)\varepsilon_{12}^1 + (\mu-\alpha)\varepsilon_{21}^1, \\ &\sigma_{21}^1 = (\mu+\alpha)\varepsilon_{21}^1 + (\mu-\alpha)\varepsilon_{12}^1 \\ &\mu_{13}^1 = (\gamma+\varepsilon)\chi_{13}^1, \quad \mu_{23}^1 = (\gamma+\varepsilon)\chi_{23}^1 \end{aligned} \tag{46}$$

Substituting kinematic relations (45) into generalized Hooke's law (46) and then the obtained result into equations of motion (44) we obtain the 1-D differential equations of micropolar elasticity in displacements and rotations for the first order theory of micropolar rods theory (41), where matrices and vectors (42) become

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^{00} & L_{12}^{00} & L_{13}^{00} & L_{11}^{01} & L_{12}^{01} & 0 \\ L_{21}^{00} & L_{22}^{00} & L_{23}^{01} & L_{21}^{01} & L_{22}^{01} & 0 \\ L_{31}^{00} & L_{32}^{00} & L_{33}^{00} & L_{31}^{01} & 0 & 0 \\ L_{11}^{10} & L_{12}^{10} & L_{13}^{10} & L_{11}^{11} & L_{12}^{11} & L_{13}^{11} \\ L_{21}^{10} & L_{22}^{10} & 0 & L_{21}^{11} & L_{22}^{11} & L_{23}^{11} \\ 0 & 0 & 0 & L_{31}^{11} & L_{32}^{11} & L_{33}^{11} \end{vmatrix}, \tag{47}$$

$$\mathbf{u} = \begin{vmatrix} u_1^0 \\ u_2^0 \\ \omega_3^0 \\ u_1^1 \\ u_2^1 \\ \omega_3^1 \end{vmatrix}, \quad \tilde{\mathbf{b}} = \begin{vmatrix} \tilde{b}_1^0 \\ \tilde{b}_2^0 \\ \tilde{m}_3^0 \\ \tilde{b}_2^1 \\ \tilde{b}_1^2 \\ \tilde{m}_3^1 \end{vmatrix}, \quad \mathbf{M}_u = \frac{\partial^2}{\partial t^2}\begin{vmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 0 & j \end{vmatrix}$$

Elements of the matrix operator $\mathbf{L}_u$ can be represented in the form

$$\begin{aligned} &L_{11}^{00}u_1^0 = \frac{(\lambda+2\mu)}{A_1^2}\frac{\partial^2 u_1^0}{\partial x_1^2} - 2\mu k_1^2 u_1^0, \\ &L_{12}^{00}u_2^0 = \frac{(\lambda+4\mu)k_1}{A_1}\frac{\partial u_2^0}{\partial x_1} - 2\mu k_1^2 u_2^0, \\ &L_{13}^{00}\omega_3^0 = -4\mu k_1\omega_3^0, \quad L_{11}^{01}u_1^1 = 2\mu k_1^2\frac{u_1^1}{h}, \\ &L_{12}^{01}u_2^1 = \frac{\lambda k_1}{A_1 h}\frac{\partial u_2^1}{\partial x_1}, \quad L_{13}^{01}\omega_3^1 = 0 \\ &L_{21}^{00}u_1^0 = -\frac{(3\mu+\alpha)k_1}{A_1}\frac{\partial u_1^0}{\partial x_1}, \\ &L_{22}^{00}u_2^0 = \frac{(\mu+\alpha)}{A_1^2}\frac{\partial^2 u_2^0}{\partial x_1^2} - \frac{(\mu-\alpha)k_1}{A_1}\frac{\partial u_2^0}{\partial x_1} - 2\mu k_1^2 u_2^0, \end{aligned} \tag{48}$$

