Research Article

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Dynamic Analysis of Single-Layered Graphene Nano-Ribbons (SLGNRs) with Variable Cross-Section Resting on Elastic Foundation

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Abstract: This article deals with free vibration of the variable cross-section (non-uniform) single-layered graphene nano-ribbons (SLGNRs) resting on Winkler elastic foundation using the Differential Quadrature Method (DQM). Here characteristic width of the cross-section is varied exponentially along the length of the nano-ribbon while the thickness of the cross section is kept constant. Euler–Bernoulli beam theory in conjunction with Eringen non-local elasticity theory is considered in this study. The numerical as well as graphical results are reported by using MATLAB codes developed by authors. Convergence of present method is explored and our results are compared with known results available in literature showing excellent agreement. Further, effects various parameters on frequency parameters are studied comprehensively.

Keywords: Graphene Nano-Ribbons; Eringen’s Nonlocal theory; DQM; Winkler Foundation; Variable Cross-Section

1 Introduction

Graphene nanoribbons [1–3] are allotropes of carbon made up of narrow strips of graphene having a width less than 50 nm. Mitsutaka Fujita has the credit to introduce theoretical model of Graphene ribbons to analyze the nanoscale effect in graphene [4–6]. The structure of GNR manifest semi-conducting as well as metallic electronic structures having tunable band gaps. GNRs possess all most all the peculiar properties Graphene and carbon nanotube along with an additional property viz. tunable band gap. It has low noise, and high thermal and elec-

trical conductivities which makes it an alternative of copper regarding integrated circuit systems. Potential application of GNRs include Field effect transistors, LED, Electro-active actuators etc. In this regards, accurate prediction of vibration characteristics of SLGNRs are very essential. For this purpose, various nonclassical continuum theories have been developed but the nonlocal elasticity theory of Eringen [7] is very popular.

Non uniform nanomaterials are very vital in reliable design of various nano devices. One can see several applications of nanomaterials viz. CNT reinforced structure in [8–11]. Also, dynamical behavior of nanobeam, nano-ribbon, nanotube etc. can be found in [12–16]. In the last decades, many studies reported the behavior of non-uniform beams in different static and dynamic manners [17–20]. Peddieson *et al.* [21] applied Eringen’s non-local elasticity theory to study Euler-Bernoulli beam. The main output from the model is that MEMS scale devices would not exhibit nonlocal effects whereas nanoscale devices could. Wang *et al.* [22] applied analytical method to investigate vibration of Euler-Bernoulli and Timoshenko nanobeams. Here rotary inertia and transverse shear deformation were taken into account for Timoshenko nanobeam. One can see bending, buckling and vibration of beam incorporated with nonlocal theories in [23, 24], and Eltaher *et al.* [25] applied finite element method to study vibration of Euler-Bernoulli nanobeams. Hashemi and Khaniki [26] analytically studied free vibration of a nonlocal nanobeam with variable cross-section.

During the past years, various elastic foundation models such as Winkler, Pasternak, and Flonenko -Borodich foundations have been introduced. It is worth mentioning that the Winkler foundation model was suggested in 1867 by Winkler which has become one of the first scenarios in this regard. Linear algebraic relationship between the normal displacement and the contact pressure [27] is assumed in the modeling of this foundation. This assumption makes it simple to get the closed-form solutions [28, 29]. On the other hand, it also gets a nonlinear behavior with fewer computations in comparison with other meth-

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ods [30–34]. One may see different studies on Winkler elastic foundation in important papers viz. [35, 36]. Finite and infinite beams on a Winkler support have been addressed by Auersch [37]. As regards, Eisenberger and Clastornik [38] studied the vibration and buckling of a beam resting on variable Winkler elastic foundation. Also, the dynamic behavior of infinite beam has been investigated by Ruge and Birk [39]. Next, Lee [40] studied the dynamic response of a Timoshenko beam subject to a moving concentrated mass. Huang and Thambiratnam [41] has used finite strip method to investigate deflection of plates having moving accelerated loads resting on Winkler foundation. Further, Oz and Pakdemirli [42] have also examined resonances of shallow beams with elastic foundations. Elastic foundations with Winkler-Pasternak problems may also be found in [43, 44]. It may be noted that structural problems with elastic foundation have also been solved by few other methods such as differential transform [45], Rayleigh-Ritz [46] and Harmonic DQ [47, 48].

Though some of the studies have been reported on Winkler elastic foundation, the current article for the first time predicate vibration characteristics of the single-layered graphene nano-ribbons with variable cross-section placed on Winkler elastic foundation.

2 Mathematical Formulation

In this study, Eringen [7] nonlocal theory is associated with Euler–Bernoulli beam theory. Here we have considered x coordinate along the length of the beam, y coordinate along the width of the beam and z coordinate from the mid-plane of the beam. One may express strain energy U as

$$U = \frac{1}{2} \int_0^L \int_A \sigma_{xx} \varepsilon_{xx} dA dx, \quad (1)$$

where σ_{xx} is the normal stress, L is the length and A is the cross-section area. The strain-displacement relation may be given as

$$\varepsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2}, \quad (2)$$

where ε_{xx} is the normal strain and w is the deflection function. Substituting Eq. (2) in Eq. (1), one may obtain

$$U = -\frac{1}{2} \int_0^L M \frac{\partial^2 w}{\partial x^2} dx, \quad (3)$$

where $M = \int_A z \sigma_{xx} dA$, is the bending moment. In this study, the free harmonic motion is considered, i.e. $w = w_0(x) e^{i\omega t}$,

where ω is the natural frequency of vibration. Using free harmonic motion in Eq. (3), we may obtain the strain energy U as

$$U = -\frac{1}{2} \int_0^L M \frac{d^2 w_0}{dx^2} dx. \quad (4)$$

The kinetic energy T is given as

$$T = \frac{1}{2} \int_0^L \rho A \omega^2 w_0^2 dx, \quad (5)$$

where ρ is the mass density and A is the area. Now the potential energy may be given as

$$V = \frac{1}{2} \int_0^L f_e w_0 dx, \quad (6)$$

where f_e is the density of reaction force of elastic foundation and is defined as $f_e = k_w w_0$. Here k_w denotes the Winkler modulus of the elastic medium. Now by using Hamilton's principle $\int_0^t \delta T - (\delta U + \delta V) dt = 0$ and setting the co-efficient of δw_0 to zero, one may obtain the governing equation as

$$\frac{d^2 M}{dx^2} - k_w w_0 = -\rho A \omega^2 w_0. \quad (7)$$

Based on Eringen's nonlocal elasticity theory [7], nonlocal constitutive relation for EBT may be expressed as

$$M - \mu \frac{d^2 M}{dx^2} = -EI \frac{d^2 w_0}{dx^2} \quad (8)$$

where μ is the nonlocal parameter which can be expressed as $\mu = (e_0 a)^2$ with e_0 and a denoting material constant and internal characteristic length respectively, I is the second moment of area and E is Young's modulus. By using Eq. (7) and Eq. (8), M may be expressed as

$$M = -EI \frac{d^2 w_0}{dx^2} + \mu \left(-\rho A \omega^2 w_0 + k_w w_0 \right) \quad (9)$$

Since we have considered that characteristic width of the cross-section is varied exponentially along the length of the nanoribbon, so from [26]

$$A(x) = A_0 e^{nx} \quad \text{and} \quad I(x) = I_0 e^{nx} \quad (10)$$

where n is the non-uniform parameter, I_0 and A_0 are the second moment of area and cross-section of nanoribbon respectively. Now by using Eq. (9) along with Eq. (10) in Eq. (7), one may obtain the governing equation in terms of displacement as

$$EI_0 \frac{d^4 w_0}{dx^4} + 2nEI_0 \frac{d^3 w_0}{dx^3} \quad (11)$$

$$+ EI_0 \left(n^2 + \frac{\rho A_0 \mu \omega^2}{EI_0} \right) \frac{d^2 w_0}{dx^2} + 2\rho A_0 \mu \omega^2 n \frac{dw_0}{dx} + \rho A_0 \omega^2 w_0 (\mu n^2 - 1) + k_w e^{-nx} \left(w_0 - \mu \frac{d^2 w_0}{dx^2} \right) = 0$$

We will now introduce non-dimensional terms for easy computation viz. $X = \frac{x}{L}$, $W = \frac{w_0}{L}$, $\lambda^2 = \frac{\rho A_0 \omega^2 L^4}{EI_0}$, $\alpha = \frac{e_0 a}{L}$, $K_w = \frac{k_w L^4}{EI_0}$ and $\eta = nL$. Substituting the given non-dimensional terms in Eq. (11), we obtain the non-dimensionalised form of the governing differential equation as

$$\frac{d^4 W}{dX^4} + 2\eta \frac{d^3 W}{dX^3} + \eta^2 \frac{d^2 W}{dX^2} + K_w e^{-\eta X} \left(W - \alpha^2 \frac{d^2 W}{dX^2} \right) = \lambda^2 \left\{ (1 - \alpha^2 \eta^2) W - \alpha^2 \frac{d^2 W}{dX^2} - 2\alpha^2 \eta \frac{dW}{dX} \right\} \quad (12)$$

3 Differential Quadrature Method

Bellman and Casti [49] have the credit to introduce this technique for the first time. This is an efficient technique to solve both linear and nonlinear equation associated with any physical model. This method is undergone many modifications. In this study we are using Quan and Chang's [50] approach along with Chebyshev-Gauss-Lobatto grid points. For more details of this method one may refer Shu [51].

The derivatives of displacement function $W(X)$ at a given discrete point i can be approximated as weighted sum of functional in the given domain as below.

$$\begin{aligned} W'_i &= \sum_{j=1}^N A_{ij} W_j \\ W''_i &= \sum_{j=1}^N B_{ij} W_j \\ W'''_i &= \sum_{j=1}^N C_{ij} W_j \\ W^{IV}_i &= \sum_{j=1}^N D_{ij} W_j \end{aligned} \quad (13)$$

where $i = 1, 2, \dots, N$ and N is the number of discrete grid points.

Here A_{ij} , B_{ij} , C_{ij} and D_{ij} are called the weighting coefficients associated with first, second, third and fourth derivatives respectively.

3.1 Determination of weighting coefficients

The weighting coefficient matrix $A = (A_{ij})$ can be computed by the following procedure.

For $i \neq j$

$$A_{ij} = \frac{1}{X_j - X_i} \prod_{\substack{k \neq i \\ k \neq j \\ k=1}}^N \frac{X_i - X_k}{X_j - X_k} \quad (14)$$

$i = 1, 2, \dots, N \quad j = 1, 2, \dots, N$

for $i = j$

$$A_{ii} = \sum_{\substack{k \neq i \\ k=1}}^N \frac{1}{X_i - X_k} \quad i = 1, 2, \dots, N. \quad (15)$$

One may note that the above weighting coefficients are obtained by using Lagrange interpolation polynomial as test function. After getting weighting coefficients of first order derivative, the weighting coefficients of higher order can be obtained as follows.

$$B = B_{ij} = \sum_{k=1}^N A_{ik} A_{kj} \quad (16)$$

$$C = C_{ij} = \sum_{k=1}^N A_{ik} B_{kj} \quad (17)$$

$$D = D_{ij} = \sum_{k=1}^N A_{ik} C_{kj} = \sum_{k=1}^N B_{ik} B_{kj}. \quad (18)$$

3.2 Application of boundary conditions

Let us now denote

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1,N-1} & A_{1,N} \\ A_{21} & A_{22} & \cdots & A_{2,N-1} & A_{2,N} \\ \vdots & \vdots & & \vdots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{N,N-1} & A_{N,N} \end{bmatrix} \\ \bar{A} &= \begin{bmatrix} 0 & A_{12} & \cdots & A_{1,N-1} & 0 \\ 0 & A_{22} & \cdots & A_{2,N-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & A_{N2} & \cdots & A_{N,N-1} & 0 \end{bmatrix} \\ \bar{A}_1 &= \begin{bmatrix} 0 & A_{1,2} & \cdots & A_{1,N} \\ 0 & A_{2,2} & \cdots & A_{2,N} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & A_{N,2} & \cdots & A_{N,N} \end{bmatrix} \\ \bar{A}_2 &= \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N-1} & 0 \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N-1,N-1} & 0 \end{bmatrix}. \end{aligned}$$

Table 1: Validation for S-S case

K_w	$\sqrt{\lambda_1}$			$\sqrt{\lambda_2}$			$\sqrt{\lambda_3}$		
	Present	[45]	[35]	Present	[45]	[35]	Present	[45]	[35]
10	3.2192	3.2192	3.2192	6.2932	6.2932	6.2932	9.4277	9.4277	9.4277
50	3.4844	3.4844	3.4844	6.3329	6.3329	6.3329	9.4396	9.4396	9.4396
100	3.7483	3.7483	3.7483	6.3816	6.3816	6.3816	9.4544	9.4544	9.4544
200	4.1527	4.1527	4.1527	6.4757	6.4757	6.4757	9.4839	9.4839	9.4839
500	4.9438	4.9438	4.9438	6.7358	6.7358	6.7358	9.5706	9.5706	9.5706
1000	5.7556	5.7556	5.7556	7.1121	7.1121	7.1121	9.7101	9.7101	9.7101
1500	6.3219	6.3219	6.3219	7.4366	7.4366	7.4366	9.8439	9.8439	9.8439
2000	6.7673	6.7673	6.7673	7.7235	7.7235	7.7235	9.9724	9.9724	9.9724

Table 2: Validation for C-C case

K_w	$\sqrt{\lambda_1}$		$\sqrt{\lambda_2}$		$\sqrt{\lambda_3}$	
	Present	[45]	Present	[45]	Present	[45]
1	4.7324	4.7324	7.8537	7.8537	10.9958	10.9958
10	4.7534	4.7534	7.8583	7.8583	10.9975	10.9975
100	4.9503	4.9503	7.9043	7.9043	11.0144	11.0144
1000	6.2239	6.2239	8.3251	8.3251	11.1790	11.1790

Now, the weighting coefficients of higher order derivatives for different edge conditions are given below.

Simply Supported-Simply Supported

$$\begin{aligned}\{W'\} &= [\bar{A}] \{W\} \\ \{W''\} &= [A] \{W'\} = [A] [\bar{A}] \{W\} = [\bar{B}] \{W\} \\ \{W'''\} &= [\bar{A}] \{W''\} = [\bar{A}] [\bar{B}] \{W\} = [\bar{C}] \{W\} \\ \{W^{IV}\} &= [A] \{W'''\} = [A] [\bar{C}] \{W\} = [\bar{D}] \{W\}\end{aligned}$$

Clamped-Simply Supported

$$\begin{aligned}\{W'\} &= [\bar{A}] \{W\} \\ \{W''\} &= [\bar{A}_1] \{W'\} = [\bar{A}_1] [\bar{A}] \{W\} = [\bar{B}] \{W\} \\ \{W'''\} &= [\bar{A}_2] \{W''\} = [\bar{A}_2] [\bar{B}] \{W\} = [\bar{C}] \{W\} \\ \{W^{IV}\} &= [A] \{W'''\} = [A] [\bar{C}] \{W\} = [\bar{D}] \{W\}\end{aligned}$$

Clamped-Clamped

$$\begin{aligned}\{W'\} &= [\bar{A}] \{W\} \\ \{W''\} &= [\bar{A}] \{W'\} = [\bar{A}] [\bar{A}] \{W\} = [\bar{B}] \{W\} \\ \{W'''\} &= [A] \{W''\} = [A] [\bar{B}] \{W\} = [\bar{C}] \{W\} \\ \{W^{IV}\} &= [A] \{W'''\} = [A] [\bar{C}] \{W\} = [\bar{D}] \{W\}\end{aligned}$$

Clamped-Free

$$\begin{aligned}\{W'\} &= [\bar{A}_1] \{W\} \\ \{W''\} &= [\bar{A}_1] \{W'\} = [\bar{A}_1] [\bar{A}_1] \{W\} = [\bar{B}] \{W\}\end{aligned}$$

$$\{W'''\} = [\bar{A}_2] \{W''\} = [\bar{A}_2] [\bar{B}] \{W\} = [\bar{C}] \{W\}$$

$$\{W^{IV}\} = [\bar{A}_2] \{W'''\} = [\bar{A}_2] [\bar{C}] \{W\} = [\bar{D}] \{W\}$$

Substituting the value of Eq. (13) into Eq. (12), one may obtain generalized eigenvalue problem as

$$[S] \{W\} = \lambda^2 [T] \{W\} \quad (19)$$

where S is the stiffness matrix and T is the mass matrix.

4 Numerical results and discussions

MATLAB program is used to solve Eq. (19) and frequency parameters $\sqrt{\lambda}$ have been obtained. Following parameters are taken from for computational purpose.

$$E = 1 \text{ TP}a, L = 10nm, h = 1 \text{ and unless mentioned } \frac{L}{h} = 10.$$

4.1 Validation

To verify the correctness of present method, we have adopted two ways. Firstly, we consider the SLGNR with nonlocal parameter $\mu = 0$ and non-uniform parameter $\eta = 0$. We compare our results of frequency parameter ($\sqrt{\lambda}$) with those available in the literature [35, 45] for different

Table 3: Validation for C-F case

K_w	$\sqrt{\lambda_1}$		$\sqrt{\lambda_2}$		$\sqrt{\lambda_3}$	
	Present	[45]	Present	[45]	Present	[45]
1	1.9119	1.9119	4.6965	4.6965	7.8552	7.8552
10	2.1746	2.1746	4.7180	4.7180	7.8599	7.8599
100	3.2557	3.2557	4.9191	4.9191	7.9058	7.9058
1000	5.6407	5.6407	6.2082	6.2082	8.3264	8.3264

Table 4: Validation for S-S case

μ	Present	[23]	[24]	[25]
0	9.8696	9.8696	9.8696	9.8696
1	9.4159	9.4159	9.4124	9.4159
2	9.0195	9.0195	9.0133	9.0195
3	8.6693	8.6693	8.6611	8.6693
4	8.3569	8.3569	8.3472	8.3569

Winkler modulus parameter and are given in Tables 1-3. Secondly, we consider the SLGNR as uniform and is not placed on any elastic foundation viz. $\eta = 0$ and $K_w = 0$. Validation of fundamental frequency parameters (λ) which are obtained by present method is accomplished by comparing with the results available in the literature [23–25] which are depicted in Tables 4. Again comparison of present results are achieved with [22] for different $\alpha = \frac{e_0 a}{L}$ which are reported in Tables 5-8. These Tables 1-8 shows the evidence of correctness of present results with those available in the literature.

4.2 Convergence

Convergence of the present study is explored by varying grid points with frequency parameters which are demonstrated in Figure 1 and in Tables 9-12. All the standard boundary conditions such as S-S, C-S, C-C and C-F are taken into consideration with $L = 10$, $\eta = 1$, $\mu = 1$ and $K_w = 10$. From these results, it is very clear that convergence is achieving fast with an increase in a number of grid points and twelve grid points are sufficient to compute the desired results.

4.3 Effect of nonlocal parameter

The impacts of nonlocal parameters on frequency parameters are noted in this subsection. The nonlocal parameters are allowed to vary from 0 to 1 nm^2 with an increment of 0.2. All the standard boundary conditions such as S-S, C-S, C-C, and C-F are taken into investigation with non-uniform pa-

rameter (η) as 0.5, Winkler modulus parameter $K_w = 50$ and $L = 10 \text{ nm}$. All the tabular and graphical results are demonstrated in Figure 2 and Tables 13-16. These results reveal that frequency parameters decrease with increase in nonlocal parameter except first fundamental frequency parameter of CF nanobeams and CC nanobeams possess having the highest frequency parameters among all other boundary conditions taken into investigation.

4.4 Effect of non-uniform parameter

This subsection deals with the variation of non-uniform parameter η on frequency parameters. The non-uniform parameters are allowed to vary from -2 to 2 with an increment of 1. Tables 17-20 and Figure 3 illustrate the response of frequency parameter with non-uniform parameter for all the standard edges and these computations are accomplished with $\mu = 1 \text{ nm}^2$, $L = 10$ and $K_w = 500$. One observation regarding the non-uniform parameters is that the frequency parameter decreases with increase in the non-uniform parameter and this decrease is more significant in case of higher modes.

4.5 Effect of aspect ratio

The effect of aspect ratio (L/h) is reported for all the standard boundary conditions for different (10, 20, 30, 40, 50). Figure 4 and Tables 21-24 represent the response in term of the graphical and tabular results which are accomplished by taking $\mu = 1$, $K_w = 50$ and . One important point is to note that frequency parameters with increase in aspect ratio for all the mode of all the standard boundary conditions except 1st mode of C-F case, where it is just reverse.

4.6 Effect of Winkler modulus parameter

Effect of Winkler modulus parameter (K_w) on the first four frequency parameters has been studied. First four frequency parameters of all the classical boundary condi-

Table 5: Validation for S-S case

$\alpha = e_0 a/L$	Mode	1 st Mode		2 nd Mode		3 rd Mode		4 th Mode		5 th Mode	
		Present	[22]	Present	[22]	Present	[22]	Present	[22]	Present	[22]
0		3.1416	3.1416	6.2832	6.2832	9.4248	9.4248	1.566	1.566	15.708	15.708
0.1		3.0685	3.0685	5.7817	5.7817	8.0400	8.0400	9.9161	9.9161	11.5111	11.5111
0.3		2.6800	2.6800	4.3013	4.3013	5.4422	5.4422	6.3630	6.3630	7.1568	7.1568

Table 6: Validation for C-S case

$\alpha = e_0 a/L$	Mode	1 st Mode		2 nd Mode		3 rd Mode		4 th Mode		5 th Mode	
		Present	[22]	Present	[22]	Present	[22]	Present	[22]	Present	[22]
0		3.9266	3.9266	7.0686	7.0686	10.2102	10.2102	13.3518	13.3518	16.4934	16.4934
0.1		3.8209	3.8209	6.4649	6.4649	8.6517	8.6517	10.469	10.469	12.018	12.018
0.3		3.2828	3.2828	4.7668	4.7668	5.8371	5.8371	6.7143	6.7143	7.4773	7.4773

Table 7: Validation for C-C case

$\alpha = e_0 a/L$	Mode	1 st Mode		2 nd Mode		3 rd Mode		4 th Mode		5 th Mode	
		Present	[22]	Present	[22]	Present	[22]	Present	[22]	Present	[22]
0		4.7300	4.7300	7.8532	7.8532	10.9956	10.9956	14.1372	14.1372	17.2787	17.2787
0.1		4.5945	4.5945	7.1402	7.1402	9.2583	9.2583	11.016	11.016	12.520	12.520
0.3		3.9184	3.9184	5.1963	5.1963	6.2317	6.2317	7.0482	7.0482	7.7955	7.7955

Table 8: Validation for C-F case

$\alpha = e_0 a/L$	Mode	1 st Mode		2 nd Mode		3 rd Mode		4 th Mode		5 th Mode	
		Present	[22]	Present	[22]	Present	[22]	Present	[22]	Present	[22]
0		1.8751	1.8751	4.6941	4.6941	7.8548	7.8548	10.9955	10.9955	14.1372	14.1372
0.1		1.8792	1.8792	4.5475	4.5475	7.1459	7.1459	9.2569	9.2569	11.016	11.016
0.3		1.9154	1.9154	3.7665	3.7665	5.2988	5.2988	6.1385	6.1385	7.1450	7.1450

Table 9: Response of frequency parameters with grid points for SS case

N	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
3	2.7361	-	-	-
4	3.1797	5.2502	-	-
5	3.0783	6.2776	7.6390	-
6	3.0861	5.7216	9.1149	9.9562
7	3.0863	5.7979	7.8481	11.6940
8	3.0863	5.7956	8.0757	9.5864
10	3.0863	5.7956	8.0510	9.9058
12	3.0863	5.7957	8.0502	9.9229
14	3.0863	5.7957	8.0502	9.9235
16	3.0863	5.7957	8.0502	9.9235

Table 10: Response of frequency parameters with grid points for CS case

N	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
3	2.4936	-	-	-
4	3.4239	4.7799	-	-
5	3.6867	6.2257	7.1386	-
6	3.7116	6.2895	9.1574	9.5255
8	3.7081	6.4102	8.6762	10.0479
10	3.7082	6.4159	8.6182	10.3505
12	3.7082	6.4159	8.6213	10.4493
14	3.7082	6.4159	8.6210	10.4469
16	3.7082	6.4159	8.6210	10.4472
18	3.7082	6.4159	8.6210	10.4472

Figure 1: Frequency parameters Vs. grid points

Table 11: Response of frequency parameters with grid points for CC case

N	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
4	4.2431	-	-	-
5	4.3489	7.1782	-	-
6	4.6461	7.0825	10.3331	-
7	4.7300	7.2439	9.8880	13.6930
8	4.6180	7.4575	9.4901	12.5849
10	4.6208	7.1344	9.2584	12.1193
12	4.6208	7.1576	9.2702	10.9375
14	4.6208	7.1561	9.2689	11.0426
16	4.6208	7.1562	9.2690	11.0210
18	4.6208	7.1562	9.2690	11.0231
20	4.6208	7.1562	9.2690	11.0230
21	4.6208	7.1562	9.2690	11.0230

Table 12: Response of frequency parameters with grid points for CF case

N	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
3	1.8796	2.5457	-	-
4	2.0202	3.8591	4.5134	-
6	2.0466	4.5329	6.7832	9.7516
8	2.0465	4.5053	7.1253	9.3260
10	2.0465	4.5064	7.1407	9.2508
12	2.0465	4.5063	7.1407	9.2596
14	2.0465	4.5063	7.1407	9.2589
16	2.0465	4.5063	7.1407	9.2589

Table 13: Response of frequency parameter on nonlocal parameter for SS case

μ	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	3.3586	6.3153	9.4357	12.5718
1	3.2999	5.8227	8.0566	9.9254
2	3.2483	5.4814	7.3224	8.8122
3	3.2025	5.2246	6.8371	8.1345
4	3.1616	5.0213	6.4813	7.6583
5	3.1248	4.8544	6.2039	7.2965

Table 14: Response of frequency parameter on nonlocal parameter for CS case

μ	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	3.9850	7.0573	10.1949	13.3380
1	3.8908	6.4640	8.6451	10.4625
2	3.8088	6.0622	7.8397	9.2792
3	3.7364	5.7640	7.3128	8.5634
4	3.6721	5.5299	6.9285	8.0618
5	3.6143	5.3389	6.6298	7.6812

tions such as SS, CS, CC and CF are given in Tables 25-28 for different K_w . In this table, $\mu = 1 \text{ nm}^2$, $L = 20$ and $\eta = 0.5$ are taken into consideration. Graphical results are depicted in Figure 6 where variation of first four frequency parameters with has been shown. In this figure, the value of (K_w) is given as 0, 50, 100, 200, 500, 1000.

Figure 2: Frequency parameters Vs nonlocal parameters

Table 15: Response of frequency parameter on nonlocal parameter for CC case

μ	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	4.8035	7.8717	11.0036	14.1417
1	4.6747	7.1643	9.2702	11.0230
2	4.5625	6.6915	8.3888	9.7636
3	4.4637	6.3434	7.8181	9.0056
4	4.3757	6.0715	7.4044	8.4753
5	4.2966	5.8505	7.0841	8.0731

Table 16: Response of frequency parameter on nonlocal parameter for CF case

μ	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	2.5691	4.7267	7.8407	10.9779
1	2.5825	4.6094	7.1657	9.2693
2	2.5965	4.4986	6.7124	8.3850
3	2.6110	4.3929	6.3838	7.8005
4	2.6262	4.2916	6.1351	7.3644
5	2.6422	4.1940	5.9423	7.0144

Table 17: Response of frequency parameter with non-uniform parameter for SS case

η	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
-2	7.8944	8.3196	9.4262	10.7616
-1	6.1705	7.0580	8.6320	10.2525
0	4.9256	6.3416	8.2704	10.0418
1	4.0576	6.0143	8.1351	9.9691
2	3.5003	5.9032	8.1059	9.9589

Table 18: Response of frequency parameter with non-uniform parameter for CS case

η	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
-2	7.9868	8.6471	9.8887	11.2541
-1	6.3321	7.5132	9.1760	10.7849
0	5.1676	6.8847	8.8385	10.5760
1	4.3842	6.5799	8.6904	10.4864
2	3.8861	6.4472	8.6322	10.4519

Table 19: Response of frequency parameter with non-uniform parameter for CC case

η	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
-2	8.0735	8.9193	10.2689	11.6639
-1	6.5266	7.9405	9.6671	11.2670
0	5.5453	7.4613	9.4119	11.1081
1	5.0224	7.2759	9.3250	11.0564
2	4.8224	7.2390	9.3174	11.0543

Table 20: Response of frequency parameter with non-uniform parameter for CF case

η	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
-2	7.8762	7.9678	8.8392	10.1762
-1	6.0870	6.4752	7.9127	9.6349
0	4.7579	5.5188	7.4662	9.4105
1	3.7891	4.9497	7.2652	9.3162
2	2.9990	4.5773	7.1568	9.2583

Table 21: Response of frequency parameter with length-to-height ratio for SS case

L/h	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
10	3.2999	5.8227	8.0566	9.9254
20	3.3432	6.1715	8.9764	11.5700
30	3.3517	6.2493	9.2172	12.0747
40	3.3547	6.2777	9.3097	12.2799
50	3.3561	6.2911	9.3540	12.3811

Table 22: Response of frequency parameter with length-to-height ratio for CS case

L/h	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
10	3.8908	6.4640	8.6451	10.4625
20	3.9601	6.8823	9.6726	12.2367
30	3.9738	6.9768	9.9456	12.7893
40	3.9787	7.0114	10.0509	13.0153
50	3.9809	7.0278	10.1016	13.1269

Table 23: Response of frequency parameter with length-to-height ratio for CC case

L/h	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
10	4.6747	7.1643	9.2702	11.0230
20	4.7695	7.6618	10.4134	12.9352
30	4.7883	7.7750	10.7211	13.5385
40	4.7949	7.8166	10.8402	13.7864
50	4.7980	7.8362	10.8976	13.9092

Table 24: Response of frequency parameter with length-to-height ratio for CF case

L/h	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
10	2.5825	4.6094	7.1657	9.2693
20	2.5724	4.6967	7.6417	10.4000
30	2.5705	4.7133	7.7491	10.7018
40	2.5699	4.7191	7.7886	10.8183
50	2.5696	4.7218	7.8071	10.8745

