Research Article

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Multi thermal waves in a thermo diffusive piezo electric functionally graded rod via refined multi-dual phase-lag model

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Abstract: In the present work, a novel analytical model is provided for wave dispersion in a piezo-thermoelastic diffusive functionally graded rod through the multi-phase lag model and thermal activation. The plain strain model for thermo piezoelectric functionally graded rod is considered. The complex characteristic equations are obtained by using normal mode method which satisfies the nonlinear boundary conditions of piezo-thermoelastic functionally graded rod. The numerical calculations are carried out for copper material. The results of the variants stress, mechanical displacement, temperature and electric distribution, frequency are explored against the geometric parameters and some special parameters graded index, concentration constants are shown graphically. The observed results will be discuss elaborate. The results can be build reasonable attention in piezo-thermoelastic materials and smart materials industry.

Keywords: Thermoelastic diffusion, generalized piezothermoelasticity, FG rod, RPL model, wave dispersion

1 Introduction

Piezo-thermoelastic materials are smart materials, which responds to mechanical, thermal and electric loads and exhibits their coupling effect simultaneously. This cou-

pling effect of thermoelastic and electric fields plays crucial role in piezo-thermoelastic materials and finds application as sensors/actuators for automatic fire control systems, infra-red (IR) detectors and devices for intruder alarming, thermal imaging and geographical mapping [1]. They also have applications in petrochemical plants, military vessels, tunnels, underground construction, solar towers, chimneys, and boilers. The sensors/ actuators are often used in the form of thin disk, hollow cylinder/rod and spherical shell structures for their best performance. When sensors/actuators designed by piezo-thermoelastic materials are used in thermomechanical/electromechanical environment the distribution of the displacement or electric potential could be controlled effectively by applying the concept of functionally grading of piezo-thermoelastic materials. The advantage of functionally graded material (FGM) is, it has continuously graded properties due to spatially varying microstructures produced by nonuniform distributions of the reinforcement phase as well as by interchanging the role of reinforcement and matrix (base) materials in a continuous manner. The continuous variation of properties may offer advantages such as local reduction of stress concentration, increased efficiency and increased bonding strength [2–4]. Functionally graded materials and structures have potential applications in space planes, rocket engine components, dentals, orthopedic implants, armour plates, bullet-proof vests, thermal barrier coatings, optoelectronic devices, automobile engine components, nuclear reactor components, turbine blades and heat exchanger [5–10].

Literatures related to piezothermoelastic materials, its functionally graded structures and their thermal diffusion problems are presented here for understanding the importance of present work. Mindlin [11] carried out detailed investigation on high frequency vibrations of a thermopiezoelectric plate by incorporating the coupling of elastic, electric and thermal fields in integral energy balance equation and obtained a uniqueness theorem to establish various face and edge-conditions sufficient to assure

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unique solutions of the two-dimensional equations. Chandrasekharaiah [12] developed a generalized linear theory for thermo piezoelectric materials by combining the thermo elasticity theory of Lebon [13] and the conventional thermopiezoelectricity theory of Mindlin [14]. He obtained an equation of energy balance and a theorem on the uniqueness of solution and explained finite speed for thermal signals in linear thermopiezoelectric materials. Rao and Sunar [15] have used finite element method (FEM) to analyse the thermal effect in thermopiezoelectric materials and its application to distributed dynamic measurement and active vibration control of advanced intelligent structures by. It is found that the thermal effects have an impact on the performance of a distributed control system. The degree of impact may vary depending on the piezoelectric material, the environment where the system operates, and the magnitude of the feedback voltage. Selvamani *et al.* [16] studied the influence of hygro, thermal and piezo fields in a functionally graded piezoelectric rod using three-dimensional elasticity equation in linear form. For the formulation of the problem, it was assumed that the stiffness and thermal conductivity of the material transport via the radial coordinate. The numerical results showed the effect of hygro, thermal and piezo fields in the physical variables via grading values and moisture constants. Poongkothai *et al.* [17] carried out analysis on thermo-electro effects on the dispersion of functionally graded piezo electric rod coupled with inviscid fluid via three-dimensional elasticity equation in linear form. Dube *et al.* [18] considered the effect of pressure, thermal and electrostatic excitation on an infinitely long, simply-supported, orthotropic, piezoelectric, flat panel in cylindrical bending and offered exact solution. They used Fourier series to expand displacement, electric potential and temperature fields to satisfy boundary conditions at the longitudinal edges. Ding *et al.* [19] reported numerical results for a spherically symmetric thermoelastic problem of a functionally graded pyroelectric hollow sphere. Ootao and Tanigawa [20] supplied uniform thermal load to a functionally graded thermopiezoelectric hollow sphere to study the transient piezothermoelastic response. They obtained one-dimensional solution for the temperature change in a transient state which explained the piezothermoelastic response of a functionally graded thermopiezoelectric hollow sphere. Wu and Huang [21] developed a modified Pagano method for a three-dimensional (3D) coupled analysis of a simply-supported, doubly curved functionally graded (FG) piezo-thermo-elastic shells under thermal loads. They studied effect of the material-property gradient index on the through-thickness distributions of various variables in the thermal, electric, and mechanical fields. Akbarzadeh and Chen [22] investigated heat conduction in one-dimensional

functionally graded material based on the dual-phase-lag theory to reveal the role of microstructural interactions in the fast transient process of heat conduction. They obtained exact expressions for the thermal wave speed in one-dimensional, functionally graded material with different geometries based on the dual-phase-lag and hyperbolic heat conduction theories. The distribution of transient temperature for various types of dynamic thermal loads was also obtained. Abouelregal and Zenkour [23] carried out detailed analysis on the vibrational characteristics of a functionally graded (FG) microbeam subjected to a ramp-type heating in the context of the dual phase lag model. They reported the effect of ramping time on the lateral vibration, temperature, displacement, stress, moment and the strain energy of the FG micro beam.

The study of thermal diffusion problems in piezothermoelastic materials has many applications in industrial engineering, thermal power plants, submarine structures, pressure vessel, aerospace, chemical engineering and metallurgy. When piezothermoelastic materials are involved in severe thermal environments, thermal stress and diffusion may cause damages and modifications in functioning of the structure. Therefore it is necessary to analyse the electric field and deformations induced by thermal loading in piezothermoelastic materials which will provide insight for proper design of functionally graded structures. Elhagary [24] has done extensive research on thermal diffusion in a two-dimensional thermoelastic thick plate subject to laser heating within the context of the theory of generalized thermoelastic diffusion with one relaxation time. Kumar and Gupta [25] analysed reflected and transmitted waves resulting from an interface between inviscid fluid half-space and a thermoelastic diffusion solid half-space using dual-phase-lag heat transfer (DPLT) and dual-phase-lag diffusion (DPLD) models. They demonstrated that the amplitude ratios and energy ratios of various reflected and transmitted waves are functions of angle of incidence, frequency of the incident wave and the thermoelastic diffusion properties of media concerned. Tripathi *et al.* [26] considered continuous supply of axisymmetric heat and an internally generated heat to a two dimensional thick circular plate of infinite extent and finite thickness within the framework of classical coupled, L-S and G-L theory. They obtained exact solutions for temperature distribution, displacement and the stress components in the Laplace transform domain. Accounting internally generated heat is significant because engineering components thick-walled pressure vessels, such as a nuclear containment vessel, a cylindrical roller etc. generates internal heat, which influences the performance of the component.

Abbas *et al.* [27] investigated on deformation produced in a micro polar thermoelastic diffusion medium due to thermal source by the use of finite element method (FEM) in the framework of Lord-Shulman (L-S) theory of thermoelasticity. They obtained the components of displacement, stress, microrotation, temperature change and mass concentration using numerical method and showed the impact of micropolarity, diffusion and relaxation times on the obtained components graphically. He *et al.* [28] studied the effect of moving heat source on the dynamic thermal and elastic responses of a piezoelectric rod fixed at both ends in the context of Lord and Shulman generalized thermo-elastic theory with one relaxation time. From the results it is found that the temperature, thermally induced displacement and stress of the rod are decrease at large source speed. Babaei and Chen [29] investigated thermo-piezoelectricity problem of a one-dimensional (1-D), finite length, and functionally graded medium excited by a moving heat source using the Lord and Shulman theory of generalized coupled thermo elasticity. Ma and He [30] worked on the dynamic response of a piezoelectric-thermo elastic rod made of piezoelectric ceramics (PZT-4) subjected to a moving heat source within the context of the fractional order theory of thermo elasticity. They investigated the effects of fractional order parameter and the velocity of heat source on the variations of the considered variables and the results showed that they have significant influence on the variations of the considered variables.

Othman *et al.* [31] studied the effect of the gravity field and the diffusion on a micro polar thermo elastic medium with dependence on the temperature properties. Further compressions were showed graphically in the presence and the absence of the gravity, the temperature-dependent properties, the diffusion and the micro polar in the context of two types of Green-Naghdi (G-N) theory II and III. Zenkour [32] recently presented a refined multi-phase-lags theory to investigate the thermo elastic response of a gravitated piezo-thermoelastic half-space. He obtained all fields like displacement, temperature, electric potential and thermo mechanical stress and demonstrated their dependency on the inclusion of gravity. Zenkour and Kutbi [33] developed a novel multi-phase-lag model to study the thermoelastic diffusion behaviour of a one-dimensional half-space. For the formation of the problem, they considered additional equation for heat of mass diffusion and additional constitutive equation for the chemical potential.

From the literature it is found that thermal diffusion is discussed elaborately in thermoelastic materials and piezoelectric materials. However in the case of piezo-thermoelastic materials thermal problems are is not discussed comprehensively. This means, the thermal diffu-

sion problems investigated so far in thermoelastic materials have not included piezoelectric equation. The present paper is dedicated to investigation on thermal diffusion behaviour of a one dimensional functionally graded piezo-thermoelastic cylindrical rod using refined multi-phase-lags (RPL) theory. The normal mode method will be used to obtain the exact solution of the coupled thermo elastic equations. The analytical expressions for these coupled equations would be the components of mechanical displacement, temperature, electric displacement, electric potential, and stresses. Numerical results for these field quantities are tabulated and illustrated graphically.

2 Basic equation and formulation of the plain stress

We consider a cylindrical rod problem. It is assumed to be made up of functionally graded (inhomogeneous) transversely isotropic, thermo elastic medium within a uniform temperature $\left| \frac{(T-T_0)}{T_0} \right| \ll 1$ and initial concentration C_0 , in the undistributed state occupying the domain $R \leq r \leq \infty$, whose state can be expressed in terms of the space variable r and the time t so that all the field functions vanish at infinity. We use cylindrical coordinates (r, θ, z) are considered with origin at $\theta = 0$, and z – axis is setting along with cylindrical axis. The density, elastic parameters and thermal conductivity of the material have been assumed to vary through the thickness according to simple power law in radial coordinate as under

$$\begin{aligned} \rho &= \rho_0 \left(\frac{r}{a} \right)^\alpha, & C_{11} &= C_{11}^0 \left(\frac{r}{a} \right)^\alpha, \\ C_{12} &= C_{12}^0 \left(\frac{r}{a} \right)^\alpha, & C_{13} &= C_{13}^0 \left(\frac{r}{a} \right)^\alpha, & K_1 &= K_1^0 \left(\frac{r}{a} \right)^\alpha \end{aligned} \quad (1)$$

Here

$$\begin{aligned} C_{11}^0 &= \lambda + 2\mu, & C_{12}^0 &= \lambda = C_{13}^0, \\ \beta_1^0 &= \beta = \beta_3^0, & K_1^0 &= K_3^0 = K \end{aligned}$$

where the exponent α represents the grading index of the material and λ, μ are Lamé's parameter's, $C_{11}^0, C_{12}^0, C_{13}^0, \beta_1^0, \beta_3^0, K_1^0, K_3^0$ are the homogeneous counterparts of the respective quantities.

The basic covering field equations and constitutive relations of generalized hexagonal piezo – thermo elastic with thermo diffusive for two-dimensional motion in r - z plane are [34]

1. The Strain- displacement- relations

$$2\varepsilon_{ij} = (u_{i,j} + u_{j,i}), \quad i, j = 1, 2 \quad (2)$$

2. The Stress-Strain-Piezo-Temperature-Concentration

$$\begin{aligned}\sigma_{ij} &= c_{ijkl}\varepsilon_{kl} - e_{kij}E_k \\ &\quad - E_\theta^3 (\gamma_{ij}(T - T_0) + b_{ij}C) \\ D_i &= e_{ijk}\varepsilon_{jk} + \varepsilon_{ij}E_j + p_i\theta \\ \rho T_0 S &= \rho C_e T + a T_0 C + \gamma_{ij} T_0 \varepsilon_{ij} + p_i E_i\end{aligned}\quad (3)$$

Where $E_i = -\phi_{,i}$, $i, j, k = 1, 2, 3$

3. The Chemical-strain-temperature- diffusion relation

$$P = -b_{kj}\varepsilon_{kl} + bC - aT \quad j, k, l = 1, 2, 3., \quad (4)$$

4. The equation of motion

$$\sigma_{ij} = \rho \ddot{u} \quad i, j = 1, 2, 3, \quad (5)$$

5. The equation of Heat conduction in the RPL model

$$\begin{aligned}E_\theta^1 (K_{ij} T_{,j})_{,i} &= E_q^1 (\rho C_e T + a T_0 C) \\ &\quad + \gamma_{ij} E_q^2 (u_{i,j} + p_i E_i), \quad i, j = 1, 2, 3\end{aligned}\quad (6)$$

6. The equation of conservation of mass-diffusion

$$E_\theta^2 \left(\frac{\partial C}{\partial t} \right) = \frac{1}{b} (D_{ij} P_{,j})_{,i}, \quad i, j = 1, 2, 3 \quad (7)$$

Where σ_{ij} are the components of the stress tensor, ε_{ij} are the components of the strain tensor, e_{kij} represent the components of piezoelectric tensor, D_i , E_i are the electric displacement, $u(r, t)$ are the displacement vector, T is the absolute temperature of the medium, T_0 is the uniform temperature of the medium, c_{ijkl} are the elastic parameter of the anisotropic medium and γ_{ij} is tensor moduli, K_{ij} are the thermal conductivity components, P denotes the chemical potential per unit mass, C is the concentration of the diffusive material in the elastic body, D_{ij} are the diffusion coefficient, a is the thermo diffusive constant, b is the diffusive constant, C_e is the specific heat at constant strain, S is the entropy per unit mass, ρ reference material of the density. We assume that the material parameters satisfy the inequalities $K, \lambda, \mu, D, T, T_0, C_e > 0$.

In the foregoing relations, the parameters E_θ^i ($i = 1, 2, 3$) and E_q^i ($i = 1, 2$) can be expressed as

$$\begin{aligned}E_\theta^i &= 1 + \iota_{0i} \frac{\partial}{\partial t} + \sum_{r=1}^{\mathfrak{R}_i-1} \frac{\iota_0^{r+1}}{(r+1)!} \frac{\partial^{r+1}}{\partial t^{r+1}}, \\ E_q^j &= \frac{\partial}{\partial t} \left(1 + \iota_{1j} \frac{\partial}{\partial t} + \sum_{s=1}^{S_j} \frac{\iota_q^{s+1}}{(s+1)!} \frac{\partial^{s+1}}{\partial t^{s+1}} \right),\end{aligned}\quad (8)$$

Here θ denotes the temperature change of a material particle, The parameters ι_{0i} , ι_{1j} , ι_q and ι_0 represent the

thermal memories in which ι_q is the multi-phase-lag (MPL) of the heat flux, ($0 \leq \iota_\theta < \iota_q$), and ι_θ is the PL of temperature gradient. Here we apply classical thermo elasticity (CTE) theory is appeared by omitting all relaxation times, i.e., $\iota_\theta = \iota_q = \iota_{0i}$.

The L-S model will be appearing when $\iota_\theta = \iota_q = \iota_{0i} = 0$ and $\iota_{11} = \iota_{22} > 0$. Also, The G-L model will be appearing when $\iota_\theta = \iota_q = \iota_{01} = \iota_{02} = 0$ and $\iota_{03} = \iota_{11} > 0$.

So, the simple dual-phase-lag (SPL) model will be given by setting $\iota_{01} = \iota_0$, $\iota_{02} = \iota_{03} = 0$, $\iota_{1j} = \iota_q$ and omitting the summations in Eq. (8). That is

$$\begin{aligned}E_\theta^1 &= 1 + \iota_\theta \frac{\partial}{\partial t}, \quad E_\theta^k = 1, \\ E_q^j &= \frac{\partial}{\partial t} + \iota_q \frac{\partial^2}{\partial t^2}, \quad k = 2, 3, 4, 5\end{aligned}\quad (9)$$

Now, the refined multi-phase-lag (RPL) theory is given by setting $\mathfrak{R}_1 = s_j = N > 1$, $\mathfrak{R}_2 = \mathfrak{R}_3 = N - 1$.

That is

$$\begin{aligned}E_\theta^i &= 1 + \sum_{r=1}^{\mathfrak{R}_i} \frac{\iota_0^r}{r!} \frac{\partial^r}{\partial t^r}, \\ E_q^j &= \frac{\partial}{\partial t} \left(1 + \iota_q \frac{\partial}{\partial t} + \sum_{s=1}^{S_j} \frac{\iota_q^{s+1}}{(s+1)!} \frac{\partial^{s+1}}{\partial t^{s+1}} \right) \\ E_q^k &= \left(\frac{\partial^2}{\partial t^2} + \frac{m_1}{r} \frac{\partial}{\partial r} + \frac{m_2}{r^2} \right)\end{aligned}\quad (10)$$

For $N = 1$, we set

$$\begin{aligned}E_\theta^1 &= 1 + \iota_\theta \frac{\partial}{\partial t}, \quad E_\theta^k = 1, \\ E_q^1 &= E_q^2 = \frac{\partial}{\partial t} + \iota_q \frac{\partial^2}{\partial t^2}, \quad k = 2, 3, 4, 5\end{aligned}\quad (11)$$

The value of N may be reaches 5 or more according to the refined multi-phase-lag (RPL) theory required.

For an isotropic medium [38], we have

$$\begin{aligned}\gamma_{ij} &= \beta_1 \delta_{ij}, \quad b_{ij} = \beta_2 \delta_{ij}, \quad D_{ij} = D \delta_{ij}, \\ C_{ijkl} &= C_{11} \varepsilon_r + C_{12} \varepsilon_\theta - E_\theta^3 (\beta_1 (T - T_0) + \beta_2 C)\end{aligned}\quad (12)$$

Where C_{11} and C_{12} are the grading elastic parameters and β_1, β_2 are the material constants given by

$\beta_1 = (3\lambda + 2\mu) \alpha_t$ and $\beta_2 = (3\lambda + 2\mu) \alpha_c$, α_t is the coefficient of linear thermal expansion, α_c is the coefficient of linear diffusion expansion D . Represents the diffusion coefficient and δ_{ij} is Kronecker's delta.

Then the constitutive Eqs. (4) and (6) can be expressed as

$$\sigma_{ij} = c_{11} \varepsilon_r + c_{12} \varepsilon_\theta - E_\theta^3 (\beta_1 (T - T_0) + \beta_2 C) + e_{ij} E_j \quad (13)$$

$$P = -\beta_2 \varepsilon_{kk} + bC - aT \quad (14)$$

Now, let us consider a cylindrical rod whose coordinates are (r, θ, z) in the context of the MPL model. All the functions are depending on the time t and the coordinate r and will be assuming to vanish as $r \rightarrow \infty$.

For the axisymmetric plane strain, The following displacement components $u_i(r, t)$ are considered as;

$$\text{i.e.,} \quad u_1 = u(r, t), \quad u_2 = u_3 = 0 \quad (15)$$

Also, The strain displacement relations are

$$\begin{aligned} e_{11} &= \frac{\partial u}{\partial r}, & e_{22} &= \frac{u}{r}, & e_{33} &= 0, \\ e_{13} &= 0, & e_{12} &= 0, & e_{23} &= 0 \end{aligned} \quad (16)$$

Then, the Equations of heat transport and heat of mass diffusion can be expressed as

$$K\epsilon_\theta^1 \nabla^2 \theta = \epsilon_q^1 (\rho C_e T + a T_0 C) + \beta_1 T_0 \epsilon_q^2 \left(\frac{\partial u}{\partial x} + p_3 E_z \right) \quad (17)$$

$$\epsilon_\theta^2 \left(\frac{\partial C}{\partial t} \right) = \frac{D}{b} \frac{\partial^2 P}{\partial x^2} \quad (18)$$

Keeping in view the equalities of these material parameters and using relations (1) and (2).

Then the relation of the constitutive Eqs. (3) can be reduced to:

$$\sigma_{11} = c_{11}^0 \left(\frac{r}{a} \right)^\alpha \left[\epsilon_\theta^3 \left(\bar{\beta}_1 (T - T_0) + \bar{\beta}_2 C \right) \right] + e_{13} \bar{E}_1 \quad (19)$$

$$\sigma_{22} = \sigma_{33} = c_{11}^0 \left(\frac{r}{a} \right)^\alpha \left[\epsilon_\theta^3 \left(\bar{\beta}_1 (T - T_0) + \bar{\beta}_2 C \right) \right] + e_{33} \bar{E}_1 \quad (20)$$

$$\sigma_{12} = \sigma_{23} = \sigma_{31} = 0$$

$$\text{Here } \bar{C}_0 = \frac{C_{12}^0}{C_{11}^0}, \bar{\beta}_1 = \frac{\beta_1}{C_{11}^0}, \bar{\beta}_2 = \frac{\beta_2}{C_{11}^0}$$

The Electric field displacement can be simplified as

$$\begin{aligned} D_1 &= e_{15} (u_{,z} + \omega_{,x}) - N_{11} E_x \\ D_3 &= e_{31} u_{,x} + e_{33} \omega_{,z} - N_{33} E_z + p_3 (T - T_0) \end{aligned} \quad (21)$$

Also, using Eqs. (1–2) in Eqs. (3) simplifying, we get the equation of motion can be expressed as

$$\begin{aligned} \epsilon_q^k [u] - \epsilon_\theta^3 \left[\frac{\partial}{\partial r} (\beta_1 (T - T_0) + \beta_2 C) \right] \\ + \epsilon_\theta^4 \left[\bar{e}_{13} \left(\frac{\partial \bar{E}_{1z}}{\partial r} + \frac{\alpha \bar{E}_{1z}}{r} \right) \right] = \rho \frac{\partial^2 u}{\partial t^2} \end{aligned} \quad (22)$$

Also, The Electrostatics equations can be reduced to

$$\begin{aligned} (e_{31} + e_{15}) \frac{\partial^2 u}{\partial r \partial z} + \frac{\partial^2}{\partial r^2} (e_{15} T - N_{11} E) \\ + \frac{\partial^2}{\partial z^2} (e_{33} T - N_{33} E) + p_3 \frac{\partial (T - T_0)}{\partial z} \end{aligned} \quad (23)$$

Further, The Heat conductive equation is reduced to

$$\begin{aligned} K\epsilon_\theta^1 \left(\frac{\partial^2 T}{\partial x^2} \right) + \frac{m_1}{r} \frac{\partial T}{\partial r} = \epsilon_q^1 [\rho C_e T + a T_0 C] \\ + \beta_1 T_0 \epsilon_q^2 \left(\frac{\partial u}{\partial x} + p_3 E_z \right) \end{aligned} \quad (24)$$

The Substitution of Eq. (14) into Eq. (17) yields the diffusion equation as

$$b\epsilon_\theta^2 \left(\frac{\partial C}{\partial t} \right) = D \frac{\partial^2}{\partial r^2} (bc - \beta_2 e - aT), \quad (25)$$