$$L_{23}^{00}\omega_3^0 = -\frac{2\alpha}{A_1}\frac{\partial\omega_3^0}{\partial x_1}, \quad L_{21}^{01}u_1^1 = \frac{(\mu-\alpha)\,k_1}{A_1 h}\frac{\partial u_1^1}{\partial x_1},$$
$$L_{22}^{01}u_2^1 = \frac{2\mu k_1}{h}u_2^1, \quad L_{23}^{01}\omega_3^1 = 0$$
$$L_{31}^{00}u_1^0 = -2\alpha k_1 u_1^0, \quad L_{31}^{00}u_2^0 = \frac{2\alpha}{A_1}\frac{\partial u_2^0}{\partial x_1} - 2\alpha k_1 u_2^0,$$
$$L_{33}^{00}\omega_3^0 = \frac{(\gamma+\varepsilon)}{A_1^2}\frac{\partial^2\omega_3^0}{\partial x_1^2} - 4\alpha\omega_3^0,$$
$$L_{31}^{01}u_1^1 = -\frac{2\alpha k_1}{h}u_1^1, \quad L_{32}^{01}u_2^1 = 0, \quad L_{33}^{01}\omega_3^1 = 0,$$
$$L_{11}^{10}u_1^0 = \frac{3\,(\mu-\alpha)}{h}k_1 u_1^0,$$
$$L_{12}^{10}u_2^0 = \frac{3\,(\mu-\alpha)}{A_1 h}\frac{\partial u_2^0}{\partial x_1} + \frac{3\,(\mu+\alpha)\,k_1}{h}u_2^0,$$
$$L_{13}^{10}\omega_3^0 = -\frac{6\alpha}{h}\omega_3^0,$$
$$L_{11}^{11}u_1^1 = \frac{(\lambda+2\mu)}{A_1^2}\frac{\partial^2 u_1^1}{\partial x_1^2} - \left(2\mu k_1^2 - \frac{3\,(\mu+\alpha)}{h^2}\right)u_1^1,$$
$$L_{12}^{11}u_2^1 = \frac{(\lambda+4\mu)\,k_1}{A_1}\frac{\partial u_2^1}{\partial x_1} - 2\mu k_1^2 u_2^1, \quad L_{13}^{11}\omega_3^1 = -4\mu k_1\omega_3^1.$$
$$L_{21}^{10}u_1^0 = -\frac{3\lambda}{A_1 h}\frac{\partial u_1^0}{\partial x_1}, \quad L_{22}^{10}u_2^0 = -\frac{3\lambda k_1}{h}u_2^0,$$
$$L_{23}^{10}\omega_3^0 = 0, \quad L_{21}^{11}u_1^1 = -(\lambda+3\mu+\alpha)\,k_1\frac{\partial u_1^1}{\partial x_1},$$
$$L_{22}^{11}u_2^1 = \frac{(\mu+\alpha)}{A_1^2}\frac{\partial^2 u_2^1}{\partial x_1^2} - \frac{(\mu-\alpha)\,k_1}{A_1}\frac{\partial u_2^1}{\partial x_1} + \left(2\mu k_1^2 - \frac{3\,(\lambda+2\mu)}{h^2}\right)u_2^1, \quad L_{23}^{11}\omega_3^1 = \frac{2\alpha}{A_1}\frac{\partial\omega_3^1}{\partial x_1},$$
$$L_{31}^{11}u_1^0 =, \quad L_{31}^{11}u_2^0 = 0, \quad L_{33}^{00}\omega_3^0 = -\frac{3(\gamma+\varepsilon)1}{A_1 h}\frac{\partial\omega_3^0}{\partial x_1},$$
$$L_{31}^{11}u_1^1 = -2\alpha k_1 u_1^1, \quad L_{32}^{11}u_2^1 = \frac{2\alpha}{A_1}\frac{\partial u_2^1}{\partial x_1},$$
$$L_{33}^{11}\omega_3^1 = (\gamma+\varepsilon)\frac{1}{A_1^2}\frac{\partial^2\omega_3^1}{\partial x_1^2} - 4\alpha\omega_3^1$$

The equations presented in this section are first order equations of micropolar curved rods theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering micropolar couple stress and rotation effects. If in above equations it is assumed that $A_1 = 1$ and $k_1 = 0$ the equations for the micropolar straight beam will be obtained in the form

$$(\lambda+2\mu)\frac{\partial^2 u_1^0}{\partial x_1^2} + \tilde{b}_1^0 = \rho\frac{\partial^2 u_1^0}{\partial t^2} \tag{49}$$
$$(\mu+\alpha)\frac{\partial^2 u_2^0}{\partial x_1^2} - 2\alpha\frac{\partial\omega_3^0}{\partial x_1} + \tilde{b}_2^0 = \rho\frac{\partial^2 u_2^0}{\partial t^2}$$
$$(\gamma+\varepsilon)\frac{\partial^2\omega_3^0}{\partial x_1^2} + 2\alpha\frac{\partial u_2^0}{\partial x_1} - 4\alpha\omega_3^0 + \tilde{m}_3^0 = \rho\frac{\partial^2\omega_3^0}{\partial t^2}$$
$$(\lambda+2\mu)\frac{\partial^2 u_1^1}{\partial x_1^2} - \frac{3\,(\mu+\alpha)}{h^2}u_1^1 - \frac{6\alpha}{h}\omega_3^0 + \frac{3\,(\mu-\alpha)}{h}\frac{\partial u_2^0}{\partial x_1} + \tilde{b}_1^1 = \rho\frac{\partial^2 u_1^1}{\partial t^2}$$
$$(\mu+\alpha)\frac{\partial^2 u_2^1}{\partial x_1^2} - 2\alpha\frac{\partial\omega_3^1}{\partial x_1} - \frac{3\,(\lambda+2\mu)}{h^2}u_2^1 - \frac{3\lambda}{h}\frac{\partial u_1^0}{\partial x_1} + \tilde{b}_2^1 = \rho\frac{\partial^2 u_2^1}{\partial t^2}$$
$$(\gamma+\varepsilon)\frac{\partial^2\omega_3^1}{\partial x_1^2} + 2\alpha\frac{\partial u_2^1}{\partial x_1} - \frac{3(\gamma+\varepsilon)1}{h}\frac{\partial\omega_3^0}{\partial x_1} - 4\alpha\omega_3^1 + \tilde{m}_3^1 = \rho\frac{\partial^2\omega_3^1}{\partial t^2}$$

