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Nonlocal theory of curved rods. 2-D, high order, Timoshenko's and Euler-Bernoulli models

DOI 10.1515/cls-2017-0015

Received Apr 17, 2017; accepted Jun 05, 2017

Abstract: New models for plane curved rods based on linear nonlocal theory of elasticity have been developed. The 2-D theory is developed from general 2-D equations of linear nonlocal elasticity using a special curvilinear system of coordinates related to the middle line of the rod along with special hypothesis based on assumptions that take into account the fact that the rod is thin. High order theory is based on the expansion of the equations of the theory of elasticity into Fourier series in terms of Legendre polynomials. First, stress and strain tensors, vectors of displacements and body forces have been expanded into Fourier series in terms of Legendre polynomials with respect to a thickness coordinate. Thereby, all equations of elasticity including nonlocal constitutive relations have been transformed to the corresponding equations for Fourier coefficients. Then, in the same way as in the theory of local elasticity, a system of differential equations in terms of displacements for Fourier coefficients has been obtained. First and second order approximations have been considered in detail. Timoshenko's and Euler-Bernoulli theories are based on the classical hypothesis and the 2-D equations of linear nonlocal theory of elasticity which are considered in a special curvilinear system of coordinates related to the middle line of the rod. The obtained equations can be used to calculate stress-strain and to model thin walled structures in micro- and nanoscales when taking into account size dependent and nonlocal effects.

Keywords: Curved rod, nonlocal elasticity, Legendre polynomial, high order theory, Timoshenko's theory, Euler-Bernoulli theory

1 Introduction

Micro and nano technology are a subject of current scientific and technological interest and development [1, 2]. The inventions of carbon nanotubes (CNTs) and the successful extraction of graphene sheets have motivated this interest. Compared to more conventional materials, these nanomaterials possess superior mechanical, thermal, electrical and electronic properties. The understanding of the mechanical response of nanoscale structures, such as bending, vibration, buckling and wave propagation, is indispensable for the development and accurate design of MEMS and NEMS, which use nanostructures such as CNTs and graphene sheets [2–10] as constituent elements.

Mathematical modeling and computer simulation of nanostructures such as CNTs, graphene and the MEMS and NEMS are important for an optimum design. The study of nanostructures can be done by experimental methods as well as by theoretical modeling. There are at least three [5, 6] main approaches for theoretically modeling of the nanomaterials: 1. atomistic modeling, 2. hybrid atomistic-continuum mechanics, 3. continuum mechanics approach.

Continuum theories describe a system in terms of a few variables such as mass, temperature, voltage, stress, deformation, etc., which are highly suited for direct measurements of these variables. Their successes, expediency, and practicality, have been demonstrated and tested throughout the history of science through explaining and predicting diverse physical phenomena. The classical linear theory of elasticity is the most popular and usable in engineering and scientific applications. It is based on the assumption that internal interactions between neighboring elements of an elastic continuum occur locally only by means of the symmetric force-based stress tensor, deformations are determined by symmetric tensor of deformation and motion of material particles are described by a position vector. However the classical theory of elasticity fails to produce acceptable results for materials and structures at nanoscale, because it cannot handle scale effects which are observed in numerous experimental studies [6, 11].

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To explain fundamental properties and behavior of material and structures at nanoscale, classical continuum models have been improved and further developed and new mathematical models have been created. The micro-continuum and nonlocal models have been developed in order to model microscopic motions of the material particles and long range material interaction. The micro-continuum (micropolar and couple stress) models have been considered in our previous publications [12, 13]. The nonlocal theory will be considered here and applied to curved rods modeling.

The classical continuum theories are based on constitutive relations which assume that the stress at a point is a function of strain at that point. On the other hand, the nonlocal continuum mechanics assumes that the stress at a point is a function of strains at all points in the continuum. These properties are introduced into the constitutive equations as material parameters in form of integral over the whole area. The nonlocal elasticity has been mostly developed in the papers Eringen's [14–16, 18]. The nonlocal theory of elasticity has been used to study lattice dispersion of elastic waves, wave propagation in composites, dislocation and fracture mechanics, etc. Precise mathematical definitions of strong and weak nonlocality were given in [11]. In [16] Eringen proposed a differential model of the nonlocal elasticity which, due to its simplicity is very popular and widely used to study bending, buckling, vibration and wave propagation, modeling size-effects in micro and nano structures, such as CNTs and graphene sheets and modeling and simulation MEMS and NEMS.

The classical Euler-Bernoulli beam theory equipped with the Eringen's nonlocal elasticity constitutive equation has been developed in [19, 20] to study nonlocal effects in bending and buckling of beams, respectively. A huge amount of publications has appeared since that time, where there has been development of new nonlocal models of beams, plates and shells for mathematical modeling and computer simulation of the materials and structures at nanoscale. In relation to the above we have to mention books [3–6] and review papers [21–24]. A variational approach was developed and applied to the nonlocal theory of elasticity in [25], CNTs in [26, 27], beams in [28] and cylindrical shells in [29] respectively. In [7, 27, 30–37] the Euler-Bernoulli, in Timoshenko's [38–42] and in high order [28, 36, 38, 39, 43] beam theories were further developed and applied to the modeling of bending, buckling and vibration of single-walled and multi-walled CNTs and to study of the size and nonlocal effects in nanostructures. Cylindrical shell models have been used in [29, 30, 44–47]. Wave propagation and dynamical nonlocal effects in

single-walled and multi-walled CNTs were studied in [5, 24, 43, 48–50].

There are several approaches to the development of the theories of thin-walled structures. One consists on the improvement of the classical physical hypothesis and the development of theories that are more accurate. Another approach consists in the expansion of the stress-strain field components into polynomials series in terms of thickness. In [51–53] the Legendre's polynomials have been used for the development of new high order theories of plates and shells. Such an approach has significant advantages since Legendre's polynomials are orthogonal [54, 55] and as result the developed equations are simple. For more information look over the extended review [56].

In our previous publications [12, 13, 57–66] the approach based on the use of Legendre's polynomials series expansion has been applied to the development of high order models of shells, plates and rods. Thermoelastic contact problems of plates and shells when mechanical and thermal conditions are changed during deformation have been considered in [57, 58]. Then, the proposed approach and methodology were further developed and extended to thermoelasticity of the laminated composite materials with the possibility of delamination along with mechanical and thermal contact in the temperature field in [59], the pencil-thin nuclear fuel rods modeling in [60], the functionally graded shells in [61, 62], modeling of MEMS and NEMS in [63, 64] as well as micropolar curved elastic rods in [12], couple stress elastic rods in [13]. An analysis and comparison with the classical theory of elastic and thermoelastic plates and shells has been done in [65, 66].

In this paper, 2-D, high order, Timoshenko's and Euler-Bernoulli models of curved rods based on the nonlocal theory of elasticity have been developed. In the 2-D model a special curvilinear system of coordinates related to the middle line of the rod and special hypothesis based on assumptions that take into account the fact that the rod is thin have been used. High order model is based on the expansion of the equations of the 2-D nonlocal theory of elasticity into Fourier series in terms of Legendre polynomials. First order and second order theories are considered in details. The Timoshenko's and Euler-Bernoulli models are based on the classical hypothesis and 2-D equations of nonlocal theory of elasticity in a special curvilinear system. The obtained equations can be used to calculate stress-strain and to model thin walled structures in micro- and nano-scales by taking into account nonlocal effects. The proposed models can be efficient in MEMS and NEMS modeling as well as in their computer simulation.

2 2-D nonlocal elasticity in orthogonal coordinates

Let's consider the size dependent material continuum theory and use it to develop an approach based on the expansion of the equations of nonlocal theory of elasticity into Fourier series in terms of Legendre polynomials and apply it to create high order curved rod theories. Thus, we consider a curved elastic rod in a 2-D Euclidian space, which occupies the domain $V = \Omega \times [-h, h]$ with a smooth boundary ∂V . Here $2h$ is thickness, $\Omega = [-L, L]$ is the middle line of the rod and $2L$ is its length. The boundary of the rod ∂V can be presented in the form $\partial V = S \cup \Omega^+ \cup \Omega^-$, where Ω^+ and Ω^- are the upper and lower sides and S denotes lateral sides.

In the nonlocal theory of elasticity the internal forces (the interaction between adjacent elements) are defined in terms of a stress tensor $\boldsymbol{\sigma}(\mathbf{x}, t)$, which depend on the strain field not only in vicinity of the point \mathbf{x} as in classical theory of elasticity, but also on the strain field at every point of the body. The displacement field is fully described by the displacements $\mathbf{u}(\mathbf{x}, t)$ vector and the symmetric strain $\boldsymbol{\varepsilon}(\mathbf{x}, t)$ tensor.

Here we consider a plane thin curved rod and assume all functions that define the stress-strain state are independent of coordinate x_3 and correspond to the so-called plane stress state. In this case, the position of any point is defined by coordinates $\mathbf{x} = (x_1, x_2)$, stress and strain tensors reduced to $\sigma_{\alpha\beta}(\mathbf{x}, t)$, $\varepsilon_{\alpha\beta}(\mathbf{x}, t)$ and $\varepsilon_{33}(\mathbf{x}, t)$ respectively and displacements vector to $u_\alpha(\mathbf{x}, t)$. Greece indices are equal to 1, 2.

Taking into account that the nonlocal theory of curved rods will be studied here, the curvilinear orthogonal system of coordinates will be used. In the orthogonal system of coordinates $\mathbf{x} = (x_1, x_2)$, the position of an arbitrary point is defined by the radius vector $\mathbf{R}(\mathbf{x}) = \mathbf{e}_\alpha x_\alpha$. Here, the unit orthogonal basic vectors and their derivatives with respect to the coordinates are equal to

$$\mathbf{e}_\alpha = \frac{1}{H_\alpha} \frac{\partial \mathbf{R}}{\partial x_\alpha}, \quad \frac{\partial \mathbf{e}_\alpha}{\partial x_\beta} = \Gamma_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \quad \alpha, \beta = 1, 2 \quad (1)$$

where H_α are Lamè coefficients and $\Gamma_{\alpha\beta}^\gamma$ are Christoffel symbols. They are calculated by the equations

$$H_\alpha = \left| \frac{\partial \mathbf{R}}{\partial x_\alpha} \right| = \sqrt{\frac{\partial \mathbf{R}}{\partial x_\alpha} \cdot \frac{\partial \mathbf{R}}{\partial x_\alpha}} \quad (2)$$

$$\Gamma_{\alpha\beta}^\gamma = -\frac{1}{H_\alpha} \frac{\partial H_\alpha}{\partial x_\beta} \delta_{\alpha\gamma} + \frac{1}{2H_\alpha H_\gamma} \left(\delta_{\beta\gamma} \frac{\partial H_\beta H_\gamma}{\partial x_\alpha} + \delta_{\alpha\gamma} \frac{\partial H_\alpha H_\gamma}{\partial x_\beta} - \delta_{\alpha\beta} \frac{\partial H_\alpha H_\beta}{\partial x_\gamma} \right) \quad (3)$$

Here $\delta_{\alpha\beta}$ is an asymmetric tensor, called Kronecker delta.

From the last equation it follows that for indices $\alpha \neq \beta \neq \gamma$, $\alpha = \beta = \gamma$ and $\alpha = \beta \neq \gamma$ the Christoffel symbols are $\Gamma_{\alpha\beta}^\gamma = 0$. The nonzero Christoffel symbols are

$$\Gamma_{\alpha\alpha}^\gamma = -\frac{1}{H_\gamma} \frac{\partial H_\alpha}{\partial x_\gamma}, \quad \Gamma_{\alpha\gamma}^\gamma = \frac{1}{H_\alpha} \frac{\partial H_\gamma}{\partial x_\alpha} \text{ for } \alpha \neq \gamma \quad (4)$$

More specifically nonzero Christoffel symbols are

$$\Gamma_{22}^1 = -\frac{1}{H_1} \frac{\partial H_2}{\partial x_1}, \quad \Gamma_{11}^2 = -\frac{1}{H_2} \frac{\partial H_1}{\partial x_2}, \quad (5)$$

$$\Gamma_{21}^1 = \frac{1}{H_2} \frac{\partial H_1}{\partial x_2}, \quad \Gamma_{12}^2 = \frac{1}{H_1} \frac{\partial H_2}{\partial x_1}$$

The differential equations of motion can be presented in the form

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \rho \ddot{\mathbf{u}}, \quad (6)$$

Here \mathbf{b} is a vector of body forces, ρ is a density of material, $\ddot{\mathbf{u}}$ is the acceleration vector.

