Research Article

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Free Vibration of Single Walled Carbon Nanotube Resting on Exponentially Varying Elastic Foundation

<https://doi.org/10.1515/cls-2018-0019>

Received May 28, 2018; accepted Jun 21, 2018

Abstract: In this article, free vibration of Single Walled Carbon Nanotube (SWCNT) resting on exponentially varying Winkler elastic foundation is investigated by using Differential Quadrature Method (DQM). Euler–Bernoulli beam theory is considered in conjunction with the nonlocal elasticity theory of Eringen. Step by step procedure is included and MATLAB code has been developed to obtain the numerical results for different scaling parameters as well as for four types of edge conditions. Obtained results are validated with known results in special cases showing good agreement. Further, numerical as well as graphical results are illustrated to show the effects of nonuniform parameter, nonlocal parameter, aspect ratio, Winkler modulus parameter and edge conditions on the frequency parameters.

Keywords: SWCNT; Euler–Bernoulli beam theory; DQM; Vibration; Exponentially Varying Elastic Foundation

1 Introduction

Application of nanomaterials has expanded in the area of physics, chemistry, engineering and nanotechnology because of their special properties like mechanical, electrical and electronic [1]. As a result of these properties, nanomaterials play very significant roles in various nanomechanical systems and nanomaterials. These materials include nanowires, nanoparticles, nanotubes, nanotube resonators [2], nanoactuators [3] etc. Various applications concerning CNT reinforced structure like reinforced beam, plate etc. can be found in the literatures [4–7]. one may also get detail information about nanobeams and

nanoplates in the book [8]. Among these nanostructures, single walled carbon nanotubes viz. nanobeams attract more attention due to their great potential in engineering applications. So, one must have appropriate knowledge about the mechanical behaviors of nanobeams for accurate prediction of vibration characteristics. So far, many researchers have been developed various nonclassical continuum theories such as couple stress theory, strain gradient theory, micropolar theory and nonlocal elasticity theory. Out of these nonclassical continuum theories, nonlocal elasticity theory which was developed by Eringen [9], has been extensively used in the vibration of nanobeams.

During the past years, various elastic foundation models such as Winkler, Pasternak and Flonenko–Borodich foundations have been introduced. It is worth mentioning that the Winkler foundation model has been suggested in 1867 by Winkler which has become one of the fundamental scenario in this regard. Linear algebraic relationship between the normal displacement and the contact pressure [10] is assumed in the modeling of this foundation. This assumption makes it simple to get the closed-form solutions [11, 12]. On the other hand, it also gets a nonlinear behavior with less computations in comparison with other methods [13–17]. One may see different studies on Winkler elastic foundation in important papers viz. [18, 19]. Finite and infinite beams on a Winkler support have been addressed by Auersch [20]. As regards, Eisenberger and Clastornik [21] studied vibration and buckling of a beam resting on variable Winkler elastic foundation. Also, dynamic behavior of infinite beam has been investigated by Ruge and Birk [22]. Next, Lee [23] studied dynamic response of a Timoshenko beam subject to a moving concentrated mass. Finite strip method has been used by Huang and Thambiratnam [24] to investigate deflection of plates having moving accelerated loads resting on Winkler foundation. Further, Oz and Pakdemirli [25] has also examined resonances of shallow beams with elastic foundations. Elastic foundations with Winkler–Pasternak problems may also be found in [26, 27]. It may be noted that structural problems with elastic foundation have also been solved

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by few other methods such as differential transform [28], Rayleigh–Ritz [29] and Harmonic DQ [30, 31].

For the first time Bellman and Casti [32] introduced Differential Quadrature (DQ) method in the year 1971. One may use this powerful technique for solving linear and nonlinear differential equations arising in various dynamic problems. Later, Bert *et al.* [33] used this method for solving dynamical problem arising in the field of structural dynamics. Since then, this method is applied by various researchers for solving different types of structural problems such as linear or nonlinear. Different authors introduced different procedures to use edge conditions in the DQ method. In this regard, for the first time Bert *et al.* [34] proposed δ technique to use edge conditions. In this case [35], one may observe that one edge condition is used at the boundary point where as other edge condition is at a distance δ from the boundary point. This δ technique may be suitable for c-c edge but this is not useful for S-S and S-C edge conditions. In order to eradicate the shortcomings of the above approach, Bert further presented a new technique in applying edge conditions which may be found in [36–40]. In this technique, one just need to implement one boundary condition numerically while the other edge condition can be obtained from the DQ weighting coefficient matrices. Similarly, Tornabene *et al.* surveyed several methods under the heading of strong formulation finite element method (SFEM) which can be found in [41].