Analysis of this system of partial differential equations shows that it splits up into three separate parts. First equation and system of second and third equations can be solved independently and after that system of rest three equations can be solved.

6 Timoshenko's theory of the micropolar curved rods

Timoshenko's beam theory is based on assumptions concerning the value and distribution of the functions that determine stress-strain state along the beam thickness. Thus, according to static assumptions $\sigma_{22} = 0$ and according to kinematic assumptions $\varepsilon_{22} = 0$.

The stress state is characterized by the normal n_{11} and shear n_{21} forces, as well as the bending m_{22} and rotating m_{13} moments. They are defined as following

$$n_{11} = \int_{-h}^{h}\sigma_{11}dx_2, \quad n_{12} = \int_{-h}^{h}\sigma_{12}dx_2, \tag{50}$$
$$n_{21} = \int_{-h}^{h}\sigma_{21}dx_2, \quad m_{11} = \int_{-h}^{h}\sigma_{11}x_2dx_2,$$
$$\tilde{\mu}_{13} = \int_{-h}^{h}\mu_{13}dx_2,$$

By integrating differential equations of motion (19) with respect to x_2 from $-h$ to h we first obtain the equations of motion presented in (51). And by multiplying the first equation of motion (19) by x_2 and integrating it with respect to x_2 from $-h$ to h we obtain the last equation of motion for Timoshenko's micropolar curved rod. The complete system of the equations of motion has the from

$$\frac{1}{A_1}\frac{\partial n_{11}}{\partial x_1} + (n_{21}+n_{12})\,k_1 + \tilde{b}_1 = \rho F\frac{\partial^2 u_1}{\partial t^2}, \tag{51}$$
$$\frac{1}{A_1}\frac{\partial n_{12}}{\partial x_1} - n_{11}k_1 + A_1\tilde{b}_2 = \rho F\frac{\partial^2 u_2}{\partial t^2},$$

$$\frac{1}{A_1}\frac{\partial \tilde{\mu}_{13}}{\partial x_1} + (n_{12} - n_{21}) + \tilde{m}_1 = jF\frac{\partial^2 \omega_3}{\partial t^2}$$
$$\frac{1}{A_1}\frac{\partial m_{11}}{\partial x_1} - n_{21} + \tilde{m}_3 = \rho J\frac{\partial^2 \gamma_1}{\partial t^2},$$

Displacements in the Timoshenko's theory of curved beans are defined by vectors $\mathbf{u}(x_1, t)$ with components u_α, $\alpha = 1, 2$ that correspond to axial and transverse displacements of the middle line in the$x_1$and x_2 directions respectively, and $\gamma_1(x_1, t)$ that is the rotation of the centroidal axis about the x_3 axis of the elements perpendicular to the middle line, rotation is defined by the vector component $\omega_3(x_1, t)$. These parameters are related to the coefficients of the displacements expansion in the first order theory in the following way

$$u_\alpha^0 \sim u_\alpha, \quad u_\alpha^1 \sim \gamma_\alpha h, \quad \omega_3^0 \sim \omega_3 \tag{52}$$

Component u_2^1 and ω_3^1 are not taken into account in the Timoshenko's theory of curved rods.

In Timoshenko's theory the strain state of the beam is determined by quantities specified on the middle surface. Deformations in the classical Timoshenko's theory are determined by the relations

$$\varepsilon_{11} = e_{11} + \kappa_{11}x_2, \quad \varepsilon_{12} = e_{12}, \quad \varepsilon_{21} = e_{21}, \tag{53}$$

Roughly speaking component e_{11} corresponds to the tension-compression deformation of the middle surface, components e_{12} to the transversal shear deformation and component κ_{11} to the bending middle line, respectively. The following formulas give us relations by corresponding quantities in the first order theory

$$\varepsilon_{\alpha\beta}^0 \sim e_{\alpha\beta}, \quad \varepsilon_{11}^1 \sim \kappa_{11} \tag{54}$$

Component ε_{33}^0 and ε_{33}^1 are not taken into account in Timoshenko's theory of beams. That also follows from the kinematic hypothesis.