Divergence of the stress tensor in the curvilinear orthogonal system of coordinates has the form

$$\nabla \cdot \boldsymbol{\sigma} = \left(\frac{1}{H_\alpha} \frac{\partial \sigma_{\alpha\beta}}{\partial x_\alpha} + \frac{\sigma_{\alpha\beta}}{H_\gamma} \Gamma_{\alpha\gamma}^\gamma + \frac{\sigma_{\alpha\gamma}}{H_\alpha} \Gamma_{\gamma\alpha}^\beta \right) \mathbf{e}_\beta \quad (7)$$

The kinematic relations have the form

$$\boldsymbol{\varepsilon} = \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \quad (8)$$

Here, the gradient of the displacements vector is

$$\nabla \mathbf{u} = \mathbf{e}_\beta \frac{1}{H_\beta} \frac{\partial u_\alpha \mathbf{e}_\alpha}{\partial x_\beta} = \mathbf{e}_\beta \mathbf{e}_\alpha \left(\frac{1}{H_\beta} \frac{\partial u_\alpha}{\partial x_\beta} + \frac{u_\gamma}{H_\beta} \Gamma_{\gamma\beta}^\alpha \right) \quad (9)$$

According to Eringen's nonlocal elasticity theory [14-18] the stress at a point \mathbf{x} in a body is functional of the strain field at every point of the body. Thus, the nonlocal stress tensor $\boldsymbol{\sigma}$ at point \mathbf{x} can be expressed by,

$$\boldsymbol{\sigma}(\mathbf{x}) = \int_V H(|\mathbf{x} - \mathbf{x}'|, \tau) \boldsymbol{\sigma}^c(\mathbf{x}') d\mathbf{x}', \quad (10)$$

$$\tau = e_0 a / l_e, \quad \mathbf{x} \in V$$

$$\boldsymbol{\sigma}^c = \bar{\lambda} (\text{tr} \boldsymbol{\varepsilon}) \mathbf{I} \delta_{ij} + 2\mu \boldsymbol{\varepsilon} \quad (11)$$

where $\bar{\lambda} = \lambda \frac{2\mu}{\lambda + 2\mu}$, λ and μ are Lamè constants of classical elasticity, $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^c$ and $\boldsymbol{\varepsilon}$ are non-local, classical stress, and strain tensors, respectively.

The kernel function $H(|\mathbf{x} - \mathbf{x}'|, \tau)$ is the nonlocal modulus incorporating into the constitutive equations the nonlocal effects at the reference point \mathbf{x} , produced by local strain at the source points \mathbf{x}' . The term $|\mathbf{x} - \mathbf{x}'|$ is the Euclidean distance and τ is a material constant that depends on internal e_0 (e.g. lattice parameter, granular size

and distance between the C-C bonds, etc.) and external l_e (e.g. crack length and wave length, etc.) characteristic lengths.

The nonlocal modulus has to satisfy the following properties.

1. It reaches maximum at $\mathbf{x} = \mathbf{x}'$ attenuating with $|\mathbf{x} - \mathbf{x}'|$.
2. When $\tau \rightarrow 0$, $H|\mathbf{x} - \mathbf{x}'|$ must revert to the Dirac delta function $\delta|\mathbf{x} - \mathbf{x}'|$ in order for a classical elasticity limit to be satisfied.

For the given material, the nonlocal modulus H can be determined by matching the dispersion curves of plain waves with those of atomic lattice dynamics or experiments. For the 2-D case nonlocal modulus is proposed by Eringen [18] in the form

$$H(|\mathbf{x}|, \tau) = \frac{1}{2\pi l_e^2 \tau^2} K_0 \left(\frac{\sqrt{\mathbf{x} \cdot \mathbf{x}}}{\tau l_e} \right) \quad (12)$$

where K_0 is the modified Bessel function. From equation (12), it can be seen that the integral of the equation over the domain yields unity. The nonlocal modulus function of equation (12) is frequently used for the analysis of materials and structures at nanoscale.

The nonlocal constitutive relations in the form (10) lead to the governing equations of nonlocal elasticity in the form of integro-differential equations, which are generally difficult to solve. To simplify the situation Eringen [16, 18] assumed that non-local modulus $H(|\mathbf{x} - \mathbf{x}'|, \tau)$ is a Green's function of the linear differential operator

$$L_0[H(|\mathbf{x} - \mathbf{x}'|, \tau)] = \delta(|\mathbf{x} - \mathbf{x}'|) \quad (13)$$

Green's function is chosen in conjunction with the properties of the non-local modulus. By applying $L[\bullet]$ to equation (10), we obtain:

$$L_0[\boldsymbol{\sigma}] = \boldsymbol{\sigma}^c \quad (14)$$

The differential operator $L_0[\bullet]$ has different forms for different expressions of the nonlocal modulus. In [20, 22] it was shown that the nonlocal modulus with kernel (12) linear differential operator $L_0[\bullet]$ has the form

$$L_0[\bullet] = (1 - \tau^2 l_e^2 \nabla^2)[\bullet] \quad (15)$$

where ∇^2 is the Laplace operator.

Therefore, according to equations (14) and (15), the nonlocal constitutive relation (10) can be expressed in a differential form as:

$$(1 - \tau^2 l_e^2 \nabla^2)\boldsymbol{\sigma} = \bar{\lambda}(\text{tr}\boldsymbol{\epsilon})\mathbf{I}\delta_{ij} + 2\mu\boldsymbol{\epsilon} \quad (16)$$

The differential equations of motion in form of displacements can be obtained by substituting kinematic relations (8) into the nonlocal constitutive relation (16) and the obtained result into the equations of motion (6). In vector form, they can be represented as the following

$$\mu \nabla^2 \mathbf{u} + (\bar{\lambda} + \mu) \nabla(\nabla \cdot \mathbf{u}) + \tilde{\mathbf{b}} = \rho \ddot{\mathbf{u}} - \rho \tau^2 l_e^2 \nabla^2 \ddot{\mathbf{u}} \quad (17)$$

where

$$\tilde{\mathbf{b}} = (1 - \tau^2 l_e^2 \nabla^2) \mathbf{b} \quad (18)$$

The expressions for differential operators in the orthogonal system of coordinates presented here can be found in [12].

The complete system of nonlocal equations of elasticity in the curvilinear orthogonal system of coordinates is considered in detail here. In the next sections to follow, it will be simplified and applied for the development of the approximate theories for plane curved rods.

3 2-D nonlocal elasticity in coordinates related to the middle line of the rod

For convenience, we will introduce curvilinear coordinates related to the middle line of the rod. In this case, coordinate x_1 is associated with the main curvature k_1 of the middle line of the rod and coordinate x_2 is perpendicular to it. The position vector $\mathbf{R}(\mathbf{x})$ of any point in domain V , occupied by material points of the rod may be presented as

$$\mathbf{R}(\mathbf{x}) = \mathbf{r}(x_1) + x_2 \mathbf{n}(x_1) \quad (19)$$

where $\mathbf{r}(x_1)$ is the position vector of the points located on the middle line of the rod, and $\mathbf{n}(x_1)$ is a unit vector normal to the middle line of the rod.

In this case the 2-D equations of nonlocal elasticity can be simplified by taking into account that Lamé coefficients and their derivatives have the simpler form

$$H_1(x_1, x_2) = A_1(x_1)(1 + k_1 x_2), \quad H_2 = 1, \quad (20)$$

$$\frac{\partial H_1}{\partial x_1} = \frac{\partial A_1}{\partial x_1}(1 + k_1 x_2), \quad \frac{\partial H_1}{\partial x_2} = k_1 A_1, \quad \frac{\partial H_2}{\partial x_\alpha} = 0$$

where $A_1(x_1) = \frac{\partial \mathbf{r}(x_1)}{\partial x_1}$ is the coefficient of the first quadratic form of the middle line.

By taking into account that we have considered relatively thin rods, the following assumptions can be made

$$1 + k_1 x_2 \approx 1 \rightarrow H_1 \approx A_1, \quad \frac{1}{H_2} \frac{\partial H_1}{\partial x_2} = k_1 A_1, \quad (21)$$

Therefore Christoffel symbols (5) become

$$\Gamma_{22}^1 = 0, \quad \Gamma_{11}^2 = -k_1 A_1, \quad \Gamma_{21}^1 = k_1 A_1, \quad \Gamma_{12}^2 = 0 \quad (22)$$

After substituting the simplified Lamé coefficients (20) and the Christoffel symbols (22) into equations of motion (6) they are simplified and the divergence of the stress tensor in the curvilinear orthogonal system of coordinates related to middle line of the rod becomes of the form

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma} &= \left(\frac{1}{A_1} \frac{\partial \sigma_{\beta 1}}{\partial x_1} + \frac{\partial \sigma_{\beta 2}}{\partial x_2} + k_1 \tilde{\sigma}_\beta \right) \mathbf{e}_\beta, \quad (23) \\ \tilde{\sigma}_1 &= \sigma_{21} + \sigma_{12}, \quad \tilde{\sigma}_2 = \sigma_{22} - \sigma_{11} \end{aligned}$$

In the same way, kinematic relations are simplified and are converted to the form

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} + k_1 u_2, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad (24) \\ \varepsilon_{12} &= \frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1 \end{aligned}$$

The nonlocal constitutive relations (16) may be presented in the form

$$\sigma_{\alpha\beta} - \tau^2 l_e^2 \nabla^2 \sigma_{\alpha\beta} = \lambda \varepsilon_{\gamma\gamma} \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta} \quad (25)$$

The operator of Laplace in curvilinear coordinates related to the middle line of the rod has the form

$$\nabla^2 = \frac{1}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial}{\partial x_1} \right) + \frac{\partial^2}{\partial x_2^2} + k_1 \frac{\partial}{\partial x_2} \quad (26)$$

Finally, the differential equations of motion in the form of displacements (17) can be rewritten in a more convenient form for our study, by using expressions for differential operators in the orthogonal system of coordinates related to middle line of the rod

$$\mathbf{L}_u \cdot \mathbf{u} + \tilde{\mathbf{b}} = \rho \ddot{\mathbf{u}} - \rho \tau^2 l_e^2 \nabla^2 \ddot{\mathbf{u}} \quad (27)$$

where \mathbf{L}_u is a matrix differential operator of the form

$$\begin{aligned} \mathbf{L}_u &= \begin{pmatrix} \mu \left(\frac{1}{A_1^2} \frac{\partial^2}{\partial x_1^2} + k_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_2^2} \right) + (\lambda + \mu) \frac{1}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial}{\partial x_1} \right) & \dots \\ (\lambda + \mu) \frac{\partial}{\partial x_2} \left(\frac{1}{A_1} \frac{\partial}{\partial x_1} \right) & \dots \end{pmatrix} \quad (28) \\ &\quad \mu \left(\frac{1}{A_1^2} \frac{\partial^2}{\partial x_1^2} + k_1 \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_2^2} \right) + (\lambda + \mu) \frac{\partial}{\partial x_2} \left(k_1 A_1 + A_1 \frac{\partial}{\partial x_2} \right) \end{aligned}$$

In this section, the system of 2-D linear nonlocal theory of elasticity in a special system of coordinates related to middle line of the rod is considered in detail. These equations will be used for the development of the approximate 1-D theories of the curved rods.

