Research Article

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The free vibration characteristics of isotropic coupled conical-cylindrical shells based on the precise integration transfer matrix method

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Abstract: Based on the transfer matrix theory and precise integration method, the precise integration transfer matrix method (PITMM) is implemented to investigate the free vibration characteristics of isotropic coupled conical-cylindrical shells. The influence on the boundary conditions, the shell thickness and the semi-vertex conical angle on the vibration characteristics are discussed. Based on the Flügge thin shell theory and the transfer matrix method, the field transfer matrix of cylindrical and conical shells is obtained. Taking continuity conditions at the junction of the coupled conical-cylindrical shell into consideration, the field transfer matrix of the coupled shell is constructed. According to the boundary conditions at the ends of the coupled shell, the natural frequencies of the coupled shell are solved by the precise integration method. An approach for studying the free vibration characteristics of isotropic coupled conical-cylindrical shells is obtained. Comparison of the natural frequencies obtained using the present method with those from literature confirms the validity of the proposed approach. The effects of the boundary conditions, the shell thickness and the semi-vertex conical angle on vibration characteristics are presented.

Keywords: Coupled conical-cylindrical shells; Precise integration; Transfer matrix; Vibration; Natural frequency

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1 Introduction

In engineering applications, especially in the field of modern military defence, the cylindrical shells, conical shells and coupled conical-cylindrical shells are basically simplified models of many types of weapons and equipment, such as aircraft, missiles, and submarines. The study of free vibration characteristics of cylindrical shells is comprehensive. Initially, researchers [1–5] investigated cylindrical shells using classic thin shell theories such as Donnell equations, Kennard equations, Flügge equations and Sander-Koiter equations. Harari, Sandman and Laulagnet were representative scholars in the field. Rayleigh [6] was a pioneer in the study of free vibration characteristics of cylindrical shells. The literary work of Leissa [7] gave general comments on the free vibration characteristics of cylindrical shells. The free vibration characteristics of a conical shells with simply-supported boundary conditions are examined using Statistical Energy Analysis by Creenwelge [8]. Talebitooti [9] and Li F. M. [10] analysed the free vibration characteristics of conical shells using the Rayleigh-Ritz method. The kp-Ritz method is used to study conical shells in the work of Liew *et al.* [11]. Guo [12] applied the multiple factor method to discuss the free vibration characteristics of conical shells. Unlike the cylindrical shells, the section radius of a conical shell will vary in the axial direction, which increases the complexity and the difficulty in studying conical shells. So far, only an approximate solution for determining the natural frequencies of conical shells has been obtained. Limited work on the analysis of free vibration characteristics of coupled conical-cylindrical shells has been carried out. Initially, the natural frequencies of the coupled conical-cylindrical shell were solved used FEM. Irie [13] investigates the natural frequencies of the coupled shell through the transfer matrix theory. Caresta [14] used the two thin theories by Donnell-Mushtari and Flügge to examine the free vibration characteristics of coupled shells.

This paper applies a new method to analyse the free vibration characteristics of isotropic coupled conical-cylindrical shells, which is different from the approach employed in previous studies. The method is referred to as PITMM. Based on the Flügge thin shell theory, equations of motion for cylindrical and conical shells are derived. The coefficient matrix in the equations of motion for cylindrical and conical shells is calculated using the precise integration method. To take into account the point transfer matrix at the junction of the coupled conical-cylindrical shell and to absorb the matrix assembly solution from FEM, the total transfer matrix of the coupled shell is constructed. According to the boundary conditions, the natural frequencies of the coupled shell are solved. Comparison of the natural frequencies are made using the method presented here, results obtained using the finite element method and with results from literature. The effects of the boundary conditions, the shell thickness and the semi-vertex conical angle on free vibration characteristics of the coupled shell have been presented in this paper.

2 Equations of motion

2.1 Equations of motion for a cylindrical shell

The shell deformation is described by the thin shell theory that is based on linear assumptions. To obtain precise results, the relatively accurate Flügge shell theory is used in this paper. The force balance equation is obtained by analysing the micro-element stress of the cylindrical shell. In this paper, the equations are based on the kinetic theory. Thus, many terms include time items. For the purpose of facilitating the writing and derivation, the dynamic response time item $e^{-i\omega t}$ is omitted in the remainder of the text. The cylindrical shell coordinates system (γ, φ, x) and displacement positive direction are shown in Figure 1.

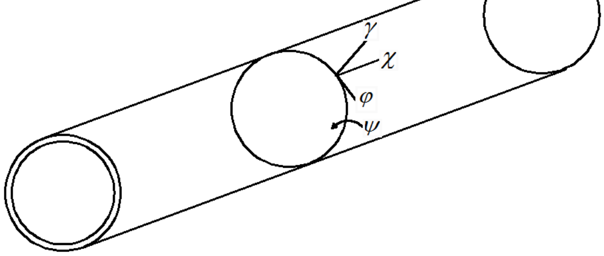

Figure 1: Coordinate system of a cylindrical shell

Based on the Flügge shell theory [15], the force balance equation of a cylindrical shell is given as follows:

$$\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta} + \rho h \omega^2 u = 0 \tag{1}$$

$$\frac{1}{R}\frac{\partial N_\phi}{\partial \theta} + \frac{\partial N_{x\theta}}{\partial x} - \frac{Q_\theta}{R} + \rho h \omega^2 v = 0 \tag{2}$$

$$\frac{N_\theta}{R} + \frac{\partial Q_x}{\partial x} + \frac{1}{R}\frac{\partial Q_\theta}{\partial \theta} - \rho h \omega^2 w = 0 \tag{3}$$

$$Q_\theta = \frac{1}{R}\frac{\partial M_\theta}{\partial \theta} + \frac{\partial M_{x\theta}}{\partial x} \tag{4}$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{1}{R}\frac{\partial M_{\theta x}}{\partial \theta} \tag{5}$$

The Kevin-Kirchhoff membrane forces, shear and all internal forces are

$$V_x = N_{x\theta} - \frac{M_{x\theta}}{R} \tag{6}$$

$$S_x = Q_x + \frac{1}{R}\frac{\partial M_{x\theta}}{\partial \theta} \tag{7}$$

$$N_x = D\left(\frac{\partial u}{\partial x} + \frac{\mu}{R}\left(\frac{\partial v}{\partial \theta} + w\right)\right) - \frac{K}{R}\frac{\partial \psi}{\partial x} \tag{8}$$

$$N_\theta = D\left(\frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right) + \mu\frac{\partial u}{\partial x}\right) + \frac{K}{R^3}\left(w + \frac{\partial^2 w}{\partial \theta^2}\right) \tag{9}$$

$$N_{x\theta} = \frac{1-\mu}{2}D\left(\frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}\right) + \frac{K}{R^2}\frac{1-\mu}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial \psi}{\partial \theta}\right) \tag{10}$$

$$N_{\theta x} = \frac{1-\mu}{2}D\left(\frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}\right) + \frac{K}{R^2}\frac{1-\mu}{2}\left(\frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial \psi}{\partial \theta}\right) \tag{11}$$

$$M_x = K\left(\frac{\partial \psi}{\partial x} + \frac{\mu}{R^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{R}\frac{\partial u}{\partial x} - \frac{\mu}{R^2}\frac{\partial v}{\partial \theta}\right) \tag{12}$$

$$M_\theta = K\left(\frac{1}{R^2}w + \frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} + \mu\frac{\partial \psi}{\partial x}\right) \tag{13}$$

$$M_{x\theta} = \frac{1-\mu}{R}K\left(\frac{\partial \psi}{\partial \theta} - \frac{\partial v}{\partial x}\right) \tag{14}$$

$$M_{\theta x} = \frac{1-\mu}{R}K\left(\frac{\partial \psi}{\partial \theta} + \frac{1}{2R}\frac{\partial u}{\partial \theta} - \frac{1}{2}\frac{\partial v}{\partial x}\right) \tag{15}$$

where K and D are the bending rigidity and membrane rigidity, respectively.

$$K=\frac{Eh^3}{12(1-\mu^2)} \tag{16}$$

$$D=\frac{Eh}{1-\mu^2} \tag{17}$$

The relationship between the radial displacement and slope is

$$\psi=\frac{\partial w}{\partial x} \tag{18}$$

There are sixteen unknown quantities in the above equations. To eliminate eight unknown quantities $(N_\theta, N_{x\theta}, N_{\theta x}, M_\theta, M_{x\theta}, M_{\theta x}, Q_x, Q_\theta)$, eight unknown quantities $(u, v, w, \psi, N_x, M_x, V_x, S_x)$ are retained, which are the sectional state vector elements of the cylindrical shell. All quantities are processed into dimensionless quantities and expanded to trigonometric series along the circumferential direction.