Though, some of the studies have been done on Winkler elastic foundation, but to the best of the present authors' knowledge, the article provides first time the frequency parameters of SWCNTs resting on an exponentially varying Winkler elastic foundation. In this article, Euler–Bernoulli beam theory in conjunction with nonlocal elasticity theory has been considered to illustrate the effects of the nonuniform parameter, nonlocal parameter, aspect ratio, Winkler modulus parameter and edge condition on the frequency parameter.

2 Review of nonlocal elastic theory

Nonlocal stress tensor σ at a given point x in conjunction to nonlocal elasticity theory can be expressed as [42]

$$\sigma(x) = \int_V K(|x' - x|, \alpha) \tau dV(x') \quad (1)$$

where τ is the classical stress tensor, $K(|x' - x|, \alpha)$ the nonlocal modulus and $|x' - x|$ the Euclidean distance. One may note that the volume integral is considered over the

region V . Here α is the material constant and it depends on both external and internal characteristic lengths.

From Hooke's law, one may have

$$\tau(x) = C(x) : \varepsilon(x) \quad (2)$$

where C is the fourth order elasticity tensor, ε is the classical strain tensor and $:$ denotes double dot product.

It may be noted that Eq. (1) is the integral constitutive relation and it is very complicated to solve. Hence we need an equivalent form of this equation which may be expressed as [42]

$$\left(1 - \alpha^2 L^2 \nabla^2\right) \sigma = \tau, \quad \alpha = \frac{e_0 a}{L} \quad (3)$$

where ∇^2 is the Laplace operator, e_0 is a material constant, a is an internal characteristic length and L is the external characteristic length of the nanostructure. Here $e_0 a$ is the nonlocal parameter which shows scale effect in on the nano-structures.

3 Formulation of non-local Euler-Bernoulli beam theory

In this study, Euler–Bernoulli beam theory along with the nonlocal elasticity theory of Eringen [9] have been considered for the investigation. For this regard, one must have adequate knowledge about energies. Here we have considered x coordinate along the length of the beam and z coordinate from the mid-plane of the beam. One may express strain energy U as

$$U = \frac{1}{2} \int_0^L \int_A \sigma_{xx} \varepsilon_{xx} dA dx, \quad (4)$$

where σ_{xx} is the normal stress, L is the length of the beam and A is the cross-section area. The strain–displacement relation may be given as

$$\varepsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2}, \quad (5)$$

where ε_{xx} is the normal strain and w is the deflection function. Substituting Eq. (5) in Eq. (4), one may obtain

$$U = -\frac{1}{2} \int_0^L M \frac{\partial^2 w}{\partial x^2} dx, \quad (6)$$

where $M = \int_A z \sigma_{xx} dA$, is the bending moment. In this study, free harmonic motion is considered, i.e. $w =$

$w_0(x) \sin \omega t$, where ω is the natural frequency of vibration. Using free harmonic motion in Eq. (6), we may obtain the strain energy U as

$$U = -\frac{1}{2} \int_0^L M \frac{d^2 w_0}{dx^2} dx. \quad (7)$$

The kinetic energy T is given as

$$T = \frac{1}{2} \int_0^L \rho A \omega^2 w_0^2 dx, \quad (8)$$

where ρ is the mass density and A is the area of beams. Now the potential energy may be given as [43]

$$V = \frac{1}{2} \int_0^L f_e w_0 dx, \quad (9)$$

where f_e is the density of reaction force of elastic foundation and is defined as $f_e = k_w w_0$. Here k_w denotes the Winkler modulus of the elastic medium. Now by using Hamilton's principle $\int_0^t \delta T - (\delta U + \delta V) dt = 0$ and setting the co-efficient of δw_0 to zero, one may obtain the governing equation as

$$\frac{d^2 M}{dx^2} - k_w w_0 = -\rho A \omega^2 w_0. \quad (10)$$

Based on Eringen's nonlocal elasticity theory, nonlocal constitutive relation for EBT nanobeam may be expressed as

$$M - \mu \frac{d^2 M}{dx^2} = -EI \frac{d^2 w_0}{dx^2} \quad (11)$$

where μ is the nonlocal parameter which can be expressed as $\mu = (e_0 a)^2$ with e_0 and a denoting material constant and internal characteristic length respectively, I is the second moment of area and E is Young's modulus. By using Eq. (10) and Eq. (11), M may be expressed as

$$M = -EI \frac{d^2 w_0}{dx^2} + \mu \left(-\rho A \omega^2 w_0 + k_w w_0 \right) \quad (12)$$

Using Eq. (12) in Eq. (10), one may obtain the governing equation in terms of displacement as

$$EI \frac{d^4 w_0}{dx^4} - \mu k_w \frac{d^2 w_0}{dx^2} + \rho A \omega^2 \mu \frac{d^2 w_0}{dx^2} + k_w w_0 = \rho A \omega^2 w_0 \quad (13)$$

Eq. (13) may be rewritten as

$$\frac{d^4 w_0}{dx^4} - \frac{\mu k_w}{EI} \frac{d^2 w_0}{dx^2} + \frac{\rho A \omega^2 \mu}{EI} \frac{d^2 w_0}{dx^2} + \frac{k_w w_0}{EI} = \frac{\rho A \omega^2 w_0}{EI} \quad (14)$$

Let us introduce the following non-dimensional terms

$$X = \frac{x}{L}$$

$$W = \frac{w_0}{L}$$

$$\lambda^2 = \frac{\rho A \omega^2 L^4}{EI_0} = \text{frequency parameter}$$

$$K = \frac{k_w L^4}{EI_0} = \text{Winkler modulus parameter.}$$

Now, the proposed exponential variation in Winkler elastic foundation has now been expressed as

$$K = (K_w) e^{-\eta X}, \quad (15)$$

where K_w is the Winkler modulus parameter and η is a positive constant.

Using the above non-dimensional terms and Eq. (15) in Eq. (14), we obtain the nondimensionalized form of the governing differential equation as

$$\frac{d^4 W}{dX^4} - \frac{\mu}{L^2} K_w e^{-\eta X} \frac{d^2 W}{dX^2} + K_w e^{-\eta X} W = \lambda^2 \left(W - \frac{\mu}{L^2} \frac{d^2 W}{dX^2} \right) \quad (16)$$

Next we introduce an overview of the differential quadrature method.

4 Differential Quadrature Method

In this present investigation, Quan and Chang's [44] approach is taken into consideration along with Chebyshev-Gauss-Lobatto grid points which is expressed as

$$X_i = \frac{1}{2} \left[1 - \cos \left(\frac{i-1}{N-1} \cdot \pi \right) \right].$$

It may also note that this is a non-uniform grid points.

One may express the derivatives of displacement function $W(X)$ at a given discrete point i as [45]

$$W'_i = \sum_{j=1}^N A_{ij} W_j \quad (17)$$

$$W''_i = \sum_{j=1}^N B_{ij} W_j$$

$$W'''_i = \sum_{j=1}^N C_{ij} W_j$$

$$W^{IV}_i = \sum_{j=1}^N D_{ij} W_j$$

where $i = 1, 2, \dots, N$ and N is the number of discrete grid points.

Here A_{ij} , B_{ij} , C_{ij} and D_{ij} are the weighting coefficients of the first, second, third and fourth derivatives respectively.

4.1 Determination of weighting coefficients

Computation of weighting coefficient matrix $A = (A_{ij})$ plays very significant role in DQ method. As per Quan and Chang's approach, matrix $A = (A_{ij})$ can be computed by the following procedure.

For $i \neq j$

$$A_{ij} = \frac{1}{X_j - X_i} \prod_{\substack{k \neq i \\ k \neq j \\ k=1}}^N \frac{X_i - X_k}{X_j - X_k} \quad i = 1, 2, \dots, N \quad (18)$$

$$j = 1, 2, \dots, N$$

for $i = j$

$$A_{ii} = \sum_{\substack{k \neq i \\ k=1}}^N \frac{1}{X_i - X_k} \quad i = 1, 2, \dots, N. \quad (19)$$

After weighting coefficients of first order derivatives are computed, one may easily get the weighting coefficients of higher order derivatives by simply matrix multiplication which is given as follows.

$$B = B_{ij} = \sum_{k=1}^N A_{ik} A_{kj} \quad (20)$$

$$C = C_{ij} = \sum_{k=1}^N A_{ik} B_{kj} \quad (21)$$

$$D = D_{ij} = \sum_{k=1}^N A_{ik} C_{kj} = \sum_{k=1}^N B_{ik} B_{kj}. \quad (22)$$

4.2 Application of boundary conditions

Four classical boundary conditions such as SS, CS, CC and CF are taken in consideration in the present study where the letters S, C, and F denote simply supported, clamped and free edge conditions respectively.

Let us now denote

$$\bar{A}_1 = \begin{bmatrix} 0 & A_{1,2} & \cdots & A_{1,N} \\ 0 & A_{2,2} & \cdots & A_{2,N} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & A_{N,2} & \cdots & A_{N,N} \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N-1} & 0 \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N-1,N-1} & 0 \end{bmatrix}.$$

Now, the weighting coefficients of higher order derivatives for different edge conditions are given as below.