4 1-D formulation of the problem

In order to reduce the 2-D problem for the nonlocal theory of elastic curved rods to a 1-D one, we expand the functions that describe the stress-strain state of the rod into the Legendre polynomials [54, 55] series along the coordinate x_2 and present them in the form

$$\begin{aligned} u_\alpha(x_1, x_2) &= \sum_{k=0}^{\infty} u_\alpha^k(x_1) P_k(\varpi), \quad u_\alpha^k(x_1) \quad (29) \\ &= \frac{2k+1}{2h} \int_{-h}^h u_\alpha(x_1, x_2) P_k(\varpi) dx_2, \\ \sigma_{\alpha\beta}(x_1, x_2) &= \sum_{k=0}^{\infty} \sigma_{\alpha\beta}^k(x_1) P_k(\varpi), \quad \sigma_{\alpha\beta}^k(x_1) \\ &= \frac{2k+1}{2h} \int_{-h}^h \sigma_{\alpha\beta}(x_1, x_2) P_k(\varpi) dx_2, \\ \varepsilon_{\alpha\beta}(x_1, x_2) &= \sum_{k=0}^{\infty} \varepsilon_{\alpha\beta}^k(x_1) P_k(\varpi), \quad \varepsilon_{\alpha\beta}^k(x_1) \\ &= \frac{2k+1}{2h} \int_{-h}^h \varepsilon_{\alpha\beta}(x_1, x_2) P_k(\varpi) dx_2, \end{aligned}$$

where $\varpi = x_2/h \in [-1, 1]$ is a normalized variable.

In general, all of the functions that are considered here also depend on time t , but to reduce typing the variable of time has been omitted.

For derivatives of the considered coefficients of the Legendre polynomial expansions with respect to x_1 the following relations take place

$$\begin{aligned} \frac{\partial u_\alpha^k(x_1)}{\partial x_1} &= \frac{2k+1}{2h} \int_{-h}^h \frac{\partial u_\alpha(x_1, x_2)}{\partial x_1} P_k(\varpi) dx_2, \quad (30) \\ \frac{\partial \sigma_{\alpha\beta}^k(x_1)}{\partial x_1} &= \frac{2k+1}{2h} \int_{-h}^h \frac{\partial \sigma_{\alpha\beta}(x_1, x_2)}{\partial x_1} P_k(\varpi) dx_2, \\ \frac{\partial^2 \sigma_{\alpha\beta}^k(x_1)}{\partial x_1^2} &= \frac{2k+1}{2h} \int_{-h}^h \frac{\partial^2 \sigma_{\alpha\beta}(x_1, x_2)}{\partial x_1^2} P_k(\varpi) dx_2, \end{aligned}$$

Derivatives of the components of the displacement vector with respect to x_2 can be represented in the form [54, 55]

$$\begin{aligned} \frac{\partial u_\alpha(x_1, x_2)}{\partial x_2} &= \sum_{k=0}^{\infty} u_\alpha^k(x_1) \frac{\partial P_k(\varpi)}{\partial x_2} P_k(\varpi) \quad (31) \\ &= \sum_{k=0}^{\infty} \frac{2k+1}{h} \left(u_\alpha^{k+1}(x_1) + u_\alpha^{k+3}(x_1) + \dots \right) P_k(\varpi) \end{aligned}$$

Representation for the derivative of the Legendre polynomial [54, 55] has been used here.

Then, coefficients of the Legendre's polynomials expansion can be represented in the form

$$\begin{aligned} \frac{\partial u_\alpha^k}{\partial x_2}(x_1) &= \frac{2k+1}{2h} \int_{-h}^h \frac{\partial u_\alpha(x_1, x_2)}{\partial x_2} P_k(\varpi) dx_2 \\ &= \underline{u}_\alpha^k(x_1), \end{aligned} \quad (32)$$

where

$$\underline{u}_\alpha^k(x_1) = \frac{2k+1}{h} \left(u_\alpha^{k+1}(x_1) + u_\alpha^{k+3}(x_1) + \dots \right), \quad (33)$$

In the same way, derivatives of the stress tensor components with respect to x_2 can be represented in the form

$$\begin{aligned} \frac{\partial \sigma_{\alpha\beta}(x_1, x_2)}{\partial x_2} &= \sum_{k=0}^{\infty} \sigma_{\alpha\beta}^k(x_1) \frac{\partial P_k(\omega)}{\partial x_2} \\ &= \sum_{k=0}^{\infty} \underline{\sigma}_{\alpha\beta}^k(x_1) P_k(\omega) \\ \frac{\partial^2 \sigma_{\alpha\beta}(x_1, x_2)}{\partial x_2^2} &= \sum_{k=0}^{\infty} \underline{\sigma}_{\alpha\beta}^k(x_1) \frac{\partial^2 P_k(\omega)}{\partial x_2^2} = \sum_{k=0}^{\infty} \underline{\underline{\sigma}}_{\alpha\beta}^k(x_1) P_k(\omega) \end{aligned} \quad (34)$$

where

$$\begin{aligned} \underline{\sigma}_{\alpha\beta}^k(x_1) &= \frac{2k+1}{h} \left(\sigma_{\alpha\beta}^{k+1}(x_1) + \sigma_{\alpha\beta}^{k+3}(x_1) + \dots \right), \\ \underline{\underline{\sigma}}_{\alpha\beta}^k(x_1) &= \frac{2k+1}{h} \left(\underline{\sigma}_{\alpha\beta}^{k+1}(x_1) + \underline{\sigma}_{\alpha\beta}^{k+3}(x_1) + \dots \right), \end{aligned} \quad (35)$$

Now, we can introduce the analogy of the Laplace operator, which acts on the coefficients of the Legendre's polynomials expansion in the following way

$$\underline{\underline{\nabla}}^2 \bullet^k = \nabla^2 \bullet^k + k_1 \underline{\bullet}^k + \underline{\bullet}^k \quad (36)$$

where $\nabla^2 \bullet = \frac{1}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial \bullet}{\partial x_1} \right)$ is 1-D Laplace operator in the orthogonal system of coordinates related to middle line of the rod.

For example, the application of the operator (36) to the coefficients of the Legendre's polynomials expansion of the stress tensor components gives us

$$\underline{\underline{\nabla}}^2 \sigma_{\alpha\beta}^k(x_1) = \nabla^2 \sigma_{\alpha\beta}^k(x_1) + k_1 \underline{\sigma}_{\alpha\beta}^k(x_1) + \underline{\underline{\sigma}}_{\alpha\beta}^k(x_1) \quad (37)$$

Multiplying the equations of motion (23) by $P_k(\varpi)$ and integrating with respect to from $-h$ to h as well as taking into account equations (29) - (30) and representation

$$\begin{aligned} \frac{2k+1}{2h} \int_{-h}^h \frac{\partial \sigma_{\alpha 2}}{\partial x_2} P_k(\varpi) dx_2 \\ = \frac{2k+1}{h} \left[\sigma_{\alpha 2}^+ - (-1)^k \sigma_{\alpha 2}^- \right] - \underline{\sigma}_{\alpha 2}^k \end{aligned} \quad (38)$$

we obtain 1-D equations of motion in the form

$$\underline{\underline{\nabla}} \cdot \underline{\sigma}^k + \underline{\underline{\mathbf{b}}}^k = \rho \underline{\underline{\mathbf{u}}}^k, \quad (39)$$

Here we introduce the analogy of the Hamilton operator which acts on coefficients of the Legendre's polynomials expansion in the following way

$$\underline{\underline{\nabla}} \cdot \underline{\sigma}^k = \left(\frac{1}{A_1} \frac{\partial \sigma_{\beta 1}^k}{\partial x_1} + k_1 \underline{\sigma}_{\beta}^k - \underline{\sigma}_{\beta 2}^k \right) \mathbf{e}_\beta, \quad (40)$$

where

$$\begin{aligned} \underline{\underline{\mathbf{b}}}_\alpha^k &= \mathbf{b}_\alpha^k + \frac{2k+1}{h} \left(\sigma_{\alpha 2}^+ - (-1)^k \sigma_{\alpha 2}^- \right), \\ \underline{\underline{\sigma}}_{2\alpha}^k(x_1) &= \frac{2k+1}{h} \left(\sigma_{2\alpha}^{k-1}(x_1) + \sigma_{2\alpha}^{k-3}(x_1) + \dots \right) \end{aligned} \quad (41)$$

In the same way by considering equations (29) - (33) the 2-D kinematic relations (24) can be transformed in the 1-D form

$$\begin{aligned} \varepsilon_{11}^k &= \frac{1}{A_1} \frac{\partial u_1^k}{\partial x_1} + k_1 u_2^k, \quad \varepsilon_{22}^k = \underline{u}_2^k, \\ 2\varepsilon_{12}^k &= \underline{u}_1^k + \frac{1}{A_1} \frac{\partial u_2^k}{\partial x_1} - k_1 u_1^k, \end{aligned} \quad (42)$$

Constitutive relations of the nonlocal theory of elasticity (25) can also be rewritten for to the coefficients of the Legendre's polynomials expansion and become

$$\sigma_{\alpha\beta}^k - \tau^2 l_e^2 \underline{\underline{\nabla}}^2 \sigma_{\alpha\beta}^k = \lambda \varepsilon_{\gamma\gamma}^k \delta_{\alpha\beta} + 2\mu \varepsilon_{\alpha\beta}^k \quad (43)$$

The substitution of the kinematic relations (42) into the constitutive relations (43) gives us the following equations

$$\begin{aligned} \sigma_{11}^k - \tau^2 l_e^2 \underline{\underline{\nabla}}^2 \sigma_{11}^k &= (\lambda + 2\mu) \left(\frac{1}{A_1} \frac{\partial u_1^k}{\partial x_1} + k_1 u_2^k \right) \\ &\quad + \lambda \underline{u}_2^k, \\ \sigma_{22}^k - \tau^2 l_e^2 \underline{\underline{\nabla}}^2 \sigma_{22}^k &= (\lambda + 2\mu) \underline{u}_2^k + \lambda \left(\frac{1}{A_1} \frac{\partial u_1^k}{\partial x_1} + k_1 u_2^k \right) \\ \sigma_{12}^k - \tau^2 l_e^2 \underline{\underline{\nabla}}^2 \sigma_{12}^k &= \mu \left(\underline{u}_1^k + \frac{1}{A_1} \frac{\partial u_2^k}{\partial x_1} - k_1 u_1^k \right), \end{aligned} \quad (44)$$

Then, the differential equations of motion for the nonlocal theory of elasticity in the form of displacements (27) are transformed into their 1-D form

$$\begin{aligned} \frac{(\lambda + 2\mu)}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial u_1^k}{\partial x_1} + k_1 u_2^k \right) \\ + \frac{\lambda}{A_1} \frac{\partial}{\partial x_1} \underline{u}_2^k + 2\mu k_1 \left(\frac{1}{A_1} \frac{\partial u_2^k}{\partial x_1} - k_1 u_1^k + \underline{u}_1^k \right) - \underline{\sigma}_{12}^k \\ + \frac{\tau^2 l_e^2}{A_1} \nabla^2 \frac{\partial^2 u_1^k}{\partial t^2} + \underline{\underline{\mathbf{b}}}_1^k = \rho \frac{\partial^2 u_1^k}{\partial t^2} - \tau^2 l_e^2 \left(k_1 \frac{\partial^2 u_1^k}{\partial t^2} + \frac{\partial^2 \underline{\underline{u}}_1^k}{\partial t^2} \right), \\ \frac{\mu}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial u_2^k}{\partial x_1} - k_1 u_1^k + \underline{u}_1^k \right) \\ - 2\mu k_1 \left(\frac{1}{A_1} \frac{\partial u_1^k}{\partial x_1} + k_1 u_2^k \right) + 2\mu k_1 \underline{u}_2^k - \underline{\sigma}_{22}^k \end{aligned} \quad (45)$$

$$+ \frac{\tau^2 l_e^2}{A_1} \nabla^2 \frac{\partial^2 u_2^k}{\partial t^2} + \tilde{b}_2^k = \rho \frac{\partial^2 u_2^k}{\partial t^2} - \tau^2 l_e^2 \left(k_1 \frac{\partial^2 u_2^k}{\partial^2 t} + \frac{\partial^2 u_2^k}{\partial^2 t} \right),$$

Now, instead of the 2-D system of the differential equations in displacements (27) we have an infinite system of 1-D differential equations for coefficients of the Legendre's polynomial series expansion (45). In order to simplify the problem an approximate theory has to be developed, where only a finite number of members have to be taken into account in the expansion of (29) as well as in all of the above relations. For example, if we consider the n -order approximate shell theory, only $n + 1$ members in the expansion (29) are taken into account

$$\begin{aligned} u_\alpha(x_1, x_2) &= \sum_{k=0}^n u_\alpha^k(x_1) P_k(\varpi), \\ \sigma_{\alpha\beta}(x_1, x_2) &= \sum_{k=0}^n \sigma_{\alpha\beta}^k(x_1) P_k(\varpi), \\ \varepsilon_{\alpha\beta}(x_1, x_2) &= \sum_{k=0}^n \varepsilon_{\alpha\beta}^k(x_1) P_k(\varpi), \end{aligned} \quad (46)$$

In this case we consider that, $\omega_3^k = 0$, $\sigma_{\alpha\beta}^k = 0$, $\mu_{\alpha 3}^k = 0$, and $\theta^k = 0$ for $k < 0$ and for $k > n$.