According to Timoshenko's beam theory, the displacement and rotation field can be represented in the form

$$u_1(x_1, x_2) = u_1(x_1) - x_2\gamma_1(x_1), \tag{55}$$
$$u_2(x_1, x_2) = u_2(x), \quad \omega_3(x_1, x_2) = \omega_3(x_1)$$

Here we use the same notations for 2-D and 1-D functions of the displacements and rotations.

By substituting expressions for displacements and rotations (55) into 2-D kinematic relations (20) we obtain kinematic relations for micropolar curved rod in the from

$$e_{11} = \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2, \quad e_{22} = 0, \tag{56}$$
$$e_{12} = \frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \omega_3, \quad \kappa_{11} = \frac{1}{A_1}\frac{\partial \gamma_1}{\partial x_1}$$
$$e_{21} = \gamma_1 - k_1 u_2 + \omega_3, \quad \chi_{13} = \frac{1}{A_1}\frac{\partial \omega_3}{\partial x_1}, \quad \chi_{23} = 0$$

By combining classical and micropolar constitutive relations of liners elasticity, we obtain generalized Hooke's law for micropolar Timoshenko's beam in the form

$$n_{11} = EFe_{11}, \quad m_{11} = EJ\kappa_{11}, \tag{57}$$
$$\mu_{13} = (\gamma + \varepsilon)F\chi_{13}^0,$$
$$n_{12} = (\mu + \alpha)Fe_{12} + (\mu - \alpha)Fe_{21},$$
$$n_{21} = (\mu + \alpha)Fe_{21} + (\mu - \alpha)Fe_{12}$$

By substituting kinematic relations (56) to the generalized Hooke's law (57) we obtain constitutive relations in the form

$$n_{11} = EF\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2\right), \quad m_{11} = EJ\frac{1}{A_1}\frac{\partial \gamma_1}{\partial x_1}, \tag{58}$$
$$n_{12} = (\mu + \alpha)F\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \omega_3\right)$$
$$+ (\mu - \alpha)F(\gamma_1 - k_1 u_2 + \omega_3),$$
$$n_{21} = (\mu + \alpha)F(\gamma_1 - k_1 u_2 + \omega_3) + (\mu - \alpha)$$
$$\cdot F\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \omega_3\right)$$
$$\tilde{\mu}_{13} = (\gamma + \varepsilon)F\frac{1}{A_1}\frac{\partial \omega_3}{\partial x_1}, \quad \mu_{23} = 0$$

By substituting generalized Hooke's law in the form (58) into equations of motion (51) we obtain the 1-D differential equations of micropolar elasticity in displacements and rotations for Timoshenko's theory of micropolar rods theory in the form (41), where matrices operators and vectors become

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^u & L_{12}^u & L_{11}^\gamma & L_{11}^\omega \\ L_{21}^u & L_{22}^u & L_{21}^\gamma & L_{21}^\omega \\ L_{31}^u & L_{32}^u & L_{31}^\gamma & L_{31}^\omega \\ L_{41}^u & L_{42}^u & L_{41}^\gamma & L_{41}^\omega \end{vmatrix}, \quad \mathbf{u} = \begin{vmatrix} u_1 \\ u_2 \\ \gamma_1 \\ \omega_3 \end{vmatrix}, \tag{59}$$
$$\tilde{\mathbf{b}} = \begin{vmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{m}_1 \\ \tilde{m}_3 \end{vmatrix}, \quad \mathbf{M}_u = \frac{\partial^2}{\partial t^2}\begin{vmatrix} \rho F & 0 & 0 & 0 \\ 0 & \rho F & 0 & 0 \\ 0 & 0 & \rho J & 0 \\ 0 & 0 & 0 & jF \end{vmatrix}$$

Elements of the matrix operator $\mathbf{L}_u$ can be represented in the form

$$L_{11}^u u_1 = \frac{EF}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2} - 2\mu F k_1^2 u_1, \tag{60}$$
$$L_{12}^u u_2 = \frac{(E + 2\mu)Fk_1}{A_1}\frac{\partial u_2}{\partial x_1} - 2\mu F k_1^2 u_2,$$
$$L_{11}^\gamma \gamma_1 = \frac{2\mu F k_1}{A_1}\gamma_1, \quad L_{11}^\omega \omega_3 = 0,$$
$$L_{21}^u u_1 = -\frac{(E + \mu + \alpha)k_1}{A_1}\frac{\partial u_1}{\partial x_1},$$