$$(u,w)=h\sum_{\partial=0}^{1}\sum_{n}h(\overline{u},\overline{w})\sin n\theta \tag{19}$$

$$v=h\sum_{\partial=0}^{1}\sum_{n}\overline{v}\cos\left(n\theta+\frac{\alpha\pi}{2}\right) \tag{20}$$

$$\psi=\frac{h}{R}\sum_{\partial=0}^{1}\sum_{n}\overline{\psi}\sin\left(n\theta+\frac{\alpha\pi}{2}\right) \tag{21}$$

$$(N_x, N_\theta, Q_x, V_x) = \frac{K}{R^2}\sum_{\partial=0}^{1}\sum_{n}(\overline{N_x},\overline{N_\theta},\overline{Q_x},\overline{V_x})\sin\left(n\theta+\frac{\alpha\pi}{2}\right) \tag{22}$$

$$(N_{x\theta}, N_{\theta x}, Q_\theta, S_x) = \frac{K}{R^2}\sum_{\partial=0}^{1}\sum_{n}(\overline{N_{x\theta}},\overline{N_{\theta x}},\overline{Q_\theta},\overline{S_x})\cos\left(n\theta+\frac{\alpha\pi}{2}\right) \tag{23}$$

$$(M_x, M_\theta)=\frac{K}{R}\sum_{\partial=0}^{1}\sum_{n}(\overline{M_x},\overline{M_\theta})\sin\left(n\theta+\frac{\alpha\pi}{2}\right) \tag{24}$$

$$(M_{x\theta}, M_{\theta x})=\frac{K}{R}\sum_{\partial=0}^{1}\sum_{n}(\overline{M_{x\theta}},\overline{M_{\theta x}})\cos\left(n\theta+\frac{\alpha\pi}{2}\right) \tag{25}$$

where n is the circumferential modal number. Other dimensionless quantities and dimensionless frequency parameter are

$$\xi=\frac{x}{l} \tag{26}$$

$$\bar{l}=\frac{l}{R} \tag{27}$$

$$\overline{h}=\frac{h}{R} \tag{28}$$

$$\lambda^2=\frac{\rho h R^2\omega^2}{D} \tag{29}$$

Through complicated simplification, a first-order matrix differential equation of the cylindrical shell is obtained.

$$\frac{d\left\{\boldsymbol{Z}(\xi)\right\}}{d\xi}=\bar{l}\boldsymbol{U}(\xi)\left\{\boldsymbol{Z}(\xi)\right\}+\left\{\boldsymbol{F}(\xi)\right\}-\left\{\boldsymbol{p}(\xi)\right\} \tag{30}$$

where $\boldsymbol{Z}(\xi)=\left\{\overline{u}\ \ \overline{v}\ \ \overline{w}\ \ \overline{\psi}\ \ \overline{M_x}\ \ \overline{V_x}\ \ \overline{S_{x\varphi}}\ \ \overline{N_x}\right\}^T$ is the state vector of the cylindrical shell. $(\overline{u},\overline{v},\overline{w})$ are the dimensionless quantities of the axial displacement (x direction), the circumferential displacement (φ direction) and the radial displacement (γ direction), respectively. $\overline{\psi}$ is a dimensionless slope, $\overline{N_x}$ is a dimensionless membrane force, $\overline{M_x}$ is a dimensionless bending moment, $(\overline{V_x},\overline{S_x})$ are the dimensionless Kelvin-Kirchhoff shear force and shear force, E and μ are Young's modulus and Poisson's ratio, respectively. $\boldsymbol{Z}(\xi)$ is the shell element's state vector and is also a function of the dimensionless variables ξ. $\boldsymbol{U}(\xi)$ is the coefficient matrix of the differential equation of the cylindrical shell and is an eight-order square matrix. There are 22 non-zero elements in $\boldsymbol{U}(\xi)$, see Appendix A.

2.2 Equations of motion for the conical shell

In a cylindrical coordinate system, the generatrix direction and radial direction of the conical shell are defined as the coordinate direction. The position of any point on a conical shell can be described as (s,θ). s is length from the top point of the conical shell to any point on the conical shell along the generatrix direction. θ is the angle of the point along the circumferential direction in a cylindrical coordinate system. The coordinate system of a conical shell is seen in Figure 2. The analysis of the conical shell force [16], the force balance equation of a conical shell is given as follows:

$$\frac{1}{s}\frac{\partial(sN_s)}{\partial s}+\frac{1}{s\sin\alpha}\frac{\partial N_{\theta s}}{\partial\theta}+\frac{N_\theta}{s}+\rho h\omega^2 u=0 \tag{31}$$

$$\frac{1}{s\sin\alpha}\frac{\partial N_\theta}{\partial\theta}+\frac{1}{s}\frac{\partial N_{s\theta}}{\partial s}+\frac{N_{s\theta}}{s}+\frac{Q_\theta}{s\tan\alpha}+\rho h\omega^2 v=0 \quad (32)$$

$$\frac{N_\theta}{s\tan\alpha}-\frac{1}{s}\frac{\partial(sQ_s)}{\partial s}-\frac{1}{s\sin\alpha}\frac{\partial Q_\theta}{\partial\theta}-\rho h\omega^2 w=0 \quad (33)$$

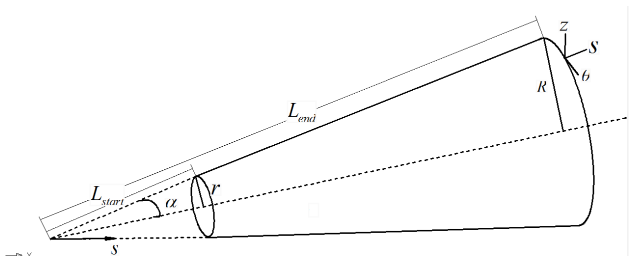

Figure 2: Coordinate system of the conical shell

The Kevin-Kirchhoff membrane forces, shear and all internal forces are

$$V_s=Q_s+\frac{1}{s\sin\alpha}\frac{\partial M_{\theta x}}{\partial\theta} \quad (34)$$

$$S_{s\theta}=N_{s\theta}+\frac{M_{\theta x}}{s\tan\alpha} \quad (35)$$

$$N_s=\frac{Eh}{1-\nu^2}\left[\frac{\partial u}{\partial s}+\frac{\nu}{s}\left(\frac{1}{\sin\alpha}\frac{\partial v}{\partial\theta}+u+\frac{w}{\tan\alpha}\right)\right] \quad (36)$$

$$N_\theta=\frac{Eh}{1-\nu^2}\left[\frac{1}{s}\left(\frac{1}{\sin\alpha}\frac{\partial v}{\partial\theta}+u+\frac{w}{\tan\alpha}\right)+\nu\frac{\partial u}{\partial s}\right] \quad (37)$$

$$N_{s\theta}=N_{\theta s}=\frac{Eh}{2(1+\nu)}\left[\frac{\partial v}{\partial s}+\frac{1}{s}\left(\frac{1}{\sin\alpha}\frac{\partial u}{\partial\theta}-v\right)\right] \quad (38)$$

$$M_\theta=\frac{Eh^3}{12(1-\nu^2)}\left[\frac{1}{s}\left(-\frac{1}{s\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}-\nu\frac{\partial^2 w}{\partial s^2}\right)-\nu\frac{\partial^2 w}{\partial s^2}\right] \quad (39)$$

$$M_{s\theta}=M_{\theta s}=\frac{Eh^3}{12(1+\nu)}\frac{1}{s\sin\alpha}\left[\frac{1}{s}\frac{\partial w}{\partial\theta}-\frac{\partial^2 w}{\partial s\partial\theta}\right] \quad (40)$$

$$Q_s=\frac{1}{s}\left(M_s+s\frac{\partial M_s}{\partial s}\right)+\frac{1}{s\sin\alpha}\frac{\partial M_{\theta x}}{\partial\theta}-\frac{M_\theta}{s} \quad (41)$$

$$Q_\phi=\frac{1}{s\sin\alpha}\frac{\partial M_\theta}{\partial\theta}+\frac{1}{s}\left(M_{s\theta}+s\frac{\partial M_{s\theta}}{\partial s}\right)+\frac{M_{\theta s}}{s} \quad (42)$$

The relationship between radial displacement and the slope of conical shell satisfies

$$\varphi=\frac{\partial w}{\partial s} \quad (43)$$

All quantities are processed into dimensionless quantities and expanded to trigonometric series along the circumferential direction.