Then the 1-D differential equations of motion for the nonlocal theory of elasticity in displacements (45) can be presented in the matrix form

$$\mathbf{L}u \cdot \mathbf{u} + \tilde{\mathbf{b}} = \rho \ddot{\mathbf{u}} - \rho \tau^2 l_e^2 \left(\nabla^2 \ddot{\mathbf{u}} + k_1 \ddot{\mathbf{u}} + \ddot{\mathbf{u}} \right) \quad (47)$$

where

$$\mathbf{L}u = \begin{pmatrix} L_{11}^{00} & L_{12}^{00} & \cdots & L_{11}^{0n} & L_{12}^{0n} \\ L_{21}^{00} & L_{22}^{00} & \cdots & L_{21}^{0n} & L_{22}^{0n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ L_{11}^{n0} & L_{12}^{n0} & \cdots & L_{11}^{nn} & L_{12}^{nn} \\ L_{21}^{n0} & L_{22}^{n0} & \cdots & L_{21}^{nn} & L_{22}^{nn} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_1^n \\ u_2^n \end{pmatrix}, \quad (48)$$

$$\tilde{\mathbf{b}} = \begin{pmatrix} \tilde{b}_1^0 \\ \tilde{b}_2^0 \\ \vdots \\ \tilde{b}_1^n \\ \tilde{b}_2^n \end{pmatrix}, \quad \ddot{\mathbf{u}} = \begin{pmatrix} \ddot{u}_1^0 \\ \ddot{u}_2^0 \\ \vdots \\ \ddot{u}_1^n \\ \ddot{u}_2^n \end{pmatrix}, \quad \underline{\ddot{\mathbf{u}}} = \begin{pmatrix} \underline{\ddot{u}}_1^0 \\ \underline{\ddot{u}}_2^0 \\ \vdots \\ \underline{\ddot{u}}_1^n \\ \underline{\ddot{u}}_2^n \end{pmatrix}$$

The order of the system of differential equations depends on the assumption regarding thickness distribution of the stress-strain parameters of the shell. The higher the order of approximation, the better accuracy of the result obtained using the proposed theory. The complete system of linear differential equations for the nonlocal theory of elasticity of any order can be obtained using the equations presented here. In the following sections, we will consider the first and second order approximation theories in detail.

5 First order approximation theory

In the case of the first order approximation theory only the first two terms of the Legendre polynomials series are considered in the expansion (3.1). In this case the functions, which describe the stress-strain state of the rod, can be presented in the form

$$\begin{aligned} \sigma_{\alpha\beta}(x_1, x_2) &= \sigma_{\alpha\beta}^0(x_1) P_0(\omega) + \sigma_{\alpha\beta}^1(x_1) P_1(\omega), \\ \varepsilon_{\alpha\beta}(x_1, x_2) &= \varepsilon_{\alpha\beta}^0(x_1) P_0(\omega) + \varepsilon_{\alpha\beta}^1(x_1) P_1(\omega), \\ u_\alpha(x_1, x_2) &= u_\alpha^0(x_1) P_0(\omega) + u_\alpha^1(x_1) P_1(\omega), \end{aligned} \quad (49)$$

All the equations presented in the previous section will be significantly simplified. For example in this case we have

$$\begin{aligned} \underline{u}_\alpha^0(x_1) &= \frac{1}{h} u_\alpha^1(x_1), \quad \underline{u}_\alpha^1(x_1) = 0, \quad \overline{\sigma_{2\alpha}^0}(x_1) = 0, \\ \overline{\sigma_{2\alpha}^1}(x_1) &= \frac{3}{h} \sigma_{2\alpha}^0(x_1), \quad \underline{\ddot{u}}_\alpha^0(x_1) = \frac{1}{h} \ddot{u}_\alpha^1(x_1), \quad \underline{\ddot{u}}_\alpha^1(x_1) = 0, \\ \underline{\ddot{u}}_\alpha^0(x_1) &= 0, \quad \underline{\ddot{u}}_\alpha^1(x_1) = 0, \quad \underline{\sigma_{\alpha\beta}^0}(x_1) = \frac{1}{h} \sigma_{\alpha\beta}^1(x_1), \\ \overline{\sigma_{\alpha\beta}^1}(x_1) &= 0, \quad \underline{\sigma_{\alpha\beta}^0}(x_1) = 0, \quad \underline{\sigma_{\alpha\beta}^1}(x_1) = 0, \\ \underline{b}_\alpha^0(x_1) &= \frac{1}{h} b_\alpha^1(x_1), \quad \underline{b}_\alpha^1(x_1) = 0, \quad \underline{b}_\alpha^0(x_1) = 0, \\ \underline{b}_\alpha^1(x_1) &= 0, \end{aligned}$$

The equations of motion (39) now have the form

$$\begin{aligned} \frac{1}{A_1} \frac{\partial \sigma_{11}^0}{\partial x_1} + k_1 (\sigma_{21}^0 + \sigma_{12}^0) + \overline{b}_1^0 &= \rho \frac{\partial^2 u_1^0}{\partial t^2}, \\ \frac{1}{A_1} \frac{\partial \sigma_{12}^0}{\partial x_1} + k_1 (\sigma_{22}^0 - \sigma_{11}^0) + \overline{b}_2^0 &= \rho \frac{\partial^2 u_2^0}{\partial t^2}, \\ \frac{1}{A_1} \frac{\partial \sigma_{11}^1}{\partial x_1} + k_1 (\sigma_{21}^1 + \sigma_{12}^1) - \frac{3}{h} \sigma_{21}^0 + \overline{b}_1^1 &= \rho \frac{\partial^2 u_1^1}{\partial t^2}, \\ \frac{1}{A_1} \frac{\partial \sigma_{12}^1}{\partial x_1} + k_1 (\sigma_{22}^1 - \sigma_{11}^1) - \frac{3}{h} \sigma_{22}^0 + \overline{b}_2^1 &= \rho \frac{\partial^2 u_2^1}{\partial t^2}, \end{aligned} \quad (51)$$

where

$$\begin{aligned} \overline{b}_i^0(x_1) &= b_i^0(x_1) \\ &+ \frac{1}{h} (\sigma_{i3}^+(x_1) + \sigma_{i3}^-(x_1)), \quad \overline{b}_i^1(x_1) = b_i^1(x_1) \\ &+ \frac{3}{h} (\sigma_{i3}^+(x_1) - \sigma_{i3}^-(x_1)) \end{aligned} \quad (52)$$

Kinematic relations (42) have form

$$\begin{aligned} \varepsilon_{11}^0 &= \frac{1}{A_1} \frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0, \quad \varepsilon_{22}^0 = \frac{1}{h} u_2^1, \\ 2\varepsilon_{12}^0 &= \frac{1}{h} u_1^1 + \frac{1}{A_1} \frac{\partial u_2^0}{\partial x_1} - k_1 u_1^0, \quad \varepsilon_{11}^1 = \frac{1}{A_1} \frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1, \\ \varepsilon_{22}^k &= 0, \quad 2\varepsilon_{12}^1 = \frac{1}{A_1} \frac{\partial u_2^1}{\partial x_1} - k_1 u_1^1, \end{aligned} \quad (53)$$

The constitutive relations for the nonlocal elasticity (43) have the form

$$\begin{aligned}
 \sigma_{11}^0 - \tau^2 l_e^2 \left(\nabla^2 \sigma_{11}^0 + k_1 \sigma_{11}^1 / h \right) &= (\lambda + 2\mu) \varepsilon_{11}^0 + \lambda \varepsilon_{22}^0, \\
 \sigma_{22}^0 - \tau^2 l_e^2 \left(\nabla^2 \sigma_{22}^0 + k_1 \sigma_{22}^1 / h \right) &= (\lambda + 2\mu) \varepsilon_{22}^0 + \lambda \varepsilon_{11}^0, \\
 \sigma_{12}^0 - \tau^2 l_e^2 \left(\nabla^2 \sigma_{12}^0 + k_1 \sigma_{12}^1 / h \right) &= 2\mu \varepsilon_{12}^0, \\
 \sigma_{11}^1 - \tau^2 l_e^2 \nabla^2 \sigma_{11}^1 &= (\lambda + 2\mu) \varepsilon_{11}^1 + \lambda \varepsilon_{22}^1, \\
 \sigma_{22}^1 - \tau^2 l_e^2 \nabla^2 \sigma_{22}^1 &= (\lambda + 2\mu) \varepsilon_{22}^1 + \lambda \varepsilon_{11}^1, \\
 \sigma_{12}^1 - \tau^2 l_e^2 \nabla^2 \sigma_{12}^1 &= 2\mu \varepsilon_{12}^1.
 \end{aligned} \tag{54}$$

By substituting kinematic relations (53) into constitutive relations (54) we obtain

$$\begin{aligned}
 \sigma_{11}^0 - \tau^2 l_e^2 \left(\nabla^2 \sigma_{11}^0 + k_1 \sigma_{11}^1 / h \right) &= (\lambda + 2\mu) \left(\frac{1}{A_1} \frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0 \right) + \lambda \frac{1}{h} u_2^1, \\
 \sigma_{22}^0 - \tau^2 l_e^2 \left(\nabla^2 \sigma_{22}^0 + k_1 \sigma_{22}^1 / h \right) &= (\lambda + 2\mu) \frac{1}{h} u_2^1 + \lambda \left(\frac{1}{A_1} \frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0 \right), \\
 \sigma_{12}^0 - \tau^2 l_e^2 \left(\nabla^2 \sigma_{12}^0 + k_1 \sigma_{12}^1 / h \right) &= 2\mu \left(\frac{1}{h} u_1^1 + \frac{1}{A_1} \frac{\partial u_2^0}{\partial x_1} - k_1 u_2^0 \right), \\
 \sigma_{11}^1 - \tau^2 l_e^2 \nabla^2 \sigma_{11}^1 &= (\lambda + 2\mu) \left(\frac{1}{A_1} \frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1 \right), \\
 \sigma_{22}^1 - \tau^2 l_e^2 \nabla^2 \sigma_{22}^1 &= \lambda \left(\frac{1}{A_1} \frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1 \right), \\
 \sigma_{12}^1 - \tau^2 l_e^2 \nabla^2 \sigma_{12}^1 &= 2\mu \left(\frac{1}{A_1} \frac{\partial u_2^1}{\partial x_1} - k_1 u_2^1 \right),
 \end{aligned} \tag{55}$$

By substituting constitutive relations in the form of (55) into equations of motion (51) we obtain the 1-D differential equations for the nonlocal theory of elasticity in displacements for the first order theory of nonlocal rods theory in the form

$$\mathbf{L}_u \cdot \mathbf{u} + \tilde{\mathbf{b}} = \rho \ddot{\mathbf{u}} - \rho \tau^2 l_e^2 \left(\nabla^2 \ddot{\mathbf{u}} + k_1 \ddot{\mathbf{u}} \right) \tag{56}$$