$$L_{22}^{u}u_2 = (\mu+\alpha)\frac{1}{A_1^2}\frac{\partial^2 u_2}{\partial x_1^2} - \frac{(\mu-\alpha)k_1}{A_1}\frac{\partial u_2}{\partial x_1} - Ek_1^2 u_2,$$
$$L_{21}^{\gamma}\gamma_1 = \frac{\mu-\alpha}{A_1}\frac{\partial \gamma_1}{\partial x_1}, \quad L_{21}^{\omega}\omega_3 = -\frac{2\alpha}{A_1}\frac{\partial \omega_3}{\partial x_1}$$
$$L_{31}^{u}u_1 = (\mu-\alpha)k_1 u_1,$$
$$L_{32}^{u}u_2 = -\frac{\mu-\alpha}{A_1}\frac{\partial u_2}{\partial x_1} + (\mu+\alpha)k_1 u_2,$$
$$L_{31}^{\gamma}\gamma_1 = E\frac{1}{A_1^2}\frac{\partial^2 \gamma_1}{\partial x_1^2} - (\mu+\alpha)\gamma_1, \quad L_{31}^{\omega}\omega_3 = -2\alpha\omega_3,$$
$$L_{41}^{u}u_1 = -2\alpha k_1 u_1, \quad L_{42}^{u}u_2 = \frac{2\alpha}{A_1}\frac{\partial u_2}{\partial x_1} + 2\alpha k_1 u_2,$$
$$L_{41}^{\gamma}\gamma_1 = -2\alpha\gamma_1, \quad L_{41}^{\omega}\omega_3 = (\gamma+\varepsilon)\frac{1}{A_1^2}\frac{\partial^2 \omega_3}{\partial x_1^2} - 4\alpha\omega_3,$$

The equations presented in this section are equations of micropolar curved rods for Timoshenko's theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering micropolar couple stress and rotation effects. If in the above equations assume $A_1 = 1$ and $k_1 = 0$ the equations for the micropolar straight beam will be obtained in the form

$$EF\frac{\partial^2 u_1}{\partial x_1^2} + \tilde{b}_1 = \rho F\frac{\partial^2 u_1}{\partial t^2} \tag{61}$$
$$(\mu+\alpha)F\frac{\partial^2 u_2}{\partial x_1^2} + (\mu-\alpha)F\frac{\partial \gamma_1}{\partial x_1} - 2\alpha F\frac{\partial \omega_3}{\partial x_1} + A_1\tilde{b}_2 = \rho\frac{\partial^2 u_2}{\partial t^2},$$
$$EJ\frac{\partial^2 \gamma_1}{\partial x_1^2} - (\mu-\alpha)J\frac{\partial u_2}{\partial x_1} - (\mu+\alpha)J\gamma_1 - 2\alpha J\omega_3 + \tilde{b}_1^1 = \rho\frac{\partial^2 \gamma_1}{\partial t^2},$$
$$2\alpha F\frac{\partial u_2}{\partial x_1} - 2\alpha F\gamma_1 + (\gamma+\varepsilon)F\frac{\partial^2 \omega_3}{\partial x_1^2} - 4\alpha F\omega_3 + \tilde{m}_3^0 = \rho\frac{\partial^2 \omega_3}{\partial t^2}$$

Analysis of this system of partial differential equations shows that it splits up into two separate parts. First equation and system of rest three equations can be solved independently. We have to mention that system (61) coincide with the one presented in [30] up to notation. Therefore analysis and verification presented in [30] take place in considered here case.

7 Euler-Bernoulli theory of the micropolar curved rods

Like Timoshenko's theory of beams, the Euler-Bernoulli theory is also based on similar assumptions concerning the value and distribution of the stress-strain state along the beam thickness. The stress state is characterized by the normal n_{11} and shear n_{21} forces, as well as the bending m_{22} and rotating m_{13} moments. They are defined by the equations (50).

Unlike to Timoshenko's theory in the Euler-Bernoulli theory of beams the rotation of the centroidal axis is not independent. It is represented through displacements by the equation

$$\gamma_1 = -\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} + k_1 u_1 \tag{62}$$

According to Euler-Bernoulli beam theory, the displacement and rotation field can be written in the following form

$$u_1(x_1,x_2,t) = u_1(x_1,t) + x_2\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1\right), \tag{63}$$
$$u_2(x_1,x_2,t) = u_2(x_1,t), \quad \omega_3(x_1,x_2,t) = \omega_3(x_1,t)$$

In Euler-Bernoulli theory the strain state of the beam is also determined by quantities specified on the middle surface.

By substituting expressions for displacements and rotations (55) into 2-D kinematic relations (20) we obtain kinematic relations for Euler-Bernoulli micropolar curved rod

$$e_{11} = \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2, \quad e_{22} = 0, \tag{64}$$
$$e_{12} = \frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \omega_3, \quad \kappa_{11} = -\frac{1}{A_1^2}\frac{\partial^2 u_2}{\partial x_1^2} + \frac{k_1}{A_1}\frac{\partial u_1}{\partial x_1}$$
$$e_{21} = \frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - k_1 u_2 + \omega_3, \quad \chi_{13} = \frac{1}{A_1}\frac{\partial \omega_3}{\partial x_1},$$
$$\chi_{23} = 0$$

Generalized Hooke's law for micropolar Euler-Bernoulli beam has the same as Timoshenko's beam form and is represented by the equations (57).

