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Couple stress theory of curved rods. 2-D, high order, Timoshenko's and Euler-Bernoulli models

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Abstract: New models for plane curved rods based on linear couple stress theory of elasticity have been developed. 2-D theory is developed from general 2-D equations of linear couple stress elasticity using a special curvilinear system of coordinates related to the middle line of the rod as well as special hypothesis based on assumptions that take into account the fact that the rod is thin. High order theory is based on the expansion of the equations of the theory of elasticity into Fourier series in terms of Legendre polynomials. First, stress and strain tensors, vectors of displacements and rotation along with body forces have been expanded into Fourier series in terms of Legendre polynomials with respect to a thickness coordinate. Thereby, all equations of elasticity including Hooke's law have been transformed to the corresponding equations for Fourier coefficients. Then, in the same way as in the theory of elasticity, a system of differential equations in terms of displacements and boundary conditions for Fourier coefficients have been obtained. Timoshenko's and Euler-Bernoulli theories are based on the classical hypothesis and the 2-D equations of linear couple stress theory of elasticity in a special curvilinear system. The obtained equations can be used to calculate stress-strain and to model thin walled structures in macro, micro and nano scales when taking into account couple stress and rotation effects.

Keywords: Curved rod; couple stress; Legendre polynomial; Timoshenko theory; Euler-Bernoulli theory; high order theory

1 Introduction

Classical continuum mechanics consider material continua as point-continua with points having three degrees of freedom, and the response of a material to the displacements of its points is characterized by a symmetric stress tensor. Such a model is insufficient for the description of certain physical phenomena especially at micro and nano scale. In 1843 St Venant remarked already that for the description of deformations of thin bodies, rotation independent of the displacements [1] must also be included. A further generalization of this idea had been done by Dugem and Voitgt. They suggested that the interaction of two parts of the body is transmitted through an area element by means not only of the force vector but also by the moment vector. Thus, besides the force stresses the moment stresses have been also defined. However, the complete theory of asymmetric elasticity was developed by the brothers F. and E. Cosserat [1–3].

Since the publication of the landmark book of Cosserat brothers [4] in literature there are various known generalizations and applications of the classical or Cauchy continuum which are summarized in many books [5–19] and papers [20–37]. Especially we must mention remarkable books of Eringen [38, 39] and Nowaski [40] and fundamental treatise of Truesdell and Toupin [3]. The research in the field of the generalized of continuous media focused on generalization of the Cosserat's model and on the creation of new simplified models that on one hand are relatively simple and on the other hand can adequately take into account the microstructure of a material. One of such simplified models is so called the Cosserat pseudo-continuum or couple stress theory. By this name we mean a continuum for which the asymmetric force stresses and moment stresses occur, however, the deformation is determined by the displacement vector only. It is interesting to notice that this model was also considered by the Cosserats who called it the model with the latent trihedron [1, 2]. The couple stress theory was developed in the sixties mostly by Aero and Kuvshinskii [21, 22], Koiter [27], Mindlin [29], Mindlin and Tiersten [30] and Toupin [35, 36]. We also have to mention recently published papers [26, 33, 37] where have been proposed modified couple stress theories which use only one additional constant that takes into account the microstructure of a material at micro and nano scale. These theories have some differences in a 3-D case which are analyzed and discussed in [41]. In the 2-D

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case they give the differences disappear, therefore we will not analyze them here and use the equations presented in [26, 37] for development of the curved rods theories.

Although micropolar models of shells and rods were first developed by Cosserat brothers in the book [4] mentioned above, only after publication of the paper [42], where by using the direct approach the Cosserat model has been extended to construction of the nonlinear mechanics of rods and shells, *i.e.* to 1-D and 2-D media, the generalized models of shells, plates and roods are extensively discussed in the literature. In most publications the considered models are based on Euler-Bernoulli and Timoshenko's hypothesis. Among the many articles on couple stress theories of beams based on Euler-Bernoulli hypothesis we mention here [43–46] and on Timoshenko's hypothesis we mention here [46-52], curved rods along with others have been considered in [4, 16, 42, 53–55], for more information see the extensive review paper [56].

Micro and nano technology are a subject of current technological interest and development. The size dependence of deformation behavior in micro and nano scales have been experimentally observed in metals, polymers and other technological materials [8, 9, 26, 37, 48, 49, etc.]. The length scales associated with nanotechnology are often sufficiently small to impugn the applicability of classical continuum models. Atomic and molecular models, while certainly conceptually valid for small length scales, are difficult to formulate accurately and are almost always computationally intensive. Generalized continuum models that take into account microstructure of material represent attempts to extend the continuum approach to smaller length scales while retaining most of its many advantages [5, 10–12, 14, 15, 18, 34, 38, 39]. It would appear that generalized continuum mechanics could potentially play a useful role in analysis related to nanotechnology applications [7–9, 13, 16, 57–60]. In micro and nano technology a problem of current interest is the development of small sensors actuators in the form of thin walled structures like beams, roods, plates and shells that exhibiting elastic response. Hence, new higher-order continuum theories beams, roods, plates and shells which take into account the size effect in the small scale structures have to be developed.

There are several approaches to the development of the theories of thin-walled structures. One consists in exploring the classical physical hypothesis and the development of theories that are more accurate and take into account the size effect in the small scale. Another approach consists in the expansion of the stress-strain field components into polynomials series in terms of thickness. Such an approach was proposed by Cauchy and Poisson and at that time it was not popular. Significant extensions and developments of that method have been done by Kil'chevskii [61]. The Legendre's polynomials have been used for development of the new high order theories of plates and shells. For more information see for example books [62–64] and the extended review [65]. Such an approach has significant advantages since Legendre's polynomials are orthogonal and as result the developed equations are simple.

In our previous publications [66–75] the approach based on the use of Legendre's polynomials series expansion has been applied to the development of high order models of shells, plates and rods. Thermoelastic contact problems of plates and shells when mechanical and thermal conditions are changed during deformation have been considered in [66, 67]. Then, the proposed approach and methodology were further developed and extended to thermoelasticity of the laminated composite materials with the possibility of delamination along with mechanical and thermal contact in the temperature field in [68], the pencil-thin nuclear fuel rods modeling in [69], modeling of MEMS and NEMS in [73, 74], the functionally graded shells in [70, 75] as well as micropolar curved elastic rods in [55]. Analysis and comparison with classical theory of elastic and thermoelastic plates and shells has been done in [71, 72].

In this paper 2-D, high order, Timoshenko's and Euler-Bernoulli models of curved rods based on couple stress theory of elasticity have been developed. In the 2-D model a special curvilinear system of coordinates related to the middle line of the rod and special hypothesis based on assumptions that take into account the fact that the rod is thin have been used. High order model is based on the expansion of the equations of the 2-D couple stress theory of elasticity into Fourier series in terms of Legendre polynomials. The Timoshenko's and Euler-Bernoulli models are based on the classical hypothesis and 2-D equations of couple stress theory of elasticity in a special curvilinear system. The obtained equations can be used to calculate stress-strain and to model thin walled structures in macro-, micro- and nano-scales by taking into account couple stress and rotation effects. The proposed models can be efficient in MEMS and NEMS modeling as well as in their computer simulation.

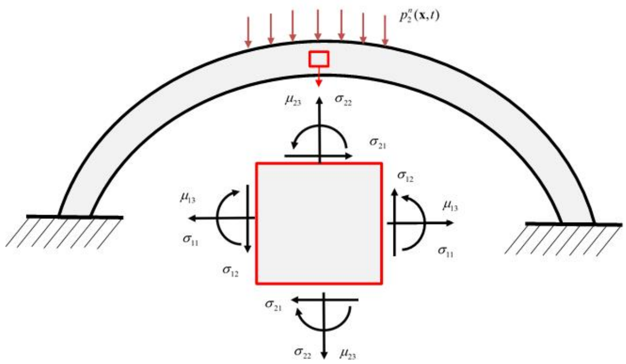

Figure 1: Curved elastic bar.

2 2-D couple stress elasticity in orthogonal coordinates

Let's consider the size dependent material continuum theory and use it to develop an approach based on the expansion of the equations of couple stress theory of elasticity into Fourier series in terms of Legendre polynomials and apply it to create high order, Timoshenko's and Euler-Bernoulli curved rod theories. Thus, we consider a curved elastic rod in a 2-D Euclidian space, which occupies the domain $V = \Omega \times [-h, h]$ with a smooth boundary ∂V. Here $2h$ is thickness, $\Omega = [-L, L]$ is the middle line of the rod and $2L$ is its length, as it is shown in Fig. 1. The boundary of the rod ∂V can be presented in the form $\partial V = S \cup \Omega^+ \cup \Omega^-$, where Ω^+ and Ω^- are the upper and lower sides and S denotes lateral sides.