Here

$$\mathbf{L}_u = \begin{pmatrix} L_{11}^{00} & L_{12}^{00} & L_{11}^{01} & L_{12}^{01} \\ L_{21}^{00} & L_{22}^{00} & L_{21}^{01} & L_{22}^{01} \\ L_{11}^{10} & L_{12}^{10} & L_{11}^{11} & L_{12}^{11} \\ L_{21}^{10} & L_{22}^{10} & L_{21}^{11} & L_{22}^{11} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_1^1 \\ u_2^1 \end{pmatrix}, \tag{57}$$

$$\tilde{\mathbf{b}} = \begin{pmatrix} \tilde{b}_1^0 \\ \tilde{b}_2^0 \\ \tilde{b}_1^1 \\ \tilde{b}_2^1 \end{pmatrix}, \quad \ddot{\mathbf{u}} = \begin{pmatrix} u_1^1/h \\ u_2^1/h \\ 0 \\ 0 \end{pmatrix}$$

Elements of the matrix operator \mathbf{L}_u can be represented in the form

$$\begin{aligned}
 L_{11}^{00} u_1^0 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial u_1^0}{\partial x_1} \right) - 2\mu k_1^2 u_1^0, \\
 L_{12}^{00} u_2^0 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial}{\partial x_1} \left(k_1 u_2^0 \right) + \frac{2\mu k_1}{A_1} \frac{\partial u_2^0}{\partial x_1}, \\
 L_{11}^{01} u_1^1 &= \frac{2\mu k_1}{h} u_1^1, \quad L_{12}^{01} u_2^1 = \frac{\lambda}{A_1 h} \frac{\partial}{\partial x_1} u_2^1, \\
 L_{21}^{00} u_2^0 &= -\frac{2\mu k_1}{A_1} \frac{\partial u_1^0}{\partial x_1} - \frac{\mu}{A_1} \frac{\partial}{\partial x_1} k_1 u_1^0, \\
 L_{22}^{00} u_2^0 &= \frac{\mu}{A_1} \frac{\partial}{\partial x_1} \frac{1}{A_1} \frac{\partial u_2^0}{\partial x_1} - 2\mu k_1^2 u_2^0, \\
 L_{21}^{01} u_1^1 &= \frac{\mu}{A_1 h} \frac{\partial}{\partial x_1} u_1^1, \quad L_{22}^{01} u_2^1 = -\frac{\lambda k_1}{h} u_2^1, \\
 L_{11}^{10} u_1^0 &= \frac{3\mu k_1}{h} u_1^0, \quad L_{12}^{10} u_2^0 = -\frac{3\mu}{A_1 h} \frac{\partial u_2^0}{\partial x_1}, \\
 L_{11}^{11} u_1^1 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial}{\partial x_1} \frac{1}{A_1} \frac{\partial u_1^1}{\partial x_1} - 2\mu k_1^2 u_1^1 - \frac{3\mu}{h^2} u_1^1, \\
 L_{12}^{11} u_2^1 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial}{\partial x_1} \left(k_1 u_2^1 \right) + \frac{2\mu k_1}{A_1} \frac{\partial u_2^1}{\partial x_1}, \\
 L_{21}^{10} u_1^0 &= -\frac{3\lambda}{A_1 h} \frac{\partial u_1^0}{\partial x_1}, \quad L_{22}^{10} u_2^0 = -\frac{3\lambda k_1}{h} u_2^0, \\
 L_{21}^{11} u_1^1 &= -\frac{\mu}{A_1} \frac{\partial}{\partial x_1} k_1 u_1^1 - \frac{2\mu k_1}{A_1} \frac{\partial u_1^1}{\partial x_1}, \\
 L_{22}^{11} u_2^1 &= \frac{\mu}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial u_2^1}{\partial x_1} \right) - 2\mu k_1^2 u_2^1 - \frac{3(\lambda + 2\mu)}{h^2} u_2^1
 \end{aligned} \tag{58}$$

The equations presented in this section are first order equations of the nonlocal curved rods theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering nonlocal effects.

If in the above equations it is assumed that $A_1 = 1$ and $k_1 = 0$ the equations for the nonlocal straight beam will be obtained in the form

$$\begin{aligned}
 (\lambda + 2\mu) \frac{\partial^2 u_1^0}{\partial x_1^2} + \frac{\lambda}{h} \frac{\partial u_2^1}{\partial x_1} + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_1^0}{\partial t^2} + \tilde{b}_1^0 &= \rho \frac{\partial^2 u_1^0}{\partial t^2} - \tau^2 l_e^2 \frac{k_1}{h} \frac{\partial^2 u_1^1}{\partial t^2} \\
 \mu \frac{\partial^2 u_2^0}{\partial x_1^2} + \frac{\mu}{h} \frac{\partial u_1^1}{\partial x_1} + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_2^0}{\partial t^2} + \tilde{b}_2^0 &= \rho \frac{\partial^2 u_2^0}{\partial t^2} \\
 &- \tau^2 l_e^2 \frac{k_1}{h} \frac{\partial^2 u_2^1}{\partial t^2}, \\
 (\lambda + 2\mu) \frac{\partial^2 u_1^1}{\partial x_1^2} - \frac{3\mu}{h} \frac{\partial u_2^0}{\partial x_1} - \frac{3\mu}{h^2} u_1^1 + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_1^1}{\partial t^2} + \tilde{b}_1^1 &= \rho \frac{\partial^2 u_1^1}{\partial t^2}, \\
 \mu \frac{\partial^2 u_2^1}{\partial x_1^2} - \frac{3\lambda}{h} \frac{\partial u_1^0}{\partial x_1} + \frac{3(\lambda + 2\mu)}{h^2} \frac{1}{h} u_2^1 + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_2^1}{\partial t^2} + \tilde{b}_2^1 &= \rho \frac{\partial^2 u_2^1}{\partial t^2},
 \end{aligned} \tag{59}$$

Analysis of this system of partial differential equations shows that in the static case it splits up into two independent parts. The first and the fourth as well as the second and the third equations form separate systems of the partial differential equations that can be solved independently.

6 Second order approximation theory

In the case of the second order approximation theory the first three terms of the Legendre polynomials series are considered in the expansion (29). In this case the functions, which describe the stress-strain state of the rod, can be presented in the form

$$\begin{aligned}\sigma_{\alpha,\beta}(x) &= \sigma_{\alpha,\beta}^0(x_1)P_0(\omega) + \sigma_{\alpha,\beta}^1(x_1)P_1(\omega) \\ &\quad + \sigma_{\alpha,\beta}^2(x_1)P_2(\omega), \\ \varepsilon_{\alpha,\beta}(x) &= \varepsilon_{\alpha,\beta}^0(x_1)P_0(\omega) + \varepsilon_{\alpha,\beta}^1(x_1)P_1(\omega) + \varepsilon_{\alpha,\beta}^2(x_1)P_2(\omega), \\ u_i(x) &= u_i^0(x_1)P_0(\omega) + u_i^1(x_1)P_1(\omega) + u_i^2(x_1)P_2(\omega).\end{aligned}\quad (60)$$

All the equations presented in the previous section have to be modified. For example in this case parameters defined by the equations (50) become

$$\begin{aligned}\underline{u}_\alpha^0(x_1) &= \frac{1}{h}u_\alpha^1(x_1), \quad \underline{u}_\alpha^1(x_1) = \frac{3}{h}u_\alpha^2(x_1), \\ \underline{u}_\alpha^2(x_1) &= 0, \quad \overline{\sigma}_{2\alpha}^0(x_1), \quad \overline{\sigma}_{2\alpha}^1(x_1) = \frac{3}{h}\overline{\sigma}_{2\alpha}^0(x_1), \\ \overline{\sigma}_{2\alpha}^2(x_1) &= \frac{5}{h}\overline{\sigma}_{2\alpha}^1(x_1), \quad \underline{\ddot{u}}_\alpha^0(x_1) = \frac{1}{h}\underline{\ddot{u}}_\alpha^1(x_1), \\ \underline{\ddot{u}}_\alpha^1(x_1) &= \frac{3}{h}\underline{\ddot{u}}_\alpha^2(x_1), \quad \underline{\ddot{u}}_\alpha^2(x_1) = 0, \quad \underline{\ddot{u}}_\alpha^0(x_1) = \frac{3}{h^2}u_\alpha^2(x_1), \\ \underline{\ddot{u}}_\alpha^1(x_1) &= 0, \quad \underline{\ddot{u}}_\alpha^2(x_1) = 0, \quad \underline{\sigma}_{\alpha,\beta}^0(x_1) = \frac{1}{h}\sigma_{\alpha,\beta}^1(x_1), \\ \underline{\sigma}_{\alpha,\beta}^1(x_1) &= \frac{3}{h}\sigma_{\alpha,\beta}^2(x_1), \quad \underline{\sigma}_{\alpha,\beta}^2(x_1) = 0, \\ \underline{\sigma}_{\alpha,\beta}^0(x_1) &= \frac{3}{h^2}\sigma_{\alpha,\beta}^2(x_1), \quad \underline{\sigma}_{\alpha,\beta}^1(x_1) = 0, \quad \underline{\sigma}_{\alpha,\beta}^2(x_1) = 0, \\ \underline{b}_\alpha^0(x_1) &= \frac{1}{h}b_\alpha^1(x_1), \quad \underline{b}_\alpha^1(x_1) = \frac{3}{h}b_\alpha^2(x_1), \\ \underline{b}_\alpha^2(x_1) &= \frac{3}{h}b_\alpha^2(x_1), \\ \underline{b}_\alpha^0(x_1) &= \frac{3}{h^2}b_\alpha^2(x_1), \quad \underline{b}_\alpha^1(x_1) = 0, \quad \underline{b}_\alpha^2(x_1) = 0,\end{aligned}\quad (61)$$

The equations of motion (39) now have the form

$$\begin{aligned}\frac{1}{A_1}\frac{\partial\sigma_{11}^0}{\partial x_1} + \sigma_{12}^0k_1 + \overline{b}_1^0 &= \rho\frac{\partial^2u_1^0}{\partial t^2}, \\ \frac{1}{A_1}\frac{\partial\sigma_{21}^0}{\partial x_1} - \sigma_{11}^0k_1 + \overline{b}_2^0 &= \rho\frac{\partial^2u_2^0}{\partial t^2}, \\ \frac{1}{A_1}\frac{\partial\sigma_{11}^1}{\partial x_1} + \sigma_{12}^1k_1 - \frac{3}{h}\sigma_{12}^0 + \overline{b}_1^1 &= \rho\frac{\partial^2u_1^1}{\partial t^2},\end{aligned}\quad (62)$$

$$\begin{aligned}\frac{1}{A_1}\frac{\partial\sigma_{21}^1}{\partial x_1} - \sigma_{11}^1k_1 - \frac{3}{h}\sigma_{22}^0 + \overline{b}_2^1 &= \rho\frac{\partial^2u_2^1}{\partial t^2}, \\ \frac{1}{A_1}\frac{\partial\sigma_{11}^2}{\partial x_1} + \sigma_{12}^2k_1 - \frac{5}{h}\sigma_{12}^1 + \overline{b}_1^2 &= \rho\frac{\partial^2u_1^2}{\partial t^2}, \\ \frac{1}{A_1}\frac{\partial\sigma_{21}^2}{\partial x_1} - \sigma_{11}^2k_1 - \frac{5}{h}A_1\sigma_{22}^1 + \overline{b}_2^2 &= \rho\frac{\partial^2u_2^2}{\partial t^2},\end{aligned}$$

where

$$\begin{aligned}\overline{b}_i^0(x_1) &= b_i^0(x_1) + \frac{1}{h}(\sigma_{i2}^+(x_1) + \sigma_{i2}^-(x_1)), \\ \overline{b}_i^1(x_1) &= b_i^1(x_1) + \frac{3}{h}(\sigma_{i2}^+(x_1) - \sigma_{i2}^-(x_1)), \\ \overline{b}_i^2(x_1) &= b_i^2(x_1) + \frac{5}{h}(\sigma_{i2}^+(x_1) + \sigma_{i2}^-(x_1)),\end{aligned}\quad (63)$$

Kinematic relations (42) have form

$$\begin{aligned}\varepsilon_{11}^0 &= \frac{1}{A_1}\frac{\partial u_1^0}{\partial x_1} + k_1u_2^0, \\ \varepsilon_{12}^0 &= \frac{1}{A_1}\frac{\partial u_2^0}{\partial x_1} - k_1u_1^0 + \frac{1}{h}u_1^1, \quad \varepsilon_{22}^0 = \frac{1}{h}u_2^1, \\ \varepsilon_{11}^1 &= \frac{1}{A_1}\frac{\partial u_1^1}{\partial x_1} + k_1u_2^1, \\ \varepsilon_{12}^1 &= \frac{1}{A_1}\frac{\partial u_2^1}{\partial x_1} - k_1u_1^1 + \frac{3}{h}u_1^2, \quad \varepsilon_{22}^1 = \frac{3}{h}u_2^2, \\ \varepsilon_{11}^2 &= \frac{1}{A_1}\frac{\partial u_1^2}{\partial x_1} + k_1u_2^2, \\ \varepsilon_{12}^2 &= \frac{1}{A_1}\frac{\partial u_2^2}{\partial x_1} - k_1u_1^0, \quad \varepsilon_{22}^2 = 0.\end{aligned}\quad (64)$$

The constitutive relations for nonlocal elasticity (43) have the form

$$\begin{aligned}\sigma_{11}^0 - \tau^2l_e^2 &\cdot \left(\nabla^2\sigma_{11}^0 + k_1(\sigma_{11}^0 + 3\sigma_{11}^1(x_1))/h + 3\sigma_{11}^2(x_1)/h^2 \right) \\ &= (\lambda + 2\mu)\varepsilon_{11}^0 + \lambda\varepsilon_{22}^0, \\ \sigma_{22}^0 - \tau^2l_e^2 &\cdot \left(\nabla^2\sigma_{22}^0 + k_1(\sigma_{22}^0 + 3\sigma_{22}^1(x_1))/h + 3\sigma_{22}^2(x_1)/h^2 \right) \\ &= (\lambda + 2\mu)\varepsilon_{22}^0 + \lambda\varepsilon_{11}^0, \\ \sigma_{12}^0 - \tau^2l_e^2 &\cdot \left(\nabla^2\sigma_{12}^0 + k_1(\sigma_{12}^0 + 3\sigma_{12}^1(x_1))/h + 3\sigma_{12}^2(x_1)/h^2 \right) \\ &= 2\mu\varepsilon_{12}^0, \\ \sigma_{11}^1 - \tau^2l_e^2 &\left(\nabla^2\sigma_{11}^1 + 3k_1\sigma_{11}^2(x_1)/h \right) \\ &= (\lambda + 2\mu)\varepsilon_{11}^1 + \lambda\varepsilon_{22}^1, \\ \sigma_{22}^1 - \tau^2l_e^2 &\left(\nabla^2\sigma_{22}^1 + 3k_1\sigma_{22}^2(x_1)/h \right) \\ &= (\lambda + 2\mu)\varepsilon_{22}^1 + \lambda\varepsilon_{11}^1, \\ \sigma_{12}^1 - \tau^2l_e^2 &\left(\nabla^2\sigma_{12}^1 + 3k_1\sigma_{12}^2(x_1)/h \right) = 2\mu\varepsilon_{12}^1,\end{aligned}\quad (65)$$

$$\begin{aligned} \sigma_{11}^2 - \tau^2 l_e^2 \nabla^2 \sigma_{11}^2 &= (\lambda + 2\mu) \varepsilon_{11}^2 + \lambda \varepsilon_{22}^2, \\ \sigma_{22}^2 - \tau^2 l_e^2 \nabla^2 \sigma_{22}^2 &= (\lambda + 2\mu) \varepsilon_{22}^2 + \lambda \varepsilon_{11}^2 \\ \sigma_{12}^2 - \tau^2 l_e^2 \nabla^2 \sigma_{12}^2 &= 2\mu \varepsilon_{12}^2. \end{aligned}$$

By substituting kinematic relations (64) into constitutive relations (65) we obtain

$$\begin{aligned} &\sigma_{11}^0 - \tau^2 l_e^2 \cdot \left(\nabla^2 \sigma_{11}^0 + k_1 \left(\sigma_{11}^1 + 3\sigma_{11}^2(x_1) \right) / h + 3\sigma_{11}^2(x_1) / h^2 \right) \\ &= (\lambda + 2\mu) \left(\frac{1}{A_1} \frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0 \right) + \lambda \frac{1}{h} u_2^1, \\ &\sigma_{22}^0 - \tau^2 l_e^2 \cdot \left(\nabla^2 \sigma_{22}^0 + k_1 \left(\sigma_{22}^1 + 3\sigma_{22}^2(x_1) \right) / h + 3\sigma_{22}^2(x_1) / h^2 \right) \\ &= (\lambda + 2\mu) \frac{1}{h} u_2^1 + \lambda \left(\frac{1}{A_1} \frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0 \right), \\ &\sigma_{12}^0 - \tau^2 l_e^2 \cdot \left(\nabla^2 \sigma_{12}^0 + k_1 \left(\sigma_{12}^1 + 3\sigma_{12}^2(x_1) \right) / h + 3\sigma_{12}^2(x_1) / h^2 \right) \\ &= \mu \left(\frac{1}{A_1} \frac{\partial u_2^0}{\partial x_1} - k_1 u_1^0 + \frac{1}{h} u_1^1 \right). \\ &\sigma_{11}^1 - \tau^2 l_e^2 \left(\nabla^2 \sigma_{11}^1 + 3k_1 \sigma_{11}^2(x_1) / h \right) \\ &= (\lambda + 2\mu) \left(\frac{1}{A_1} \frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1 \right) + \lambda \frac{3}{h} u_2^2, \\ &\sigma_{22}^1 - \tau^2 l_e^2 \left(\nabla^2 \sigma_{22}^1 + 3k_1 \sigma_{22}^2(x_1) / h \right) \\ &= (\lambda + 2\mu) \frac{3}{h} u_2^2 + \lambda \left(\frac{1}{A_1} \frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1 \right), \\ &\sigma_{12}^1 - \tau^2 l_e^2 \left(\nabla^2 \sigma_{12}^1 + 3k_1 \sigma_{12}^2(x_1) / h \right) \\ &= \mu \left(\frac{1}{A_1} \frac{\partial u_2^1}{\partial x_1} - k_1 u_1^1 + \frac{3}{h} u_1^2 \right). \\ &\sigma_{11}^2 - \tau^2 l_e^2 \nabla^2 \sigma_{11}^2 = (\lambda + 2\mu) \left(\frac{1}{A_1} \frac{\partial u_1^2}{\partial x_1} + k_1 u_2^2 \right), \\ &\sigma_{22}^2 - \tau^2 l_e^2 \nabla^2 \sigma_{22}^2 = \lambda \left(\frac{1}{A_1} \frac{\partial u_2^2}{\partial x_1} + k_1 u_2^2 \right), \\ &\sigma_{12}^2 - \tau^2 l_e^2 \nabla^2 \sigma_{12}^2 = \mu \left(\frac{1}{A_1} \frac{\partial u_2^2}{\partial x_1} - k_1 u_1^2 \right). \end{aligned} \tag{66}$$

By substituting constitutive relations in the form of (66) into equations of motion (62) we obtain the 1-D differential equations for the nonlocal theory of elasticity in displacements for the second order theory of nonlocal rods theory in the form (47), where matrices and vectors (48) become

$$\mathbf{L}_u = \begin{pmatrix} L_{11}^{00} & L_{12}^{00} & L_{11}^{01} & L_{12}^{01} & 0 & 0 \\ L_{21}^{00} & L_{22}^{00} & L_{21}^{01} & L_{22}^{01} & 0 & 0 \\ L_{11}^{10} & L_{12}^{10} & L_{11}^{11} & L_{12}^{11} & 0 & L_{12}^{12} \\ L_{21}^{10} & L_{22}^{10} & L_{21}^{11} & L_{22}^{11} & 0 & L_{22}^{12} \\ 0 & 0 & L_{11}^{21} & L_{12}^{21} & L_{11}^{22} & L_{12}^{22} \\ 0 & 0 & L_{21}^{21} & L_{22}^{21} & L_{21}^{22} & L_{22}^{22} \end{pmatrix}, \tag{67}$$

$$\mathbf{u} = \begin{pmatrix} u_1^0 \\ u_2^0 \\ u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{pmatrix}, \quad \tilde{\mathbf{b}} = \begin{pmatrix} \tilde{b}_1^0 \\ \tilde{b}_2^0 \\ \tilde{b}_1^1 \\ \tilde{b}_2^1 \\ \tilde{b}_1^2 \\ \tilde{b}_2^2 \end{pmatrix}, \quad \underline{\mathbf{u}} = \begin{pmatrix} u_1^1/h \\ u_2^1/h \\ 3u_1^2/h \\ 3u_2^2/h \\ 0 \\ 0 \end{pmatrix}, \quad \underline{\underline{\mathbf{u}}} = \begin{pmatrix} 3u_1^2/h^2 \\ 3u_2^2/h^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Elements of the matrix operator \mathbf{L}_u can be represented in the form

$$\begin{aligned} L_{11}^{00} u_1^0 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial u_1^0}{\partial x_1} \right) - 2\mu k_1^2 u_1^0, \\ L_{12}^{00} u_2^0 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial (k_1 u_2^0)}{\partial x_1} + \frac{2\mu k_1}{A_1} \frac{\partial u_2^0}{\partial x_1}, \quad L_{11}^{01} u_1^1 = \frac{2\mu k_1}{h} u_1^1, \\ L_{12}^{01} u_2^1 &= \frac{\lambda}{h} \frac{\partial u_2^1}{\partial x_1}, \quad L_{11}^{02} u_2^2 = 0, \quad L_{12}^{02} u_2^2 = 0. \\ L_{21}^{00} u_2^0 &= \frac{2\mu k_1}{A_1} \frac{\partial u_1^0}{\partial x_1} - \frac{\mu}{A_1} \frac{\partial (k_1 u_1^0)}{\partial x_1}, \\ L_{22}^{00} u_2^0 &= \frac{\mu}{A_1} \frac{\partial}{\partial x_1} \frac{1}{A_1} \frac{\partial u_2^0}{\partial x_1} - 2\mu k_1^2 u_2^0, \quad L_{21}^{01} = \frac{\mu}{A_1 h} \frac{\partial u_1^1}{\partial x_1}, \\ L_{22}^{01} &= -\frac{\lambda k_1}{h} u_2^1, \quad L_{21}^{02} u_2^2 = 0, \quad L_{22}^{02} u_2^2 = 0. \\ L_{11}^{10} u_1^0 &= \frac{3\mu k_1}{h} u_1^0, \quad L_{12}^{10} u_2^0 = -\frac{3\mu}{A_1 h} \frac{\partial u_2^0}{\partial x_1}, \\ L_{11}^{11} u_1^1 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial}{\partial x_1} \frac{1}{A_1} \frac{\partial u_1^1}{\partial x_1} - 2\mu k_1^2 u_1^1 - \frac{3\mu}{h^2} u_1^1, \\ L_{12}^{11} u_2^1 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial (k_1 u_2^1)}{\partial x_1} + \frac{2\mu k_1}{A_1} \frac{\partial u_2^1}{\partial x_1}, \quad L_{11}^{12} u_1^2 = 0, \\ L_{12}^{12} u_2^2 &= \frac{3\lambda}{h} \frac{\partial u_2^2}{\partial x_1}. \\ L_{21}^{10} u_1^0 &= -\frac{3\lambda}{A_1 h} \frac{\partial u_1^0}{\partial x_1}, \quad L_{22}^{10} u_2^0 = -\frac{3\lambda k_1}{h} u_2^0, \\ L_{21}^{11} u_1^1 &= -\frac{\mu}{A_1} \frac{\partial (k_1 u_1^1)}{\partial x_1} - \frac{2\mu k_1}{A_1} \frac{\partial u_1^1}{\partial x_1}, \\ L_{22}^{11} u_2^1 &= \frac{\mu}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial u_2^1}{\partial x_1} \right) - 2\mu k_1^2 u_2^1 - \frac{3(\lambda + 2\mu)}{h^2} u_2^1, \\ L_{21}^{12} u_1^2 &= 0, \quad L_{22}^{12} u_2^2 = -\frac{3\lambda k_1}{h} u_2^2. \\ L_{11}^{20} u_1^0 &= 0, \quad L_{12}^{20} u_2^0 = 0, \quad L_{11}^{21} u_1^1 = \frac{5\mu k_1}{h} u_1^1, \\ L_{22}^{21} u_2^1 &= -\frac{5\lambda}{A_1 h} \frac{\partial u_2^1}{\partial x_1}, \\ L_{11}^{22} u_1^2 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial}{\partial x_1} \frac{1}{A_1} \frac{\partial u_1^2}{\partial x_1} - 2\mu k_1^2 u_1^2 - \frac{15\mu}{h^2} u_1^2, \\ L_{12}^{22} u_2^2 &= \frac{\lambda + 2\mu}{A_1} \frac{\partial (k_1 u_2^2)}{\partial x_1} + \frac{2\mu k_1}{A_1} \frac{\partial u_2^2}{\partial x_1}, \quad L_{21}^{20} u_1^0 = 0, \\ L_{22}^{20} u_2^0 &= 0, \quad L_{12}^{21} u_1^1 = -\frac{5\lambda}{A_1 h} \frac{\partial u_1^1}{\partial x_1}, \quad L_{22}^{21} u_2^1 = -\frac{5\lambda k_1}{h^2} u_2^1, \\ L_{21}^{22} u_1^2 &= -\mu \frac{\partial (k_1 u_1^2)}{\partial x_1} - \frac{2\mu k_1}{A_1} \frac{\partial u_1^2}{\partial x_1}, \\ L_{22}^{22} u_2^2 &= \frac{\mu}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial u_2^2}{\partial x_1} \right) - \frac{2\mu k_2^2}{A_1} u_2^2 - \frac{15(\lambda + 2\mu)}{h^2} u_2^2 \end{aligned} \tag{68}$$

