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Efficiency and Optimality of Some Weighted-Residual Error Estimator for Adaptive 2D Boundary Element Methods

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Dedicated to Professor Ernst Peter Stephan on the occasion of his 65th birthday

Abstract — We prove convergence and quasi-optimality of a lowest-order adaptive boundary element method for a weakly-singular integral equation in 2D. The adaptive mesh-refinement is driven by the weighted-residual error estimator. By proving that this estimator is not only reliable, but under some regularity assumptions on the given data also efficient on locally refined meshes, we characterize the approximation class in terms of the Galerkin error only. In particular, this yields that no adaptive strategy can do better, and the weighted-residual error estimator is thus an optimal choice to steer the adaptive mesh-refinement. As a side result, we prove a weak form of the saturation assumption.

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1. Introduction and Outline

Recently, there was a major breakthrough in the thorough mathematical understanding of convergence and quasi-optimality of h-adaptive FEM (AFEM) for second-order elliptic

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PDEs. Following the pioneering works [9,15,30] which analyzed quasi-optimality of AFEM for homogeneous Dirichlet problems, the successors included non-symmetric problems [16, 21], inhomogeneous Dirichlet/Neumann conditions [4, 23], and even nonlinearities [7, 21] into the AFEM analysis. However, many of the ingredients which appear in their proofs were mathematically open for adaptive BEM (ABEM). Only very recently, the works [22, 31] proved quasi-optimal convergence for certain BEM model problems like the weakly-singular and hypersingular integral equations for the 3D Laplacian. To the best of our knowledge, the approximation classes $\mathbb{A}_s$ involved in the quasi-optimality results have only been characterized for AFEM for the Laplace equation in terms of regularity of the unknown solution and the given data [9]. For general operators, the approximation classes involved are characterized by the optimal decay of the total error which consists of energy norm error plus certain oscillations. The latter arise typically from inverse estimates and incorporate the computed discrete solutions, see, e.g., [15,22,31]. Put differently, since the total error is equivalent to the error estimator used, these results for AFEM/ABEM guarantee the quasi-optimal convergence rate for the error estimator. This is somewhat unsatisfactory, since the error could even decay with a better rate than the estimator. In this work, we overcome these restrictions by proving that the error estimator is, under some regularity assumptions on the given data, equivalent to the energy norm error. We consider Symm's integral equation on a domain $\Omega \subset \mathbb{R}^2$,

$$V\phi = (K + \tfrac{1}{2})g \quad \text{on the boundary } \partial\Omega$$

for some given boundary data $g \in H^{1/2}(\partial\Omega)$. To steer a usual adaptive algorithm of the type

$$\boxed{\texttt{solve}} \to \boxed{\texttt{estimate}} \to \boxed{\texttt{mark}} \to \boxed{\texttt{refine}} \tag{1.1}$$

we use the weighted-residual error estimator proposed by Carstensen and Stephan [14] and later sharpened in [11]. This allows to build upon the arguments from [22] and additionally prove efficiency of the error estimator under some regularity assumptions on the given boundary data only. Prior to this, the only efficiency result for the weighted-residual error estimator was [10], where slightly stronger regularity assumptions and globally quasi-uniform meshes are required. Instead, we prove that the weighted-residual error estimator is also efficient on locally refined meshes up to certain higher-order terms which do not depend on the error estimator or the discrete solution, but only on the given data. This efficiency estimate allows to characterize the approximation class in terms of the Galerkin error only. In particular, this yields that the weighted-residual error estimator is optimal and that no other estimator can perform better in the sense of asymptotic convergence rates of the Galerkin error.

The remainder of the work is organized as follows: In Section 2, we formulate the model problem and the adaptive algorithm. Moreover, we present the main results of this work in detail. Section 3 is devoted to an optimal 1D mesh refining strategy. The proof of the efficiency estimate for the weighted-residual error estimator is found in Section 4. Finally, Section 5 concludes the optimality proof. We underline the theoretical results with numerical experiments in Section 6 and conclude the work with some remarks on the saturation assumption in a short appendix.

Throughout the work, we use the notation $\lesssim$ which indicates $\leqslant$ up to a multiplicative constant which is clear from the context.

2. Model Problem and Main Results

2.1. Model Problem

We consider Symm's integral equation

$$V\phi = f := (K + \tfrac{1}{2})g \quad \text{on } \Gamma, \tag{2.1}$$

where $\Gamma := \partial\Omega$ is the boundary of a polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ with diameter $\operatorname{diam}(\Omega) < 1$. With $n(x) \in \mathbb{R}^2$ denoting the exterior normal unit field at $x \in \Gamma$ and the fundamental solution of the 2D Laplacian

$$G(z) := -\tfrac{1}{2\pi}\log|z| \quad \text{for all } z \in \mathbb{R}^2 \setminus \{0\}, \tag{2.2}$$

the simple-layer potential V and the double-layer potential K formally read

$$(V\phi)(x) := \int_\Gamma G(x-y)\phi(y)\,dy \quad \text{and} \quad (Kg)(x) := \text{p.v.}\int_\Gamma \partial_{n(y)}G(x-y)g(y)\,dy \tag{2.3}$$

for all $x \in \Gamma$. Here, p.v. $\int_\Gamma$ denotes Cauchy's principal value. