Research Article

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# On the deformation of laminated composite and sandwich curved beams

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**Abstract:** Plenty of research articles are available on the static deformation analysis of laminated straight beams using refined shear deformation theories. However, research on the deformation of laminated curved beams with simply supported boundary conditions is limited and needs more attention nowadays. With this objective, the present study deals with the static analysis of laminated composite and sandwich beams curved in elevation using a new quasi-3D polynomial type beam theory. The theory considers the effects of both transverse shear and normal strains, i.e. thickness stretching effects. In the present theory, axial displacement has expanded up to the fifth-order polynomial in terms of thickness coordinates to effectively account for the effects of curvature and deformations. The present theory satisfies the zero traction boundary condition on the top and bottom surfaces of the beam. Governing differential equations and associated boundary conditions are established by using the Principal of virtual work. Navier's solution technique is used to obtain displacements and stresses for simply supported beams curved in elevation and subjected to uniformly distributed load. The present results can be benefited to the upcoming researchers.

**Keywords:** fifth-order polynomial, laminated and sandwich, curved beams, static analysis

## 1 Introduction

Laminated composite curved beams/arches are widely used in aerospace, automobile, ships, civil and mechanical industries due to their superior properties such as high stiffness-to-weight ratios as well as the high strength-to-weight ratio. Therefore, it is significantly important to investigate the accurate static behaviour of curved beams subjected to static loading. Various laminated beam theories are available in the literature which can be extended for the analysis of curved beams such as classical beam theory [1], Timoshenko beam theory [2], Higher order beam theories [3, 4, 5], etc. In this section literature on various beam theories available in the literature and their applications in various problems are reviewed.

Reddy [6] has developed a well-known third order shear deformation theory for the analysis of laminated composite beams which is further extended by Khdeir and Reddy [7, 8] for the analysis of cross-ply laminated beams/arches. Kant and Manjunath [9] have presented a static analysis of symmetric and unsymmetric laminated composite and sandwich beams based on a $C^0$ continuity finite element using various refine higher order beam theories. Kant *et al.* [10] have developed semi-analytical elasticity bending solutions for laminated composite beams. Li *et al.* [11] have studied the free vibration analysis of laminated composite beams of different boundary conditions using various higher-order beam theories and spectral finite element method which is also used by Nanda and Kapuria [12] for the wave propagation analysis of laminated composite curved beams. Luu *et al.* [13] investigated non-dimensional deflection and critical buckling loads of shear deformable laminated composite curved beams using the NURBS-based isogeometric method. Ye *et al.* [14] and Mohamad *et al.* [15] studied vibration analysis of laminated composite curved beams of various boundary conditions. Jianghua *et al.* [16] have presented the static and vibration analysis of laminated composite curved beams using a domain decomposition approach. Zenkour [17] has developed a new shear and normal deformation theory for the static analysis of cross-ply laminated composite and sandwich beams. Karama *et al.* [18] have developed an exponential shear deformation theory for the bending, buckling, and free vibration analysis of multi-layered laminated composite beams. Piovan *et al.* [19] have analyzed composite thin-walled curved beams with open and closed cross-sections. Hajianmaleki and Qatu [20] have applied Timoshenko beam theory for the static and free vibration analysis of generally laminated

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deep curved beams with boundary conditions. Tessler *et al.* [21] have presented the analysis of laminated composites and sandwich beams using a zig-zag theory. Carrera *et al.* [22] have investigated the static response of laminated composite beams by using various polynomial and non-polynomial beam theories. Dogruoglu and Komurcu [23] have presented the static analysis of planar curved beams by using a finite element method. Bhimaraddi *et al.* [24] have investigated the static and dynamic analysis of isoparametric thick laminated curved beams by using a finite element method. Bouclier *et al.* [25] have developed shear and membrane locking free isogeometric formulation for the analysis of curved beams. Fraternali and Bilotti [26] have presented the stress analysis of composite curved beams using one-dimensional theory and finite element method. Kurtaran [27] applied a differential quadrature method for the static and transient analysis of functionally graded curved beams using first order shear deformation theory. Matsunaga [28] has investigated the displacements and stresses of the laminated and sandwich curved beams under mechanical/thermal loading by using higher order theory. Qian *et al.* [29] have studied the displacements and stresses of the laminated arches, under thermal load by using two-dimensional thermo-elastic theory. Malekzadeh [30] has presented the static analysis of thick laminated deep circular arches with various boundary conditions by using the two-dimensional theory of elasticity. Casari and Gornet [31] have presented the static analysis of composite curved sandwich beams under thermo-mechanical load. Kress *et al.* [32] have investigated the stress distributions of thick and singly curved laminates by using finite element method. Marur and Kant [33] have analyzed the static behaviour of laminated arches using transverse shear and normal deformation theory. Sayyad and his co-authors' [34-38] have presented various polynomial, and non-polynomial type beam theories for the static, vibration, and buckling analysis of isotropic, functionally graded, laminated, and sandwich beams. Recently, Sayyad and Ghugal [39] presented static analysis sandwich curved beams using a sinusoidal shear deformation theory considering the effects of thickness stretching. Fazzolari *et al.* [40] presented a comparative study between two computational techniques (hierarchical ritz formulation and generalized differential quadrature) to evaluate the natural frequencies of composite beams and shells. Dimitri *et al.* [41] also presented the analysis of various curved segments using generalized differential quadrature method. Fantuzzi and Tornabene [42] presented analysis of arbitrarily shaped plates using strong finite element formation and differential quadrature method. Tornabene *et al.* [43] have presented the through-the-thickness distribution of strains and stresses for doubly-curved panels using various single layer equivalent and layerwise theories based on Carrera's unified formulation. Tornabene *et al.* [44] presented natural frequencies of several doubly-curved shells with variable thickness using various higher-order equivalent single layer theories including the Murakami's function to capture the zig-zag effect. Pagani *et al.* [45] applied various classical and refined plate theories for the analysis of laminates and sandwich structures using finite element method. Carrera *et al.* [46, 47] have developed a new finite element for the analysis of metallic and composite plates and shells.

## 1.1 Scientific soundness of the topic

1) Beams curved in elevation are widely used in many engineering industries such as arch type bridges, chain links, crane hooks, pipe bends, and curved segments of machine tool frames. Curved beam segments used in these structures are often subjected to the static forces where it needs to analyze and design accurately. This forced the authors to consider this topic as an area of research.
2) Based on the aforementioned literature review, it is observed that a lot of research has been carried out by researchers on static analysis of straight beams using higher-order shear deformation theories. However, research on static analysis of curved beams is limited in the literature.
3) From the literature review, it is also observed that, in the higher order theories available in the literature, thickness coordinates are expanded up to third order only. However, for the accurate prediction of static behaviour laminated curved beams, it is necessary to expand thickness coordinates up to minimum fifth order. Therefore, in this paper, a fifth-order shear deformation theory is developed for the curved beam.
4) It is recommended by Carrera *et al.* [22] that the transverse normal strain plays an important role in predicting the accurate bending behavior of thick composite beams. Therefore, the present theory also considers the effects of transverse normal strain i.e. thickness stretching effects for the modeling and analysis of curved laminated beams.
5) In this paper, the static analysis of laminated composites and sandwich beams curved in elevation is analyzed under uniform load.
6) The present theory and the approach can be extended to analyze the curved beams subjected to dynamic loading, thermal loading, nonlinear problems, and

the curved beams accounting for variable radii of curvatures and thicknesses.

## 2 Mathematical formulation of the present theory for curved beam

Consider a laminated curved beam with radius of curvature $R$ in the Cartesian coordinate system as shown in Figure 1. The beam has curved length $L$, and rectangular cross-section $b \times h$ where $b$ is the width ($b$=1) and $h$ is the total thickness ($0 \leq x \leq L$; $-b/2 \leq y \leq b/2$; $-h/2 \leq z \leq h/2$). The beam is composed of $N$ number of layers perfectly bonded together and made up of fibrous composite materials. The bond between the two adjacent layers is of zero thickness. The beam is subjected to uniform load ($q$).

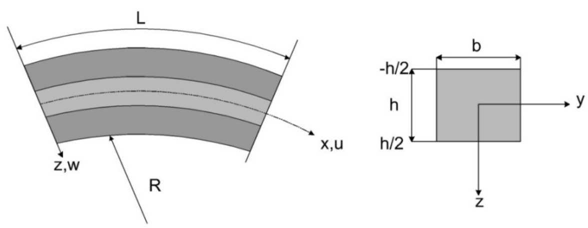


**Figure 1:** Geometry ad coordinates system of laminated curved beam.

Following are the kinematic assumptions made for the development of the displacement field of the present theory. 1) The $x$-directional displacement contains the extension, bending, and shear components. Theories that consider the effects of bending deformation, shear deformation, and normal deformation (thickness-stretching) are called quasi-3D theories. 2) The extension component is presented interms of the radius of curvature. 3) The $z$-directional displacement considered the effects of transverse normal deformations i.e. $\varepsilon_z \neq 0$. 4) The present theory consists of fifth order polynomial to account for the traction-free boundary conditions at the top and bottom surfaces of the beam. Based on the before mentioned assumptions, the displacement field of the present theory is written as follows:

$$\begin{aligned} u(x,z) &= \left(1+\frac{z}{R}\right)u_0(x) - z\frac{\partial w_0}{\partial x} + \\ &+ \left(z-\frac{4z^3}{3h^2}\right)\phi_x(x) + \left(z-\frac{16z^5}{5h^4}\right)\psi_x(x) \\ w(x,z) &= w_0(x) + \left(1-\frac{4z^2}{h^2}\right)\phi_z(x) + \\ &+ \left(1-\frac{16z^4}{h^4}\right)\psi_z(x) \end{aligned} \quad (1)$$

where $(u, u_0)$ and $(w, w_0)$ are the $x$- and $z$- directional displacements, respectively; $\phi_x$, $\psi_x$, $\phi_z$, $\psi_z$are the unknown rotations to be determined. In the case of plates, the displacement model of the theory is modeled by considering displacement components in all three directions of the plate. But, in the case of beam, rotation in the y-direction is assumed as zero and the width of the beam is taken as unity in the y-direction. Hence, the displacement in the y-direction is neglected. For the curved beam, the non-zero strain components are determined using the following strain-displacement relationship for the theory of elasticity.

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{w}{R}, \quad \varepsilon_Z = \frac{\partial w}{\partial z}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} - \frac{u_0}{R} \quad (2)$$

Using displacements from Eq. (1), the following are the expressions for normal strains and transverse shear strain at any point of the beam.

$$\begin{aligned} \varepsilon_x &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} + f_1(z)\frac{\partial \phi_x}{\partial x} + \\ &+ f_2(z)\frac{\partial \psi_x}{\partial x} + \frac{w_0}{R} + f_1'(z)\frac{\phi_z}{R} + f_2'(z)\frac{\psi_z}{R} \\ \varepsilon_z &= f_1''(z)\phi_z + f_2''(z)\psi_z \\ \gamma_{xz} &= f_1'(z)\phi_x + f_2'(z)\psi_x + f_1'(z)\frac{\partial \phi_z}{\partial x} + f_2'(z)\frac{\partial \psi_z}{\partial x} \end{aligned} \quad (3)$$

where

$$\begin{aligned} f_1(z) &= \left(z-\frac{4z^3}{3h^2}\right), f_2(z) = \left(z-\frac{16z^5}{5h^4}\right) \\ f_1'(z) &= \left(1-\frac{4z^2}{h^2}\right), f_2'(z) = \left(1-\frac{16z^4}{h^4}\right) \end{aligned} \quad (4)$$

The prime ($'$) indicates the derivative with respect to the $z$-coordinate. A generalized Hooke's law is used to determine the stresses in the $k^{\text{th}}$ layer of laminated curved beams.

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{13} & 0 \\ C_{13} & C_{33} & 0 \\ 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix} \quad (5)$$

where $C_{\text{ij}}$ are the reduced stiffness coefficients.

$$\begin{aligned} C_{11} &= \left(\frac{E_1}{1-\mu_{13}\mu_{31}}\right), C_{33} = \left(\frac{E_3}{1-\mu_{13}\mu_{31}}\right), \\ C_{13} &= \left(\frac{\mu_{13}E_1}{1-\mu_{13}\mu_{31}}\right), C_{55} = G_{13} \end{aligned} \quad (6)$$

where $E_1$ and $E_3$ represent elastic moduli, $G_{13}$ represent shear modulus and $\mu_{13}$, $\mu_{31}$ represent Poisson's ratios. Axial force, bending moment and shear force resultants associated with the present theory can be derived using the following relations.

$$
\begin{aligned}
&N_x = \int_{-h/2}^{h/2} \sigma_x dz, M_x^b = \int_{-h/2}^{h/2} \sigma_x z dz, \\
&M_x^{s1} = \int_{-h/2}^{h/2} \sigma_x f_1(z) dz, M_x^{s2} = \int_{-h/2}^{h/2} \sigma_x f_2(z) dz \\
&V_x^1 = \int_{-h/2}^{h/2} \sigma_x f_1'(z) dz, \quad V_x^2 = \int_{-h/2}^{h/2} \sigma_x f_2'(z) dz, \\
&Q_z^1 = \int_{-h/2}^{h/2} \sigma_z f_1''(z) dz \\
&Q_z^2 = \int_{-h/2}^{h/2} \sigma_z f_2''(z) dz, \quad Q_{xz}^1 = \int_{-h/2}^{h/2} \tau_{xz} f_1'(z) dz, \\
&Q_{xz}^2 = \int_{-h/2}^{h/2} \tau_{xz} f_2'(z) dz
\end{aligned} \tag{7}
$$

Substitution of stresses from Eq. (5) into Eq. (7) one can get the final expressions for stress resultants as follows:

$$
\begin{aligned}
&N_x = AA_{11}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - AB_{11}\frac{\partial^2 w_0}{\partial x^2} + AC_{11}\frac{\partial \phi_x}{\partial x} + AD_{11}\frac{\partial \psi_x}{\partial x} + AE_{11}\frac{\phi_z}{R} + \\
&+ AF_{11}\frac{\psi_z}{R} + AG_{13}\phi_z + AH_{13}\psi_z \\
&M_x^b = AB_{11}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - AI_{11}\frac{\partial^2 w_0}{\partial x^2} + AJ_{11}\frac{\partial \phi_x}{\partial x} + AK_{11}\frac{\partial \psi_x}{\partial x} + AL_{11}\frac{\phi_z}{R} + \\
&+ AM_{11}\frac{\psi_z}{R} + AN_{13}\phi_z + AO_{13}\psi_z \\
&M_x^{s1} = AC_{11}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - AJ_{11}\frac{\partial^2 w_0}{\partial x^2} + AP_{11}\frac{\partial \phi_x}{\partial x} + AQ_{11}\frac{\partial \psi_x}{\partial x} + AR_{11}\frac{\phi_z}{R} + \\
&+ AS_{11}\frac{\psi_z}{R} + AT_{13}\phi_z + AU_{13}\psi_z \\
&M_x^{s2} = AD_{11}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - AK_{11}\frac{\partial^2 w_0}{\partial x^2} + AQ_{11}\frac{\partial \phi_x}{\partial x} + AV_{11}\frac{\partial \psi_x}{\partial x} + AW_{11}\frac{\phi_z}{R} + \\
&+ AX_{11}\frac{\psi_z}{R} + AY_{13}\phi_z + AZ_{13}\psi_z \\
&V_x^1 = AE_{11}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - AL_{11}\frac{\partial^2 w_0}{\partial x^2} + AR_{11}\frac{\partial \phi_x}{\partial x} + AW_{11}\frac{\partial \psi_x}{\partial x} + BA_{11}\frac{\phi_z}{R} + \\
&+ BB_{11}\frac{\psi_z}{R} + BC_{13}\phi_z + BD_{13}\psi_z \\
&V_x^2 = AF_{11}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - AM_{11}\frac{\partial^2 w_0}{\partial x^2} + AS_{11}\frac{\partial \phi_x}{\partial x} + AX_{11}\frac{\partial \psi_x}{\partial x} + BB_{11}\frac{\phi_z}{R} + \\
&+ BE_{11}\frac{\psi_z}{R} + BF_{13}\phi_z + BG_{13}\psi_z \\
&Q_z^1 = AG_{13}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - AN_{13}\frac{\partial^2 w_0}{\partial x^2} + AT_{13}\frac{\partial \phi_x}{\partial x} + AY_{13}\frac{\partial \psi_x}{\partial x} + BC_{13}\frac{\phi_z}{R} + \\
&+ BF_{13}\frac{\psi_z}{R} + BH_{33}\phi_z + BI_{33}\psi_z \\
&Q_z^2 = AH_{13}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - AO_{13}\frac{\partial^2 w_0}{\partial x^2} + AU_{13}\frac{\partial \phi_x}{\partial x} + AZ_{13}\frac{\partial \psi_x}{\partial x} + BD_{13}\frac{\phi_z}{R} + \\
&+ BG_{13}\frac{\psi_z}{R} + BI_{33}\phi_z + BJ_{33}\psi_z \\
&Q_{xz}^1 = BK_{55}\left(\phi_x + \frac{\partial \phi_z}{\partial x}\right) + BL_{55}\left(\psi_x + \frac{\partial \psi_z}{\partial x}\right) \text{ and } Q_{xz}^2 = \\
&= BL_{55}\left(\phi_x + \frac{\partial \phi_z}{\partial x}\right) + BM_{55}\left(\psi_x + \frac{\partial \psi_z}{\partial x}\right)
\end{aligned} \tag{8}
$$

Integration constants appeared in Eq. (8) are defined as follows:

$$
\begin{aligned}
&(AA_{11}, AB_{11}, AC_{11}, AD_{11}, AE_{11}, AF_{11}) = \\
&C_{11} \int_{-h/2}^{h/2} \left[1, z, f_1(z), f_2(z), f_1'(z), f_2'(z)\right] dz \\
&(AI_{11}, AJ_{11}, AK_{11}, AL_{11}, AM_{11}) = \\
&C_{11} \int_{-h/2}^{h/2} \left[z, f_1(z), f_2(z), f_1'(z), f_2'(z)\right] zdz \\
&(AG_{13}, AH_{13}) = \\
&C_{13} \int_{-h/2}^{h/2} \left[f_1^n(z), f_2''(z)\right] dz \, (AN_{13}, AO_{13}) = \\
&C_{13} \int_{-h/2}^{h/2} \left[f_1^n(z), f_2^n(z)\right] zdz \\
&(AP_{11}, AQ_{11}, AR_{11}, AS_{11}) = \\
&C_{11} \int_{-h/2}^{h/2} \left[f_1(z), f_2(z), f_1'(z), f_2'(z)\right] f_1(z) dz \\
&(AT_{13}, AU_{13}) = \\
&C_{13} \int_{-h/2}^{h/2} \left[f_1''(z), f_2''(z)\right] f_1(z) dz \\
&(AV_{11}, AW_{11}, AX_{11}) = \\
&C_{11} \int_{-h/2}^{h/2} \left[f_2(z), f_1'(z), f_2'(z)\right] f_2(z) dz
\end{aligned} \tag{9}
$$

$$(AY_{13}, AZ_{13}) = C_{13}\int_{-h/2}^{h/2}\left[f_1''(z), f_2''(z)\right] f_2(z)dz$$

$$(BA_{11}, BB_{11}) = C_{11}\int_{-h/2}^{h/2}\left[f_1'(z), f_2'(z)\right] f_1'(z)dz$$

$$(BC_{13}, BD_{13}) = C_{13}\int_{-h/2}^{h/2}\left[f_1''(z), f_2''(z)\right] f_1^1(z)dz$$

$$(BF_{13}, BG_{13}) = C_{13}\int_{-h/2}^{h/2}\left[f_1''(z), f_2''(z)\right] f_2'(z)dz, BE_{11} = C_{11}\int_{-h/2}^{h/2}\left[f_2'(z)\right]^2 dz$$

$$(BH_{33}, BI_{33}) = C_{33}\int_{-h/2}^{h/2}\left[f_1^n(z), f_2^n(z)\right] f_1''(z)\cdot dz \quad BJ_{33} = C_{33}\int_{-h/2}^{h/2}\left[f_2''(z)\right]^2 dz$$

$$(BK_{55}, BL_{55}) = C_{55}\int_{-h/2}^{h/2}\left[f_1'(z), f_2'(z)\right] f_1'(z)dzBM_{55} = C_{55}\int_{-h/2}^{h/2}\left[f_2'(z)\right]^2 dz$$

The principle of virtual work is employed to derive the governing equation and boundary condition associated with the present theory.

$$\int_0^L\int_{-h/2}^{h/2}\left(\sigma_x\delta\varepsilon_x + \sigma_z\delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz}\right)dzdx - \int_0^L q\delta w dx = 0 \tag{10}$$

where $\delta$ is the variational operator. Eq. (10) can be written as follows using strains from Eqs. (3) and stress resultants from Eq. (8).

$$\begin{aligned}
&\int_0^L\left(N_x\frac{\partial\delta u_0}{\partial x}\right)dx + \int_0^L\left(N_x\frac{\delta w_0}{R} - M_x^b\frac{\partial^2\delta w_0}{\partial x^2}\right)dx + \\
&\int_0^L\left(M_x^{s1}\frac{\partial\delta\phi_x}{\partial x} + Q_{x=}^1\delta\phi_x\right)dx \\
&+\int_0^L\left(M_x^{s2}\frac{\partial\delta\psi_x}{\partial x} + Q_{xz}^2\delta\psi_x\right)dx \\
&+\int_0^L\left(V_x^1\frac{\delta\phi_z}{R} + Q_z^1\delta\phi_z + Q_{x=}^1\frac{\partial\delta\phi_z}{\partial x}\right)dx \\
&+\int_0^L\left(V_x^2\frac{\delta\psi_z}{R} + Q_z^2\delta\psi_z + Q_{xz}^2\frac{\partial\delta\psi_z}{\partial x}\right) - \\
&\int_0^L q\left(\delta w_0 + f_1'(z)\delta\phi_z + f_2'(z)\delta\psi_z\right) = 0
\end{aligned} \tag{11}$$

Integrating Eq. (11) by parts, collecting the coefficients of $\delta u_0$, $\delta w_0$, $\delta\phi_x$, $\delta\psi_x$, $\delta\phi_z$, $\delta\psi_z$ and setting them equal to zero, the following six governing differential equations are obtained.

$$\begin{aligned}
\delta u_0 &: \frac{\partial N_x}{\partial x} = 0 \\
\delta w_0 &: \frac{\partial^2 M_x^b}{\partial x^2} - \frac{N_x}{R} + q = 0 \\
\delta\phi_x &: \quad \frac{\partial M_x^{s1}}{\partial x} - Q_{xz}^1 = 0 \\
\delta\psi_x &: \frac{\partial M_x^{s2}}{\partial x} - Q_{xz}^2 = 0 \\
\delta\phi_z &: \quad \frac{\partial Q_{xz}^1}{\partial x} - \frac{V_x^1}{R} - Q_z^1 + qf_1'(z) = 0 \\
\delta\psi_z &: \frac{\partial Q_{xz}^2}{\partial x} - \frac{V_x^2}{R} - Q_z^2 + qf_2'(z) = 0
\end{aligned} \tag{12}$$

The boundary conditions at the supports ($x$=0 and $x$=$L$) are presented in Table 1.

Using Eq. (8), in terms of unknown variables, Eq. (12) can be written as

$$\begin{aligned}
&\delta u_0 : -AA_{11}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{1}{R}\frac{\partial w_0}{\partial x}\right) + AB_{11}\frac{\partial^3 w_0}{\partial x^3} - \\
&AC_{11}\frac{\partial^2\phi_x}{\partial x^2} - AD_{11}\frac{\partial^2\psi_x}{\partial x^2} - \frac{AE_{11}}{R}\frac{\partial\phi_z}{\partial x} \\
&-\frac{AF_{11}}{R}\frac{\partial\psi_z}{\partial x} - AG_{13}\frac{\partial\phi_z}{\partial x} - AH_{13}\frac{\partial\psi_z}{\partial x} = 0
\end{aligned} \tag{13}$$

**Table 1:** Natural and essential boundary conditions associated with the present theory.

| | Natural | | Essential |
|---|---|---|---|
| given | $N_x \,\vert x = 0, L$ | or | $u_0 \,\vert x = 0, L$ |
| given | $M_x^b \,\vert x = 0, L$ | or | $\frac{\partial w_0}{\partial x} \,\vert x = 0, L$ |
| given | $\frac{\partial M_x^b}{\partial x} \,\vert x = 0, L$ | or | $w_0 \,\vert x = 0, L$ |
| given | $M_x^{s1} \,\vert x = 0, L$ | or | $\phi_x \,\vert x = 0, L$ |
| given | $M_x^{s2} \,\vert x = 0, L$ | or | $\psi_x \,\vert x = 0, L$ |
| given | $Q_{xz}^1 \,\vert x = 0, L$ | or | $\phi_z \,\vert x = 0, L$ |
| given | $Q_{xz}^2 \,\vert x = 0, L$ | or | $\psi_z \,\vert x = 0, L$ |

$$\begin{aligned}\delta w_0 :\ & AA_{11}\left(\frac{1}{R}\frac{\partial u_0}{\partial x} + \frac{w_0}{R^2}\right) - AB_{11}\left(\frac{\partial^3 u_0}{\partial x^3} + \frac{2}{R}\frac{\partial^2 w_0}{\partial x^2}\right) \\ & + AI_{11}\frac{\partial^4 w_0}{\partial x^4} - AJ_{11}\frac{\partial^3 \phi_x}{\partial x^3} + \\ & \frac{AD_{11}}{R}\frac{\partial \psi_x}{\partial x} - AK_{11}\frac{\partial^3 \psi_x}{\partial x^3} + \frac{AE_{11}}{R}\phi_z + \frac{AL_{11}}{R}\frac{\partial^2 \phi_z}{\partial x^2} - \\ & AN_{13}\frac{\partial^2 \phi_z}{\partial x^2} + \frac{AF_{11}}{R^2}\psi_z + \frac{AG_{13}}{R}\phi_z \\ & + \frac{AH_{13}}{R}\psi_z - \frac{AM_{11}}{R}\frac{\partial^2 \psi_z}{\partial x^2} - AO_{13}\frac{\partial^2 \psi_z}{\partial x^2} = q\end{aligned} \tag{14}$$

$$\begin{aligned}\delta \phi_x :\ & -AC_{11}\left(\frac{\partial^2 u_0}{\partial^2 x} + \frac{1}{R}\frac{\partial w_0}{\partial x}\right) + AJ_{11}\frac{\partial^3 w_0}{\partial x^3} - \\ & AP_{11}\frac{\partial^2 \phi_x}{\partial x^2} + BK_{55}\phi_x - AQ_{11}\frac{\partial^2 \psi_z}{\partial x^2} + BL_{55}\psi_x \\ & - \frac{AR_{11}}{R}\frac{\partial \phi_z}{\partial x} - AT_{13}\frac{\partial \phi_Z}{\partial x} + BK_{55}\frac{\partial \phi_z}{\partial x} - \\ & \frac{AS_{11}}{R}\frac{\partial \psi_z}{\partial x} - AU_{13}\frac{\partial \psi_z}{\partial x} + BL_{55}\frac{\partial \psi_z}{\partial x} = 0\end{aligned} \tag{15}$$

$$\begin{aligned}\delta \psi_x :\ & -AD_{11}\left(\frac{\partial^2 u_0}{\partial^2 x} + \frac{1}{R}\frac{\partial w_0}{\partial x}\right) + AK_{11}\frac{\partial^3 w_0}{\partial x^3} - \\ & AQ_{11}\frac{\partial^2 \phi_x}{\partial x^2} + BL_{55}\phi_x - AV_{11}\frac{\partial^2 \psi_z}{\partial x^2} + BM_{55}\psi_x \\ & - \frac{AW_{11}}{R}\frac{\partial \phi_z}{\partial x} - AY_{13}\frac{\partial \phi_z}{\partial x} + BL_{55}\frac{\partial \phi_z}{\partial x} - \\ & \frac{AX_{11}}{R}\frac{\partial \psi_z}{\partial x} - AZ_{13}\frac{\partial \psi_Z}{\partial x} + BM_{55}\frac{\partial \psi_z}{\partial x} = 0\end{aligned} \tag{16}$$

$$\begin{aligned}\delta \phi_z :\ & AE_{11}\left(\frac{1}{R}\frac{\partial u_0}{\partial x} + \frac{w_0}{R^2}\right) + AG_{13}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - \\ & \frac{AL_{11}}{R}\frac{\partial^2 w_0}{\partial x^2} - AN_{13}\frac{\partial^2 w_0}{\partial x^2} + \frac{AR_{11}}{R}\frac{\partial^2 \phi_x}{\partial x^2} + AT_{13}\frac{\partial \phi_x}{\partial x} \\ & - BK_{55}\frac{\partial \phi_x}{\partial x} + \frac{AW_{11}}{R}\frac{\partial \psi_x}{\partial x} + AY_{13}\frac{\partial \psi_x}{\partial x} - BL_{55}\frac{\partial \psi_x}{\partial x} \\ & + \frac{BA_{11}}{R}\phi_Z + 2\frac{BC_{13}}{R}\phi_z + BH_{33}\phi_Z + \frac{BB_{11}}{R}\psi_Z - \\ & BK_{55}\frac{\partial^2 \phi_Z}{\partial x^2} + \frac{BD_{13}}{R}\psi_Z + \frac{BF_{13}}{R}\psi_z + BI_{33}\psi_Z - \\ & BL_{55}\frac{\partial^2 \psi_Z}{\partial x^2} = qf_1'(z)\end{aligned} \tag{17}$$

$$\begin{aligned}\delta \psi_z :\ & AF_{11}\left(\frac{1}{R}\frac{\partial u_0}{\partial x} + \frac{w_0}{R^2}\right) + AH_{13}\left(\frac{\partial u_0}{\partial x} + \frac{w_0}{R}\right) - \\ & \frac{AM_{11}}{R}\frac{\partial^2 w_0}{\partial x^2} - AO_{13}\frac{\partial^2 w_0}{\partial x^2} + \frac{AS_{11}}{R}\frac{\partial^2 \phi_x}{\partial x^2} + AU_{13}\frac{\partial \phi_x}{\partial x} \\ & - BL_{55}\frac{\partial \phi_x}{\partial x} + \frac{AX_{11}}{R}\frac{\partial \psi_x}{\partial x} + AZ_{13}\frac{\partial \psi_x}{\partial x} - BM_{55}\frac{\partial \psi_x}{\partial x} \\ & + \frac{BB_{11}}{R}\phi_Z + \frac{BF_{13}}{R}\phi_Z + \frac{BD_{13}}{R}\phi_z + BI_{33}\phi_z - \\ & - BL_{55}\frac{\partial^2 \phi_Z}{\partial x^2} + \frac{BE_{11}}{R^2}\psi_Z + 2\frac{BG_{13}}{R}\psi_Z + BJ_{33}\psi_Z - \\ & BM_{55}\frac{\partial^2 \psi_Z}{\partial x^2} = qf_2'(z)\end{aligned} \tag{18}$$

# 3 The Navier solution

Analytical solution of above mentioned six governing equations is obtained using Navier's solution technique. The solution is obtained to investigate the static behaviour of simply-supported laminated composites and sandwich beams curved in elevation. Following are the boundary conditions at the simple supports of the curved beams.

$$N_x = 0, w_0 = 0, M_x^b = 0, M_x^{s1} = 0, M_x^{s2} = 0 \phi_z = 0, \psi_z = 0 \quad \text{at } x = 0 \text{ and } x = L \tag{19}$$

According to Navier's technique, unknown variables are expanded in the single trigonometric series satisfying simply-supported boundary conditions stated in Eq. (19). The following trigonometric form is assumed for the unknown variables.

$$u_0 = \sum_{m=1}^{\infty} u_m \cos\left(\frac{m\pi x}{L}\right), w_0 = \sum_{m=1}^{\infty} w_m \sin\left(\frac{m\pi x}{L}\right)$$
$$\phi_x = \sum_{m=1}^{\infty} \phi_{xm} \cos\left(\frac{m\pi x}{L}\right), \phi_z = \sum_{m=1}^{\infty} \phi_{zm} \sin\left(\frac{m\pi x}{L}\right)$$
$$\psi_x = \sum_{m=1}^{\infty} \psi_{xm} \cos\left(\frac{m\pi x}{L}\right), \psi_z = \sum_{m=1}^{\infty} \psi_{zm} \sin\left(\frac{m\pi x}{L}\right) \tag{20}$$

where $u_m$, $w_m$, $\phi_{\mathrm{xm}}$, $\psi_{\mathrm{xm}}$, $\phi_{\mathrm{zm}}$, $\psi_{\mathrm{zm}}$ are the unknown coefficients to be determined; the transverse uniform load ($q$) acting on the beam is also expanded in the single trigonometric form:

$$q = \sum_{m=1}^{\infty} \left(\frac{4q_0}{m\pi}\right) \sin\left(\frac{m\pi x}{L}\right) \tag{21}$$

where $q_0$ is the maximum intensity of uniform load. By substituting unknown variables from Eq. (20) and transverse load from Eq. (21) into governing equations (13)-(18), the following equation is derived.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \times \begin{Bmatrix} u_0 \\ w_0 \\ \phi_x \\ \psi_x \\ \phi_z \\ \psi_z \end{Bmatrix} = \tag{22}$$

$$= \frac{4q_0}{m\pi} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ f_1'(z) \\ f_2'(z) \end{Bmatrix}$$

where,

$$K_{11} = AA_{11}\alpha^2, \quad K_{12} = K_{21} = \left(-\frac{AA_{11}}{R}\alpha - AB_{11}\alpha^3\right),$$
$$K_{13} = K_{31} = AC_{11}\alpha^2, \quad K_{14} = K_{41} = \left(AD_{11}\alpha^2\right),$$
$$K_{15} = K_{51} = \left(-\frac{AE_{11}}{R}\alpha - AG_{13}\alpha\right),$$
$$K_{16} = K_{61} = -\frac{AF_{11}}{R}\alpha - H_{13}\alpha,$$
$$K_{22} = \left(\frac{AA_{11}}{R^2} + AI_{11}\alpha^4 + 2\frac{AB_{11}}{R}\alpha^2\right),$$
$$K_{23} = K_{32} = \left(-\frac{AC_{11}}{R}\alpha - AJ_{11}\alpha^3\right),$$
$$K_{24} = K_{42} = \left(-\frac{AD_{11}}{R}\alpha - AK_{11}\alpha^3\right),$$
$$K_{25} = K_{52} = \left(\frac{AE_{11}}{R^2} + \frac{AG_{13}}{R} + \frac{AL_{11}}{R}\alpha^2 + AN_{13}\alpha^2\right),$$
$$K_{26} = K_{62}2 = \left(\frac{AF_{11}}{R^2} + \frac{AH_{13}}{R} + \frac{AM_{11}}{R}\alpha^2 + AO_{13}\alpha^2\right),$$
$$K_{33} = \left(AP_{11}\alpha^2 + BK_{55}\right), K_{34} = K_{43} = \left(AQ_{11}\alpha^2 + BL_{55}\right),$$
$$K_{35} = K_{53} = \left(-\frac{AR_{11}}{R}\alpha - AT_{13}\alpha + BK_{55}\alpha\right),$$
$$K_{36} = K_{63} = \left(-\frac{AS_{11}}{R}\alpha - AU_{13}\alpha + BL_{55}\alpha\right),$$
$$K_{44} = \left(AV_{11}\alpha^2 + BM_{55}\right),$$
$$K_{45} = K_{54} = \left(-\frac{AW_{11}}{R}\alpha - AY_{13}\alpha + BL_{55}\alpha\right), \tag{23}$$

$$K_{46} = K_{64} = \left(-\frac{AX_{11}}{R}\alpha - AZ_{13}\alpha + BM_{55}\alpha\right),$$
$$K_{55} = \left(\frac{BA_{11}}{R^2} + 2\frac{BC_{13}}{R} + BH_{33} + BK_{55}\alpha^2\right)$$
$$K_{65} = K_{56} =$$
$$= \left(\frac{BB_{11}}{R^2} + \frac{BD_{13}}{R} + \frac{BF_{13}}{R} + BI_{33} + BL_{55}\alpha^2\right)$$
$$K_{66} = \left(\frac{BE_{11}}{R^2} + 2\frac{BG_{13}}{R} + BJ_{33} + BM_{55}\alpha^2\right)$$

## 4 Illustrative problems

In this section, static analysis of simply supported laminated composites and sandwich beams curved in elevation is presented to prove the efficiency and accuracy of the present theory. The numerical results for the static analysis of laminated curved beams are not available in the literature, therefore, the present theory is validated with straight beam results available in the literature. Three types of lamination schemes ($0^0/90^0$, $0^0/90^0/0^0$, $0^0$/core/$0^0$) have been solved in the present study. The following material properties have been used to present the numerical results for laminated curved beams.

**M1:**$E_1$= 172.4 GPa, $E_3$= 6.89 GPa, $G_{13}$= 3.45 GPa, $G_{31}$= 1.378 GPa, $\mu_{13}$=0.25.
**M2:**$E_1$= 0.276 GPa, $E_3$= 3.45 GPa, $G_{13}$= 3.45 GPa, $G_{31}$= 1.378 GPa,$\mu_{13}$=0.25.

The thickness of the beam is assumed as a unity ($h$=1.0) and other dimensions depend on $L/h$ and $R/h$ ratios. The numerical results are expressed in the following normalized form in Tables 2 through 5.

$$\bar{w} = 100\frac{E_3 h^3}{q_0 L} w\left(\frac{L}{2}\right), \bar{u} = 100\frac{E_3 h^3}{q_0 L} u\left(0, -\frac{h}{2}\right),$$
$$\bar{\sigma}_x = \frac{h}{q_0 L}\sigma_x\left(0, -\frac{h}{2}\right), \bar{\tau}_{xz} = \frac{h}{q_0 L}\tau_{xz}(0,0) \quad (24)$$
where $E_3 = 6.89\text{GPa}$ (24)

Tables 2 through 5 show the normalized displacements and stresses of laminated composite and sandwich curved beams subjected to uniform load. In the case of laminated composite beams, all layers are of equal thickness. However, in the case of sandwich beams, each face sheet is of thickness 0.1*h* and the core is of thickness 0.8*h* where *h* is the overall thickness of the beam. Laminated composite beams are made up of material M1. In the case of sandwich beams, each face sheet is made up of material M1 and the core is made up of material M2. The numerical results of straight beams are compared with those presented by Reddy [6], Kant *et al.* [10], and Sayyad and Ghugal [37]. The normalized displacement and stresses are obtained for $L/h$ = 4, 10 and $R/h$=5, 10, 20. The straight beam ($R$=∞) results are compared with previously published results.

Table 1 shows the comparison of non-dimensional vertical displacement for laminated composite and sandwich beams curved in elevation. Examination of Table 1 reveals that the present results are in good agreement with those presented by Reddy [6], Kant *et al.* [10], and Sayyad and Ghugal [37] when $R$=∞. The minimum value of non-dimensional vertical displacement is observed for $0^0/90^0/0^0$ due to the absence of extension-bending coupling stiffness. Also, it is observed that the non-dimensional vertical displacement is maximum for deep curvature and minimum for shallow curvature i.e. vertical displacement decreases with respect to an increase in radius of curvature. Figure 2 shows the through-the-thickness distribution of vertical displacement. Due to the consideration of thickness stretching, i.e. the effect of transverse normal strain, vertical displacement is not constant through the thickness. In the well-known theory of Reddy [6] also this effect is neglected.

Table 3 shows a comparison of normalized axial displacement of laminated and sandwich curved beams subjected to uniform load. The present results are in good agreement with other theories for $R$=∞. Through-the-thickness distributions for all lamination schemes are plotted in Figure 3. From the figures is observed that the axial displacement is zero at $z$=+0.369*h* for $0^0/90^0$ scheme, however, in the case of $0^0/90^0/0^0$ and $0^0$/Core/$0^0$ lamination schemes, axial displacement does not change its sign and remains positive throughout the thickness of the beam.

Table 4 presents the normalized values of bending stresses in laminated and sandwich curved beams subjected to uniform load. Bending stresses of the top fiber i.e ($z$=-$h$/2) are summarized in Table 4. From the comparison of results, it is observed that the present results are closely matched with other theories. Bending stresses are increasing with respect to an increase in radius of curvature i.e. bending stress is maximum at $R$=∞. Figure 4 shows through-the-thickness distributions of bending stresses in laminated and sandwich curved beams. It is observed that bending stresses are maximum in $0^0$ layer and minimum in $90^0$ layer.

Numerical values of normalized transverse shear stresses in laminated and sandwich curved beams subjected to uniform load are compared in Table 5 and through-the-thickness distributions are plotted in Figure 5. When transverse shear stresses are obtained using the constitutive relations (CR), it shows a discontinuity at the layer interfaces which is practically not acceptable. Therefore, transverse shear stresses are recovered using direct integration (DI) of equilibrium equations of the theory of elasticity to achieve continuity at the layer interfaces.

$$\tau_{xz}^k = \int_{-h/2}^{h/2}\left(-\frac{\partial \sigma_x^k}{\partial x}\right) dz + C$$

where integration constants are determined after imposing boundary conditions of top, bottom surfaces and continuity at the layer interface. Numerical results are in good agreement with previously published results of the straight beam ($R$=∞). It is observed that the transverse shear stresses are more or less the same for all curvature values. Figure 5 shows that traction-free conditions are satisfied along with continuity of shear stresses at the layer interface. Transverse shear stresses are found maximum in $0^0$ layer.

## 5 Conclusions

In the present study, a higher order shear and normal deformation theory is developed and applied to investigate the static analysis of laminated composite and sandwich beams curved in elevation subjected to uniform load. The present theory considers the effects of both transverse shear and normal deformations. A simply-supported boundary condition is analyzed using Navier's solution technique. A close agreement with other theories for straight beams is ob-

**Table 2:** Normalized vertical displacement ($\bar{w}$) of simply-supported laminated composite and sandwich curved beams.

| *R/h* | Theory | | ***L/h*=4** | | | ***L/h*=10** | |
|---|---|---|---|---|---|---|---|
| | | $[0^0/90^0]$ | $[0^0/90^0/0^0]$ | $[0^0/\text{Core}/0^0]$ | $[0^0/90^0]$ | $[0^0/90^0/0^0]$ | $[0^0/\text{Core}/0^0]$ |
| 5 | Present | 5.8578 | 3.5166 | 12.0776 | 3.7370 | 1.1345 | 3.0862 |
| 10 | Present | 5.8565 | 3.5151 | 12.0777 | 3.7381 | 1.1346 | 3.0863 |
| 20 | Present | 5.8571 | 3.5151 | 12.0777 | 3.7381 | 1.1346 | 3.0863 |
| ∞ | Present | 5.8578 | 3.5151 | 12.0777 | 3.7390 | 1.1346 | 3.0863 |
| | Reddy [6] | 5.5900 | 3.3680 | 12.4550 | 3.6970 | 1.0980 | 3.0920 |
| | Kant *et al.* [10] | 5.9000 | 3.6050 | 13.7500 | 3.7440 | 1.1710 | 3.3300 |
| | Sayyad and Ghugal [37] | 5.5230 | 3.3940 | 12.4630 | 3.6830 | 1.1060 | 3.1000 |

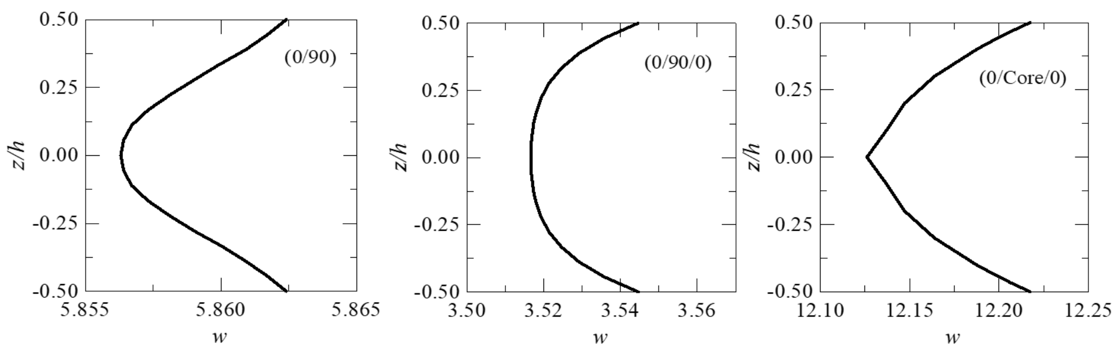


**Figure 2:** Through-the-thickness variations of vertical displacement for laminated and sandwich beams curved in elevation (*L/h*=4, *R/h*=5).

**Table 3:** Normalized axial displacement ($\bar{u}$) of simply-supported laminated composite and sandwich curved beams.

| *R/h* | Theory | | ***L/h*=4** | | | ***L/h*=10** | |
|---|---|---|---|---|---|---|---|
| | | $[0^0/90^0]$ | $[0^0/90^0/0^0]$ | $[0^0/\text{Core}/0^0]$ | $[0^0/90^0]$ | $[0^0/90^0/0^0]$ | $[0^0/\text{Core}/0^0]$ |
| 5 | Present | 5.7738 | 3.5159 | 9.8338 | 248.0907 | 78.7989 | 205.205 |
| 10 | Present | 4.0330 | 2.5155 | 6.3123 | 144.7630 | 47.4498 | 119.291 |
| 20 | Present | 3.0898 | 1.9712 | 4.3962 | 88.5963 | 30.3921 | 72.5438 |
| ∞ | Present | 2.0979 | 1.3976 | 2.3764 | 29.4110 | 12.4121 | 23.2686 |
| | Reddy [6] | 2.2580 | 1.1620 | 2.3650 | 29.8050 | 11.7340 | 23.2400 |
| | Sayyad and Ghugal [37] | 2.2680 | 1.1950 | 2.3910 | 29.7990 | 11.8910 | 23.0300 |

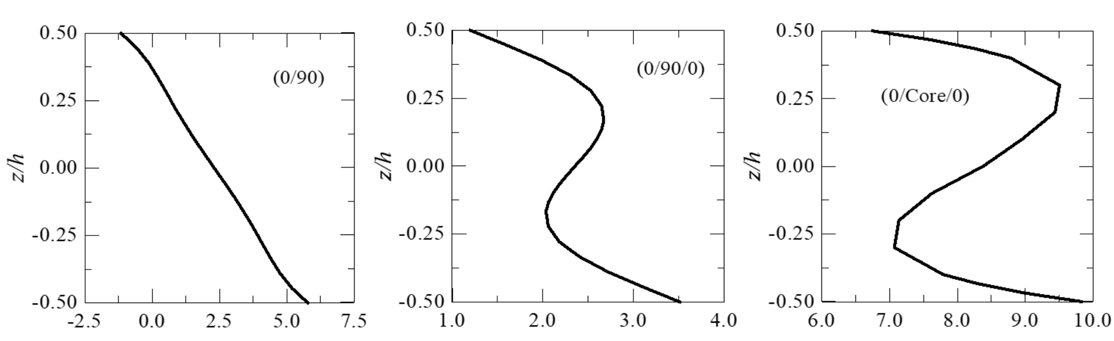


**Figure 3:** Through-the-thickness variations of axial displacement for laminated and sandwich beams curved in elevation (*L/h*=4, *R/h*=5).