$$(u,w)=h\sum_{\alpha=0}^{1}\sum_{n}(\tilde{u},\tilde{w})\sin\left(n\theta+\frac{\alpha\pi}{2}\right) \quad (44)$$

$$v=h\sum_{\alpha=0}^{1}\sum_{n}\tilde{v}\cos\left(n\theta+\frac{\alpha\pi}{2}\right) \quad (45)$$

$$\varphi=\frac{h}{R}\sum_{\alpha=0}^{1}\sum_{n}\tilde{\varphi}\sin\left(n\theta+\frac{\alpha\pi}{2}\right) \quad (46)$$

$$(M_s,M_\theta)=\frac{K}{R}\sum_{\alpha=0}^{1}\sum_{n}\left(\tilde{M}_s,\tilde{M}_\theta\right)\sin\left(n\theta+\frac{\alpha\pi}{2}\right) \quad (47)$$

$$(M_{s\theta},M_{\theta s})=\frac{K}{R}\sum_{\alpha=0}^{1}\sum_{n}\left(\tilde{M}_{s\theta},\tilde{M}_{\theta s}\right)\cos\left(n\theta+\frac{\alpha\pi}{2}\right) \quad (48)$$

$$(N_{s\theta},N_{\theta s},Q_\theta,S_{s\theta})=\frac{K}{R^2}\sum_{\alpha=0}^{1}\sum_{n}\left(\tilde{N}_{s\theta},\tilde{N}_{\theta s},\tilde{Q}_\theta,\tilde{S}_{s\theta}\right)\cos\left(n\theta+\frac{\alpha\pi}{2}\right) \quad (49)$$

$$(N_s,N_\theta,Q_s,V_s)=\frac{K}{R^2}\sum_{\alpha=0}^{1}\sum_{n}\left(\tilde{N}_s,\tilde{N}_\theta,\tilde{Q}_s,\tilde{V}_s\right)\sin\left(n\theta+\frac{\alpha\pi}{2}\right) \quad (50)$$

where bending rigidity is $K=\frac{Eh^3}{12(1-\nu^2)}$, Young's modulus and Poisson's ratio are E and ν, respectively. n is the circumferential modal number. $\alpha=1$ and $\alpha=0$ are the symmetric or the antisymmetric modal, respectively. R is the radius at the larger end of the conical shell. h is the thickness of the conical shell. Other dimensionless quantities are presented as

$$\xi=\frac{s}{R} \quad (51)$$

$$\xi_1=\frac{L_{start}}{R} \quad (52)$$

$$\xi_2=\frac{L_{end}}{R} \quad (53)$$

$$\tilde{h}=\frac{h}{R} \quad (54)$$

$$\lambda^2 = \frac{\rho h R^2 \omega^2}{D} \tag{55}$$

s, L_{start}, L_{end} are described in Figure 2. ρ, ω, λ are, respectively, material density, circular frequency and the dimensionless frequency parameter. There are sixteen unknown quantities in the above equations. To eliminate eight unknown quantities ($M_\theta, M_{s\theta}, M_{\theta s}, N_{s\theta}, N_{\theta s}, Q_\theta, N_\theta, Q_s$), eight unknown quantities ($u, v, w, \varphi, M_s, V_s, N_s, S_{s\theta}$) are retained, which are the sectional state vector elements of the conical shell. Then, the first-order matrix differential equation of the conical shell is obtained.

$$\frac{d\left\{\boldsymbol{Z}(\xi)\right\}}{d\xi} = \boldsymbol{U}(\xi)\left\{\boldsymbol{Z}(\xi)\right\} + \left\{\boldsymbol{F}(\xi)\right\} - \left\{\boldsymbol{p}(\xi)\right\} \tag{56}$$

where $\left\{\boldsymbol{Z}(\xi)\right\} = \left\{\tilde{u}, \tilde{v}, \tilde{w}, \tilde{\varphi}, \tilde{M}_s, \tilde{V}_s, \tilde{N}_s, \tilde{S}_{s\theta}\right\}^T$ is the state vector of the conical shell. $\boldsymbol{U}(\xi)$ is the variable coefficient matrix, $\left\{\boldsymbol{F}(\xi)\right\} - \left\{\boldsymbol{p}(\xi)\right\}$ are exciting loads. The non-zero elements in $\boldsymbol{U}(\xi)$ are shown in Appendix B.

3 Solutions to equations of motion

Assuming that the exciting loads of Eqs. (30), (56) are zero, the equations of motion are simplified to

$$\frac{d\left\{\boldsymbol{Z}_{cy}(\xi)\right\}}{d\xi} = \mathbf{U_{cy}}(\xi)\left\{\boldsymbol{Z}_{cy}(\xi)\right\} \tag{57}$$

$$\frac{d\left\{\boldsymbol{Z}_{co}(\xi)\right\}}{d\xi} = \mathbf{U_{co}}(\xi)\left\{\boldsymbol{Z}_{co}(\xi)\right\} \tag{58}$$

Eqs. (57) and (58) are the equations of motion for the cylindrical and conical shell, respectively, which are dealt with as follows

$$\frac{d\left\{\boldsymbol{Z}(\xi)\right\}}{d\xi} = \mathbf{U}(\xi)\left\{\boldsymbol{Z}(\xi)\right\} \tag{59}$$

$$\int_{\xi_1}^{\xi} \frac{d\left\{\boldsymbol{Z}(\xi)\right\}}{\left\{\boldsymbol{Z}(\xi)\right\}} = \int_{\xi_1}^{\xi} \mathbf{U}(\tau)d\tau \tag{60}$$

$$\ln \boldsymbol{Z}(\xi)\Big|_{\xi_1}^{\xi} = \int_{\xi_1}^{\xi} \mathbf{U}(\tau)d\tau \tag{61}$$

$$\ln\left(\frac{\boldsymbol{Z}(\xi)}{\boldsymbol{Z}(\xi_1)}\right) = \int_{\xi_1}^{\xi} \mathbf{U}(\tau)d\tau \tag{62}$$

$$\boldsymbol{Z}(\xi) = \exp(\int_{\xi_1}^{\xi} \mathbf{U}(\tau)d\tau)\boldsymbol{Z}(\xi_1) \tag{63}$$

In the following chapters, the solution for the coefficient matrix $\exp\left(\int_{\xi_1}^{\xi} U(\tau)d\tau\right)$ using the precise integration method is presented.

3.1 Solutions for the coefficient matrix of the cylindrical shell

In the numerical calculation, the cylindrical shell is divided into a series of segments. The node coordinate of a segment is $\xi_k, k = i+1, i+2, i+3, \ldots$. Any coordinates of contiguous nodes are ξ_k and ξ_{k+1}, where $\xi_{k+1} = \xi_k + \Delta\xi$. The coefficient matrix $U(\xi)$ for the cylindrical shell is independent of ξ. Thus, the coefficient matrix in Eq. (63) can be written as

$$e^{U\Delta\xi} = \exp\left(\int_{\xi_k}^{\xi_{k+1}} \mathbf{U}(\tau)d\tau\right) \tag{64}$$

Assuming that

$$\boldsymbol{\Phi}_0(\Delta\xi) = e^{U\Delta\xi} = \exp(\boldsymbol{H})^{2^s} \tag{65}$$

where $\boldsymbol{H} = \boldsymbol{U}\frac{\Delta\xi}{2^s}$, and a value of 20 is recommended for s. $\exp(\boldsymbol{H})$ can be expressed in terms of the Taylor series by

$$\exp(\boldsymbol{H}) = \boldsymbol{I}_8 + \sum_{k=1}^{\infty}\frac{\boldsymbol{H}^k}{k!} = \boldsymbol{I}_8 + \boldsymbol{T}_a \tag{66}$$

where $\boldsymbol{I}_8$ is an eight-order unit matrix. Using the addition theorem directly to add $\boldsymbol{I}_8$ and $\boldsymbol{T}_a$ when $\boldsymbol{T}_a$ is small relative to $\boldsymbol{I}_8$, an error in the mantissa will occur due to computer rounding errors and leads to loss of precision. Therefore, this paper uses an addition theorem to calculate $\boldsymbol{T}_a$

$$\boldsymbol{\Phi}_0(\Delta\xi) = \left[(\boldsymbol{I}_8 + \boldsymbol{T}_a)(\boldsymbol{I}_8 + \boldsymbol{T}_a)\right]^{2^{s-1}} = \left[\boldsymbol{I}_8 + 2\boldsymbol{T}_a + \boldsymbol{T}_a^2\right]^{2^{s-1}} \tag{67}$$

$\boldsymbol{T}_a$ can be assumed as

$$\boldsymbol{T}_a = 2\boldsymbol{T}_a + \boldsymbol{T}_a^2 \tag{68}$$

After the N circulating assignment of Eq. (68), Eq. (67) can be written as

$$\boldsymbol{\Phi}_0(\Delta\xi) = e^{U\Delta\xi} = \boldsymbol{I}_8 + \boldsymbol{T}_a \tag{69}$$

Assuming segment coefficient matrix $\boldsymbol{T}_{k+1} = e^{U\Delta\xi}$, the relationship of the state vector of each node can be described as

$$\boldsymbol{Z}(\xi_{i+1}) = \boldsymbol{T}_{i+1}\boldsymbol{Z}(\xi_i) \tag{70}$$

$$\mathbf{Z}(\xi_{i+2}) = \boldsymbol{T}_{i+2}\boldsymbol{Z}(\xi_{i+1}) \tag{71}$$

$$\vdots$$

$$\vdots$$

$$\boldsymbol{Z}(\xi_{k+1}) = \boldsymbol{T}_{k+1}\boldsymbol{Z}(\xi_k) \tag{72}$$

$$\vdots$$

$$\vdots$$

$$\boldsymbol{Z}(\xi_n) = \boldsymbol{T}_n\boldsymbol{Z}(\xi_{n-1}) \tag{73}$$

3.2 Solutions for the coefficient matrix of the conical shell

To facilitate the numerical calculation, the conical shell is split into a series of segments along the generatrix direction. Eq. (63) can be written as

$$\boldsymbol{Z}(\xi_1) = \exp\left[\int_{\xi_0}^{\xi_1} \boldsymbol{U}(\tau)d\tau\right]\boldsymbol{Z}(\xi_0) \tag{74}$$

$$\boldsymbol{Z}(\xi_2) = \exp\left[\int_{\xi_1}^{\xi_2} \boldsymbol{U}(\tau)d\tau\right]\boldsymbol{Z}(\xi_1) \tag{75}$$

$$\vdots$$

$$\vdots$$

$$\boldsymbol{Z}(\xi_{j+1}) = \exp\left[\int_{\xi_j}^{\xi_{j+1}} \boldsymbol{U}(\tau)d\tau\right]\boldsymbol{Z}(\xi_j) \tag{76}$$

$$\vdots$$

$$\vdots$$

$$\boldsymbol{Z}(\xi_i) = \exp\left[\int_{\xi_{i-1}}^{\xi_i} \boldsymbol{U}(\tau)d\tau\right]\boldsymbol{Z}(\xi_{i-1}) \tag{77}$$