By substituting kinematic relations (64) to the generalized Hooke's law (57) we obtain constitutive relations in the form

$$n_{11} = EF\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2\right), \tag{65}$$
$$m_{11} = EJ\left(\frac{1}{A_1^2}\frac{\partial^2 u_2}{\partial x_1^2} - \frac{k_1}{A_1}\frac{\partial u_1}{\partial x_1}\right),$$
$$n_{12} = 2\mu F\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1\right) - (\mu-\alpha)Fk_1 u_2 - 2\alpha\omega_3$$
$$n_{21} = 2\mu F\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1\right) + (\mu+\alpha)Fk_1 u_2 + 2\alpha\omega_3$$
$$\mu_{13} = (\gamma+\varepsilon)F\frac{1}{A_1}\frac{\partial \omega_3}{\partial x_1},$$

Taking into account that in Euler-Bernoulli beam theory rotational inertial is neglected and therefore $\frac{\partial^2 \gamma_1}{\partial t^2} = 0$, from the last equation of motion (51) we have

$$n_{12} = \frac{1}{A_1}\frac{\partial m_{11}}{\partial x_1} - \tilde{m}_3 \tag{66}$$

Then the equations of motion for Euler-Bernoulli curved rod can be presented in the form

$$\frac{1}{A_1}\frac{\partial n_{11}}{\partial x_1}+(n_{21}+n_{12})\,k_1+\tilde{b}_1=\rho F\frac{\partial^2 u_1}{\partial t^2}, \tag{67}$$
$$\frac{1}{A_1^2}\frac{\partial^2 m_{11}}{\partial x_1^2}-\frac{(\mu-\alpha)F}{A_1}\frac{\partial u_2}{\partial x_1}-\frac{2\alpha}{A_1}\frac{\partial\omega_3}{\partial x_1}-n_{11}k_1+\tilde{b}_2-\frac{\partial\tilde{m}_3}{\partial x_1}$$
$$=\rho F\frac{\partial^2 u_2}{\partial t^2},$$
$$\frac{1}{A_1}\frac{\partial\tilde{\mu}_{13}}{\partial x_1}+(n_{12}-n_{21})+\tilde{m}_1=\rho F\frac{\partial^2\omega_3}{\partial t^2}$$

By substituting kinematic relations in the form (65) in the equations of motion (67) we obtain a differential equation of motion in the form of displacements and rotation for the micropolar Euler-Bernoulli curved rod in the form

$$\frac{EF}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2}-4\mu k_1^2Fu_1+\frac{(E+4\mu)k_1F}{A_1}\frac{\partial u_2}{\partial x_1} \tag{68}$$
$$-2\alpha k_1^2Fu_2+\tilde{b}_1=\rho F\frac{\partial^2 u_1}{\partial t^2}$$
$$\frac{EJ}{A_1^4}\frac{\partial^4 u_2}{\partial x_1^4}-\frac{EJk_1}{A_1^3}\frac{\partial^3 u_1}{\partial x_1^3}-\frac{EFk_1}{A_1}\frac{\partial u_1}{\partial x_1}-\frac{(\mu-\alpha)Fk_1}{A_1}\frac{\partial u_2}{\partial x_1}$$
$$-EFk_1^2u_2-\frac{2\alpha}{A_1}\frac{\partial\omega_3}{\partial x_1}+\tilde{b}_2-\frac{\partial\tilde{m}_3}{\partial x_1}=\rho F\frac{\partial^2 u_2}{\partial t^2}$$
$$(\gamma+\varepsilon)F\frac{1}{A_1^2}\frac{\partial^2\omega_3}{\partial x_1^2}+2\mu Fk_1u_2-4\alpha\omega_3+\tilde{m}_1=\rho F\frac{\partial^2\omega_3}{\partial t^2}$$

These can be easy converted to the form (41) with matrix operators and vectors in the form

$$\mathbf{L}_u=\begin{vmatrix}L_{11}^u & L_{12}^u & L_{12}^\omega\\ L_{21}^u & L_{22}^u & L_{22}^\omega\\ L_{31}^u & L_{32}^u & L_{31}^\omega\end{vmatrix},\quad \mathbf{u}=\begin{vmatrix}u_1\\ u_2\\ \omega_3\end{vmatrix}, \tag{69}$$
$$\mathbf{f}=\begin{vmatrix}\tilde{b}_1\\ \tilde{b}_2\\ \tilde{m}_3\end{vmatrix},\quad \mathbf{M}_u=\frac{\partial^2}{\partial t^2}\begin{vmatrix}\rho F & 0 & 0\\ 0 & \rho J & 0\\ 0 & 0 & jF\end{vmatrix}$$