In the couple stress elasticity it is assumed that the transfer of the interaction occurs between two particles of the body through a surface element dS with unit normal vector n_i by means of a force vector p_i^n and a moment vector μ_i^n. Surface forces and couples are then represented by the generally non-symmetric force-stress $\sigma_{ij}(\mathbf{x}, t)$ and couple-stress $\mu_{ij}(\mathbf{x}, t)$ tensors. The displacement field and the deformation are fully described by the displacements $\mathbf{u}(\mathbf{x}, t)$ and rotation $\boldsymbol{\omega}(\mathbf{x}, t)$ vectors and the asymmetric strain $\varepsilon_{ij}(\mathbf{x}, t)$ and curvature $\kappa_{ij}(\mathbf{x}, t)$ tensors. The deformations are fully described by the displacements gradient $\nabla\mathbf{u}(\mathbf{x}, t)$, which can be decomposed into symmetric and skew-symmetric parts

$$\nabla_j u_i = e_{ij} + \omega_{ij}, \tag{1}$$

where

$$e_{ij} = \frac{1}{2}\left(\nabla_j u_i + \nabla_i u_j\right) \tag{2}$$

$$\omega_{ij} = \frac{1}{2}\left(\nabla_j u_i - \nabla_i u_j\right) \tag{3}$$

The rotation vector ω_i dual to the rotation tensor ω_{ij} is defined by

$$\omega_i = \frac{1}{2}\in_{ijk}\omega_{kj} = \frac{1}{2}\in_{ijk}\nabla_j u_k, \quad \boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{u} \tag{4}$$

Here $\in_{ijk}$ is skew symmetric Levi-Civita tensor.

The curvature $\kappa_{ij}(\mathbf{x}, t)$ tensor is skew-symmetric and can be expressed though the rotation vector in the form

$$\kappa_{ij} = \frac{1}{2}(\nabla_j \omega_i - \nabla_i \omega_j) \tag{5}$$

The curvature vector κ_i dual to this tensor is defined by

$$\kappa_i = \frac{1}{2}\in_{ijk}\kappa_{kj} = \frac{1}{2}\in_{ijk}\nabla_j \omega_k, \quad \boldsymbol{\kappa} = \frac{1}{2}\nabla \times \boldsymbol{\omega} \tag{6}$$

It also can be expressed as

$$\begin{aligned} &\boldsymbol{\kappa} = \frac{1}{2}\nabla \times \boldsymbol{\omega}, \\ &\boldsymbol{\kappa} = \frac{1}{4}\nabla \times (\nabla \times \mathbf{u}) = \frac{1}{4}\nabla(\nabla \cdot \mathbf{u}) - \frac{1}{4}\nabla^2 \mathbf{u} \\ &\kappa_i = \frac{1}{4}\nabla_i \nabla_k u_k - \frac{1}{4}\nabla_k \nabla_k u_i = \frac{1}{4}\nabla_i \nabla_k u_k - \frac{1}{4}\nabla^2 u_i \end{aligned} \tag{7}$$

All of the relations represented here are valid in the case of the small deformation theory, which requires the components of the strain tensor and mean curvature vector to be infinitesimal. These conditions can be written as

$$\left|\varepsilon_{ij}\right| \ll 1, \quad |\kappa_i| \ll \frac{1}{l_s} \tag{8}$$

where l_s is the smallest characteristic length in the body.

Here, we consider a plane thin curved rod and assume all functions that define a stress-strain state are independent of coordinate x_3 and correspond to the so-called plane stress state. As it was shown in [13, 31] the above mentioned functions have the forms

$$\begin{aligned} &\boldsymbol{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \boldsymbol{\mu} = \begin{vmatrix} 0 & 0 & \mu_{13} \\ 0 & 0 & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{vmatrix}, \\ &\boldsymbol{\varepsilon} = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{vmatrix}, \quad \boldsymbol{\kappa} = \begin{vmatrix} 0 & 0 & \kappa_{13} \\ 0 & 0 & \kappa_{23} \\ 0 & 0 & 0 \end{vmatrix}, \\ &\mathbf{u} = \begin{vmatrix} u_1 \\ u_2 \\ 0 \end{vmatrix}, \quad \boldsymbol{\omega} = \begin{vmatrix} 0 \\ 0 \\ \omega_3 \end{vmatrix} \end{aligned} \tag{9}$$

These quantities are not independent, they are related by the equations of linear couple stress theory of elasticity [18, 38, 40]. Taking into account that the couple stress theory of curved rods will be studied here, the curvilinear orthogonal system of coordinates will be used for that.

In the orthogonal system of coordinates $\mathbf{x} = (x_1, x_2)$, the position of an arbitrary point is defined by the radius vector $\mathbf{R}(\mathbf{x}) = \mathbf{e}_\alpha x_\alpha$. Here unit orthogonal basic vectors and their derivatives with respect to the coordinates are equal to

$$\mathbf{e}_\alpha = \frac{1}{H_\alpha}\frac{\partial \mathbf{R}}{\partial x_\alpha}, \quad \frac{\partial \mathbf{e}_\alpha}{\partial x_\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{e}_\gamma, \quad \alpha, \beta = 1, 2 \tag{10}$$

where H_α are Lamè coefficients and $\Gamma^\gamma_{\alpha\beta}$ are Christoffel symbols.

The equations of motion have the form

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}}, \quad \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\sigma}_\times = 0, \tag{11}$$

Here $\mathbf{b}$ is a vector of body forces, ρ is a density of material, $\ddot{\mathbf{u}}$ is the acceleration vector, $\boldsymbol{\sigma}_\times$ is the vector invariant of force-stress defined by the equation [12]

$$\boldsymbol{\sigma}_\times = \left(\sigma_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)_\times = \sigma_{\alpha\beta}\mathbf{e}_\alpha \times \mathbf{e}_\beta. \tag{12}$$

Divergence of the stress tensor in the curvilinear orthogonal system of coordinates has the form

$$\nabla \cdot \boldsymbol{\sigma} = \left(\frac{1}{H_\alpha}\frac{\partial \sigma_{\alpha\beta}}{\partial x_\alpha} + \frac{\sigma_{\alpha\beta}}{H_\gamma}\Gamma^\gamma_{\alpha\gamma} + \frac{\sigma_{\alpha\gamma}}{H_\alpha}\Gamma^\beta_{\gamma\alpha}\right)\mathbf{e}_\beta \tag{13}$$

The ordered pair $\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ is called the dyad, and $\mathbf{e}_\alpha \times \mathbf{e}_\beta$ is the cross product of the basis vectors.

By considering the symmetry relations required for an isotropic linear elastic material the quadratic positive definite stored energy density can be presented in the form [26]

$$E(\mathbf{e}, \boldsymbol{\kappa}) = \frac{1}{2}\bar{\lambda}e_{kk}e_{kk} + \mu e_{ij}e_{ij} + 8\eta\kappa_i\kappa_i \tag{14}$$

where $\bar{\lambda} = \lambda\frac{2\mu}{\lambda+2\mu}$, λ and μ are Lamè constants of classical elasticity, $\eta$is additional elastic constant that accounts for couple stress effects in an isotropic material.

The elastic constants have to satisfy the following restrictions for positive definite stored energy

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad \eta > 0 \tag{15}$$

Then, the constitutive relations can be written in the form

$$\sigma_{(\alpha\beta)} = \lambda e_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}, \quad \mu_\alpha = -8\eta\kappa_\alpha \tag{16}$$

Here $\delta_{\alpha\beta}$ is asymmetric tensor, called Kronecker's delta.

Taking into account that in 2-D case kinematic relations have the form

$$\nabla_\beta u_\alpha = \varepsilon_{\alpha\beta} + \omega_{\alpha\beta}, \quad \varepsilon_{\alpha\beta} = \frac{1}{2}(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta), \tag{17}$$
$$\omega_3 = \omega_{21} = \frac{1}{2}(\nabla_2 u_1 - \nabla_1 u_2)$$

and

$$\kappa_1 = -\kappa_{23} = \frac{1}{2}\nabla_2\omega_3, \quad \kappa_2 = -\kappa_{13} = \nabla_1\omega_3 \tag{18}$$

Derivatives in (17) and (18) have the form

$$\nabla_\beta u_\alpha = \frac{1}{H_\beta}\frac{\partial u_\alpha}{\partial x_\beta} + \frac{u_\gamma}{H_\beta}\Gamma^\alpha_{\gamma\beta}, \quad \nabla_\beta\omega_3 = \frac{1}{H_\beta}\frac{\partial \omega_3}{\partial x_\beta} \tag{19}$$

The constitutive relations can be rewritten in the form

$$\sigma_{(\alpha\beta)} = \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2\mu\varepsilon_{i\alpha\beta}, \quad \mu_\alpha = -4\eta\nabla_\beta\omega_{\beta\alpha} \tag{20}$$

and by considering (17) in the form

$$\sigma_{\alpha\beta} = \lambda\nabla_\gamma u_\gamma\delta_{\alpha\beta} + \mu\left(\nabla_\beta u_\alpha + \nabla_\alpha u_\beta\right) - \eta\nabla^2\left(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta\right) \tag{21}$$

with Laplace operator in the form

$$\Delta = \frac{1}{H_1H_2}\left(\frac{\partial}{\partial x_1}\left(\frac{H_2}{H_1}\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{H_1}{H_2}\frac{\partial}{\partial x_2}\right)\right) \tag{22}$$

By substituting the equations for generalized Hooke's law (21) and the obtained result into the equations of motion (11) the differential equations of motion in form of displacements can be represented as the following

$$(\lambda + 2\mu)\nabla_\alpha\nabla_\beta u_\beta - \eta\nabla^2\nabla_\beta\left(\nabla_\beta u_\alpha - \nabla_\alpha u_\beta\right) + b_\alpha = \rho\ddot{u}_\alpha \tag{23}$$

Let's simplify the above equation for the development of approximate theories of the curved rods. For convenience we introduce curvilinear coordinates related to the middle line of the rod. In this case coordinate x_1 is associated with the main curvature k_1 of the middle line of the rod and coordinate x_2 is perpendicular to it. The position vector $\mathbf{R}(\mathbf{x})$ of any point in domain V, occupied by material points of the rod may be presented as

$$\mathbf{R}(\mathbf{x}) = \mathbf{r}(x_1) + x_2\mathbf{n}(x_1) \tag{24}$$

where $\mathbf{r}(x_1)$ is the position vector of the points located on the middle line of the rod, and $\mathbf{n}(\mathbf{x}_\alpha)$ is a unit vector normal to the middle line of the rod.