Assuming

$$\boldsymbol{T}_{j+1} = \exp\left[\int_{\xi_j}^{\xi_{j+1}} \boldsymbol{U}(\tau)d\tau\right] \tag{78}$$

Eqs. (75)-(78) can be described as

$$Z(\xi_1) = \boldsymbol{T}_1\boldsymbol{Z}(\xi_0) \tag{79}$$

$$\mathbf{Z}(\xi_2) = \boldsymbol{T}_2\boldsymbol{Z}(\xi_1) \tag{80}$$

$$\vdots$$

$$\vdots$$

$$Z(\xi_{j+1}) = \boldsymbol{T}_{j+1}\boldsymbol{Z}(\xi_j) \tag{81}$$

$$\vdots$$

$$\vdots$$

$$Z(\xi_i) = \boldsymbol{T}_i\boldsymbol{Z}(\xi_{i-1}) \tag{82}$$

The coefficient matrix $\mathbf{U}(\xi)$ for the conical shell is dependent on ξ. Therefore, the transfer matrix $\exp\left[\int_{\xi_j}^{\xi_{j+1}} U(\tau)d\tau\right]$ cannot be calculated like the transfer matrix for a cylindrical shell. This paper calculates the transfer matrix $\boldsymbol{T}_{j+1}$ for a conical shell by precise integration. Segments $\Delta\xi$ of the conical shell are divided into a precise integral step $\Delta\varsigma(\Delta\varsigma = \frac{\Delta\xi}{s})$. A value of 5 is recommended for s. For the segment $\xi_j \sim \xi_{j+1}$ of the conical shell, the integral step node is $\varsigma_k = \xi_j + k\left(\xi_{j+1} - \xi_j\right)/s = \xi_j + k\Delta\varsigma$, $k = 0, 1, .., s$. In a precise integral step, assuming $\tau = (\varsigma_{k-1} + \varsigma_k)/2$, $U(\tau)$ can be recognized to be the constant coefficient matrix, which is independent of ς. The variable coefficient matrix $\boldsymbol{T}_{j+1}$ in segment of the conical shell can be calculated through the constant coefficient matrix of integral steps tiered multiplication

$$\begin{aligned}\boldsymbol{T}_{j+1} &= \prod_{k=1}^{s} \exp\left[\boldsymbol{U}(\tau_k)(\varsigma_k - \varsigma_{k-1})\right] \\ &= \prod_{k=1}^{s} \exp\left[\boldsymbol{U}(\tau_k)\Delta\varsigma\right] = \prod_{k=1}^{s} \boldsymbol{T}_{j+1}^{k+1}\end{aligned} \tag{83}$$

The constant coefficient matrix $\boldsymbol{T}_{j+1}^{k+1} = \exp\left[\boldsymbol{U}(\tau_k)\Delta\varsigma\right]$ of the precise integral step can be solved like the reference method for solving $e^{U\Delta\xi}$ in section 3.1.

3.3 Solutions for the point matrix at the junction of the coupled conical-cylindrical shell

The displacements, slopes, forces and moments at the junction of the coupled conical-cylindrical shell should satisfy the following deformation compatibility conditions.

(a) At the junction of the coupled shell, the displacements and slopes $u_{co}, v_{co}, w_{co}, \varphi_{co}$ of the conical shell are in continuity with $u_{cy}, v_{cy}, w_{cy}, \varphi_{cy}$ of the cylindrical shell at the three coordinate axis.

(b) At the junction of the coupled shell, the forces and moments $N^s_{co}, S^{s\varphi}_{co}, N^s_{co}, M^s_{co}$ of the conical shell are equal to $M^s_{co}, S^{s\varphi}_{cy}, V^s_{cy}, M^s_{cy}$ of the cylindrical shell at the three coordinate axis.

According to the positive directions shown in the Figure 3. and Figure 4, the displacements, slopes, forces and moments of the conical and cylindrical shells at the junction satisfy

$$u_{cy} = u_{co} \cos\alpha - w_{co} \sin\alpha \tag{84}$$

$$v_{cy} = v_{co} \tag{85}$$

$$w_{cy} = u_{co} \sin\alpha + w_{co} \cos\alpha \tag{86}$$

$$\varphi_{cy} = \varphi_{co} \tag{87}$$

$$N^s_{cy} = N^s_{co} \cos\alpha + V^s_{co} \sin\alpha \tag{88}$$

$$S^{s\varphi}_{cy} = S^{s\varphi}_{co} \tag{89}$$

$$V^s_{cy} = -N^s_{co} \sin\alpha + V^s_{co} \cos\alpha \tag{90}$$

$$M^s_{cy} = M^s_{co} \tag{91}$$

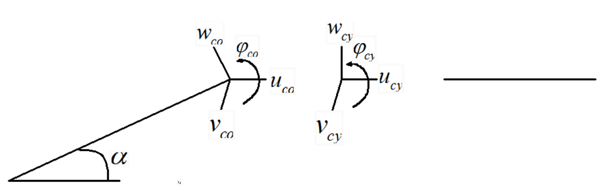

Figure 3: The positive directions for displacements and slopes of conical and cylindrical shells

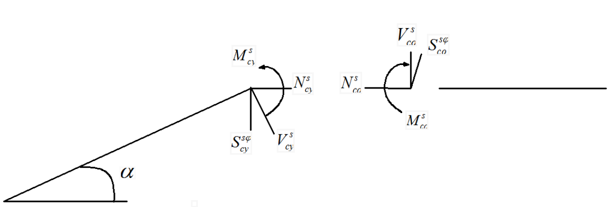

Figure 4: The positive directions for forces and moments of conical and cylindrical shells

To take into consideration displacements, slopes, forces and moments satisfying the continuity conditions at the junction, the relationship of the left end state vector and the right end state vector at the junction is

$$\boldsymbol{Z}(s = L^L_{cy}) = \mathbf{P}^{\mathbf{co}\to\mathbf{cy}} \boldsymbol{Z}(s = L^R_{co}) \tag{92}$$

Eq. (92) can be written as

$$\boldsymbol{Z}(\xi^{cy}_i) = \mathbf{P}^{\mathbf{co}\to\mathbf{cy}} \boldsymbol{Z}(\xi^{co}_i) \tag{93}$$

The point transfer matrix $\mathbf{P}^{\mathbf{co}\to\mathbf{cy}}$ can be described as

$$\mathbf{P}^{\mathbf{co}\to\mathbf{cy}} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos\alpha & 0 & -\sin\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{94}$$

3.4 Solutions for the coefficient matrix of the coupled shell

An illustration of the coupled conical-cylindrical shell is seen in Figure 5, where α is the semi-vertex conical angle. R is the radius of the cylindrical shell, which is also the larger end radius of the conical shell. L_s is the length from the top point of the conical shell to the smaller end of conical shell along the generatrix direction. L_e is the length from the top point of the conical shell to the larger end of the conical shell along the generatrix direction. The length of conical shell is $L_{co} = L_e - L_s$ and the length of the cylindrical shell is L_{cy}. The thickness of the coupled shell is h.

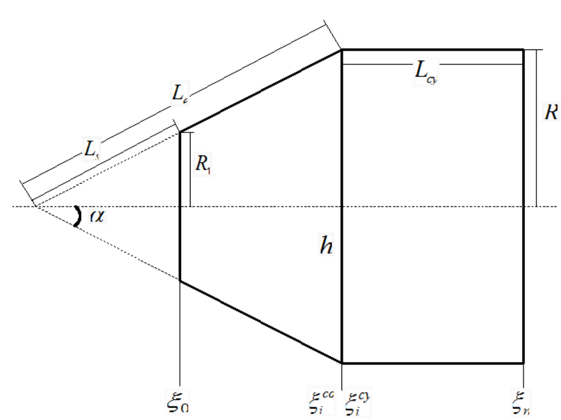

Figure 5: Illustration for the coupled conical-cylindrical shell

According to sections 3.1, 3.2, and 3.3, the state vector of the segment nodes from the coupled conical-cylindrical shell satisfy

$$\boldsymbol{Z}(\xi_1) = \boldsymbol{T}_1\boldsymbol{Z}(\xi_0) \tag{95}$$

$$\boldsymbol{Z}(\xi_2) = \boldsymbol{T}_2\boldsymbol{Z}(\xi_1) \tag{96}$$

$$\vdots$$

$$\vdots$$

$$\boldsymbol{Z}(\xi_i^{co}) = \boldsymbol{T}_i\boldsymbol{Z}(\xi_{i-1}) \tag{97}$$

$$\boldsymbol{Z}(\xi_i^{cy}) = \mathbf{P}^{\mathbf{co}\to\mathbf{cy}}\boldsymbol{Z}(\xi_i^{co}) \tag{98}$$

$$\boldsymbol{Z}(\xi_{i+1}) = \boldsymbol{T}_{i+1}\boldsymbol{Z}(\xi_i^{cy}) = \boldsymbol{T}_{i+1}\mathbf{P}^{\mathbf{co}\to\mathbf{cy}}\boldsymbol{Z}(\xi_i^{co}) \tag{99}$$

$$\vdots$$

$$\vdots$$

$$\boldsymbol{Z}(\xi_n) = \boldsymbol{T}_n\boldsymbol{Z}(\xi_{n-1}) \tag{100}$$

Eqs. (95)-(100) can be written in term of a matrix as follows

$$\begin{bmatrix} -\boldsymbol{T}_1 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\boldsymbol{T}_2 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\boldsymbol{T}_3 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\boldsymbol{T}_{i+1}\mathbf{P} & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\boldsymbol{T}_n & \boldsymbol{I} \end{bmatrix}_{(8n,8n+8)} \tag{101}$$

$$\cdot \begin{Bmatrix} \boldsymbol{Z}(\xi_0) \\ \boldsymbol{Z}(\xi_1) \\ \boldsymbol{Z}(\xi_2) \\ \vdots \\ \boldsymbol{Z}(\xi_i) \\ \vdots \\ \boldsymbol{Z}(\xi_n) \end{Bmatrix}_{(8n+8,1)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