Where $e = \epsilon_{kk}$, dilatation. For our convenience, we introduce the following non-dimensionless variables in the next part

$$\begin{aligned} \{x', u'\} &= C_0' \eta \{x, u\} \\ \{t_1', t_{0i}', t_\theta', t_q'\} &= \eta C_0'^2 \{t_1, t_{1j}, t_\theta, t_q\} \\ \theta' &= \frac{\beta_1 (T - T_0)}{\rho' C_0'^2}, \quad C_1' = \frac{\beta_2 C}{\rho' C_0'^2} \\ \sigma_{11}^* &= \frac{\sigma_{11}}{C_{11}^0}, \quad \sigma_{22}^* = \frac{\sigma_{22}}{C_{11}^0}, \quad U = \frac{u}{a}, \quad x = \frac{r}{a}, \\ e_{13} &= e_{33} = e_{15}, \quad N_{11} = N_{33}, \quad T = (T - T_0) \end{aligned}$$

$$\begin{aligned} \theta' &= \frac{\bar{\beta}_1 (T - T_0)}{p' C_0'^2}, \quad \bar{\beta}_1 = \frac{\beta_1 T_0}{\lambda + 2\mu}, \quad C_1' = \frac{\beta_2 C}{p' C_0'^2}, \\ C_0'^2 &= \frac{(\lambda + 2\mu)}{\rho} \bar{\beta}_2 = \frac{\beta_2 C_0}{\lambda + 2\mu}, \quad P' = \frac{P}{\bar{\beta}_2} \\ \eta &= \frac{\rho C_e}{K}, \quad \bar{e}_T = \frac{\beta_1^2 T_0}{\rho C_e (\lambda + 2\mu)}, \quad e_{13} = e_{33} \end{aligned} \quad (26)$$

All the governing equations, with above non-dimensionless variables are reduced to

$$e = \frac{\partial u}{\partial x},$$

$$\sigma_{11}^* = X^\theta \left[\frac{\partial U}{\partial x} + S_1 \frac{U}{x} - \epsilon_\theta^3 (\theta' + C_1') \right] + \bar{e}_{33} \bar{E}_1 \quad (27)$$

$$\sigma_{22}^* = X^\theta \left[S_1 \frac{\partial U}{\partial x} + \frac{U}{x} - \epsilon_\theta^3 (\theta' + C_1') \right] + e_{33} \bar{E}_1 \quad (28)$$

$$\begin{aligned} \epsilon_q^k [U] - \epsilon_\theta^3 \left[\frac{\partial}{\partial x} ((\theta' + C_1')) \right] \\ + \epsilon_\theta^4 \left[\bar{e}_{13} \left(\frac{\partial \bar{E}_{1z}}{\partial x} + \frac{\alpha \bar{E}_{1z}}{x} \right) \right] = \frac{\partial^2 U}{\partial t^2} \end{aligned} \quad (29)$$

$$\epsilon_{\theta}^1 \left(\frac{\partial^2 \theta'}{\partial x^2} \right) + \frac{m_1}{x} \frac{\partial \theta'}{\partial x} = \epsilon_q^1 [\theta' + \bar{\epsilon}_T S_2 C_1'] \quad (30)$$

$$+ \bar{\epsilon}_T \epsilon_q^2 (e + p_3 \bar{E}_z)$$

$$\zeta_1^* \left(\frac{\partial^2 U}{\partial x \partial z} \right) + \epsilon_{\theta}^5 \left[\zeta_2^* \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \right] = p_3 \bar{E}_z \quad (31)$$

$$S_3 \epsilon_{\theta}^2 \left(\frac{\partial C}{\partial t} \right) - \frac{\partial^2}{\partial x^2} (S_4 C_1' - e S_2 - \theta'), \quad (32)$$

$$P = S_4 C_1' - S_2 \theta' - e \quad (33)$$

Where

$$S_1 = X^g \left(\frac{C_{12}^0}{C_{11}^0} \right), \quad S_2 = X^g \frac{a C_{11}^0}{\beta_1^0 \beta_2^0}, \quad (34)$$

$$S_3 = X^g \frac{b C_{11}^0}{\beta_2^2 \eta D}, \quad S_4 = X^g \frac{b C_{11}^0}{\beta_2^2},$$

$$\zeta_1^* = (e_{31} + e_{15}), \quad \zeta_2^* = (e_{33} T - N_{11} E),$$

$$\bar{E}_z = \frac{\partial T}{\partial z}, \quad \bar{\epsilon}_T = \frac{\beta_1^2 T_0}{C_{11}^0 \eta D}$$

3 The solution of the problem

In this section, we apply the normal mode analysis, which gives exact solutions without any assumed restrictions on the displacement, temperature, dilation, concentration, stress, and chemical potential. It is applied to a wide range of problem in different branches Othman *et al.* [35–37]

We will firstly apply the following initial conditions:

$$U(x, 0) = \left[\frac{\partial U}{\partial t} \right]_{t=0}, \quad \theta'(x, 0) = \left[\frac{\partial \theta'}{\partial t} \right]_{t=0}, \quad (35)$$

$$C'(x, 0) = \left[\frac{\partial C'}{\partial t} \right]_{t=0}, \quad \zeta(x, 0) = - \left[\frac{\partial \phi}{\partial t} \right]_{t=0}$$

The appropriate solution that satisfied the above initial condition in terms of the normal mode.

Of the following forms:

$$\{U, \theta', C', \zeta\}(x, t) = \{u^*, \omega^*, v^*, \varphi^*\}(x) h_t(t) \quad (36)$$

Where $h_t(t)$ will be chosen such that the temperature, dilatation, displacement and concentration and their derivatives should be $t = 0$.

Then we gets

$$e^* = Du^* \quad (37)$$

$$\sigma_{11}^* = X^g \left[(e^* + S_1 m) - (\omega^* + v^*) h_2(t) \right] + \varphi^*$$

$$\sigma_{22}^* = \sigma_{33}^* = X^g \left[(S_1 e^* + m) - (\omega^* + v^*) h_2(t) \right] + \varphi^*$$

$$\zeta_1^* e^* + (\omega^* + v^*) \zeta_2^* - \varphi^* = 0$$

$$P = S_4 C^* - e^* - S_2 \omega^*$$

Taking the divergence of Eq. (28) and using Eqs. (37) and (38).

We obtain

$$\left[D^2 - h_1(t) \right] e^* - h_2(t) D^2 (\omega^* + v^*) + \varphi^* = 0 \quad (38)$$

And the equations of heat conduction, Electric displacement, and mass diffusion became

$$\left[D^2 - h_3(t) \right] \omega^* + \bar{\epsilon}_T \left[\frac{h_4(t) e^* +}{S_2 h_3(t) v^*} \right] + h_6(t) \varphi^* = 0 \quad (39)$$

$$\left[S_4 D^2 - S_3 h_5(t) \right] \varphi^* - S_2 D^2 \omega^* - D^2 e^* + h_7(t) \varphi^* = 0 \quad (40)$$

$$h_{11}(t) \varphi^* + \left[\frac{D^2 +}{S_4 h_9(t)} \right] \omega^* + \left[\frac{S_4 h_9(t) D^2 +}{h_{10}(t)} \right] \varphi^* + h_8(t) D^2 e^* = 0 \quad (41)$$

Where

$$D = \frac{d}{dx}, \quad (42)$$

$$h_1 = \epsilon_q^k \left[\frac{1}{h_t} \frac{d^2 h_t}{dt^2} + \frac{m_1}{h_t} \frac{dh_t}{dt} + \frac{2m_2 h_t}{h_t^2} \right],$$

$$h_2 = \frac{1}{h_t} \epsilon_{\theta}^3 (h_t),$$

$$\varphi^* = \epsilon_{\theta}^4 \left[\bar{\epsilon}_{13} \left(\frac{d \bar{E}_{1z}}{dt} + \frac{\alpha \bar{E}_{1z}}{t} \right) \right] - \frac{d^2 h_t}{dt^2}$$

$$h_3 = \frac{1}{\epsilon_{\theta}^1 (h_t)} \epsilon_q^1 (h_t), \quad h_4 = \frac{1}{\epsilon_{\theta}^1 (h_t)} \epsilon_q^2 (h_t),$$

$$h_5 = \frac{1}{h_t} \epsilon_{\theta}^2 \left(\frac{dh_t}{dt} \right)$$

$$h_6 = \epsilon_{\theta}^4 \left[\bar{\epsilon}_{13} \bar{E}_{1z} \left(\frac{d}{dt} + \frac{\alpha}{t} \right) \right],$$

$$h_7 = \bar{\epsilon}_T \epsilon_q^2 (e + p_3 \bar{E}_z),$$

$$h_8 = - \frac{m_n (e_{15} + e_{31})}{N_{33}}$$

$$h_9 = - \frac{e_{15} m_n^2}{N_{33}},$$

$$h_{10} = - \frac{N_{11} m_n^2}{N_{33}} = - \frac{N_{33} m_n^2}{N_{11}},$$

$$h_{11} = - \frac{p_3 \bar{E}_z}{N_{33}}$$

The system of equations appeared in Eqs. (38–41) can be expressed in eight-order ordinary homogeneous differential equations in the amplitude $u^*(x)$, $\omega^*(x)$, $v^*(x)$, $\varphi^*(x)$

which can be written as:

$$\left(\begin{array}{c} D^8 + E_3 D^6 + \\ E_2 D^4 + E_1 D^2 + E_0 \end{array} \right) \left\{ \left(u^*, \omega^*, v^*, \varphi^* \right) (x) \right\} = 0 \quad (43)$$

Eq. (43) can be factored as

$$\left(D^2 - K_1^2 \right) \left(D^2 - K_2^2 \right) \left(D^2 - K_3^2 \right) \left(D^2 - K_4^2 \right) \left\{ u^*(x), \omega^*(x), v^*(x), \varphi^*(x) \right\} = 0 \quad (44)$$

where K_1, K_2, K_3, K_4 are the roots with positive real parts of the characteristic equation.

$$K_n^8 - E_3 K_n^6 + E_2 K_n^4 - E_1 K_n^2 + E_0 = 0 \quad (45)$$

Where the coefficient $E_i, i = 0, 1, 2, 3$ are given by

$$K_n^8 - E_3 K_n^6 + E_2 K_n^4 - E_1 K_n^2 + E_0 = 0 \quad (46)$$

$$E_0 = \frac{S_3 h_1 h_3 h_5 h_9}{S_4 - h_2}$$

$$E_1 = \frac{(S_4 + \bar{\epsilon}_T S_2^2) h_1 h_3 + (S_4 + h_9) h_{10} + S_3 h_5 (\bar{\epsilon}_T h_3 + h_1 + h_3)}{S_4 - h_2}$$

$$E_2 = \frac{\bar{\epsilon}_T S_2^2 h_3 + h_2 h_3 (\bar{\epsilon}_T S_2 - 1) + \bar{\epsilon}_T h_2 h_4 (S_2 + S_4) + S_4 (h_1 + h_3) + S_3 h_5}{S_4 - h_2}$$

$$E_3 = \frac{\bar{\epsilon}_T S_2^2 h_3 + h_8 h_9 h_{10} (\bar{\epsilon}_T S_2 + 1) + \bar{\epsilon}_T h_{10} h_{11} (S_3 - S_4) + S_4 (h_1 - h_3) + S_3 h_8}{S_4 - h_2}$$

The complete solution of Eq. (44) is given by

$$\left\{ u^*(x), \omega^*(x), v^*(x), \varphi^*(x) \right\} = \sum_{j=1}^4 \left\{ C_j^1, C_j^2, C_j^3, C_j^4 \right\} e^{-m_n x} \quad (47)$$