**Table 4:** Normalized bending stresses ($\bar{\sigma}_x$) of simply-supported laminated composite and sandwich curved beams.

| *R/h* | Theory | *L/h*=4 | | | *L/h*=10 | | |
|---|---|---|---|---|---|---|---|
| | | [0⁰/90⁰] | [0⁰/90⁰/0⁰] | [0⁰/Core/0⁰] | [0⁰/90⁰] | [0⁰/90⁰/0⁰] | [0⁰/Core/0⁰] |
| 5 | Present | 40.9198 | 22.2994 | 40.1524 | 219.8111 | 88.1728 | 169.2986 |
| 10 | Present | 40.9294 | 22.3754 | 40.1892 | 219.8409 | 88.1083 | 169.1772 |
| 20 | Present | 40.9363 | 22.4434 | 40.2073 | 219.8508 | 88.0762 | 169.1176 |
| ∞ | Present | 40.9352 | 22.5115 | 40.2254 | 219.8585 | 88.0442 | 169.0576 |
| | Reddy [6] | 40.2390 | 19.6710 | 39.1610 | 221.0170 | 85.0300 | 168.1300 |
| | Kant *et al.* [10] | 36.6780 | 21.5680 | 43.4880 | 217.3300 | 89.1200 | 172.6000 |
| | Sayyad and Ghugal [37] | 40.4970 | 20.2880 | 39.5690 | 221.4020 | 85.6640 | 168.7600 |

**Table 5:** Normalized shear stress ($\bar{\tau}_{xz}$) of simply-supported laminated composite and sandwich curved beams.

| *R/h* | Theory | *L/h*=4 | | | *L/h*=10 | | |
|---|---|---|---|---|---|---|---|
| | | [0⁰/90⁰] | [0⁰/90⁰/0⁰] | [0⁰/Core/0⁰] | [0⁰/90⁰] | [0⁰/90⁰/0⁰] | [0⁰/Core/0⁰] |
| 5 | Present | 3.6812 | 1.9855 | 2.0462 | 8.9175 | 5.6263 | 5.8645 |
| 10 | Present | 3.6832 | 1.9852 | 2.0462 | 8.9175 | 5.6255 | 5.8645 |
| 20 | Present | 3.6851 | 1.9850 | 2.0462 | 8.9178 | 5.6249 | 5.8645 |
| ∞ | Present | 3.6860 | 1.9848 | 2.0462 | 8.9181 | 5.6243 | 5.8645 |
| | Reddy [6] | 5.0240 | 1.8310 | 2.6620 | 11.5440 | 6.0690 | 5.2870 |
| | Kant *et al.* [10] | 3.8480 | 2.4880 | 2.2800 | 10.7380 | 6.1500 | 5.2400 |
| | Sayyad and Ghugal [37] | 5.0780 | 1.7610 | 2.7970 | 11.5860 | 6.0160 | 5.2650 |

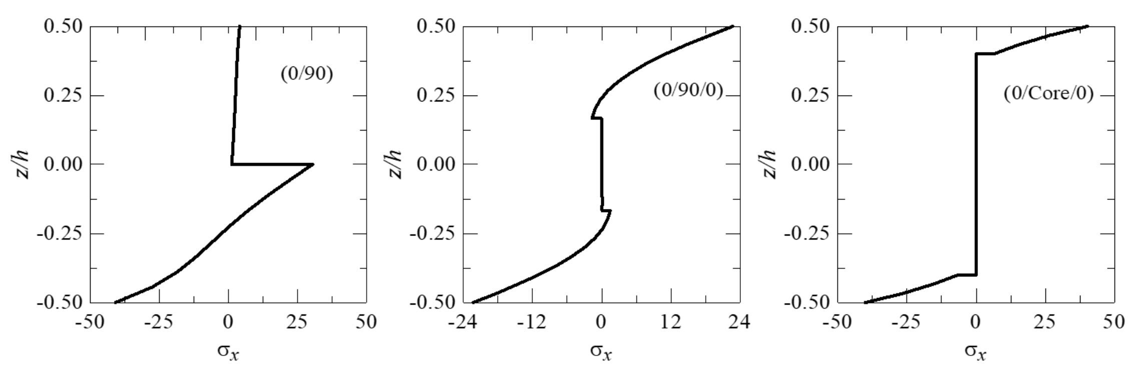


**Figure 4:** Through-the-thickness variations of bending stresses for laminated and sandwich beams curved in elevation (*L/h*=4, *R/h*=5)).

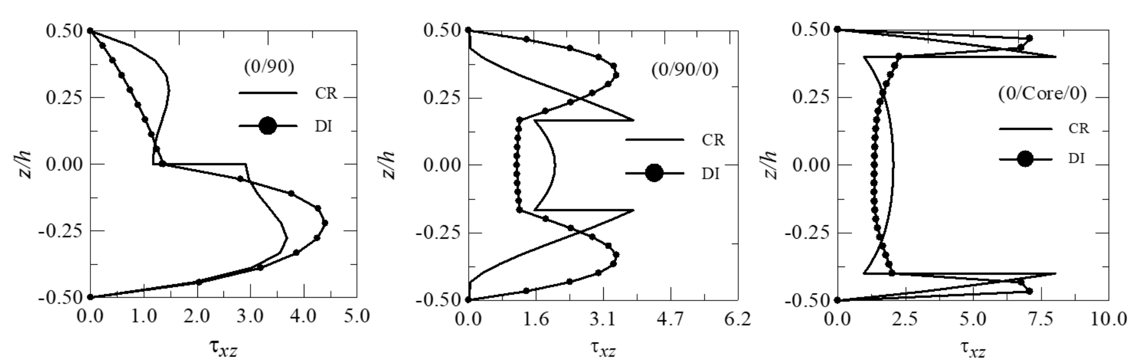


**Figure 5:** Through-the-thickness variations of transverse shear stresses for laminated and sandwich beams curved in elevation (*L/h*=4, *R/h*=5).

served. The solution presented herein for curved beams can be observed as a benchmark solution for future researchers.

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