The boundary conditions at the ends of the coupled conical-cylindrical shell are given by
(Note: F is a free end. S is a simply supported end. C is a clamped end.)
F-F: $\overline{M_x} = \overline{V_x} = \overline{S_{x\varphi}} = \overline{N_x} = 0$

$$\boldsymbol{Z}(\xi_0) = \left\{\overline{u}^0 \quad \overline{v}^0 \quad \overline{w}^0 \quad \overline{\psi}^0 \quad 0 \quad 0 \quad 0 \quad 0\right\}^T \tag{102}$$

$$\boldsymbol{Z}(\xi_n) = \left\{\overline{u}^n \quad \overline{v}^n \quad \overline{w}^n \quad \overline{\psi}^n \quad 0 \quad 0 \quad 0 \quad 0\right\}^T \tag{103}$$

S-S: $\overline{v} = \overline{w} = \overline{M_x} = \overline{N_x} = 0$

$$\boldsymbol{Z}(\xi_0) = \left\{\overline{u}^0 \quad 0 \quad 0 \quad \overline{\psi}^0 \quad 0 \quad \overline{V_x}^0 \quad \overline{S_{x\varphi}}^0 \quad 0\right\}^T \tag{104}$$

$$\boldsymbol{Z}(\xi_n) = \left\{\overline{u}^n \quad 0 \quad 0 \quad \overline{\psi}^n \quad 0 \quad \overline{V_x}^n \quad \overline{S_{x\varphi}}^n \quad 0\right\}^T \tag{105}$$

C-C: $\overline{u} = \overline{v} = \overline{w} = \overline{\psi} = 0$

$$\boldsymbol{Z}(\xi_0) = \left\{0 \quad 0 \quad 0 \quad 0 \quad \overline{M_x}^0 \quad \overline{V_x}^0 \quad \overline{S_{x\varphi}}^0 \quad \overline{N_x}^0\right\}^T \tag{106}$$

$$\boldsymbol{Z}(\xi_n) = \left\{0 \quad 0 \quad 0 \quad 0 \quad \overline{M_x}^n \quad \overline{V_x}^n \quad \overline{S_{x\varphi}}^n \quad \overline{N_x}^n\right\}^T \tag{107}$$

F-S:

$$\boldsymbol{Z}(\xi_0) = \left\{\overline{u}^0 \quad \overline{v}^0 \quad \overline{w}^0 \quad \overline{\psi}^0 \quad 0 \quad 0 \quad 0 \quad 0\right\}^T \tag{108}$$

$$\boldsymbol{Z}(\xi_n) = \left\{\overline{u}^n \quad 0 \quad 0 \quad \overline{\psi}^n \quad 0 \quad \overline{V_x}^n \quad \overline{S_{x\varphi}}^n \quad 0\right\}^T \tag{109}$$

F-C:

$$\boldsymbol{Z}(\xi_0) = \left\{\overline{u}^0 \quad \overline{v}^0 \quad \overline{w}^0 \quad \overline{\psi}^0 \quad 0 \quad 0 \quad 0 \quad 0\right\}^T \tag{110}$$

$$\boldsymbol{Z}(\xi_n) = \left\{0 \quad 0 \quad 0 \quad 0 \quad \overline{M_x}^n \quad \overline{V_x}^n \quad \overline{S_{x\varphi}}^n \quad \overline{N_x}^n\right\}^T \tag{111}$$

S-F

$$\boldsymbol{Z}(\xi_0) = \left\{\overline{u}^0 \quad 0 \quad 0 \quad \overline{\psi}^0 \quad 0 \quad \overline{V_x}^0 \quad \overline{S_{x\varphi}}^0 \quad 0\right\}^T \tag{112}$$

$$\boldsymbol{Z}(\xi_n) = \left\{\overline{u}^n \quad \overline{v}^n \quad \overline{w}^n \quad \overline{\psi}^n \quad 0 \quad 0 \quad 0 \quad 0\right\}^T \tag{113}$$

S-C

$$\boldsymbol{Z}(\xi_0) = \left\{\overline{u}^0 \quad 0 \quad 0 \quad \overline{\psi}^0 \quad 0 \quad \overline{V_x}^0 \quad \overline{S_{x\varphi}}^0 \quad 0\right\}^T \tag{114}$$

$$\boldsymbol{Z}(\xi_n) = \left\{0 \quad 0 \quad 0 \quad 0 \quad \overline{M_x}^n \quad \overline{V_x}^n \quad \overline{S_{x\varphi}}^n \quad \overline{N_x}^n\right\}^T \tag{115}$$

C-F

$$\boldsymbol{Z}(\xi_0) = \left\{0 \quad 0 \quad 0 \quad 0 \quad \overline{M_x}^0 \quad \overline{V_x}^0 \quad \overline{S_{x\varphi}}^0 \quad \overline{N_x}^0\right\}^T \tag{116}$$

$$\boldsymbol{Z}(\xi_n) = \left\{\overline{u}^n \quad \overline{v}^n \quad \overline{w}^n \quad \overline{\psi}^n \quad 0 \quad 0 \quad 0 \quad 0\right\}^T \tag{117}$$

C-F

$$\boldsymbol{Z}(\xi_0) = \left\{ 0 \quad 0 \quad 0 \quad 0 \quad \overline{M_x}^0 \quad \overline{V_x}^0 \quad \overline{S_{\varphi x}}^0 \quad \overline{N_x}^0 \quad \right\}^T \tag{118}$$

$$\boldsymbol{Z}(\xi_{\text{n}}) = \left\{ \overline{u}^n \quad 0 \quad 0 \quad \overline{\psi}^n \quad 0 \quad \overline{V_x}^n \quad \overline{S_{x\varphi}}^n \quad 0 \right\}^T \tag{119}$$

According to the given boundary conditions at the ends of the coupled shell, row numbers where elements of the state vector are zero are found. Then, to delete corresponding columns of coefficient matrix, Eq. (101) can be written as

$$\begin{bmatrix} -\hat{\boldsymbol{T}}_1 & \boldsymbol{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\boldsymbol{T}_2 & \boldsymbol{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\boldsymbol{T}_3 & \boldsymbol{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -\boldsymbol{T}_{i+1}\mathbf{P} & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\boldsymbol{T}_n & \hat{\boldsymbol{I}} \end{bmatrix}_{(8n,8n)} \tag{120}$$

$$\cdot \begin{Bmatrix} \hat{\boldsymbol{Z}}(\xi_0) \\ \hat{\boldsymbol{Z}}(\xi_1) \\ \hat{\boldsymbol{Z}}(\xi_2) \\ \vdots \\ \hat{\boldsymbol{Z}}(\xi_i) \\ \vdots \\ \hat{\boldsymbol{Z}}(\xi_n) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{Bmatrix}_{(8n,1)}$$

Since the state vectors cannot all be zero vectors, the determinant of the coefficient matrix must be zero. The following equation is obtained.

$$\begin{vmatrix} -\hat{\boldsymbol{T}}_1 & \boldsymbol{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\boldsymbol{T}_2 & \boldsymbol{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\boldsymbol{T}_3 & \boldsymbol{I} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\boldsymbol{T}_{i+1}\mathbf{P} & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\boldsymbol{T}_n & \hat{\boldsymbol{I}} \end{vmatrix}_{(8n,8n)} = 0 \tag{121}$$

The natural frequency ω of the coupled conical-cylindrical shell is the only unknown quantity in the matrix **T** and is obtained through solving the frequency characteristic Eq. (121). By substituting the natural frequency into Eq. (120), the proportional relationship of state vectors can be obtained. Then, the modes of the coupled shell will be acquired in the given boundary condition.