Simply supported-simply supported

$$\{W'\} = [\bar{A}] \{W\}.$$

$$\{W''\} = [A] [\bar{A}] \{W\} = [\bar{B}] \{W\} \text{ with } \bar{B} = [A] [\bar{A}].$$

$$\{W'''\} = [\bar{A}] \{W''\} = [\bar{A}] [\bar{B}] \{W\} = [\bar{C}] \{W\}.$$

$$\{W^{IV}\} = [A] \{W'''\} = [A] [\bar{C}] \{W\} = [\bar{D}] [\bar{B}] \{W\} \\ = [\bar{D}] \{W\}$$

where $[\bar{D}] = [\bar{B}] [\bar{B}]$ or $[\bar{D}] = [A] [\bar{C}]$.

Clamped-simply supported:

$$\{W'\} = [\bar{A}] \{W\}$$

$$\{W''\} = [\bar{A}_1] \{W'\} = [\bar{A}_1] [\bar{A}] \{W\} \\ = [\bar{B}] \{W\} \text{ with } \bar{B} = [\bar{A}_1] [\bar{A}].$$

$$\{W'''\} = [\bar{A}_2] \{W''\} = [\bar{A}_2] [\bar{B}] \{W\} \\ = [\bar{C}] \{W\} \text{ with } \bar{C} = [\bar{A}_2] [\bar{B}].$$

$$\{W^{IV}\} = [A] \{W'''\} = [A] [\bar{C}] \{W\} \\ = [\bar{D}] \{W\} \text{ with } \bar{D} = [A] [\bar{C}].$$

Clamped-clamped:

$$\{W'\} = [\bar{A}] \{W\}$$

$$\{W''\} = [\bar{A}] \{W'\} = [\bar{A}] [\bar{A}] \{W\} \\ = [\bar{B}] \{W\} \text{ with } \bar{B} = [\bar{A}] [\bar{A}].$$

$$\{W'''\} = [A] \{W''\} = [A] [\bar{B}] \{W\} \\ = [\bar{C}] \{W\} \text{ with } \bar{C} = [A] [\bar{B}].$$

$$\{W^{IV}\} = [A] \{W'''\} = [A] [\bar{C}] \{W\} \\ = [\bar{D}] \{W\} \text{ with } \bar{D} = [A] [\bar{C}].$$

Clamped-free:

$$\{W'\} = [\bar{A}_1] \{W\}$$

$$\{W''\} = [\bar{A}_1] \{W'\} = [\bar{A}_1] [\bar{A}_1] \{W\}$$

Table 1: Comparisons of frequency parameter $(\sqrt{\lambda})$ for SS case

K_w	$\sqrt{\lambda_1}$			$\sqrt{\lambda_2}$			$\sqrt{\lambda_3}$		
	Present	Ref. [28]	Ref. [18]	Present	Ref. [28]	Ref. [18]	Present	Ref. [28]	Ref. [18]
10	3.2192	3.2192	3.2192	6.2932	6.2932	6.2932	9.4277	9.4277	9.4277
50	3.4844	3.4844	3.4844	6.3329	6.3329	6.3329	9.4396	9.4396	9.4396
100	3.7483	3.7483	3.7483	6.3816	6.3816	6.3816	9.4544	9.4544	9.4544
200	4.1527	4.1527	4.1527	6.4757	6.4757	6.4757	9.4839	9.4839	9.4839
500	4.9438	4.9438	4.9438	6.7358	6.7358	6.7358	9.5706	9.5706	9.5706
1000	5.7556	5.7556	5.7556	7.1121	7.1121	7.1121	9.7101	9.7101	9.7101
1500	6.3219	6.3219	6.3219	7.4366	7.4366	7.4366	9.8439	9.8439	9.8439
2000	6.7673	6.7673	6.7673	7.7235	7.7235	7.7235	9.9724	9.9724	9.9724

$$\begin{aligned}
 &= [\bar{B}] \{W\} \text{ with } [\bar{B}] = [\bar{A}_1] [\bar{A}_1]. \\
 \{W'''\} &= [\bar{A}_2] \{W''\} = [\bar{A}_2] [\bar{B}] \{W\} \\
 &= [\bar{C}] \{W\} \text{ with } [\bar{C}] = [\bar{A}_2] [\bar{B}]. \\
 \{W^{IV}\} &= [\bar{A}_2] \{W'''\} = [\bar{A}_2] [\bar{C}] \{W\} \\
 &= [\bar{D}] \{W\} \text{ with } [\bar{D}] = [\bar{A}_2] [\bar{C}].
 \end{aligned}$$

Substituting the value of Eq. (17) into Eq. (16), one may obtain generalized eigenvalue problem as

$$[S] \{W\} = \lambda^2 [T] \{W\} \quad (23)$$

where S is the stiffness matrix and T is the mass matrix.