Elements of the matrix operator $\mathbf{L}_u$ can be represented in the form

$$L_{11}^u u_1=\frac{EF}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2}-4\mu k_1^2Fu_1, \tag{70}$$
$$L_{12}^u u_2=\frac{(E+4\mu)k_1F}{A_1}\frac{\partial u_2}{\partial x_1}-2\alpha k_1^2Fu_2,\quad L_{11}^\omega\omega_3=0$$
$$L_{21}^u u_1=-\frac{EJk_1}{A_1^3}\frac{\partial^3 u_1}{\partial x_1^3}-\frac{k_1EF}{A_1}\frac{\partial u_1}{\partial x_1},$$
$$L_{22}^u u_2=\frac{EJ}{A_1^4}\frac{\partial^4 u_2}{\partial x_1^4}-\frac{(\mu-\alpha)Fk_1}{A_1}\frac{\partial u_2}{\partial x_1}-EFk_1^2u_2,$$
$$L_{21}^\omega\omega_3=-\frac{2\alpha}{A_1}\frac{\partial\omega_3}{\partial x_1}\quad L_{31}^u u_1=0,\quad L_{32}^u u_2=2\mu Fk_1u_2,$$
$$L_{31}^\omega\omega_3=(\gamma+\varepsilon)F\frac{1}{A_1^2}\frac{\partial^2\omega_3}{\partial x_1^2}-4\alpha\omega_3$$

The equations presented in this section are equations of micropolar curved rods for Euler-Bernoulli theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering micropolar couple stress and rotation effects. If in the above equations assume $A_1=1$ and $k_1=0$ the equations for the micropolar straight beam will be obtained in the form

$$EF\frac{\partial^2 u_1}{\partial x_1^2}+\tilde{b}_1=\rho F\frac{\partial^2 u_1}{\partial t^2} \tag{71}$$
$$EJ\frac{\partial^4 u_2}{\partial x_1^4}-2\alpha F\frac{\partial\omega_3}{\partial x_1}+\tilde{b}_2-\frac{\partial\tilde{m}_3}{\partial x_1}=\rho F\frac{\partial^2 u_2}{\partial t^2}$$
$$(\gamma+\varepsilon)\frac{\partial^2\omega_3}{\partial x_1^2}-4\alpha\omega_3+\tilde{m}_1=\rho\frac{\partial^2\omega_3}{\partial t^2}$$

Analysis of this system of partial differential equations shows that it splits up into three separate parts. Each equation can be solved independently. Inconsistencis of the Euler-Bernoulli beam or Kirchhoff plate micropolar theories have been discussed in [15, 30] respectively

8 Conclusions

In this paper new theories for micropolar plane curved rods have been developed. 2-D theory is developed from general 2-D equations of linear micropolar elasticity using special curvilinear system of coordinates related to the middle line of the rod and a special assumption based on assumptions that take into account the fact that the rod is thin. High order theory is based on the expansion of the equations of theory of elasticity into Fourier series in terms of Legendre polynomials in a thickness coordinate. All the functions that define the stress-strain state of the rod including stress and strain tensors, vectors of displacements and rotation and body forces have been expanded into Fourier series in terms of Legendre polynomials with respect to a thickness coordinate. Thereby, all equations of elasticity including Hooke's law have been transformed to the corresponding equations for Fourier coefficients of the expansion. Then, for Fourier coefficients the system of differential equations of motion in term of displacements and rotations has been obtained in the same way as in the theory of elasticity. The Timoshenko's and Euler-Bernoulli theories have been developed based on the classical hypothesis and 2-D equations of linear micropolar elasticity in a special curvilinear system of coordinates. In the same way the system of differential equations of motion in term of displacements and rotations has been developed for all considered here cases.

Analysis of the systems of partial differential equations (47), (59) and (69) show that all of them are coupled and related longitudinal, flexural and rotational deformation modes. The first order approximation theory is more complete and all quantities are approximated by linear functions. The theory based on Timoshenko's hypothesis is less accurate, but simple and take into account shear deformation, which is important for dynamic analysis. The theory based on Euler-Bernoulli hypothesis is less accurate compared with the previous ones, but it is the simplest one and couple all the deformation modes. It is free of inconsistencies that take place in the case strait beams and can be used for modeling and analysis of the curved rod with considering micropolar couple stress and rotation effects.

As special case, the equations for micropolar straight beam can be derived from the equations presented here. The obtained equations can be used to stress-strain calculation as well as modeling thin structures in macro, micro and nano scales by taking into account micropolar couple stress and rotation effects. Specially proposed models can be efficient in MEMS and NEMS modeling as well as computer simulation.