4 Confirmation of the PITMM and numerical analysis

4.1 Confirmation of the PITMM

4.1.1 Results comparison of the PITMM to the previous literature

To verify the validity of the PITMM that is used to research free vibration characteristics of isotropic coupled conical-cylindrical shells, the results from the PITMM are compared with the results from the references [13, 14, 17]. The geometrical parameter of the coupled conical-cylindrical shells is $L/a = 1, h/a = 0.01, R_1/a = 0.4226, \alpha = 30°$, where L is the length of the cylindrical shell, is the radius of the smaller end of the conical shell, a is the radius of the cylindrical shell, and $\alpha$is the semi-vertex angle. $\lambda = a\omega(\rho h/D)^{\frac{1}{2}}$ is the dimensionless frequency parameter. The material parameters are Poisson's ratio $\upsilon = 0.3$, Young's modulus $E = 2.11 \times 10^{11} N/m^2$, and Density $\rho = 7800 kg/m^3$. The boundary condition of the coupled conical-cylindrical shells is free-simply.

The axial modal number is m=1, the circumferential modal number is n=0:5, and the dimensionless frequency parameter of the coupled conical-cylindrical shells are presented in Table 1. The values of the frequency parameter from the PITMM agree well with the references [13, 14, 17]. In Table 1, m=T represents a purely torsional modal. However, in reference [13], Irie did not take it into account. M. Caresta [14] and E. Efraim [17] show a purely torsional modal. The frequency parameter corresponding to the purely torsional modal is solved by the PITMM in this paper.

4.1.2 Comparison of the PITMM and FEM

Comparing the results from the PITMM with the FEM, the validity of the PITMM proposed in this paper is further verified, which is examined in Figure 9. It can be observed from Figure 9 that the results from the PITMM match almost perfectly with the results from the FEM. The results of the PITMM and FEM only show a slight difference at n=0, n=5. Therefore, the PITMM given by this paper can accurately calculate the natural frequencies of coupled conical-cylindrical shells. Furthermore, the PITMM is not limited by the boundary conditions at the ends of the coupled shell.

The author established the finite model for the coupled conical-cylindrical shell by using ABAQUS. In three

Table 1: Frequency parameters for the coupled shells in a free-clamped boundary condition

Modal number		Frequency parameter λ			
n	m	Reference [13]	Reference [14] (Flügge)	Reference [17]	PITMM
0	1	0.5047	0.505354	0.503779	0.5037
	T	-	0.609816	0.609852	0.6098
	2	0.9312	0.930904	0.930942	0.9309
	3	0.9566	0.956292	0.956379	0.9563
	4	0.9718	0.971538	0.971634	0.9715
	5	1.0122	1.011873	1.012090	1.0119
1	1	0.2930	0.293357	0.292875	0.2928
	2	0.6368	0.636844	0.635834	0.6356
	3	0.8116	0.811434	0.811454	0.8113
	4	0.9316	0.931458	0.931565	0.9310
	5	0.9528	0.952120	0.952178	0.9520
	6	0.9922	0.991936	0.992175	0.9909
2	1	0.1010	0.100087	0.099968	0.1005
	2	0.5032	0.502819	0.502701	0.5020
	3	0.6916	0.691353	0.691305	0.6910
	4	0.8592	0.858971	0.859114	0.8570
	5	0.9164	0.915877	0.915870	0.9153
	6	0.9608	0.960429	0.960702	0.9562
3	1	0.09076	0.087330	0.087603	0.0879
	2	0.3921	0.391450	0.391569	0.3900
	3	0.5148	0.514424	0.514478	0.5135
	4	0.7537	0.753295	0.753402	0.7492
	5	0.7970	0.796557	0.796590	0.7942
	6	0.9197	0.919369	0.919635	0.9190
4	1	0.1477	0.144478	0.144619	0.1422
	2	0.3312	0.330177	0.330354	0.3368
	3	0.3965	0.395495	0.395649	0.3929
	4	0.6473	0.646548	0.646678	0.6408
	5	0.6932	0.692690	0.692805	0.6925
	6	0.8720	0.871532	0.871812	0.8719
5	1	0.2021	0.199540	0.199546	0.1993
	2	0.2966	0.295939	0.296020	0.2970
	3	0.3730	0.370707	0.370901	0.3754
	4	0.5805	0.579581	0.579750	0.5802
	5	0.6138	0.613231	0.613363	0.6139
	6	0.8187	0.818014	0.817951	0.8170

different boundary conditions, clamped-free, free-simply, and simply-clamped, corresponding to axial modal number m=1 and circumferential modal number n=0:5, the frequency parameters and modal shapes are shown in Figures 6-8.

4.2 Numerical analysis

4.2.1 Effect of boundary conditions on the free vibration characteristic

The influences of boundary conditions on the free vibration characteristics of a coupled shell are shown in Figure 9. In three different boundary conditions, free-simply,

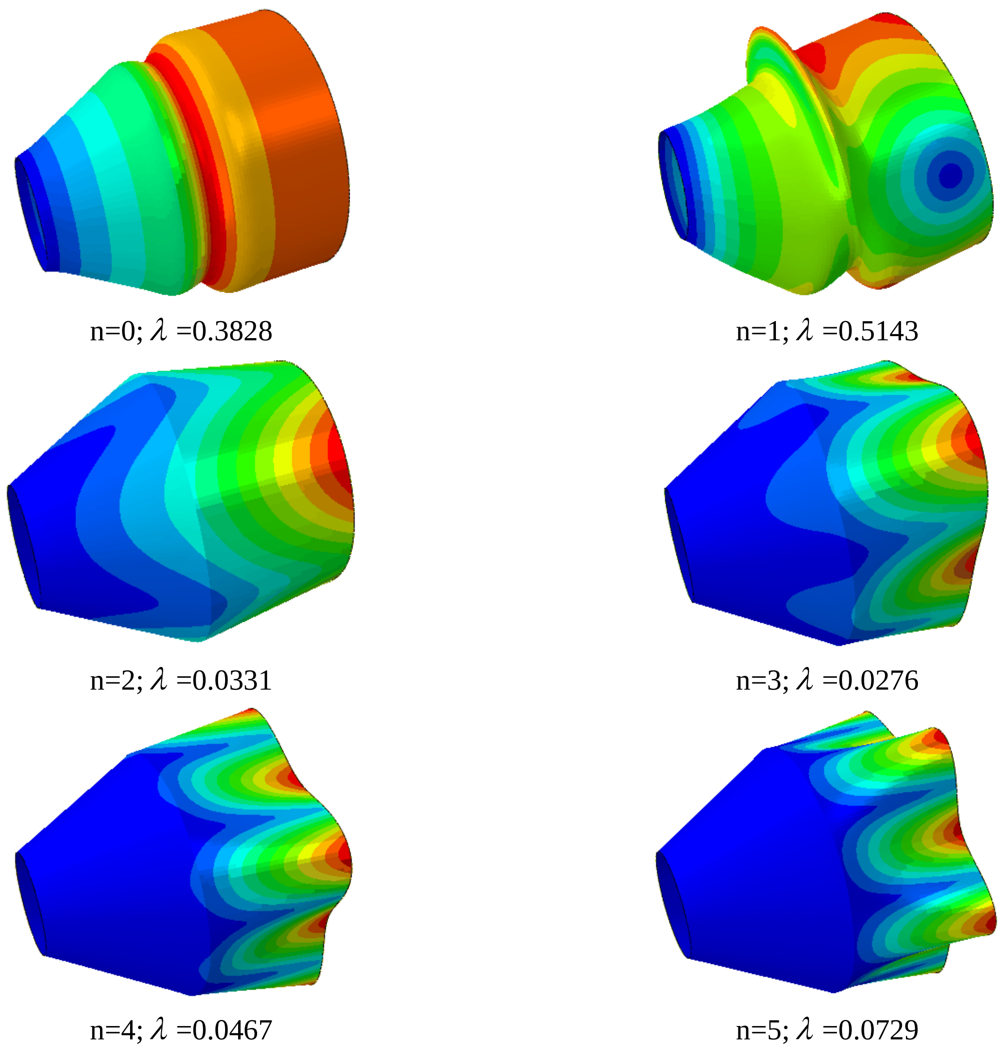

Figure 6: m=1, n=0:5 modal shape in C-S condition

simply-clamped, and clamped-free, corresponding to axial modal number m=1, the values of the frequency parameter varying with circumferential modal number n are plotted. It can be seen that the frequency parameter curve is relatively flat and that the frequency parameter decreases with an n increasing in the simply-clamped boundary condition. In the free-simply and clamped-free boundary conditions, the frequency parameter curves intensely vary with n and reach the minimum value at n=2. After n=2, both of the curves rise. Corresponding to n=1:5, in the free-simply and clamped-free boundary conditions, almost all the frequency parameter values are less than the values in the simply-clamped boundary condition.

4.2.2 Effect of shell thickness on the free vibration characteristic

Figure 10 presents the influence of the shell thickness h of the coupled shell on the frequency parameter in the clamped-clamped boundary condition. With the increasing shell thickness, the frequency parameter slowly increases corresponding to the circumferential modal number n=0. However, the frequency parameter rapidly increases corresponding to n=5. The stiffness increases with the increasing of the shell thickness. The stiffness effect dominates. Therefore, the frequency parameter of the coupled conical-cylindrical shell increases with the increase in the shell thickness.