5 Mode Shapes

Proper understanding of mode-shapes is an essential aspect of vibrational characteristics which helps in designing various engineering systems like nanoelectromechanical system. In this regards, mode-shapes are plotted in Figures 6–17 for different modes of SS, CS, CC, and CF. Mode-shapes are demonstrated by taking, $L = 10$, $\mu = 1$, $K_w = 500$ and grid points $N = 100$. These graphs are

Table 25: Response of frequency parameter with Winkler modulus parameter for SS case

K_w	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	3.1187	6.1390	8.9659	11.5651
50	3.3432	6.1715	8.9764	11.5700
100	3.5299	6.2035	8.9869	11.5749
200	3.8332	6.2661	9.0077	11.5847
500	4.4661	6.4433	9.0693	11.6138
1000	5.1457	6.7097	9.1693	11.6619

Table 26: Response of frequency parameter with Winkler modulus parameter for CS case

K_w	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	3.8320	6.8590	9.6642	12.2326
50	3.9601	6.8823	9.6726	12.2367
100	4.0769	6.9055	9.6809	12.2409
200	4.2843	6.9511	9.6976	12.2491
500	4.7727	7.0827	9.7471	12.2738
1000	5.3547	7.2871	9.8279	12.3146

Table 27: Response of frequency parameter with Winkler modulus parameter for CC case

K_w	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	4.6981	7.6449	10.4067	12.9317
50	4.7695	7.6618	10.4134	12.9352
100	4.8379	7.6786	10.4201	12.9387
200	4.9666	7.7119	10.4335	12.9457
500	5.3023	7.8092	10.4733	12.9667
1000	5.7507	7.9637	10.5387	13.0013

Table 28: Response of frequency parameter with Winkler modulus parameter for CF case

K_w	$\sqrt{\lambda_1}$	$\sqrt{\lambda_2}$	$\sqrt{\lambda_3}$	$\sqrt{\lambda_4}$
0	1.9098	4.6213	7.6246	10.3932
50	2.5724	4.6967	7.6417	10.4000
100	2.9356	4.7686	7.6587	10.4068
200	3.4102	4.9034	7.6924	10.4203
500	4.2233	5.2523	7.7909	10.4604
1000	4.9960	5.7139	7.9473	10.5263

plotted by varying non-uniform parameter η as -2 , 0 and 2 . From these figures, we may note that the amplitudes of displacements are affected significantly by varying non-uniform parameters.

Figure 3: Frequency parameters Vs. non-uniform parameters

Figure 4: Frequency parameters Vs. Aspect ratio

Figure 5: Frequency parameters Vs. Winkler modulus parameters

Figure 6: SS-1st Mode Figure 7: SS-2nd Mode Figure 8: SS-3rd Mode Figure 9: CS-1st Mode

6 Conclusions

Present investigation deals with the study of free vibration of variable cross-section single-layered graphene nano-

Figure 10: CS-2nd Mode

Figure 11: CS-3rd Mode

Figure 12: CC-1st Mode

Figure 13: CC-2nd Mode

Figure 14: CC-3rd Mode

Figure 15: CF-1st Mode

Figure 16: CF-2nd Mode

Figure 17: CF-3rd Mode

ribbons (SLGNRs) which is placed on Winkler elastic foundation using the Differential Quadrature Method (DQM). Eringen's nonlocal theory is employed along with EBT for this study. The tabular and graphical results are obtained by solving the generalized Eigen value problem using MATLAB codes. The impacts of several scaling parameters such as the Winkler modulus parameter, non-uniform parameter, nonlocal parameter, and aspect ratio are studied. Following are the main observations.

- The frequency parameter decreases with increase in the non-uniform parameter and this decrease is more significant in the case of higher modes.
- The Clamped-Clamped case possesses the highest frequency parameters whereas the Clamped-Free case possesses the lowest among all other types of boundary conditions.
- The effect of the nonlocal parameter is more significant in higher modes. Frequency parameters decrease with increase in nonlocal parameter except fundamental frequency parameter of CF case.
- Frequency parameters with increase in aspect ratio for all the mode of all the standard boundary conditions except 1st mode of C-F case, where it is just reverse.
- The frequency parameter increases with increase in the Winkler modulus parameter and this increase is more significant in the case of lower modes.
- The amplitudes of displacements are affected significantly by varying non-uniform parameters.

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