$$L_{22}^{11}u_2^1 = \frac{\mu}{A_1} \frac{\partial}{\partial x_1} \left(\frac{1}{A_1} \frac{\partial u_2^1}{\partial x_1} \right) - 2\mu k_1^2 u_2^1 - \frac{3(\lambda + 2\mu)}{h^2} u_2^1$$

The equations presented in this section are second order equations of the nonlocal curved rods theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering nonlocal effects.

If in the above equations it is assumed that $A_1 = 1$ and $k_1 = 0$ the equations for the couple stress straight rod will be obtained in the form

$$\begin{aligned} & (\lambda + 2\mu) \frac{\partial^2 u_1^0}{\partial x_1^2} + \frac{\lambda}{h} \frac{\partial u_2^1}{\partial x_1} + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_1^0}{\partial t^2} + \bar{b}_1^0 \quad (69) \\ & = \rho \frac{\partial^2 u_1^0}{\partial t^2} - \tau^2 l_e^2 \left(\frac{k_1}{h} \frac{\partial^2 u_1^1}{\partial t^2} + \frac{3}{h^2} \frac{\partial^2 u_1^2}{\partial t^2} \right), \\ & \mu \frac{\partial^2 u_1^0}{\partial x_1^2} + \frac{\mu}{h} \frac{\partial u_1^1}{\partial x_1} + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_1^0}{\partial t^2} + \bar{b}_2^0 = \rho \frac{\partial^2 u_2^0}{\partial t^2} \\ & - \tau^2 l_e^2 \left(\frac{k_1}{h} \frac{\partial^2 u_2^1}{\partial t^2} + \frac{3}{h^2} \frac{\partial^2 u_2^2}{\partial t^2} \right), \\ & (\lambda + 2\mu) \frac{\partial^2 u_1^1}{\partial x_1^2} + \frac{3\lambda}{h} \frac{\partial u_2^2}{\partial x_1} - \frac{3\mu}{h} \frac{\partial u_2^0}{\partial x_1} - \frac{3\mu}{h^2} u_1^1 + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_1^1}{\partial t^2} \\ & + \bar{b}_1^1 = \rho \frac{\partial^2 u_1^k}{\partial t^2} - \tau^2 l_e^2 \frac{3k_1}{h} \frac{\partial^2 u_1^2}{\partial t^2}, \\ & \mu \frac{\partial^2 u_2^1}{\partial x_1^2} - \lambda \frac{3}{h} \frac{\partial u_1^0}{\partial x_1} - \frac{3(\lambda + 2\mu)}{h^2} u_2^1 + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_2^1}{\partial t^2} + \bar{b}_2^1 \\ & = \rho \frac{\partial^2 u_2^k}{\partial t^2} - \tau^2 l_e^2 \frac{3k_1}{h} \frac{\partial^2 u_2^2}{\partial t^2}, \\ & (\lambda + 2\mu) \frac{\partial^2 u_1^2}{\partial x_1^2} - \frac{5\mu}{h} \frac{\partial u_2^1}{\partial x_1} - \frac{15}{h^2} u_1^2 + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_1^2}{\partial t^2} + \bar{b}_1^{-2} \\ & = \rho \frac{\partial^2 u_1^2}{\partial t^2}, \\ & \mu \frac{\partial^2 u_2^2}{\partial x_1^2} - \frac{5\lambda}{h} \frac{\partial u_1^1}{\partial x_1} - \frac{15(\lambda + 2\mu)}{h^2} u_2^2 + \tau^2 l_e^2 \nabla^2 \frac{\partial^2 u_2^2}{\partial t^2} + \bar{b}_2^{-2} \\ & = \rho \frac{\partial^2 u_2^2}{\partial t^2}, \end{aligned}$$

Analysis of this system of partial differential equations shows that it is coupled and cannot be split up into independent parts.

7 Nonlocal theory of Timoshenko's curved rod

Timoshenko's curved rod theory is based on assumptions concerning the value and distribution of the functions that determine the stress-strain state along the rod thickness. Thus, according to static assumptions $\sigma_{22} = 0$ and according to kinematic assumptions $\varepsilon_{22} = 0$.

The stress state is characterized by the normal n_{11} and shear n_{21} forces, as well as the bending m_{11} moment. They

are defined as following

$$\begin{aligned} n_{11} &= \int_{-h}^h \sigma_{11} dx_2, & n_{12} &= \int_{-h}^h \sigma_{12} dx_2, \\ m_{11} &= \int_{-h}^h \sigma_{11} x_2 dx_2, \end{aligned} \quad (70)$$

By integrating differential equations of motion (6) and (23) with respect to x_2 from $-h$ to h we first obtain the first two equations of motion presented in (71). And by multiplying the first equation of motion (23) by x_2 and integrating it with respect to x_2 from $-h$ to h we obtain the third equation of motion for Timoshenko's curved rod. The complete system of the equations of motion has the form

$$\begin{aligned} \frac{1}{A_1} \frac{\partial n_{11}}{\partial x_1} + n_{12} k_1 + \bar{b}_1 &= \rho F \frac{\partial^2 u_1}{\partial t^2}, \\ \frac{1}{A_1} \frac{\partial n_{12}}{\partial x_1} - n_{11} k_1 + \bar{b}_2 &= \rho F \frac{\partial^2 u_2}{\partial t^2}, \\ \frac{1}{A_1} \frac{\partial m_{11}}{\partial x_1} - n_{12} + \bar{m}_3 &= \rho J \frac{\partial^2 \gamma_1}{\partial t^2}, \end{aligned} \quad (71)$$

Displacements in Timoshenko's theory of curved beams are defined by vectors $\mathbf{u}(x_1, t)$ with components $u_\alpha(x_1, t)$ that correspond to axial and transverse displacements of the middle line in the x_1 and x_2 directions respectively, and $\gamma_1(x_1, t)$ that is the rotation of the centroidal axis about the axis of the elements perpendicular to the middle line. These parameters are related to the coefficients of the displacements expansion in the first order theory in the following way

$$u_\alpha^0 \sim u_\alpha, \quad u_\alpha^1 \sim \gamma_\alpha h, \quad (72)$$

Component u_2^1 is not taking into account in the Timoshenko's theory of curved rods.

In Timoshenko's theory the strain state of the rod is determined by quantities specified on the middle surface. Deformations in the classical Timoshenko's theory are determined by the relations

$$\varepsilon_{11} = e_{11} + \kappa_{11} x_2, \quad \varepsilon_{12} = e_{12}, \quad \varepsilon_{21} = e_{21}, \quad (73)$$

Roughly speaking, component $W2LOK$ corresponds to the tension-compression deformation of the middle surface, whereas component e_{12} corresponds to the transversal shear deformation and component κ_{11} to the bending middle line, respectively. The following formulas give us relations by corresponding quantities in the first order theory

$$\varepsilon_{\alpha\beta}^0 \sim e_{\alpha\beta}, \quad \varepsilon_{11}^1 \sim \kappa_{11} \quad (74)$$

Component ε_{33}^0 and ε_{33}^1 are not taken into account in Timoshenko’s theory of rods. Which also follows from the kinematic hypothesis.

According to Timoshenko’s rod theory, the displacement field can be represented in the form

$$u_1(x_1, x_2) = u_1(x_1) - x_2\gamma_1(x_1), \quad u_2(x_1, x_2) = u_2(x) \quad (75)$$

where $u_1(x_1)$ and $u_2(x_1)$ are axial and transverse displacements in the x_1 and x_2 directions respectively. Here we use the same notations for 2-D and 1-D functions of the displacements.

By substituting expressions for displacements (75) into 2-D kinematic relations (24) we obtain kinematic relations for Timoshenko’s curved rod in the form

$$\begin{aligned} e_{11} &= \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} + k_1 u_2, \quad e_{22} = 0, \\ e_{12} &= \frac{1}{2} \left(\frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1 - \gamma_1 \right), \quad \kappa_{11} = \frac{1}{A_1} \frac{\partial \gamma_1}{\partial x_1} \end{aligned} \quad (76)$$

Constitutive relations for the nonlocal Timoshenko’s rod theory can be obtained from (25) in the form

$$\begin{aligned} n_{11} - l^2 \tau^2 \nabla^2 n_{11} &= EF e_{11}, \\ m_{11} - l^2 \tau^2 \nabla^2 m_{11} &= EJ \kappa_{11}, \quad n_{12} - l^2 \tau^2 \nabla^2 n_{12} = \mu F e_{12} \end{aligned} \quad (77)$$

By substituting kinematic relations (76) to the constitutive relations (77) we obtain constitutive relations for nonlocal Timoshenko’s rod in the form of displacements and rotation

$$\begin{aligned} n_{11} - l^2 \tau^2 \nabla^2 n_{11} &= EF \left(\frac{1}{A_1} \frac{\partial u_1}{\partial x_1} + k_1 u_2 \right), \\ m_{11} - l^2 \tau^2 \nabla^2 m_{11} &= EJ \frac{1}{A_1} \frac{\partial \gamma_1}{\partial x_1}, \\ n_{12} - l^2 \tau^2 \nabla^2 n_{12} &= \mu F \left(\frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1 \right), \end{aligned} \quad (78)$$

By substituting these constitutive relations into equations of motion (71) we obtain the 1-D differential equations of motion for nonlocal Timoshenko’s curved rods in the form

$$\mathbf{L}_u \cdot \mathbf{u} + \tilde{\mathbf{b}} = \rho(\ddot{\mathbf{u}} - \tau^2 l_e^2 \nabla^2 \ddot{\mathbf{u}}) \quad (79)$$

where matrices operators, vectors of deformation and body forces become

$$\mathbf{L}_u = \begin{bmatrix} L_{11}^u & L_{12}^u & L_{13}^\gamma \\ L_{21}^u & L_{22}^u & L_{23}^\gamma \\ L_{31}^u & L_{32}^u & L_{33}^\gamma \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \gamma_1 \end{bmatrix}, \quad \tilde{\mathbf{b}} = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{m}_1 \end{bmatrix}, \quad (80)$$

and

$$\tilde{b}_\alpha = (1 - \tau^2 l_e^2 \nabla^2) \bar{b}_\alpha, \quad \tilde{m}_1 = (1 - \tau^2 l_e^2 \nabla^2) \bar{m}_1 \quad (81)$$

Elements of the matrix operator \mathbf{L}_u can be represented in the form

$$\begin{aligned} L_{11}^u u_1 &= \frac{EF}{A_1^2} \frac{\partial^2 u_1}{\partial x_1^2} - 2k_1^2 \mu F u_1, \\ L_{12}^u u_2 &= \left(\frac{k_1 EF}{A_1} + \frac{2k_1 \mu F}{A_1} \right) \frac{\partial u_2}{\partial x_1}, \quad L_{11}^\gamma \gamma_1 = -2k_1 \mu F \gamma_1, \\ L_{21}^u u_1 &= - \left(\frac{k_1 EF}{A_1} + \frac{k_1 \mu F}{A_1} \right) \frac{\partial u_1}{\partial x_1}, \\ L_{22}^u u_2 &= \frac{\mu F}{A_1^2} \frac{\partial^2 u_2}{\partial x_1^2} + k_1^2 EF u_2, \quad L_{23}^\gamma \gamma_1 = - \frac{\mu F}{A_1} \frac{\partial \gamma_1}{\partial x_1}, \\ L_{31}^u u_1 &= -2k_1 \mu F u_1, \quad L_{32}^u u_2 = - \frac{\mu F}{A_1} \frac{\partial u_2}{\partial x_1}, \\ L_{33}^\gamma \gamma_1 &= \frac{EJ}{A_1^2} \frac{\partial^2 \gamma_1}{\partial x_1^2} + \mu F \gamma_1, \end{aligned} \quad (82)$$

The equations presented in this section are equations of the Timoshenko’s nonlocal curved rods theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering nonlocal effects.