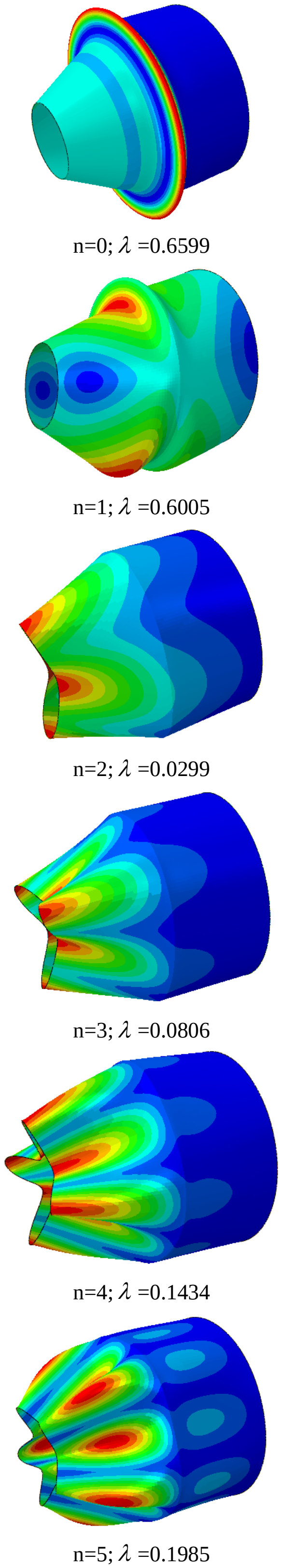

Figure 7: m=1, n=0:5 modal shape in F-S condition

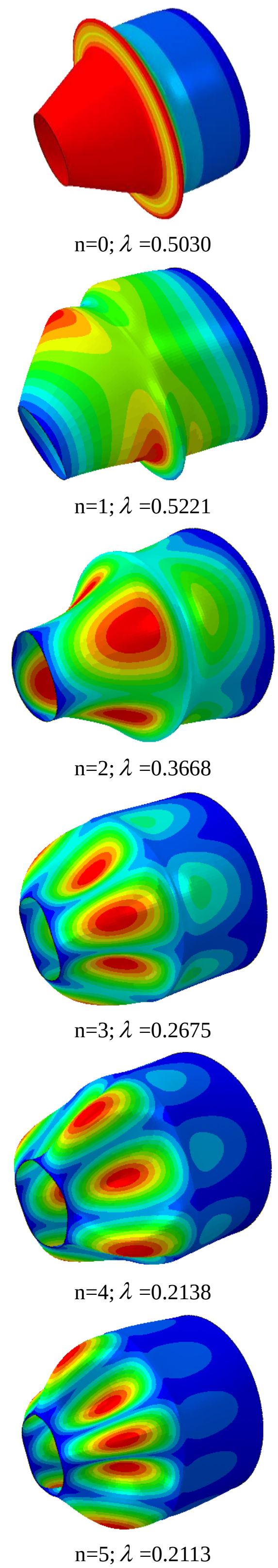

Figure 8: m=1, n=0:5 modal shape in S-C condition

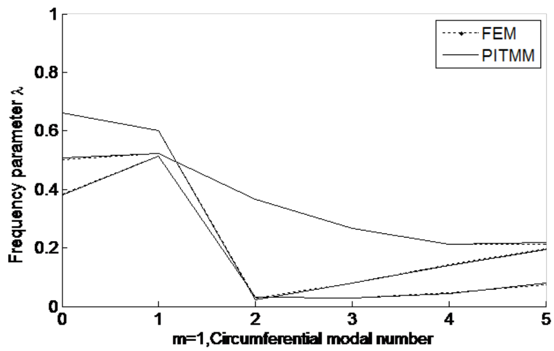

Figure 9: Comparison of the frequency parameterλ from the PITMM and FEM in different boundary conditions

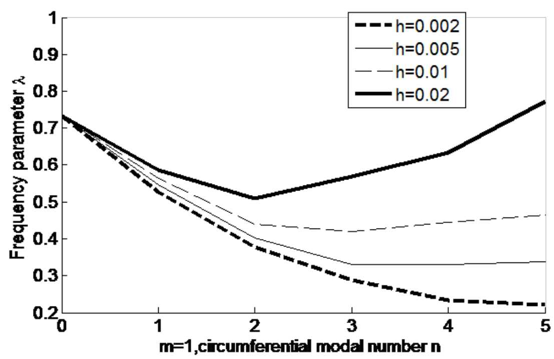

Figure 10: Effects of shell thickness on the frequency parameter λ in the C-C boundary condition

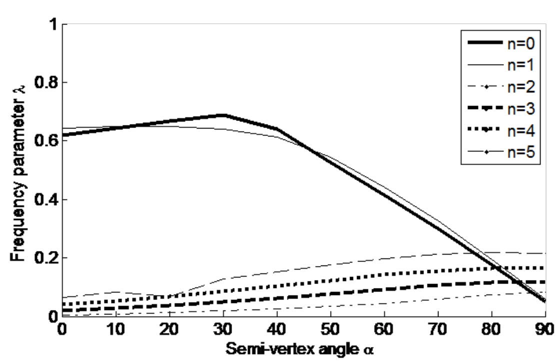

Figure 11: m=1 frequency parameter λ varying with the semi-vertex angle in the F-S boundary condition

4.2.3 Effect of the semi-vertex angle on the free vibration characteristic

When the influence of the semi-vertex conical angle on the frequency parameter is investigated, the structural parameters for the coupled conical-cylindrical shells are $a = 0.2m$, $h = 0.002m$, $L_{cy} = 0.2m, L_{co} = 0.1m$, and $\alpha = 0^\circ - 90^\circ$. The material parameters are Poisson's ratio ,

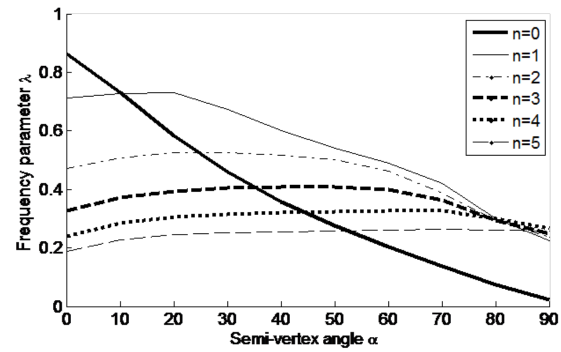

Figure 12: m=1 frequency parameter λ varying with the semi-vertex angle in the C-S boundary condition

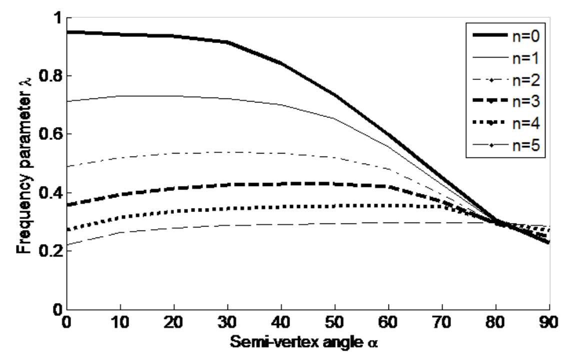

Figure 13: m=1 frequency parameter λ varying with the semi-vertex angle in the C-C boundary condition

Young's modulus $E = 2.11 \times 10^{11} N/m^2$, and density $\rho = 7800 kg/m^3$. Figures 11-13 present the effect of the semi-vertex conical angle on the frequency parameter for the coupled conical-cylindrical shells in free-simply, clamped-simply, and clamped-clamped boundary conditions. The conical shell is recognized as a cylindrical shell and the circular plate at the extreme semi-vertex conical angle $\alpha = 0^\circ$ and $\alpha = 90^\circ$. The values of the frequency parameter correspond to axial modal number m=1. In Figure 11 of the free-simply boundary condition, n=0:1, with a semi-vertex angle α varying from 0° to, the frequency parameter gradually increases and then decreases after $\alpha = 30^\circ$. A decrease in axial stiffness occurs resulting in a decrease in the frequency parameter. A similar behaviour to the n=0 mode is observed for the n=1 bending mode. The curve of the frequency parameter gradually drops, for n=2:5. In Figures 12-13 of the clamped-simply and clamped-clamped boundary conditions, n=0, the frequency parameter quickly decreases and reaches a minimum value at $\alpha = 90^\circ$. The frequency parameter curves initially rise and then decline, corresponding to n=1:5. For n=1:5 as light increase of the frequency parameter with α is observed, showing a small mass effect. A stiffening effect then dominates after $\alpha = 70^\circ$. The higher order circumferential modes result in a wavelike behaviour due to greater shape complexity.

5 Conclusions

A new method, PITMM, is introduced in this paper to research the free vibration characteristics of isotropic coupled conical-cylindrical shells. Based on the traditional transfer matrix and precise integration methods, the PITMM is constructed. The method not only retains the traditional transfer matrix methods' advantages of formula regularity and easy programming but also obtains the high accuracy from the precise integration methods. The accuracy of results solved by the PITMM rises, which can be observed from the comparison of the results from the previous research, FEM, and PITMM. Based on the PITMM, the effects of the boundary conditions, the shell thickness and the semi-vertex conical angle on the free vibration characteristics of the coupled conical-cylindrical shells are examined. The following observations can be made:

1. Corresponding to n=1:5, in the free-simply and clamped-free boundary conditions, almost all the values of the frequency parameter are less than the values in the simply-clamped boundary condition.
2. The frequency parameter of the coupled conical-cylindrical shell increases with an increase in the shell thickness.
3. The semi-vertex angle significantly impacts the frequency parameter corresponding to n=0:1. However, it has little effect on the frequency parameter corresponding to n=2:5. Furthermore, when $\alpha = 80^\circ$, the frequency parameter values tend towards fixed values corresponding to n=1:5.