Where $C_j^i, (i, j = 1, 2, 3, 4)$ are the integration parameters and $m_n, (n = 1, 2, 3, 4)$ are the positive roots of the Characteristic equation

$$m_n^8 + E_3 m_n^6 + E_2 m_n^4 + E_1 m_n^2 + E_0 = 0 \quad (48)$$

Here

$$m_1 = \frac{1}{\sqrt{6E_2}} \sqrt{(2E_2^2 + E_3)^2 - 2E_3 E_4 - 12E_1 - 4E_4^2} \quad (49)$$

$$m_{2,3} = \frac{1}{2\sqrt{3}E_3} \sqrt{4E_2 E_4 - 4(1 \pm i\sqrt{3})(2E_2^2 - 3E_1) - (1 \mp i\sqrt{3})E_3^2}$$

Further,

$$E_4 = \sqrt[3]{108E_0 + 12E_5 + 8E_2^3 - 36E_1 E_2} \quad (50)$$

$$E_5 = \sqrt{81E_0^2 + 6E_0 E_2 (2E_2^2 - 9E_1) + E_1^2 (4E_1 - E_2^2)}$$

The relations between the parameters $C_j^1, C_j^2, C_j^3, C_j^4$, can be obtained by using Eqs. (46) into

$$\left\{ C_j^2, C_j^3, C_j^4 \right\} = \{ R_{1n}, R_{2n}, R_{3n} \} C_j^1 \quad (51)$$

Where

$$R_{1n} = \frac{h_2 m_n^2 (m_n^2 + \bar{\epsilon}_T S_2 h_3 - h_3)}{[(S_2 h_3 + h_2 h_4) m_n^2 - S_2 h_1 h_3] \bar{\epsilon}_T} \quad (52)$$

$$R_{2n} = \frac{m_n^4 - m_n^2 (\bar{\epsilon}_T h_2 h_4 + h_1 + h_3) + h_1 h_3}{[(S_2 h_3 + h_2 h_4) m_n^2 - S_2 h_1 h_3] \bar{\epsilon}_T}$$

$$R_{3n} = \frac{h_8 m_n^4 (m_n^4 h_{10} h_{11} + \bar{\epsilon}_T S_4 h_9) + h_7 h_8}{[(S_2 h_7 + h_2 h_4) m_n^2 - S_2 h_8 h_3] \bar{\epsilon}_T}$$

Now, the final form of the displacement temperature components is given by

$$\left\{ \theta^*, e^*, C, \varphi^* \right\} (x, t) \quad (53)$$

$$= h_t(t) \sum_{j=1}^4 C_j^1 \{ 1, R_{1n}, R_{2n}, R_{3n} \} e^{-m_n x}$$

By integrating Eqs. (47)₃, we get

$$u(x, t) = h_t(t) \sum_{j=1}^4 C_j^1 R_{4n} e^{-m_n x} \quad (54)$$

$$R_{4n} = - \frac{h_2 m_n (m_n^2 + \bar{\epsilon}_T S_2 h_3 - h_3) - h_6}{[(S_2 h_3 + h_2 h_4) m_n^2 - S_2 h_1 h_3] \bar{\epsilon}_T} \quad (55)$$

Then, we get the final result of plane stress, displacement and chemical potential

$$\sigma_{11} = \sum_{n=1}^4 C_j^1 \left[X^9 \left(\frac{R_{1n}}{S_1 h_2 (1 + R_{2n}) + h_6} \right) \right] e^{-m_n x} \quad (56)$$

$$\sigma_{22} = \sigma_{33}$$

$$= \sum_{n=1}^4 C_j^1 \left[X^9 \left(\frac{S_1 R_{1n}}{h_2 (1 + R_{2n}) + h_6} \right) \right] e^{-m_n x}$$

$$D_1 = \sum_{j=1}^4 C_j^1 [h_9 ((1 + R_{3n}) - h_{10})] e^{-m_n x}$$

$$D_3 = \sum_{j=1}^4 C_j^1 [(h_8 - h_{10} (1 + R_{3n}) + h_{11})] e^{-m_n x}$$

3.1 The Boundary conditions

In this section we used following boundary conditions for obtaining required results.

3.1.1 Mechanical conditions

The time dependent periodic force with magnitude σ_0 is assumed to be acting normal direction on the medium.

That is

$$\sigma_{11}(0, t) = \sigma_{33}(0, t) = p_0 h_t(t) = -\sigma_0 \quad (57)$$

3.1.2 Thermal conditions

A thermal boundary condition on the surface of the half-space subjected to thermal shock

$$\theta(0, t) = \theta_0 h_t(t) = \theta_0 \quad (58)$$

3.1.3 Electric conditions

Also, the medium is free from the normal electric field E_{13} at $x = 0$, so that

$$E_{13}(x, 0, t) = -\frac{\partial \phi}{\partial z} = \varphi_0 h_t(t) = 0 \quad (59)$$

3.1.4 Concentration conditions

A concentration can be applied on the surface of the half space $x = 0$ and taken the value C_0 of in the normal direction

$$C(0, t) = \frac{\partial C}{\partial Z} = C_0 h_t(t) = 0 \quad (60)$$

Therefore using equations [27] and (53)₁, (53)₂, (53)₃ & (56) are the parameters C_j^1 .

It can be determinant by solving the following system

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ R_{21} & R_{22} & R_{23} & R_{24} \\ KR_{31} & KR_{32} & KR_{33} & KR_{34} \\ S_{11} & S_{12} & S_{13} & S_{14} \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{Bmatrix} = \begin{Bmatrix} \theta_0 \\ 0 \\ 0 \\ -\sigma_0 \end{Bmatrix} \quad (61)$$

Where

$$S_{1n} = R_{1n} - h_2(1 + R_{2n}) \quad (62)$$

After determining the parameters, the final form of the physical fields of the problem under the investigation can be obtained.