In this case the 2-D equations of couple stress theory of elasticity can be simplified by taking into account that Lamè coefficients and their derivatives have the simpler form

$$H_1(x_1, x_2) = A_1(x_1)(1 + k_1x_2), \quad H_2 = 1, \tag{25}$$
$$\frac{\partial H_1}{\partial x_1} = \frac{\partial A_1}{\partial x_1}(1 + k_1x_2), \quad \frac{\partial H_1}{\partial x_2} = k_1A_1, \quad \frac{\partial H_2}{\partial x_\alpha} = 0$$

where $A_1(x_1) = \frac{\partial \mathbf{r}(x_1)}{\partial x_1}$ is the coefficient of the first quadratic form of the middle line.

Taking into account that we consider relatively thin rods, the following assumptions can be applied

$$1 + k_1x_2 \approx 1 \rightarrow H_1 \approx A_1, \quad \frac{1}{H_2}\frac{\partial H_1}{\partial x_2} = k_1A_1, \tag{26}$$

Therefore Christoffel symbols become [55]

$$\Gamma^1_{22} = 0, \quad \Gamma^2_{11} = -k_1 A_1, \quad \Gamma^1_{21} = k_1 A_1, \quad \Gamma^2_{12} = 0 \quad (27)$$

After substitution of the simplified Lamè coefficients (25) and the Christoffel symbols (27) into equations of motion (11) they are simplified and take the form

$$\begin{gathered}\nabla_\beta \sigma_{\beta 1} + k_1 (\sigma_{21} + \sigma_{12}) + b_1 = \rho \frac{\partial^2 u_1}{\partial t^2} \\ \nabla_\beta \sigma_{\beta 2} + k_1 (\sigma_{22} - \sigma_{11}) + b_2 = \rho \frac{\partial^2 u_2}{\partial t^2} \\ \nabla_\beta \mu_{\beta 3} + (\sigma_{12} - \sigma_{21}) = 0\end{gathered} \quad (28)$$

where ∇_β are the differential operators of the form $\nabla_1 = \frac{1}{A_1}\frac{\partial}{\partial x_1}$ and $\nabla_2 = \frac{\partial}{\partial x_2}$.

From the last equation of equilibrium follows

$$\sigma_{[\alpha\beta]} = \frac{1}{2}\left(\nabla_\alpha \mu_\beta - \nabla_\beta \mu_\alpha\right) \quad (29)$$

Therefore constitutive equations (20) can be presented in the form

$$\begin{aligned}&\sigma_{11} = \lambda (\varepsilon_{11} + \varepsilon_{22}) + 2\mu\varepsilon_{11}, \\ &\sigma_{22} = \lambda (\varepsilon_{11} + \varepsilon_{22}) + 2\mu\varepsilon_{22}, \\ &\sigma_{12} = 2\mu\varepsilon_{12} - 2\eta\nabla^2\omega_3 \quad \sigma_{21} = 2\mu\varepsilon_{21} + 2\eta\nabla^2\omega_3\end{aligned} \quad (30)$$

where

$$\nabla^2 = \frac{1}{A_1}\frac{\partial}{\partial x_1}\left(\frac{1}{A_1}\frac{\partial}{\partial x_1}\right) + \frac{\partial^2}{\partial x_2^2} + k_1\frac{\partial}{\partial x_2} \quad (31)$$

In the same way kinematic relations simplify and have the form

$$\begin{aligned}&\varepsilon_{11} = \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \\ &\varepsilon_{12} = \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} - k_1 u_1\right) \\ &\omega_3 = \omega_{21} = \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} - k_1 u_1\right)\end{aligned} \quad (32)$$

By substituting these kinematic relations into the constitutive relations (30) we obtain

$$\begin{aligned}\sigma_{11} &= \lambda\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2 + \frac{\partial u_2}{\partial x_2}\right) \\ &\quad + 2\mu\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2\right), \\ \sigma_{22} &= \lambda\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2 + \frac{\partial u_2}{\partial x_2}\right) + 2\mu\frac{\partial u_2}{\partial x_2}, \\ \sigma_{12} &= \mu\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} - k_1 u_1\right) \\ &\quad - \eta\nabla^2\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} - k_1 u_1\right), \\ \sigma_{21} &= \mu\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} - k_1 u_1\right) \\ &\quad + \eta\nabla^2\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} - k_1 u_1\right)\end{aligned} \quad (33)$$

Finally the differential equations of motion in the form of displacements (23) can be represented in the simpler form.

$$\begin{aligned}&\lambda\frac{1}{A_1}\frac{\partial}{\partial x_1}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2 + \frac{\partial u_2}{\partial x_2}\right) \\ &+ 2\mu\frac{1}{A_1}\frac{\partial}{\partial x_1}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2\right) \\ &+ \mu\left(\frac{1}{A_1}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2} - k_1\frac{\partial u_1}{\partial x_2}\right) \\ &+ \eta\nabla^2\left(\frac{1}{A_1}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} - \frac{\partial^2 u_1}{\partial x_2^2} - k_1\frac{\partial u_1}{\partial x_2}\right) \\ &+ 2\mu k_1\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} - k_1 u_1\right) + b_1 = \rho\frac{\partial^2 u_1}{\partial t^2} \\ &\mu\frac{1}{A_1}\frac{\partial}{\partial x_1}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} - k_1 u_1\right) \\ &- \eta\nabla^2\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1^2} - \frac{\partial u_1}{\partial x_1 \partial x_2} - k_1\frac{\partial u_1}{\partial x_1}\right) \\ &+ \lambda\frac{\partial}{\partial x_2}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2 + \frac{\partial u_2}{\partial x_2}\right) + 2\mu\frac{\partial^2 u_2}{\partial x_2^2} \\ &+ 2\mu k_1\left(\frac{\partial u_2}{\partial x_2} - \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} - k_1 u_2\right) + b_2 = \rho\frac{\partial^2 u_2}{\partial t^2}\end{aligned} \quad (34)$$

In this section the complete system of linear couple stress theory of elasticity is considered in detail. These equations will be used for the development of the approximate theories of the curved rods.

3 1-D formulation of the problem

In order to reduce the 2-D problem for the couple stress theory of elastic curved rods to a 1-D one, we expand the functions that describe the stress-strain state of the rod into the Legendre polynomials [76] series along the coordinate x_2 and present them in the form

$$\begin{aligned}&u_\alpha(x_1, x_2) = \sum_{k=0}^{\infty} u_\alpha^k(x_1) P_k(\varpi), \\ &u_\alpha^k(x_1) = \frac{2k+1}{2h}\int_{-h}^{h} u_\alpha(x_1, x_2) P_k(\varpi) dx_2, \\ &\omega_3(x_1, x_2) = \sum_{k=0}^{\infty} \omega_3^k(x_1) P_k(\varpi), \\ &\omega_3^k(x_1) = \frac{2k+1}{2h}\int_{-h}^{h} \omega_3(x_1, x_2) P_k(\varpi) dx_2,\end{aligned} \quad (35)$$

$$\sigma_{\alpha\beta}(x_1,x_2)=\sum_{k=0}^{\infty}\sigma_{\alpha\beta}^{k}(x_1)P_k(\varpi),$$
$$\sigma_{\alpha\beta}^{k}(x_1)=\frac{2k+1}{2h}\int_{-h}^{h}\sigma_{\alpha\beta}(x_1,x_2)P_k(\varpi)dx_2,$$
$$\mu_{\alpha3}(x_1,x_2)=\sum_{k=0}^{\infty}\mu_{\alpha3}^{k}(x_1)P_k(\varpi),$$
$$\mu_{\alpha3}^{k}(x_1)=\frac{2k+1}{2h}\int_{-h}^{h}\mu_{\alpha3}(x_1,x_2)P_k(\varpi)dx_2,$$
$$\varepsilon_{\alpha\beta}(x_1,x_2)=\sum_{k=0}^{\infty}\varepsilon_{\alpha\beta}^{k}(x_1)P_k(\varpi),$$
$$\varepsilon_{\alpha\beta}^{k}(x_1)=\frac{2k+1}{2h}\int_{-h}^{h}\varepsilon_{\alpha\beta}(x_1,x_2)P_k(\varpi)dx_2,$$

where $\varpi=x_2/h\in[-1,1]$ is a normalized variable.

Generally, all of the functions that are considered here also depend on time t, but to reduce typing the variable of time has been omitted.