5 Numerical results and discussions

Eq. (23) is solved by using MATLAB program which is developed by the authors and frequency parameters $\sqrt{\lambda}$ have been obtained. DQ method has been implemented properly along with the boundary conditions in the coefficient matrix. Following parameters are taken from [46] for computational purpose.

$$E = 30 \times 10^6 \text{ Pa}, L = 10 \text{ nm}, \text{Poisson's ratio } (\nu) = 0.3,$$

$$K_s = \frac{5}{6} \text{ and unless mentioned } \frac{L}{h} = 10.$$

5.1 Validation

Validation of the present method is done in two ways. Firstly, we consider Euler-Bernoulli beam with nonlocal parameter $\mu = 0$ and non-uniform parameter $\eta = 0$. we compare our results of frequency parameter $(\sqrt{\lambda})$ with those available in the literature [18, 28] for different Winkler modulus parameter and are given in Tables 1-3. Secondly, we consider a nanobeam not resting on any elastic

Table 2: Comparisons of frequency parameter $(\sqrt{\lambda})$ for CC case

K_w	$\sqrt{\lambda_1}$		$\sqrt{\lambda_2}$		$\sqrt{\lambda_3}$	
	Present	Ref. [28]	Present	Ref. [28]	Present	Ref. [28]
1	4.7324	4.7324	7.8537	7.8537	10.9958	10.9958
10	4.7534	4.7534	7.8583	7.8583	10.9975	10.9975
100	4.9503	4.9503	7.9043	7.9043	11.0144	11.0144
1000	6.2239	6.2239	8.3251	8.3251	11.1790	11.1790

Table 3: Comparisons of frequency parameter $(\sqrt{\lambda})$ for CF case

K_w	$\sqrt{\lambda_1}$		$\sqrt{\lambda_2}$		$\sqrt{\lambda_3}$	
	Present	Ref. [28]	Present	Ref. [28]	Present	Ref. [28]
1	1.9119	1.9119	4.6965	4.6965	7.8552	7.8552
10	2.1746	2.1746	4.7180	4.7180	7.8599	7.8599
100	3.2557	3.2557	4.9191	4.9191	7.9058	7.9058
1000	5.6407	5.6407	6.2082	6.2082	8.3264	8.3264

foundation viz. $\eta = 0$ and . our results of fundamental frequency parameter (λ) are now compared with Reddy [46], Aydogu [47] and Eltaher [48] for different nonlocal parameters (μ) which are presented in Tables 4-5. In Table 4, aspect ratio (L/h) is taken as 10. Similarly, in Table 5, fundamental frequency parameter (λ) for SS nanobeam is compared with Aydogu [47] and Eltaher [48] with aspect ratio (L/h) as 20. From these Tables 1-5, one may observe close agreement of results with those available in the literatures.

5.2 Convergence

Minimum number of grid points have been obtained by studying convergence to obtain the final results. To show how the solution is affected by the grid points, variations of the frequency parameters $(\sqrt{\lambda})$ with the number of grid points are shown in Figure 1 and in Tables 6-9 for SS,CS,CC

Figure 1: Variation of frequency parameters with number of terms for SS, CS, CC and CF conditions

Table 4: Comparisons of first fundamental frequency parameter (λ) for SS Nano beam

μ	Present	Reddy [46]	Aydogu [47]	Eltaher [48]
0	9.8696	9.8696	9.8696	9.8696
1	9.4159	9.4159	9.4124	9.4159
2	9.0195	9.0195	9.0133	9.0195
3	8.6693	8.6693	8.6611	8.6693
4	8.3569	8.3569	8.3472	8.3569

Table 5: Comparisons of first fundamental frequency parameter (λ) for SS Nano beam

μ	Aydogu [47]	Eltaher [48]	Present
0	9.8696	9.8798	9.8696
1	9.7498	9.4238	9.7500
2	9.6343	9.0257	9.63473
3	9.5228	8.6741	9.5234
4	9.4150	8.3606	9.4158

and CF cases. Here, we have considered $L = 10$, $\eta = 0.2$, $\mu = 1$ and $K_w = 10$. The convergence is shown for SS, CS, CC and CF conditions. From this figure, one may note that with increase in number of grid points, the convergence is achieving fast. One may also observe that twelve grid points are sufficient to compute the desired results.

5.3 Effect of nonlocal parameter

In this subsection, first four frequency parameters ($\sqrt{\lambda}$) of nanobeam are obtained for different nonlocal parameters. We have taken the values of nonlocal parameter as 0, 0.2, 0.4, 0.6, 0.8, 1 nm^2 . In this study, all the classical boundary conditions such as SS, CS, CC and CF are considered for investigation. Both tabular and graphical results are depicted by taking other parameters such as non-uniform parameter (η) as 0.5, Winkler modulus parameter $K_w = 50$ and $L = 10$ nm. First four frequency parameters of SS, CS, CC and CF edges for different nonlocal parameters are presented Tables 10-13. From these tables, one may observe that frequency parameters decrease with increase in nonlocal parameter except first fundamental frequency parameter of CF nanobeams. One may also notice that frequency parameters increase with increase in mode number. Here one important point to note that CC nanobeams are having highest frequency parameters than other set of boundary conditions. Figure 2 show variation of frequency ratio with nonlocal parameters for different edges such as SS, CS, CC and CF.