If in the above equations it is assumed that $A_1 = 1$ and $k_1 = 0$ the equations for the nonlocal straight beam will be obtained in the form

$$\begin{aligned} EF \frac{\partial^2 u_1}{\partial x_1^2} + \tilde{b}_1 &= \rho(1 - \tau^2 l_e^2 \nabla^2) \frac{\partial^2 u_1}{\partial t^2}, \\ \mu F \left(\frac{\partial^2 u_2}{\partial x_1^2} - \frac{\partial \gamma_1}{\partial x_1} \right) + \tilde{b}_2 &= \rho(1 - \tau^2 l_e^2 \nabla^2) \frac{\partial^2 u_2}{\partial t^2}, \\ EJ \frac{\partial^2 \gamma_1}{\partial x_1^2} - \mu F \left(\frac{\partial u_2}{\partial x_1} - \gamma_1 \right) + \tilde{m}_1 &= \rho(1 - \tau^2 l_e^2 \nabla^2) \frac{\partial^2 \gamma_1}{\partial t^2}, \end{aligned} \quad (83)$$

Analysis of this system of partial differential equations shows that it splits up into two separate parts. The first equation corresponds to the tension-compression mode and the system of the remaining two equations corresponds to the bending mode. They can be solved independently. We have to mention that the system of the equations (83) coincides with the one presented in [5, 6] up to notation. Therefore the analysis and verification presented in [5, 6] take place in the case that is considered here.

8 Nonlocal theory of Euler–Bernoulli curved rod

Like Timoshenko’s theory of curved rods, the Euler–Bernoulli theory is also based on similar assumptions concerning the value and distribution of the stress-strain state along the rod thickness. Thus, according to static assumptions $\sigma_{22} = 0$ and according to kinematic assumptions . The stress state is characterized by the normal n_{11} and

shear n_{21} forces, as well as the bending m_{11} . They are defined by the equations (70).

Unlike Timoshenko's theory in Euler-Bernoulli theory of rods the rotation of the centroidal axis is not independent. It is represented through displacements by the equation

$$\gamma_1 = -\frac{1}{A_1} \frac{\partial u_2}{\partial x_1} + k_1 u_1 \quad (84)$$

According to Euler-Bernoulli rod theory, the displacement field can be written in the following form

$$u_1(x_1, x_2) = u_1(x_1) + x_2 \left(\frac{1}{A_1} \frac{\partial u_2(x_1)}{\partial x_1} - k_1 u_1(x_1) \right), \quad (85)$$

$$u_2(x_1, x_2) = u_2(x_1)$$

In the Euler-Bernoulli theory the strain state of the curved rod is also determined by quantities specified on the middle surface. By substituting expressions for displacements into 2-D kinematic relations (2.6) we obtain kinematic relations for Euler-Bernoulli curved rod in the form

$$e_{11} = \frac{1}{A_1} \frac{\partial u_1}{\partial x_1} + k_1 u_2, \quad e_{22} = 0, \quad (86)$$

$$e_{12} = \frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1 \quad \kappa_{11} = -\frac{1}{A_1^2} \frac{\partial^2 u_2}{\partial x_1^2} + \frac{k_1}{A_1} \frac{\partial u_1}{\partial x_1}$$

Constitutive relations for the nonlocal Euler-Bernoulli rod are the same as Timoshenko's rod and are represented by the equations (77). By substituting kinematic relations (86) to the constitutive relations (77) we represent them in the form

$$n_{11} - l^2 \tau^2 \nabla^2 n_{11} = EF \left(\frac{1}{A_1} \frac{\partial u_1}{\partial x_1} + k_1 u_2 \right), \quad (87)$$

$$m_{11} - l^2 \tau^2 \nabla^2 m_{11} = EJ \left(\frac{1}{A_1^2} \frac{\partial^2 u_2}{\partial x_1^2} - \frac{k_1}{A_1} \frac{\partial u_1}{\partial x_1} \right),$$

$$n_{12} - l^2 \tau^2 \nabla^2 n_{12} = 2\mu F \left(\frac{1}{A_1} \frac{\partial u_2}{\partial x_1} - k_1 u_1 \right)$$

Taking into account that in Euler-Bernoulli rod theory, rotational inertial is neglected and therefore $\frac{\partial^2 \gamma_1}{\partial t^2} = 0$, from the last equation of motion (71) we have

$$n_{12} = \frac{1}{A_1} \frac{\partial m_{11}}{\partial x_1} - \tilde{m}_3 \quad (88)$$

Then the equations of motion for Euler-Bernoulli curved rod can be presented in the form

$$\frac{1}{A_1} \frac{\partial n_{11}}{\partial x_1} + \frac{k_1}{A_1} \frac{\partial m_{11}}{\partial x_1} + \bar{b}_1 - k_1 \tilde{m}_3 = \rho F \frac{\partial^2 u_1}{\partial t^2}, \quad (89)$$

$$\frac{1}{A_1^2} \frac{\partial^2 m_{11}}{\partial x_1^2} - k_1 n_{11} + \bar{b}_2 - \frac{1}{A_1} \frac{\partial m_3}{\partial x_1} = \rho F \frac{\partial^2 u_2}{\partial t^2}$$

By substituting kinematic relations in the form (87) in the equations of motion (89) we obtain a differential equation of motion in the form of displacements for the nonlocal Euler-Bernoulli curved rod in the form (79), where matrices operators and vectors become

$$\mathbf{L}_u = \begin{vmatrix} L_{11}^u & L_{12}^u \\ L_{21}^u & L_{22}^u \end{vmatrix}, \quad \mathbf{u} = \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}, \quad \tilde{\mathbf{b}} = \begin{vmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{vmatrix}, \quad (90)$$

and

$$\tilde{b}_1 = (1 - \tau^2 l_e^2 \nabla^2) \bar{b}_1, \quad (91)$$

$$\tilde{b}_2 = (1 - \tau^2 l_e^2 \nabla^2) \left(\bar{b}_2 - \frac{1}{A_1} \frac{\partial^2 \tilde{m}_3}{\partial x_1^2} \right)$$

Elements of the matrix operator \mathbf{L}_u can be represented in the form

$$L_{11}^u u_1 = \left(\frac{EF}{A_1^2} + \frac{k_1^2 EJ}{A_1^2} \right) \frac{\partial^2 u_1}{\partial x_1^2}, \quad (92)$$

$$L_{12}^u u_2 = -\frac{k_1 EJ}{A_1^3} \frac{\partial^3}{\partial x_1^3} + \frac{EFk_1}{A_1} \frac{\partial u_2}{\partial x_1},$$

$$L_{21}^u u_1 = -\frac{k_1 EJ}{A_1^3} \frac{\partial^3 u_1}{\partial x_1^3} + \frac{EFk_1}{A_1} \frac{\partial u_1}{\partial x_1},$$

$$L_{22}^u u_2 = \frac{EJ}{A_1^4} \frac{\partial^4 u_2}{\partial x_1^4} + EFk_1^2 u_2,$$

The equations presented in this section are equations of the Euler-Bernoulli nonlocal curved rods theory. They can be used for modeling and stress-strain calculations of plane curved rods by considering nonlocal effects.

If in the above equations it is assumed that $A_1 = 1$ and the equations for the nonlocal straight beam will be obtained in the form

$$EF \frac{\partial^2 u_1}{\partial x_1^2} + \tilde{b}_1 = \rho(1 - \tau^2 l_e^2 \nabla^2) \frac{\partial^2 u_1}{\partial t^2} \quad (93)$$

$$EJ \frac{\partial^4 u_2}{\partial x_1^4} + \tilde{b}_1 = \rho(1 - \tau^2 l_e^2 \nabla^2) \frac{\partial^2 u_2}{\partial t^2}$$

Analysis of this system of partial differential equations shows that it splits up into two separate parts. The first equation corresponds to the tension-compression mode and the second one corresponds to the bending mode. They can be solved independently. We have to mention that the system (93) coincides with the one presented in [5, 6] up to notation. Therefore the analysis and verification presented in [5, 6] take place in the case that is considered here.

9 Conclusions

In this paper new theories for the nonlocal plane curved rods have been developed. The 2-D theory is developed

from general 2-D equations of the nonlocal theory of elasticity using a special curvilinear system of coordinates related to the middle line of the rod and a special assumption based on assumptions that take into account the fact that the rod is thin. High order theory is based on the expansion of the equations of the theory of elasticity into Fourier series in terms of Legendre polynomials in a thickness coordinate. All of the functions that define the stress-strain state of the rod including stress and strain tensors, vectors of displacements and rotation and body forces have been expanded into Fourier series in terms of Legendre polynomials with respect to a thickness coordinate. Thereby, all equations of elasticity including nonlocal constitutive relations have been transformed to the corresponding equations for Fourier coefficients of the Legendre polynomials expansion. Then, for Fourier coefficients the system of differential equations of motion in terms of displacements and rotations has been obtained in the same way as in the local theory of elasticity. First order and second order theories are considered in details. All differential equations including equations of motion in displacements have been developed and presented here. The Timoshenko's and Euler-Bernoulli theories have been developed based on the classical hypothesis and the 2-D equations of linear couple stress theory of elasticity in a special curvilinear system of coordinates. In the same way, the system of differential equations of motion in terms of displacements and rotations has been developed for all the cases that have been considered here. The equations for the nonlocal theory of the straight beam can be derived from the equations presented here as a special case. The obtained equations can be used to calculate the stress-strain as well as to model thin structures in nanoscales by taking into account size and nonlocal effects.

Analysis of the systems of partial differential equations (56), (67), (79) and (90) show that all of them are coupled and related longitudinal and flexural deformation modes and take into account effects of nonlocal deformations. The second order approximation theory is more complete and all quantities are approximated by quadratic functions. In the first order approximation theory all quantities are approximated by linear functions. The theory based on Timoshenko's hypothesis is less accurate, but simple and take into account shear deformation, which is important for dynamic analysis. The theory based on Euler-Bernoulli hypothesis is less accurate compared with the previous ones, but it is the simplest one and couple all the deformation modes and takes into account effects of nonlocal deformations.

As special case, the equations for straight beam can be derived from the equations presented here. The ob-

tained equations can be used to stress-strain calculation as well as modeling thin structures in macro, micro and nano scales with takes into account effects of nonlocal deformations. Specially proposed models can be efficient in MEMS and NEMS modeling as well as computer simulation.

Acknowledgement: The work presented in this paper was supported by the Committee of Science and Technology of Mexico (CONACYT) by the Research Grant, (Ciencia Basica, Reference No 256458), which is gratefully acknowledged.

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