For derivatives of the considered functions with respect to x_1 the following relations take place

$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial u_\alpha(x_1,x_2)}{\partial x_1}P_k(\varpi)\,dx_2=\frac{\partial u_\alpha^k(x_1)}{\partial x_1},\tag{36}$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial \omega_3(x_1,x_2)}{\partial x_1}P_k(\varpi)\,dx_2=\frac{\partial \omega_3^k(x_1)}{\partial x_1},$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial \sigma_{\alpha\beta}(x_1,x_2)}{\partial x_1}P_k(\varpi)\,dx_2=\frac{\partial \sigma_{\alpha\beta}^k(x_1)}{\partial x_1},$$

Derivatives of the displacements and rotation with respect to x_2 following [62, 63] can be represented in the form

$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial u_\alpha}{\partial x_2}P_k(\varpi)\,dx_2=\underline{u_\alpha^k}(x_1),\tag{37}$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial \omega_3}{\partial x_2}P_k(\varpi)\,dx_2=\underline{\omega_3^k}(x_1)$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial^2 \omega_3}{\partial x_2^2}P_k(\varpi)\,dx_2=\underline{\underline{\omega_3^k}}$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial \sigma_{\alpha2}}{\partial x_2}P_k(\varpi)\,dx_2=\frac{2k+1}{h}\left[\sigma_{\alpha2}^{+}-(-1)^k\sigma_{\alpha2}^{-}\right]$$
$$-\underline{\sigma_{2\alpha}^k}(x_1)$$

where

$$\underline{u_\alpha^k}(x_1)=\frac{2k+1}{h}\left(u_\alpha^{k+1}(x_1)+u_\alpha^{k+3}(x_1)+\ldots\right),\tag{38}$$
$$\underline{\omega_3^k}(x_1)=\frac{2k+1}{h}\left(\omega_3^{k+1}(x_1)+\omega_3^{k+3}(x_1)+\ldots\right),$$
$$\underline{\underline{\omega_3^k}}(x_1)=\frac{2k+1}{h}\left(\underline{\omega_3^{k+1}}(x_1)+\underline{\omega_3^{k+3}}(x_1)+\ldots\right),$$
$$\underline{\sigma_{2\alpha}^k}(x_1)=\frac{2k+1}{h}\left(\sigma_{2\alpha}^{k-1}(x_1)+\sigma_{2\alpha}^{k-3}(x_1)+\ldots\right)$$

By multiplying the equations of motion (28) by $P_k(\varpi)$ and integrating with respect to x_2 from $-h$ to h as well as taking into account (35)–(38) we obtain 1-D equations of motion in the form

$$\frac{1}{A_1}\frac{\partial\sigma_{11}^k}{\partial x_1}+k_1\left(\sigma_{21}^k+\sigma_{12}^k\right)-\underline{\sigma_{21}^k}+b_1^k=\rho\frac{\partial^2u_1^k}{\partial t^2},\tag{39}$$
$$\frac{1}{A_1}\frac{\partial\sigma_{12}^k}{\partial x_1}+k_1\left(\sigma_{22}^k-\sigma_{11}^k\right)-\underline{\sigma_{22}^k}+b_2^k=\rho\frac{\partial^2u_2^k}{\partial t^2},$$

where

$$\tilde b_\alpha^k(x_1)=b_\alpha^k(x_1)+\frac{2k+1}{h}\left(\sigma_{\alpha2}^{+}(x_1)-(-1)^k\sigma_{\alpha2}^{-}(x_1)\right)\tag{40}$$

In the same way by considering (35)–(38) the 2-D kinematic relations (32) can be found in the form

$$\varepsilon_{11}^k=\frac{1}{A_1}\frac{\partial u_1^k}{\partial x_1}+k_1u_2^k,\ \varepsilon_{22}^k=\underline{u_2^k},\tag{41}$$
$$2\varepsilon_{12}^k=\underline{u_1^k}+\frac{1}{A_1}\frac{\partial u_2^k}{\partial x_1}-k_1u_1^k,$$
$$\omega_3^k=\frac{1}{2}\left(\frac{1}{A_1}\frac{\partial u_2^k}{\partial x_1}-\underline{u_1^k}-k_1u_1^k\right)$$

Constitutive relations (generalized Hooke's law) (30) become

$$\sigma_{11}^k=(\lambda+2\mu)\varepsilon_{11}^k+\lambda\varepsilon_{22}^k,\tag{42}$$
$$\sigma_{22}^k=(\lambda+2\mu)\varepsilon_{22}^k+\lambda\varepsilon_{11}^k$$
$$\sigma_{12}^k=2\mu\varepsilon_{12}^k-2\eta\frac{1}{A_1^2}\frac{\partial^2\omega_3^k}{\partial x_1^2}-2\eta\underline{\underline{\omega_3^k}},$$
$$\sigma_{21}^k=2\mu\varepsilon_{21}^k+2\eta\frac{1}{A_1^2}\frac{\partial^2\omega_3^k}{\partial x_1^2}+2\eta\underline{\underline{\omega_3^k}}$$

The substitution of the kinematic relations (41) into the generalized Hooke's law (42) gives us the following equations

$$\sigma_{11}^k=(\lambda+2\mu)\left(\frac{1}{A_1}\frac{\partial u_1^k}{\partial x_1}+k_1u_2^k\right)+\lambda\underline{u_2^k},\tag{43}$$
$$\sigma_{22}^k=(\lambda+2\mu)\underline{u_2^k}+\lambda\left(\frac{1}{A_1}\frac{\partial u_1^k}{\partial x_1}+k_1u_2^k\right)$$
$$\sigma_{12}^k=\mu\left(\underline{u_1^k}+\frac{1}{A_1}\frac{\partial u_2^k}{\partial x_1}-k_1u_1^k\right)$$
$$-\eta\frac{1}{A_1^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{A_1}\frac{\partial u_2^k}{\partial x_1}-\underline{u_1^k}-k_1u_1^k\right)-\eta\underline{\underline{\omega_3^k}},$$

$$\sigma_{21}^k = \mu\left(\underline{u_1^k} + \frac{1}{A_1}\frac{\partial u_2^k}{\partial x_1} - k_1 u_1^k\right) + \eta\frac{1}{A_1^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{A_1}\frac{\partial u_2^k}{\partial x_1} - \underline{u_1^k} - k_1 u_1^k\right) + \eta\underline{\underline{\omega_3^k}}$$

Then, the differential equations of motion for couple stress theory of elasticity in the form of displacements (34) are transformed into their 1-D form

$$\frac{\lambda+2\mu}{A_1^2}\frac{\partial^2 u_1^k}{\partial x_1^2} + \frac{k_1(\lambda+4\mu)}{A_1}\frac{\partial u_2^k}{\partial x_1} + \lambda\frac{\partial \underline{u_2^k}}{\partial x_1} + 2k_1\mu\underline{u_1^k} - 2k_1^2\mu u_2^k - \underline{\sigma_{21}^k} + b_1^k = \rho\frac{\partial^2 u_1^k}{\partial t^2}, \tag{44}$$

$$\frac{\mu}{A_1^2}\frac{\partial^2 u_2^k}{\partial x_1^2} - \frac{\mu k_1}{A_1}\frac{\partial u_2^k}{\partial x_1} + \frac{\mu}{A_1}\frac{\partial \underline{\underline{u_1^k}}}{\partial x_1} - \frac{\eta}{A_1^4}\frac{\partial^4 u_2^k}{\partial x_1^4} + \frac{\eta}{A_1^3}\frac{\partial^3 \underline{u_1^k}}{\partial x_1^3} + \frac{\eta k_1}{A_1^3}\frac{\partial^3 u_1^k}{\partial x_1^3} - \frac{\eta}{A_1^3}\frac{\partial^3 \underline{\underline{\omega_3^k}}}{\partial x_1^3} - \frac{2\mu k_1}{A_1}\frac{\partial u_1^k}{\partial x_1} - 2\mu k_1^2 u_2^k + 2\mu\underline{u_2^k} - \underline{\sigma_{22}^k} + b_2^k = \rho\frac{\partial^2 u_2^k}{\partial t^2},$$

Now, instead of the 2-D system of the differential equations in displacements (34) we have an infinite system of 1-D differential equations for coefficients of the Legendre's polynomial series expansion (44). In order to simplify the problem an approximate theory has to be developed where only a finite number of members have to be taken into account in the expansion of (35) and in all of the above relations. For example, if we consider n-order approximate shell theory, only $n+1$ members in the expansion (35) are taken into account

$$u_\alpha(x_1,x_2) = \sum_{k=0}^{n} u_\alpha^k(x_1)P_k(\varpi), \tag{45}$$
$$\omega_3(x_1,x_2) = \sum_{k=0}^{n} \omega_3^k(x_1)P_k(\varpi),$$
$$\sigma_{\alpha\beta}(x_1,x_2) = \sum_{k=0}^{n} \sigma_{\alpha\beta}^k(x_1)P_k(\varpi),$$
$$\mu_{\alpha 3}(x_1,x_2) = \sum_{k=0}^{n} \mu_{\alpha 3}^k(x_1)P_k(\varpi),$$
$$\varepsilon_{\alpha\beta}(x_1,x_2) = \sum_{k=0}^{n} \varepsilon_{\alpha\beta}^k(x_1)P_k(\varpi),$$
$$\kappa_{\alpha 3}(x_1,x_2) = \sum_{k=0}^{n} \kappa_{\alpha 3}^k(x_1)P_k(\varpi),$$

In this case we consider that $u_\alpha^k = 0$, $\omega_3^k = 0$, $\sigma_{\alpha\beta}^k = 0$, $\mu_{\alpha 3}^k = 0$, and $\theta^k = 0$ for $k < 0$ and for $k > n$.