5.4 Effect of nonuniform parameter

Effect of non-uniform parameter η on first four frequency parameters is analyzed by taking non-uniform parameter η as 0, 0.2, 0.4, 0.6, 0.8 and 1. One may note that η takes positive values only. Tables 14-17 and Figure 3 illustrate the

Table 6: Frequency parameter $(\sqrt{\lambda})$ for SS case for different grid points

N	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
8	3.137047387663272	5.792189727676567	8.069652183108396	9.587980174560933
10	3.137043626122775	5.792222858427450	8.044736703926970	9.900828395069310
12	3.137043645950507	5.792229270220762	8.043907944491545	9.917670536656008
14	3.137043645870416	5.792229198324495	8.043924442382881	9.918214267400069
16	3.137043645874682	5.792229199028369	8.043923930463707	9.918206815221069
18	3.137043645875629	5.792229199022920	8.043923941298058	9.918207137096605
20	3.137043645883244	5.792229199024242	8.043923941117477	9.918207128737370

Table 7: $(\sqrt{\lambda})$ for CS case for different grid points

N	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
8	3.856973271397841	6.463369136400239	8.698387847145867	10.013373349121141
10	3.857068900732693	6.472653477925265	8.652930571005422	10.358046845755311
12	3.857068049264370	6.472440609260722	8.655034716695576	10.473304478306719
14	3.857068057194151	6.472445866438307	8.654843259979375	10.470245167091482
16	3.857068057157146	6.472445777886075	8.654851649036990	10.470550548000791
18	3.857068057156827	6.472445779054758	8.654851382273554	10.470535930174004
20	3.857068057112573	6.472445779046844	8.654851388706513	10.470536499911651

Table 8: Frequency parameter $(\sqrt{\lambda})$ for CC case for different grid points

N	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
8	4.614872016555520	7.460588192516093	9.486498247290566	12.609873904602038
10	4.615425147178145	7.121835740601477	9.253522431196084	12.111957988099142
12	4.615417617988975	7.147480265640297	9.261700624304588	10.931617747978734
14	4.615417714547347	7.145802024795920	9.260855837455789	11.037275335647557
16	4.615417713845297	7.145867516572760	9.260896551844526	11.015322124620903
18	4.615417713847023	7.145865717988663	9.260895055673364	11.017480223039190
20	4.615417713848704	7.145865754113659	9.260895097840340	11.017322575498191

Table 9: Frequency parameter $(\sqrt{\lambda})$ for CF case for different grid points

N	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
8	2.131906821758265	4.568180334350084	7.135038616372697	9.294609304241696
10	2.131906968161213	4.569107989776002	7.151439435905257	9.246361354117974
12	2.131906968641838	4.569093872033319	7.151500240014390	9.260430132289168
14	2.131906968623912	4.569094037626583	7.151498157051591	9.259396249267780
16	2.131906968654128	4.569094036365876	7.151498264697817	9.259455049663973
18	2.131906968600308	4.569094036328919	7.151498262389548	9.259452763225076
20	2.131906968944130	4.569094036409303	7.151498262428689	9.259452830507044

Figure 2: Variation of frequency parameter with nonlocal parameter

Table 10: Frequency parameter for SS case for different nonlocal parameter

μ	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	3.3618	6.3135	9.4338	12.5701
0.2	3.3493	6.1970	9.0574	11.7377
0.4	3.3372	6.0908	8.7463	11.1241
0.6	3.3254	5.9935	8.4825	10.6438
0.8	3.3139	5.9037	8.2545	10.2523
1.0	3.3027	5.8205	8.0545	9.9238

Table 11: Frequency parameter for CS case for different nonlocal parameter

μ	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	4.0462	7.0899	10.2172	13.3549
0.2	4.0259	6.9473	9.7876	12.4373
0.4	4.0061	6.8181	9.4358	11.7684
0.6	3.9869	6.7004	9.1397	11.2487
0.8	3.9682	6.5923	8.8854	10.8274
1.0	3.9501	6.4927	8.6633	10.4753

variation of frequency parameter with nonuniform parameter η for all the classical boundary conditions such as SS, CS, CC and CF. Here, computation is done with nonlocal parameter $\mu = 1\text{nm}^2$, $L = 10\text{nm}$ and $K_w = 100$. It may also be noted from the graph that the frequency parameter decreases with increase in the non-uniform parameter and this decrease is more significant in case of higher modes.