Based on the characteristics of the PITMM, the method can also be extended to solve the free vibration problem about variable thickness of the cylindrical shell, reinforced cone shells and other rotating body structures.

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References

[1] Harari A. Wave propagation in cylindrical shells with finite regions of structural discontinuity [J]. Journal of the Acoustical Society of America, 1977, 62: 1196-1205.
[2] Harari A. Wave propagation in a cylindrical shell with joint discontinuity [J]. Shock and Vibration. Bulletin, 1978, 48: 52-61.
[3] Harari A., Sandman B.E. Radiation and vibrational properties of submerged stiffened cylindrical shells [J]. Journal of the Acoustical Society of America, 1990, 88(4):1817-1830.
[4] Sandman B. E. Fluid-loaded influence coefficients for a finite cylindrical shell [J]. Journal of the Acoustical Society of America, 1976, 60(6):1256-1264.
[5] Koutunvov VB. Dynamic stability and nonlinear parametric vibration of cylindrical shells [J]. Comput Struct,1993,46(1):149-56.
[6] Rayleigh J. The theory of Sound [M]. New York: Dover Publication, 1945.
[7] Leissa A. W. Vibration of Shells, NASA SP-288[R]. Washington: U.S. Government Printing Office, 1973.
[8] Greenwelge Jr. O. E., Muster D. Free vibrations of ring-and-stringer-stiffened conical shells [J]. Journal of the Acoustical Society of America, 1969, 46 (1, 2): 176-185.
[9] Talebitooti M., Ghayour M., Ziaei-Rad S., et al. Free vibrations of rotating composite conical shells with stringer and ring stiffeners [J]. Archive of Applied Mechanics, 2010, 8: 1-15.
[10] Li F. M., Kishimoto K., Huang W. H. The calculations of natural frequencies and forced vibration responses of conical shell using the rayleigh-ritz method [J]. Mechanics Research Communications, 2009, 36 (5): 595-602.
[11] Liew K. M., Ng T. Y., Zhao X. Free vibration analysis of conical shells via the element-free kp-ritz method [J]. Journal of Sound and Vibration, 2005, 281(3-5): 627-645.
[12] Guo Y. P. Normal mode propagation on conical shells [J]. Journal of the Acoustical Society of America, 1994, 96(1): 256-264.
[13] Irie T., Yamada G., Muramoto Y. Free vibration of joined conical-cylindrical shells [J]. Journal of Sound and Vibration, 1994, 95(l): 31-39.
[14] M.Caresta, N.J.Kessissoglou [J]. Journal of Sound and Vibration, 329(2010): 733-751.
[15] W.Flugge. Stresses in Shells [M]. Berlin: Siegfried Publication, 1973.
[16] Trie T., Yamada G., Kaneko Y. Free vibration of a conical shell with variable thickness [J]. Journal of Sound and Vibration, 1982, 82(1): 83-94.
[17] E.Efraim, M.Eisenberger. Exact vibration characteristics of laminated composite jointed conical-cylindrical shells [J]. Journal of Sound and Vibration, 237(2000): 920-930.

Appendix A. Coefficient matrix $U(\xi)$ of the cylindrical shell

$$U_{12} = U_{75} = -U_{78} = -\mu n \quad (A.1)$$

$$U_{13} = U_{68} = -\mu \quad (A.2)$$

$$U_{18} = \frac{\tilde{h}}{12} \quad (A.3)$$

$$U_{21} = -U_{87} = n \quad (A.4)$$

$$U_{24} = -\frac{n\tilde{h}^2}{6} \quad (A.5)$$

$$U_{27} = \frac{\tilde{h}}{6(1-\mu)} \quad (A.6)$$

$$U_{34} = U_{56} = 1 \quad (A.7)$$

$$U_{43} = U_{65} = \mu n^2 \quad (A.8)$$

$$U_{45} = \frac{1}{\tilde{h}} \quad (A.9)$$

$$U_{54} = 2(1-\mu)n^2\tilde{h} \quad (A.10)$$

$$U_{62} = -\frac{12(1-\mu^2)n}{\tilde{h}} \quad (A.11)$$

$$U_{63} = -(12 + n^4\tilde{h}^2)\frac{1-\nu^2}{\tilde{h}} + \frac{12\lambda^2}{\tilde{h}} \quad (A.12)$$

$$U_{72} = \frac{12}{\tilde{h}}\left\{(1-\nu^2)n^2 - \lambda^2\right\} \quad (A.13)$$

$$U_{73} = (12 + n^2\tilde{h}^2)\frac{(1-\nu^2)n}{\tilde{h}} \quad (A.14)$$

$$U_{81} = -\frac{12\lambda^2}{\tilde{h}} \quad (A.15)$$

$$U_{84} = (1-\nu)n^2\tilde{h} \quad (A.16)$$

Appendix B. Coefficient matrix of the conical shell

$$U_{11} = U_{44} = -\nu\frac{1}{\xi} \quad (B.1)$$

$$U_{12} = -U_{78} = -\frac{\nu n}{\sin\alpha}\frac{1}{\xi} \quad (B.2)$$

$$U_{13} = U_{68} = -\frac{\nu}{\xi\tan\alpha} \quad (B.3)$$

$$U_{18} = \frac{h}{12} \quad (B.4)$$

$$U_{21} = -U_{87} = \frac{n}{\xi\sin\alpha} \quad (B.5)$$

$$U_{22} = -U_{66} = \frac{1}{\xi} \quad (B.6)$$

$$U_{23} = \frac{nh^2}{6\xi^3\tan\alpha\sin\alpha} \quad (B.7)$$

$$U_{24} = -\frac{nh^2}{6\xi^2\tan\alpha\sin\alpha} \quad (B.8)$$

$$U_{27} = \frac{h}{6(1-\nu)} \quad (B.9)$$

$$U_{34} = U_{56} = 1 \quad (B.10)$$

$$U_{43} = U_{65} = \frac{\nu n^2}{\xi^2\sin^2\alpha} \quad (B.11)$$

$$U_{43} = \frac{1}{h} \quad (B.12)$$

$$U_{53} = -U_{64} = -\frac{(3+\nu)(1-\nu)n^2h}{\xi^3\sin^2\alpha} \quad (B.13)$$

$$U_{54} = (1-\nu)(1+\nu+\frac{2n^2}{\sin^2\alpha})\frac{h}{\xi^2} \quad (B.14)$$

$$U_{55} = U_{88} = -(1-\nu)\frac{1}{\xi} \quad (B.15)$$

$$U_{61} = -\frac{12(1-\nu^2)}{\xi^2 h\tan\alpha} \quad (B.16)$$

$$\boldsymbol{U}_{62} = -\frac{12n(1-\nu^2)}{\xi^2 h \tan\alpha \sin\alpha} \tag{B.17}$$

$$\boldsymbol{U}_{63} = -\frac{12(1-\nu^2)}{\xi^2 h \tan^2\alpha} - \left\{2 + \frac{n^2(1+\nu^2)}{\sin^2\alpha}\right\} \frac{(1-\nu^2)n^2 h}{\xi^4 \sin^2\alpha} + \frac{12\lambda^2}{h} \tag{B.18}$$

$$\boldsymbol{U}_{71} = \boldsymbol{U}_{82} = \frac{12(1-\nu^2)n}{\xi^2 h \sin\alpha} \tag{B.19}$$

$$\boldsymbol{U}_{72} = \frac{12(1-\nu^2)n^2}{\xi^2 h \sin\alpha} - \frac{12\lambda^2}{h} \tag{B.20}$$

$$\boldsymbol{U}_{73} = \frac{(1-\nu)n}{\xi^2 \tan\alpha \sin\alpha} \left[\frac{12(1+\nu)}{h} + \left\{1 + \frac{(1+\nu)n^2}{\sin^2\alpha}\right\} \frac{h}{\xi^2}\right] \tag{B.21}$$

$$\boldsymbol{U}_{74} = -\frac{nh(1-\nu)(2+\nu)}{\xi^3 \tan\alpha \sin\alpha} \tag{B.22}$$

$$\boldsymbol{U}_{75} = -\frac{\nu n}{\xi^2 \tan\alpha \sin\alpha} \tag{B.23}$$

$$\boldsymbol{U}_{77} = -\frac{2}{\xi} \tag{B.24}$$

$$U_{81} = \left\{(1-\nu^2)\frac{1}{\xi^2} - \lambda^2\right\} \frac{12}{h} \tag{B.25}$$

$$U_{83} = \left\{\frac{12(1+\nu)}{h} - \frac{n^2 h}{\xi^2 \sin^2\alpha}\right\} \frac{1-\nu}{\xi^2 \tan\alpha} \tag{B.26}$$

$$U_{84} = \frac{(1-\nu)n^2 h}{\xi^3 \tan\alpha \sin^2\alpha} \tag{B.27}$$