Then the 1-D differential equations of couple stress theory of elasticity in displacements and rotations (44) can be presented in the matrix form

$$\mathbf{L}_u\cdot\mathbf{u} + \mathbf{f} = \rho\frac{\partial^2}{\partial t^2}\mathbf{I}_u\cdot\mathbf{u} \tag{46}$$

where

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^{00} & L_{12}^{00} & \cdots & L_{11}^{0n} & L_{12}^{0n} \\ L_{21}^{00} & L_{22}^{00} & \cdots & L_{21}^{0n} & L_{22}^{0n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ L_{11}^{n0} & L_{12}^{n0} & \cdots & L_{11}^{nn} & L_{12}^{nn} \\ L_{21}^{n0} & L_{22}^{00} & \cdots & L_{21}^{nn} & L_{22}^{nn} \end{vmatrix}, \quad \mathbf{u} = \begin{vmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_1^n \\ u_2^n \end{vmatrix}, \tag{47}$$
$$\tilde{\mathbf{b}} = \begin{vmatrix} \tilde{b}_1^0 \\ \tilde{b}_2^0 \\ \vdots \\ \tilde{b}_1^n \\ \tilde{b}_2^n \end{vmatrix}$$

Here $\mathbf{I}_u$ is the matrix operator that has on the main diagonal elements equal to one.

The order of the system of differential equations depends on the assumption regarding thickness distribution of the stress-strain parameters of the shell. The higher the order of approximation, the better accuracy of the result obtained using the proposed theory. The complete system of linear differential equations for the couple stress theory of elasticity of any order can be obtained using the equations presented here. In the next section, we will consider the first order approximation theory in detail.

4 First order approximation theory

In the case of the first order approximation theory only the first two terms of the Legendre polynomials series are considered in the expansion (35). In this case the functions, which describe the stress-strain state of the rod, can be presented in the form

$$u_\alpha(x_1,x_2) = u_\alpha^0(x_1)P_0(\omega) + u_\alpha^1(x_1)P_1(\omega), \tag{48}$$
$$\omega_3(x_1,x_2) = \omega_3^0(x_1)P_0(\omega) + \omega_3^1(x_1)P_1(\omega)$$
$$\sigma_{\alpha\beta}(x_1,x_2) = \sigma_{\alpha\beta}^0(x_1)P_0(\omega) + \sigma_{\alpha\beta}^1(x_1)P_1(\omega),$$
$$\mu_{\alpha 3}(x_1,x_2) = \mu_{\alpha 3}^0(x_1)P_0(\omega) + \mu_{\alpha 3}^1(x_1)P_1(\omega)$$
$$\varepsilon_{\alpha\beta}(x_1,x_2) = \varepsilon_{\alpha\beta}^0(x_1)P_0(\omega) + \varepsilon_{\alpha\beta}^1(x_1)P_1(\omega),$$
$$\kappa_{\alpha 3}(x_1,x_2) = \kappa_{\alpha 3}^0(x_1)P_0(\omega) + \kappa_{\alpha 3}^1(x_1)P_1(\omega)$$

All the equations presented in the previous section will be significantly simplified. For example in this case we have

$$\underline{u_\alpha^0}(x_1) = \frac{1}{h}u_\alpha^1(x_1), \quad \underline{\underline{u_\alpha^0}}(x_1) = 0, \tag{49}$$
$$\underline{\sigma_{2\alpha}^0}(x_1) = 0, \quad \underline{\sigma_{2\alpha}^1}(x_1) = \frac{3}{h}\sigma_{2\alpha}^0(x_1)$$
$$\underline{\omega_3^0}(x_1) = \frac{1}{h}\omega_3^1(x_1),$$

$$\underline{\omega_3^1}(x_1) = 0, \quad \underline{\underline{\omega_3^0}} = \frac{1}{h}\underline{\omega_3^1} = 0, \quad \underline{\underline{\omega_3^1}} = 0$$

The equations of motion (39) now have the form

$$\begin{aligned}
&\frac{1}{A_1}\frac{\partial \sigma_{11}^0}{\partial x_1} + k_1\left(\sigma_{21}^0 + \sigma_{12}^0\right) + b_1^0 = \rho\frac{\partial^2 u_1^0}{\partial t^2}, \\
&\frac{1}{A_1}\frac{\partial \sigma_{12}^0}{\partial x_1} + k_1\left(\sigma_{22}^0 - \sigma_{11}^0\right) + b_2^0 = \rho\frac{\partial^2 u_2^0}{\partial t^2}, \\
&\frac{1}{A_1}\frac{\partial \sigma_{11}^1}{\partial x_1} + k_1\left(\sigma_{21}^1 + \sigma_{12}^1\right) - \frac{3}{h}\sigma_{21}^0 + b_1^k = \rho\frac{\partial^2 u_1^k}{\partial t^2}, \\
&\frac{1}{A_1}\frac{\partial \sigma_{12}^1}{\partial x_1} + k_1\left(\sigma_{22}^1 - \sigma_{11}^1\right) - \frac{3}{h}\sigma_{22}^0 + b_2^k = \rho\frac{\partial^2 u_2^k}{\partial t^2},
\end{aligned} \tag{50}$$

Kinematic relations (41) have form

$$\begin{aligned}
&\varepsilon_{11}^0 = \frac{1}{A_1}\frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0, \quad \varepsilon_{22}^0 = \frac{1}{h}u_2^1, \\
&2\varepsilon_{12}^0 = \frac{1}{h}u_1^1 + \frac{1}{A_1}\frac{\partial u_2^0}{\partial x_1} - k_1 u_1^0, \\
&\omega_3^0 = \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial u_2^0}{\partial x_1} - \frac{1}{h}u_\alpha^1(x_1) - k_1 u_1^0\right) \\
&\varepsilon_{11}^1 = \frac{1}{A_1}\frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1, \quad \varepsilon_{22}^k = 0, \\
&2\varepsilon_{12}^1 = \frac{1}{A_1}\frac{\partial u_2^1}{\partial x_1} - k_1 u_1^1, \quad \omega_3^1 = \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial u_2^1}{\partial x_1} - k_1 u_1^1\right)
\end{aligned} \tag{51}$$

The generalized Hooke's law for couple stress elasticity (42) has the form

$$\begin{aligned}
\sigma_{11}^0 &= (\lambda + 2\mu)\varepsilon_{11}^0 + \lambda\varepsilon_{22}^0, \\
\sigma_{22}^0 &= (\lambda + 2\mu)\varepsilon_{22}^0 + \lambda\varepsilon_{11}^0 \\
\sigma_{12}^0 &= 2\mu\varepsilon_{12}^0 - 2\eta\frac{1}{A_1^2}\frac{\partial^2 \omega_3^0}{\partial x_1^2}, \\
\sigma_{21}^0 &= 2\mu\varepsilon_{21}^0 + 2\eta\frac{1}{A_1^2}\frac{\partial^2 \omega_3^0}{\partial x_1^2} \\
\sigma_{11}^1 &= (\lambda + 2\mu)\varepsilon_{11}^1 + \lambda\varepsilon_{22}^1, \\
\sigma_{22}^1 &= (\lambda + 2\mu)\varepsilon_{22}^1 + \lambda\varepsilon_{11}^1 \\
\sigma_{12}^1 &= 2\mu\varepsilon_{12}^1 - 2\eta\frac{1}{A_1^2}\frac{\partial^2 \omega_3^1}{\partial x_1^2}, \\
\sigma_{21}^1 &= 2\mu\varepsilon_{21}^1 + 2\eta\frac{1}{A_1^2}\frac{\partial^2 \omega_3^1}{\partial x_1^2}.
\end{aligned} \tag{52}$$

By substituting kinematic relations (51) into equation of motion (52) we obtain

$$\begin{aligned}
\sigma_{11}^0 &= (\lambda + 2\mu)\left(\frac{1}{A_1}\frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0\right) + \lambda\frac{1}{h}u_2^1, \\
\sigma_{22}^0 &= (\lambda + 2\mu)\frac{1}{h}u_2^1 + \lambda\left(\frac{1}{A_1}\frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0\right) \\
\sigma_{12}^0 &= \mu\left(\frac{1}{h}u_1^1 + \frac{1}{A_1}\frac{\partial u_2^0}{\partial x_1} - k_1 u_2^0\right) \\
&\quad - \eta\frac{1}{A_1^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{A_1}\frac{\partial u_2^0}{\partial x_1} - \frac{1}{h}u_1^1 - k_1 u_1^0\right), \\
\sigma_{21}^0 &= \mu\left(\frac{1}{h}u_1^1 + \frac{1}{A_1}\frac{\partial u_2^0}{\partial x_1} - k_1 u_2^0\right) \\
&\quad + \eta\frac{1}{A_1^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{A_1}\frac{\partial u_2^0}{\partial x_1} - \frac{1}{h}u_1^1 - k_1 u_1^0\right) \\
\sigma_{11}^1 &= (\lambda + 2\mu)\left(\frac{1}{A_1}\frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1\right), \\
\sigma_{22}^1 &= \lambda\left(\frac{1}{A_1}\frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1\right) \\
\sigma_{12}^1 &= \mu\left(\frac{1}{A_1}\frac{\partial u_2^1}{\partial x_1} - k_1 u_2^1\right) - \eta\frac{1}{A_1^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{A_1}\frac{\partial u_2^1}{\partial x_1} - k_1 u_1^1\right), \\
\sigma_{21}^1 &= \mu\left(\frac{1}{A_1}\frac{\partial u_2^1}{\partial x_1} - k_1 u_2^1\right) + \eta\frac{1}{A_1^2}\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{A_1}\frac{\partial u_2^1}{\partial x_1} - k_1 u_1^1\right).
\end{aligned} \tag{53}$$