Table 12: Frequency parameter ($\sqrt{\lambda}$) for CC case for different non-local parameter

μ	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	4.8001	7.8688	11.0013	14.1398
0.2	4.7726	7.6985	10.5166	13.1348
0.4	4.7460	7.5450	10.1229	12.4094
0.6	4.7202	7.4056	9.7937	11.8495
0.8	4.6951	7.2781	9.5124	11.3976
1.0	4.6707	7.1609	9.2678	11.0214

Table 13: Frequency parameter ($\sqrt{\lambda}$) for CF case for different nonlocal parameter

μ	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	2.5561	4.7657	7.8703	11.0012
0.2	2.5564	4.7363	7.7006	10.5164
0.4	2.5567	4.7076	7.5478	10.1225
0.6	2.5570	4.6797	7.4092	9.7931
0.8	2.5573	4.6525	7.2827	9.5114
1.0	2.5577	4.6260	7.1665	9.2664

5.5 Effect of length-to-height ratio

In this subsection, effect of length-to-height ratio (L/h) on the first four frequency parameters has been analyzed for all the classical boundary conditions such as SS, CS, CC and CF. First four frequency parameters of nanobeam are given in Tables 18-21 for different L/h (10, 20, 30, 40,

Figure 3: Variation of frequency parameter with non-uniform or non-homogeneous parameter

Table 14: Frequency parameter ($\sqrt{\lambda}$) for SS case for different non-uniform parameter

η	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	3.7061	5.9068	8.0876	9.9416
0.2	3.6136	5.8847	8.0790	9.9370
0.4	3.5323	5.8664	8.0720	9.9332
0.6	3.4613	5.8513	8.0662	9.9301
0.8	3.3997	5.8389	8.0615	9.9276
1.0	3.3466	5.8286	8.0576	9.9255

Table 16: Frequency parameter ($\sqrt{\lambda}$) for CC case for different non-uniform parameter

η	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	4.8330	7.2079	9.2896	11.0344
0.2	4.7923	7.1958	9.2840	11.0310
0.4	4.7582	7.1858	9.2793	11.0283
0.6	4.7298	7.1776	9.2755	11.0260
0.8	4.7061	7.1709	9.2724	11.0241
1.0	4.6864	7.1653	9.2698	11.0226

Table 15: Frequency parameter ($\sqrt{\lambda}$) for CS case for different non-uniform parameter

η	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	4.2066	6.5554	8.6900	10.4904
0.2	4.1443	6.5393	8.6831	10.4865
0.4	4.0912	6.5260	8.6774	10.4833
0.6	4.0461	6.5150	8.6728	10.4806
0.8	4.0080	6.5060	8.6689	10.4785
1.0	3.9760	6.4986	8.6658	10.4767

Table 17: Frequency parameter ($\sqrt{\lambda}$) for CF case for different non-uniform parameter

η	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	3.2565	4.7927	7.2134	9.2882
0.2	3.1165	4.7510	7.2013	9.2825
0.4	2.9860	4.7160	7.1913	9.2779
0.6	2.8647	4.6868	7.1832	9.2741
0.8	2.7525	4.6624	7.1764	9.2710
1.0	2.6492	4.6422	7.1709	9.2684

50). In this table, computation is done with $\mu = 1\text{nm}^2$, $K_w = 50$ and $\eta = 0.5$. Graphical results are presented in Figure 4, where variation of first four frequency parameters with L/h has been shown. In this figure, L/h varies from 10 to 50.

5.6 Effect of boundary condition

One needs to have adequate knowledge about boundary conditions for designing any engineering structures. It helps designers to gather some important information without carrying out detailed experimental investigation. Therefore, it is quite important to study the effect of boundary

Figure 4: Variation of frequency parameter with length-to-height ratio

Table 18: Frequency parameter ($\sqrt{\lambda}$) for SS case for different length-to-height ratio

L/h	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
10	3.3027	5.8205	8.0545	9.9238
20	3.3463	6.1696	8.9744	11.5682
30	3.3548	6.2474	9.2152	12.0729
40	3.3579	6.2759	9.3077	12.2782
50	3.3593	6.2893	9.3521	12.3794

Table 19: Frequency parameter ($\sqrt{\lambda}$) for CS case for different length-to-height ratio