4 Numerical discussion and results

The purpose of numerical analysis and discussion consider the copper material. The Copper material properties are given according to the following values of parameters [4, 14, 17].

$$C_{11}^0 = 0.4040 \times 10^{10} \text{ Nm}^{-2}, \quad C_{12}^0 = 0.212 \times 10^{10} \text{ Nm}^{-2},$$

$$C_{13}^0 = 0.0105 \times 10^{10} \text{ Nm}^{-2},$$

$$T_0 = 0.850 \text{ K}, \quad \beta_1^0 = 2.3620 \times 10^6 \text{ Nm}^{-2} \text{ deg}^{-1},$$

$$\omega_1^* = 1.9890 \times 10^{13} \text{ S}^{-1},$$

$$K_1^0 = 0.3000 \times 10^2 \text{ Wm}^{-1} \text{ deg}^{-1}, \quad \rho_0 = 0.1910 \text{ Kg m}^{-3},$$

$$k = 2, \quad \varepsilon_1^0 = 0.04162, \quad a = 10^{-3} \text{ m},$$

$$\lambda = 7.76 \times 10^{10} \text{ Nm}^{-2}, \quad \mu = 3.86 \times 10^{10} \text{ Nm}^{-2},$$

$$K = 386 \text{ Wm}^{-1} \text{ K}^{-1}, \quad \rho = 8954 \text{ kgm}^{-3},$$

$$\alpha_t = 1.78 \times 10^{-5} \text{ K}^{-1}, \quad \alpha_c = 1.98 \times 10^{-4} \text{ m}^3 \text{ kg}^{-1}.$$

For suitability, the absolute values of the following thermo elastic variables have been adopted to represent the results; $\bar{\theta} = 10\theta(x, t)$, $\bar{e} = 10e(x, t)$, $\bar{u} = 10^2 u(x, t)$, $\{\sigma_1, \sigma_2\} = 10\{\sigma_{11}, \sigma_{22}\}(x, t)$, $\bar{C} = 10C(x, t)$, $\bar{P} = \frac{1}{10}P(x, t)$. Numerical results are obtained for $p_0 = 1$, $\theta_0 = 10$, $\omega_0 = 1.95$, $\omega_1 = 0.05$, $\tau_{1j} = \tau_q = 0.1$ and $\tau_{0i} = \tau_\theta = 0.05$.

Figure 1 and Figure 2 shows distribution of radial stress in thermo-piezoelectric functionally graded rod against the radius of rod with respect to various parameters such as graded index and concentration condition. In Figure 1 the radial stress for different values of graded index against the increasing values of radius of rod is observed. From this, the distribution of radial stress is initially increased up to certain values of radius ($r = 0.2$) after decreased up to ($r = 0.6$) and follows unique nature for higher values of radius of rod in several values of graded index because of their wavelength. The increasing values of the graded index make quiet reasonable attendance in distribution of

Figure 1: Distribution of radial stress with the radius

Figure 2: Distribution of radial stress with the radius

radial stress is additionally noticed. In Figure 2 the radial stress for several values of concentration constant against the increasing values of radius of rod is detected. Since the spreading of radial stress is originally rise up to certain values of radius ($r = 0.2$) after decreased up to ($r = 0.4$) and monitors liner nature for upper values of radius of rod because of their wavelength. The increasing values of the concentration constant generates quiet reasonable attendance in spreading of radial stress is also perceived.

The spread of radial stress against the radius of rod is observed for different thermoelasticity theories in Figure 3. From this observation the spread of radial stress is follows grown nature in small values of radius and follows diminish nature for certain values of radius after follows linear nature for higher values of radius of rod in all theories of thermoelasticity. Additionally its observed RPL model generates excessive impact in distribution of radial stress.

Figure 3: Distribution of radial stress with the radius via-RPL $\Re = 1$

The displacement stress in thermopiezoelectric functionally graded rod against the radius of rod for diverse values of concentration constants in RPL model is detected in Figures 4 and 5. The displacement stress is follows diminish nature in minor values of radius of rod and follows unique nature for higher values of radius for diverse concentration constants in both RPL ($\Re = 2$ and $\Re = 4$) models. The displacement stress distribution makes reasonable attention for increasing value of in RPL model. The temperature

Figure 4: Distribution of displacement with the radius via RPL $\Re = 2$

Figure 5: Distribution of displacement stress with the radius via RPL $\Re = 4$

distribution in thermopiezoelectric functionally graded rod against the radius of rod for different values of time parameter t in RPL model is detected in Figure 6 and Figure 7. The temperature distribution is follows diminish nature for increasing values of radius of rod for different values of time parameter t in both RPL ($\Re = 2$ and $\Re = 4$) models. The temperature distribution creates sensible consideration for increasing value of in RPL model.

Figure 6: Distribution of temperature with the radius via RPL $\Re = 2$

Figure 7: Distribution of temperature with the radius via RPL $\Re = 4$

The distribution of electric displacement in thermopiezoelectric functionally graded rod against the radius of rod for different values of concentration constants in RPL model is noticed in Figures 8 and 9. The distribution of electric displacement follows oscillating nature for increasing values of radius for different concentration constants. The distribution of electric displacement creates reasonable attention for increasing value of $\Re$ in RPL model. The frequency in thermopiezoelectric functionally graded rod

Figure 8: Distribution of electric displacement with the radius via RPL $\mathfrak{R} = 2$

Figure 9: Distribution of electric displacement with the radius via RPL $\mathfrak{R} = 4$

against the radius of rod for different values of parameter t in RPL model is detected in Figures 10 and 11. The frequency distribution follows growing nature for increasing values of radius of rod for different values of parameter t in both RPL ($\mathfrak{R} = 2$ and $\mathfrak{R} = 4$) models. The temperature distribution creates sensible consideration for increasing value of $\mathfrak{R}$ in RPL model.

Figure 10: Distribution of frequency with the radius via RPL $\mathfrak{R} = 2$

Figure 11: Distribution of frequency with the radius via RPL $\mathfrak{R} = 4$

5 Conclusions

The present work is generated analytical model for wave dispersion in a thermally activated chemico diffusive piezoelectric functionally graded rod through refined multi-dual phase-lag model. The normal mode method and suitable boundary conditions used to obtain the exact solution of the coupled thermopiezoelectric equations. The analytical expressions for these coupled equations would be the components of mechanical displacement, temperature, electric displacement, electric potential, and stresses. Numerical results for these field quantities are illustrated graphically. The exact findings from this graphically illustration are discussed elaborate, for instance impacts of graded index, concentration constants and different RPL models. The present results may be applicable to a broad range of piezothermoelastic materials and smart materials industry.

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