By substituting generalized Hooke's law in the form of (53) into equations of motion (50) we obtain the 1-D differential equations for the couple stress theory of elasticity in displacements and rotations for the first order theory of couple stress rods theory in the form (46), where matrices and vectors (47) become

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^{00} & L_{12}^{00} & L_{11}^{01} & L_{12}^{01} \\ L_{21}^{00} & L_{22}^{00} & L_{21}^{01} & L_{22}^{01} \\ L_{11}^{10} & L_{12}^{01} & L_{11}^{11} & L_{12}^{11} \\ L_{21}^{10} & L_{22}^{10} & L_{21}^{11} & L_{22}^{11} \end{vmatrix}, \quad \mathbf{u} = \begin{vmatrix} u_1^0 \\ u_2^0 \\ u_1^1 \\ u_2^1 \end{vmatrix}, \tag{54}$$

$$\tilde{\mathbf{b}} = \begin{vmatrix} \tilde{b}_1^0 \\ \tilde{b}_2^0 \\ \tilde{b}_1^1 \\ \tilde{b}_2^1 \end{vmatrix},$$

Elements of the matrix operator $\mathbf{L}_u$ can be represented in the form

$$\begin{aligned}
L_{11}^{00}u_1^0 &= \frac{\lambda + 2\mu}{A_1^2}\frac{\partial^2 u_1^0}{\partial x_1^2}, \\
L_{12}^{00}u_2^0 &= \frac{k_1(\lambda + 4\mu)}{A_1}\frac{\partial u_2^0}{\partial x_1} - 2k_1^2\mu u_2^0, \\
L_{11}^{01}u_1^1 &= \frac{2\mu k_1}{h}u_1^1, \quad L_{11}^{01}u_2^1 = \frac{\lambda}{A_1 h}\frac{\partial u_2^1}{\partial x_1}, \\
L_{21}^{00}u_1^0 &= -\frac{2\mu k_1}{A_1}\frac{\partial u_1^0}{\partial x_1} + \frac{\eta k_1}{A_1^3}\frac{\partial^3 u_1^0}{\partial x_1^3}, \\
L_{22}^{00}u_1^0 &= \frac{\mu}{A_1^2}\frac{\partial^2 u_2^0}{\partial x_1^2} - \frac{\mu k_1}{A_1}\frac{\partial u_2^0}{\partial x_1} - 2\mu k_1^2 u_2^0 - \frac{\eta}{A_1^4}\frac{\partial^4 u_2^0}{\partial x_1^4} \\
L_{21}^{01}u_1^1 &= \frac{\mu}{A_1 h}\frac{\partial u_1^1}{\partial x_1} + \frac{\eta}{A_1^3 h}\frac{\partial^3 u_1^1}{\partial x_1^3}, \\
L_{22}^{01}u_2^1 &= \frac{2\mu k_1}{h}u_2^1, \quad L_{11}^{10}u_1^0 = 0, \\
L_{12}^{10}u_2^0 &= -\frac{3\mu}{A_1 h}\frac{\partial u_2^0}{\partial x_1} + \frac{3\mu k_1}{h}u_2^0 - \frac{3\eta}{A_1^3 h}\frac{\partial^3 u_2^0}{\partial x_1^3} - \frac{3\eta k_1}{A_1^2 h}\frac{\partial^2 u_2^0}{\partial x_1^2}, \\
L_{11}^{11}u_1^1 &= \frac{\lambda + 2\mu}{A_1^2}\frac{\partial^2 u_1^1}{\partial x_1^2} - \frac{3\mu}{h^2}u_1^1 + \frac{3\eta}{A_1^2 h^2}\frac{\partial^2 u_1^1}{\partial x_1^2}, \\
L_{12}^{11}u_2^1 &= \frac{(\lambda + 4\mu)k_1}{A_1}\frac{\partial u_2^1}{\partial x_1} - 2\mu k_1^2 u_2^1,
\end{aligned} \tag{55}$$

$$L_{21}^{10}u_1^0=-\frac{3\lambda}{A_1h}\frac{\partial u_1^0}{\partial x_1},\quad L_{22}^{10}u_2^0=-\frac{3\lambda k_1}{h}u_2^0,$$
$$L_{21}^{11}u_1^1=-\frac{2\mu k_1}{A_1}\frac{\partial u_1^1}{\partial x_1}+\frac{\eta k_1}{A_1^3}\frac{\partial^3 u_1^1}{\partial x_1^3},$$
$$L_{22}^{11}u_2^1=\frac{\mu}{A_1^2}\frac{\partial^2 u_2^1}{\partial x_1^2}-\frac{\mu k_1}{A_1}\frac{\partial u_2^1}{\partial x_1}-2\mu k_1^2u_2^1-\frac{3(\lambda+2\mu)}{h^2}u_2^1$$
$$-\frac{\eta}{A_1^4}\frac{\partial^4 u_2^1}{\partial x_1^4}.$$

The equations presented in this section are first order equations of the couple stress curved rods theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering couple stress and rotation effects. If in the above equations it is assumed that $A_1 = 1$ and $k_1 = 0$ the equations for the couple stress straight beam will be obtained in the form

$$\lambda+2\mu\frac{\partial^2 u_1^0}{\partial x_1^2}+\frac{\lambda}{h}\frac{\partial u_2^1}{\partial x_1}+b_1^0=\rho\frac{\partial^2 u_1^0}{\partial t^2},\tag{56}$$
$$\mu\frac{\partial^2 u_2^0}{\partial x_1^2}+\frac{\mu}{h}\frac{\partial u_1^1}{\partial x_1}-\eta\frac{\partial^4 u_2^0}{\partial x_1^4}+\frac{\eta}{h}\frac{\partial^3 u_1^1}{\partial x_1^3}+b_2^0=\rho\frac{\partial^2 u_2^0}{\partial t^2},$$
$$-\frac{3\mu}{h}\frac{\partial u_2^0}{\partial x_1}-\frac{3\eta}{h}\frac{\partial^3 u_2^0}{\partial x_1^3}+\lambda+2\mu\frac{\partial^2 u_1^1}{\partial x_1^2}-\frac{3\mu}{h^2}u_1^1+\frac{3\eta}{h^2}\frac{\partial^2 u_1^1}{\partial x_1^2}$$
$$+b_1^1=\rho\frac{\partial^2 u_1^1}{\partial t^2},$$
$$-\frac{3\lambda}{h}\frac{\partial u_1^0}{\partial x_1}+\mu\frac{\partial^2 u_2^1}{\partial x_1^2}-\eta\frac{\partial^4 u_2^1}{\partial x_1^4}-\frac{3(\lambda+2\mu)}{h^2}u_2^1+b_2^1=\rho\frac{\partial^2 u_2^1}{\partial t^2},$$

Analysis of this system of partial differential equations shows that it splits up into two independent parts. First and fours and also second and third equations form separate systems of the partial differential equations that can be solved independently.

5 Timoshenko's couple stress theory of the curved rods

Timoshenko's theory of the curved rods is based on assumptions concerning the value and distribution of the parameters that define the stress-strain state of the rods. In this theory the stress-strain state of the beams is determined by quantities specified on the middle surface. Thus, according to static assumptions the stress in the direction perpendicular to the middle line $\sigma_{22} = 0$ and according to kinematic assumptions deformations in the same direction $\varepsilon_{22} = 0$. The stress state is characterized by the normal n_{11} and shear n_{21} forces, as well as the bending m_{22} and rotating m_{13} moments, which are defined as following

$$n_{11}=\int_{-h}^{h}\sigma_{11}dx_2,\quad n_{12}=\int_{-h}^{h}\sigma_{12}dx_2,\tag{57}$$
$$n_{21}=\int_{-h}^{h}\sigma_{21}dx_2,\quad m_{11}=\int_{-h}^{h}\sigma_{11}x_2dx_2,$$
$$m_{13}=\int_{-h}^{h}\mu_{13}dx_2,$$

Displacements in the Timoshenko's theory of curved beans are defined by vectors $\mathbf{u}(x_1, t)$ with components u_α, $\alpha = 1, 2$ that correspond to displacements of the middle line and $\gamma_1(x_1, t)$ that is rotation of the centroidal axis about the x_3 axis of the elements perpendicular to the middle line. These parameters are related to the coefficients of the displacements expansion in the first order theory in the following way

$$u_i^0\sim u_i,\quad u_\alpha^1\sim\gamma_\alpha h\tag{58}$$

Component u_3^1 is not taken into account in the Timoshenko's theory of shells.

Deformations in Timoshenko's theory are determined by the relations

$$\varepsilon_{11}=e_{11}+\kappa_{11}x_2,\quad \varepsilon_{12}=e_{12},\quad \varepsilon_{21}=e_{21},\tag{59}$$

Roughly speaking, component e_{11} corresponds to the tension-compression deformation of the middle surface, components e_{12} to the transversal shear deformation and component κ_{11} to the bending middle line, respectively. The following formulas give us relations with corresponding quantities in the first order theory

$$\varepsilon_{\alpha\beta}^0\sim e_{\alpha\beta},\quad \varepsilon_{11}^1\sim\kappa_{11}\tag{60}$$

Component ε_{33}^0 and ε_{33}^1 are not taken into account in the Timoshenko's theory of shells. That follows also from the kinematic hypothesis.