L/h	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
10	3.9501	6.4927	8.6633	10.4753
20	4.0209	6.9138	9.6934	12.2519
30	4.0348	7.0088	9.9672	12.8053
40	4.0398	7.0437	10.0728	13.0317
50	4.0421	7.0602	10.1236	13.1435

conditions on frequency parameter. So, the effect of boundary condition on the frequency parameter is studied in this subsection. Figure 5 illustrates variation of frequency parameter with nonlocal parameter for different boundary conditions. The values of parameters are taken as $L = 10\text{nm}$, $K_w = 50$ and $\eta = 0.5$ for plotting the figure. One may observe from the figure that CC nanobeams are having highest frequency parameter whereas CF nanobeams possess lowest frequency parameter. It may also be noted

Table 20: Frequency parameter ($\sqrt{\lambda}$) for CC case for different length-to-height ratio

L/h	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
10	4.6707	7.1609	9.2678	11.0214
20	4.7659	7.6587	10.4109	12.9332
30	4.7847	7.7719	10.7187	13.5365
40	4.7914	7.8136	10.8378	13.7844
50	4.7945	7.8332	10.8953	13.9072

Table 21: Frequency parameter ($\sqrt{\lambda}$) for CF case for different length-to-height ratio

L/h	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
10	2.5577	4.6260	7.1665	9.2664
20	2.5565	4.7290	7.6610	10.4107
30	2.5562	4.7492	7.7738	10.7186
40	2.5561	4.7564	7.8153	10.8377
50	2.5561	4.7597	7.8349	10.8952

that frequency parameters decrease with increase in nonlocal parameter for SS, CS, and CC edge but in case of first mode of CF nanobeams, it is exactly opposite.

5.7 Effect of Winkler modulus parameter

Effect of Winkler modulus parameter (K_w) on the first four frequency parameters has been studied. First four fre-

Figure 5: Variation of frequency parameter with boundary condition

Table 22: Frequency parameter ($\sqrt{\lambda}$) for SS nanobeam for different Winkler modulus parameter

K	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
20	3.2176	6.1501	8.9681	11.5652
40	3.3050	6.1631	8.9723	11.5672
60	3.3860	6.1760	8.9765	11.5692
80	3.4616	6.1888	8.9807	11.5711
100	3.5325	6.2016	8.9848	11.5731

Table 23: Frequency parameter ($\sqrt{\lambda}$) for CS nanobeam for different Winkler modulus parameter

K	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
20	3.9490	6.9000	9.6884	12.2495
40	3.9973	6.9092	9.6918	12.2511
60	4.0440	6.9184	9.6951	12.2528
80	4.0891	6.9275	9.6984	12.2544
100	4.1327	6.9366	9.7017	12.2561

quency parameters of all the classical boundary conditions such as SS, CS, CC and CF are given in Tables [22–25] for different K_w . In this table, $\mu = 1 \text{ nm}^2$, $L = 20$ and $\eta = 0.5$ are taken into consideration. Graphical results are depicted in Figure 6 where variation of first four frequency parameters with (K_w) has been shown. In this figure, (K_w) ranges from 20 to 100 with an increment of 20.

Table 24: Frequency parameter ($\sqrt{\lambda}$) for CC nanobeam for different Winkler modulus parameter

K	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
20	4.7233	7.6485	10.4069	12.9311
40	4.7518	7.6553	10.4095	12.9325
60	4.7799	7.6620	10.4122	12.9339
80	4.8074	7.6688	10.4149	12.9353
100	4.8345	7.6755	10.4176	12.9367

Table 25: Frequency parameter ($\sqrt{\lambda}$) for CF nanobeam for different Winkler modulus parameter

K	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
20	2.2252	4.6854	7.6508	10.4067
40	2.4604	4.7146	7.6576	10.4093
60	2.6427	4.7433	7.6643	10.4120
80	2.7936	4.7715	7.6711	10.4147
100	2.9234	4.7991	7.6778	10.4174

6 Conclusions

Vibration analysis of Single Walled Carbon Nanotube (SWCNT) resting on an exponentially varying Winkler elastic foundation is investigated by using Differential Quadrature Method (DQM). The numerical as well as graphical results are presented to show the effects of the Winkler modulus parameter, nonuniform parameter, non-

Figure 6: Variation of frequency parameter with Winkler modulus parameter

local parameter, L/h and the boundary conditions on the frequency parameters. One may observe from the graph that the frequency parameter decreases with increase in the non-uniform parameter and this decrease is more significant in case of higher modes. It is found that clamped(CC) nanobeam possesses highest frequency parameters and cantilever(CF) nanobeam possesses lowest among all other types of boundary conditions. It is also found that the effect of nonlocal parameter is more in higher modes. One of the interesting observation is that fundamental frequency parameter of cantilever(CF) nanobeam does not decrease with increase in nonlocal parameter.

Acknowledgement: The authors are thankful to Defence Research & Development Organization(DRDO), Ministry of Defence, New Delhi, India (Sanction Code: DG/TM/ERIPR/GIA/17-18/0129/020) for the support and funding to carry out the present research work.

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