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Appendix. Differential operators in orthogonal system of coordinates.

For orthogonal system of coordinates in general form

Operator of Hamilton

$$\nabla = \mathbf{e}_\beta \frac{1}{H_\beta}\frac{\partial}{\partial x_\beta} = \frac{\mathbf{e}_1}{H_1}\frac{\partial}{\partial x_1} + \frac{\mathbf{e}_2}{H_2}\frac{\partial}{\partial x_2} \tag{A.1}$$

Divergence of the displacements vector

$$\nabla\cdot\mathbf{u} = \nabla\cdot u_\alpha \mathbf{e}_\alpha = \frac{1}{H_\alpha}\frac{\partial u_\alpha}{\partial x_\alpha} + \frac{u_\alpha}{H_\beta}\Gamma^\beta_{\alpha\beta} = \frac{1}{H_1H_2}\left(\frac{\partial(H_2u_1)}{\partial x_1} + \frac{\partial(H_1u_2)}{\partial x_2}\right) \tag{A.2}$$

Gradient of the displacements vector

$$\nabla\mathbf{u} = \mathbf{e}_\beta \frac{1}{H_\beta}\frac{\partial u_\alpha \mathbf{e}_\alpha}{\partial x_\beta} = \mathbf{e}_\beta\mathbf{e}_\alpha\left(\frac{1}{H_\beta}\frac{\partial u_\alpha}{\partial x_\beta} + \frac{u_\gamma}{H_\beta}\Gamma^\alpha_{\gamma\beta}\right) \tag{A.3}$$

Rotor of the displacements vector

$$\nabla\times\mathbf{u} = \left(\frac{1}{H_\beta}\frac{\partial u_\alpha}{\partial x_\beta}\in_{\beta\alpha\sigma} + \frac{u_\alpha}{H_\beta}\Gamma^\gamma_{\alpha\beta}\in_{\beta\gamma\sigma}\right)\mathbf{e}_\sigma \tag{A.4}$$

In particular

$$(\nabla\times\mathbf{u})_3 = \frac{1}{H_1H_2}\left(\frac{\partial(H_2u_2)}{\partial x_1} - \frac{\partial(H_1u_1)}{\partial x_2}\right) \tag{A.5}$$

Operator of Laplace

$$\Delta = \frac{1}{H_1H_2}\left(\frac{\partial}{\partial x_1}\left(\frac{H_2}{H_1}\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{H_1}{H_2}\frac{\partial}{\partial x_2}\right)\right) \tag{A.6}$$

Gradient of divergence of the displacements vector

$$\nabla(\nabla\cdot\mathbf{u}) = \frac{1}{H_\alpha}\frac{\partial}{\partial x_\alpha}\left(\frac{1}{H_1H_3}\frac{\partial(H_2u_1)}{\partial x_1} + \frac{1}{H_1H_2}\frac{\partial(H_1u_2)}{\partial x_2}\right) \tag{A.7}$$

For orthogonal system of coordinates related to middle line of the rod

Operator of Hamilton

$$\nabla = \mathbf{e}_\beta \frac{1}{H_\beta}\frac{\partial}{\partial x_\beta} = \frac{\mathbf{e}_1}{A_1}\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} \tag{A.8}$$

Divergence of the displacements vector

$$\nabla\cdot\mathbf{u} = \nabla\cdot u_\alpha\mathbf{e}_\alpha = \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1u_2 + \frac{\partial u_2}{\partial x_2} \tag{A.9}$$

Gradient of the displacements vector

$$\nabla\mathbf{u} = \mathbf{e}_\beta\mathbf{e}_\alpha\begin{vmatrix} \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1u_2 & \frac{\partial u_1}{\partial x_2} \\ \frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1u_2 & \frac{\partial u_2}{\partial x_2}\end{vmatrix} \tag{A.10}$$

Rotor of the displacements vector in particular case

$$(\nabla\times\mathbf{u})_3 = \frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1u_1 - \frac{\partial u_1}{\partial x_2} \tag{A.11}$$

Operator of Laplace

$$\Delta = \frac{1}{A_1}\frac{\partial}{\partial x_1}\left(\frac{1}{A_1}\frac{\partial}{\partial x_1}\right) + \frac{\partial^2}{\partial x_2^2} + k_1\frac{\partial}{\partial x_2} \tag{A.12}$$

Gradient of divergence of the displacements vector

$$\nabla(\nabla\cdot\mathbf{u})_1 = \frac{1}{A_1}\frac{\partial}{\partial x_1}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1u_2 + \frac{\partial u_2}{\partial x_2}\right), \tag{A.13}$$

$$\nabla(\nabla\cdot\mathbf{u})_2 = \frac{\partial}{\partial x_2}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1u_2 + \frac{\partial u_2}{\partial x_2}\right)$$