According to Timoshenko beam theory, the displacement and rotation field can be written as

$$u_1(x_1,x_2,t)=u_1(x_1,t)-x_2\gamma_1(x_1,t),\tag{61}$$
$$u_2(x_1,x_2,t)=u_2(x_1,t),$$
$$\omega_3(x_1,x_2,t)=\omega_3(x_1,t)$$

where $u_1(x_1, t)$ and $u_2(x_1, t)$ are axial and transverse displacements in the x_1 and x_2 directions respectively, $\omega_3(x_1, t)$ is rotation of the centroidal axis about the x_3 axis.

By substituting expressions for displacements and rotations (61) into 2-D kinematic relations (32) we obtain kinematic relations for the couple stress theory of curved rod in the from

$$e_{11} = \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2, \quad e_{22} = 0, \tag{62}$$
$$e_{12} = \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \gamma_1\right) \quad \kappa_{11} = \frac{1}{A_1}\frac{\partial \gamma_1}{\partial x_1}$$
$$\omega_3 = \omega_{21} = \frac{1}{2}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 + \gamma_1\right)$$

Constitutive relations of liners couple stress elasticity for couple stress Timoshenko's curved rod theory can be obtained in the form generalized Hooke's law

$$n_{11} = EFe_{11}, \quad m_{11} = EJ\kappa_{11}, \tag{63}$$
$$n_{12} = 2\mu Fe_{12} - 2\eta\frac{F}{A_1^2}\frac{\partial^2 \omega_3}{\partial x_1^2},$$
$$n_{21} = 2\mu Fe_{12} + 2\eta\frac{F}{A_1^2}\frac{\partial^2 \omega_3}{\partial x_1^2}$$

By substituting kinematic relations (62) to the generalized Hooke's law (63) we obtain constitutive relations in the form

$$n_{11} = EF\left(\frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + k_1 u_2\right), \quad m_{11} = EJ\frac{1}{A_1}\frac{\partial \gamma_1}{\partial x_1}, \tag{64}$$
$$n_{12} = \mu F\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \gamma_1\right)$$
$$- \frac{\eta F}{A_1^2}\left(\frac{1}{A_1}\frac{\partial^3 u_2}{\partial x_1^3} - k_1\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 \gamma_1}{\partial x_1^2}\right),$$
$$n_{21} = \mu F\left(\frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - k_1 u_1 - \gamma_1\right)$$
$$+ \frac{\eta F}{A_1^2}\left(\frac{1}{A_1}\frac{\partial^3 u_2}{\partial x_1^3} - k_1\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 \gamma_1}{\partial x_1^2}\right)$$

By integrating differential equations of motion (28) with respect to x_2 from $-h$ to h we first obtain the equations of motion presented in (65). And by multiplying the first equation of motion (28) by x_2 and integrating it with respect to x_2 from $-h$ to h we obtain the last equation of motion for Timoshenko's couple stress curved rod theory. The complete system of the equations of motion has the from

$$\frac{1}{A_1}\frac{\partial n_{11}}{\partial x_1} + (n_{21} + n_{12})\,k_1 + \tilde{b}_1 = \rho F\frac{\partial^2 u_1}{\partial t^2}, \tag{65}$$
$$\frac{1}{A_1}\frac{\partial n_{12}}{\partial x_1} - n_{11}k_1 + \tilde{b}_2 = \rho F\frac{\partial^2 u_2}{\partial t^2},$$
$$\frac{1}{A_1}\frac{\partial m_{11}}{\partial x_1} - n_{21} + \tilde{m}_3 = \rho J\frac{\partial^2 \gamma_1}{\partial t^2},$$

By substituting constitutive relations in the form (64) into the equations of motion (65) we obtain the 1-D differential equations of couple stress theory of elasticity in the form of displacements for Timoshenko's couple stress rods theory of curved rods in the form (46), where matrices and vectors (47) become

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^u & L_{12}^u & L_{11}^\gamma \\ L_{21}^u & L_{22}^u & L_{21}^\gamma \\ L_{31}^u & L_{32}^u & L_{31}^\gamma \end{vmatrix}, \quad \mathbf{u} = \begin{vmatrix} u_1 \\ u_1 \\ \gamma_1 \end{vmatrix}, \quad \mathbf{b} = \begin{vmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{m}_1 \end{vmatrix}, \tag{66}$$

Elements of the matrix operator $\mathbf{L}_u$ can be represented in the form

$$L_{11}^u u_1 = \frac{EF}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2} - 2\mu Fk_1^2 u_1, \tag{67}$$
$$L_{12}^u u_2 = (E + 2\mu)\,Fk_1\frac{\partial u_2}{\partial x_1},$$
$$L_{11}^\gamma \gamma_1 = -2\mu Fk_1\gamma_1,$$
$$L_{21}^u u_1 = -\frac{(E+\mu)Fk_1}{A_1}\frac{\partial u_1}{\partial x_1} + \frac{\eta Fk_1}{A_1^3}\frac{\partial^3 u_1}{\partial x_1^3},$$
$$L_{22}^u u_2 = \frac{\mu F}{A_1^2}\frac{\partial^2 u_2}{\partial x_1^2} - EFk_1^2 u_2 - \frac{\eta F}{A_1^4}\frac{\partial^4 u_2}{\partial x_1^4},$$
$$L_{21}^\gamma \gamma_1 = -\frac{\mu F}{A_1}\frac{\partial \gamma_1}{\partial x_1} - \frac{\eta F}{A_1^3}\frac{\partial^3 \gamma_1}{\partial x_1^3},$$
$$L_{31}^u u_1 = \mu Fk_1 u_1 - \frac{\eta Fk_1}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2},$$
$$L_{32}^u u_2 = -\frac{\mu F}{A_1}\frac{\partial u_2}{\partial x_1} + \frac{\eta F}{A_1^3}\frac{\partial^3 u_2}{\partial x_1^3},$$
$$L_{31}^\gamma \gamma_1 = \frac{EJ}{A_1^2}\frac{\partial^2 \gamma_1}{\partial x_1^2} + \mu F\gamma_1 + \frac{\eta F}{A_1^2}\frac{\partial^2 \gamma_1}{\partial x_1^2}.$$

The equations presented in this section are equations of the Timoshenko's couple stress curved rods theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering couple stress and rotation effects. If in the above equations it is assumed that $A_1 = 1$ and $k_1 = 0$ the equations for the couple stress straight beam will be obtained in the form

$$EF\frac{\partial^2 u_1}{\partial x_1^2} + \tilde{b}_1 = \rho\frac{\partial^2 u_1}{\partial t^2} \tag{68}$$
$$\mu F\frac{\partial^2 u_2}{\partial x_1^2} - \eta F\frac{\partial^4 u_2}{\partial x_1^4} - \mu F\frac{\partial \gamma_1}{\partial x_1} - \eta F\frac{\partial^3 \gamma_1}{\partial x_1^3} + \tilde{b}_2 = \rho\frac{\partial^2 u_2}{\partial t^2}$$
$$-\mu F\frac{\partial u_2}{\partial x_1} + \mu F\gamma_1 + \eta F\frac{\partial^3 u_2}{\partial x_1^3} + EJ\frac{\partial^2 \gamma_1}{\partial x_1^2} + \eta F\frac{\partial^2 \gamma_1}{\partial x_1^2} + \tilde{m}_3$$
$$= \rho\frac{\partial^2 \gamma_1}{\partial t^2}$$

Analysis of this system of partial differential equations shows that it splits up into two separate parts. First equation and system of the rest two equations can be solved independently. We have to mention that system (68) coincide with the one presented in [47, 50, 51] up to notation. Therefore analysis and verification presented in [47, 50] take place in considered here case.

6 Euler-Bernoulli couple stress theory of the curved rods

Euler-Bernoulli theory of beams is based on assumptions concerning the value and distribution of the parameters that define the stress-strain state of the rods. Thus, according to static assumptions $\sigma_{22} = 0$ and according to kinematic assumptions $\varepsilon_{22} = 0$. In this theory the stress-strain state of beams is determined by quantities specified on the middle surface. The stress state is characterized by the normal n_{11} and shear n_{21} forces, as well as the bending m_{22} and rotating m_{13} moments. They are defined as the following

$$n_{11} = \int_{-h}^{h} \sigma_{11} dx_2, \quad n_{12} = \int_{-h}^{h} \sigma_{12} dx_2, \tag{69}$$
$$n_{21} = \int_{-h}^{h} \sigma_{21} dx_2, \quad m_{11} = \int_{-h}^{h} \sigma_{11} x_2 dx_2,$$
$$m_{13} = \int_{-h}^{h} \mu_{13} dx_2,$$

Displacements in the Euler-Bernoulli theory of curved rods are defined by vectors $\mathbf{u}(x_1, t)$ with components $u_\alpha, \alpha = 1, 2$ that correspond to displacements of the middle line and $\gamma_1 (x_1, t)$ that is the rotation of the centroidal axis about the x_3 axis of the elements perpendicular to the middle line.

Unlike the Timoshenko's theory, in the Euler-Bernoulli theory of beams the rotation of the centroidal axis is not independent. It is represented through displacements by the equation

$$\gamma_1 = \frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1 \tag{70}$$

Deformations in Euler-Bernoulli theory are determined by the relations

$$\varepsilon_{11} = e_{11} + \kappa_{11} x_2, \quad \varepsilon_{12} = e_{12}, \quad \varepsilon_{21} = e_{21}, \tag{71}$$

Roughly speaking component e_{11} corresponds to the tension-compression deformation of the middle surface, components e_{12} to the transversal shear deformation and component κ_{11} to the bending middle line, respectively. The following formulas give us relations with corresponding quantities in the first order theory

$$\varepsilon^0_{\alpha\beta} \sim e_{\alpha\beta}, \quad \varepsilon^1_{11} \sim \kappa_{11} \tag{72}$$

Component ε^0_{33} and ε^1_{33} are not taken into account in the Euler-Bernoulli theory of rods. That also follows from the kinematic hypothesis.

According to Euler-Bernoulli beam theory, the displacement and rotation field can be written in the following form

$$u_1(x_1, x_2, t) = u_1(x_1, t) - x_2 \left(\frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1 \right), \tag{73}$$
$$u_2(x_1, x_2, t) = u_2(x_1, t), \quad \omega_3(x_1, x_2, t) = \omega_3(x_1, t)$$

where $u_1(x_1, t)$ and $u_2(x_1, t)$ are axial and transverse displacements in the x_1 and x_2 directions respectively, $\omega_3(x_1, t)$ rotation of the centroidal axis about the x_3 axis.

By substituting expressions for displacements and rotations (73) into 2-D kinematic relations (32) we obtain kinematic relations for the couple stress theory of curved rod in the from

$$e_{11} = \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} + k_1 u_2, \quad e_{22} = 0, \tag{74}$$
$$e_{12} = 0, \quad \kappa_{11} = \frac{1}{A_1^2} \frac{\partial^2 u_2}{\partial x_1^2} - \frac{k_1}{A_1} \frac{\partial u_1}{\partial x_1},$$
$$\omega_3 = \omega_{21} = \frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1$$

Constitutive relations of liner couple stress elasticity for couple stress Euler-Bernoulli curved rod theory can be obtained in the form generalized Hooke's law

$$n_{11} = EFe_{11}, \quad m_{11} = EJ\kappa_{11}, \tag{75}$$
$$n_{12} = -2\eta \frac{F}{A_1^2} \frac{\partial^2 \omega_3}{\partial x_1^2}, \quad n_{21} = 2\eta \frac{F}{A_1^2} \frac{\partial^2 \omega_3}{\partial x_1^2}$$

By substituting kinematic relations (74) to the generalized Hooke's law (75) we obtain constitutive relations in the form

$$n_{11} = EF \left(\frac{1}{A_1} \frac{\partial u_1}{\partial x_1} + k_1 u_2 \right), \tag{76}$$
$$m_{11} = EJ \left(\frac{1}{A_1^2} \frac{\partial^2 u_2}{\partial x_1^2} - \frac{k_1}{A_1} \frac{\partial u_1}{\partial x_1} \right),$$
$$n_{12} = -2\eta \frac{F}{A_1^2} \frac{\partial^2}{\partial x_1^2} \left(\frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1 \right)$$
$$= -\frac{2\eta F}{A_1^3} \frac{\partial^3 u_2}{\partial x_1^3} + \frac{2\eta F k_1}{A_1^2} \frac{\partial^2 u_1}{\partial x_1^2},$$
$$n_{21} = 2\eta \frac{F}{A_1^2} \frac{\partial^2}{\partial x_1^2} \left(\frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1 \right)$$
$$= \frac{2\eta F}{A_1^3} \frac{\partial^3 u_2}{\partial x_1^3} - \frac{2\eta F k_1}{A_1^2} \frac{\partial^2 u_1}{\partial x_1^2}$$

Taking into account that in Euler-Bernoulli beam theory rotational inertial is neglected and therefore $\frac{\partial^2 \gamma_1}{\partial t^2} = 0$, the equations of motion for Euler-Bernoulli curved rod can be presented in the form

$$\frac{1}{A_1} \frac{\partial n_{11}}{\partial x_1} + (n_{21} + n_{12}) k_1 + \tilde{b}_1 = \rho F \frac{\partial^2 u_1}{\partial t^2}, \tag{77}$$

$$\frac{1}{A_1}\frac{\partial n_{12}}{\partial x_1} - n_{11}k_1 + \tilde{b}_2 = \rho F\frac{\partial^2 u_2}{\partial t^2},$$
$$\frac{1}{A_1}\frac{\partial m_{11}}{\partial x_1} - n_{21} + \tilde{m}_3 = 0,$$

From the last equation of motion (77) we have

$$n_{12} = \frac{1}{A_1}\frac{\partial m_{11}}{\partial x_1} - \tilde{m}_3, \tag{78}$$

By substituting kinematic relations in the form (76) in the equations of motion (77) by considering (78) and generalized Hooke's law (77) we obtain a differential equation of motion in the form of displacements for the Euler-Bernoulli couple stress curved rod theory in the form

$$\frac{EF}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2} + \frac{EFk_1}{A_1}\frac{\partial u_2}{\partial x_1} + \tilde{b}_1 = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{79}$$
$$-\frac{EJk_1}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2} + \frac{2\eta Fk_1}{A_1^3}\frac{\partial^3 u_1}{\partial x_1^3} - \frac{EFk_1}{A_1}\frac{\partial u_1}{\partial x_1} + \frac{EJ - 2\eta F}{A_1^4}\frac{\partial^4 u_2}{\partial x_1^4}$$
$$- EFk_1^2 u_2 + \tilde{b}_2 = \rho\frac{\partial^2 u_2}{\partial t^2},$$

where

$$\tilde{b}_2(x_1) = b_2(x_1) + \frac{2k+1}{h}\left(\sigma_{22}^+(x_1) - (-1)^k\sigma_{22}^-(x_1)\right) - \frac{\partial\tilde{m}_3}{\partial x_1}(x_1) \tag{80}$$

These can be easy converted to the form (46) with matrix operators and vectors (47) in the form

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^u & L_{12}^u \\ L_{21}^u & L_{22}^u \end{vmatrix}, \quad \mathbf{u} = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}, \quad \tilde{\mathbf{b}} = \begin{vmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{vmatrix}, \tag{81}$$

Elements of the matrix operator $\mathbf{L}_u$ can be represented in the form

$$L_{11}^u u_1 = \frac{EF}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2}, \quad L_{12}^u u_2 = \frac{EFk_1}{A_1}\frac{\partial u_2}{\partial x_1}, \tag{82}$$
$$L_{21}^u u_1 = -\frac{EJk_1}{A_1^2}\frac{\partial^2 u_1}{\partial x_1^2} + \frac{2\eta Fk_1}{A_1^3}\frac{\partial^3 u_1}{\partial x_1^3} - \frac{EFk_1}{A_1}\frac{\partial u_1}{\partial x_1},$$
$$L_{22}^u u_2 = +\frac{EJ - 2\eta F}{A_1^4}\frac{\partial^4 u_2}{\partial x_1^4} - EFk_1^2 u_2,$$

The equations presented in this section are for Euler-Bernoulli couple stress curved rods theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering couple stress and rotation effects. If in the above equations assume $A_1 = 1$ and $k_1 = 0$ the equations for the couple stress straight beam theory will be obtained in the form

$$EF\frac{\partial^2 u_1}{\partial x_1^2} + \tilde{b}_1 = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{83}$$
$$(EJ - 2\eta F)\frac{\partial^4 u_2}{\partial x_1^4} + \tilde{b}_2 = \rho\frac{\partial^2 u_2}{\partial t^2},$$

Analysis of this system of partial differential equations shows that it splits up into three separate parts. Each equation can be solved independently. We have to mention that system (83) coincide with the one presented in [43–45] up to notation. Therefore analysis and verification presented there take place in considered here case.

7 Conclusions

In this paper new theories for couple stress theory of the plane curved rods have been developed. 2-D theory is developed from general 2-D equations of linear couple stress theory of elasticity using a special curvilinear system of coordinates related to the middle line of the rod and a special assumption based on assumptions that take into account the fact that the rod is thin. High order theory is based on the expansion of the equations of theory of elasticity into Fourier series in terms of Legendre polynomials in a thickness coordinate. All the functions that define the stress-strain state of the rod including stress and strain tensors, vectors of displacements and rotation and body forces have been expanded into Fourier series in terms of Legendre polynomials with respect to a thickness coordinate. Thereby, all equations of elasticity including Hooke's law have been transformed to the corresponding equations for Fourier coefficients of the Legendre polynomials expansion. Then, for Fourier coefficients the system of differential equations of motion in terms of displacements and rotations has been obtained in the same way as in the theory of elasticity. The Timoshenko's and Euler-Bernoulli theories have been developed based on the classical hypothesis and 2-D equations of linear couple stress theory of elasticity in a special curvilinear system of coordinates. In the same way, the system of differential equations of motion in term of displacements and rotations has been developed for all the cases that have been considered here. The equations for couple stress theory of the straight beam can be derived from the equations presented here as special case. The obtained equations can be used to calculate the stress-strain as well as to model thin structures in macro, micro and nano scales by taking into account couple stress and rotation effects. In particular, the proposed models can be efficient in modeling, as well as in computer simulation of the MEMS and NEMS.

Analysis of the systems of partial differential equations (54), (55) for first order theory, (66), (67) for Timoshenko's theory and (81), (82) for Euler-Bernoulli theory show that all of them are coupled and related longitudinal, flexural and rotational deformation modes. The first order approximation theory is more complete and all quantities are approximated by linear functions. The theory based on Timoshenko's hypothesis is less accurate, but id it simpler and take into account shear deformation, which is important for dynamic analysis. The theory based on Euler-Bernoulli hypothesis is less accurate compared with the previous ones, but it is the simplest one and couple all the deformation modes. It is free of inconsistencies and can be used for modeling and analysis of the curved rod with considering micropolar couple stress effects.

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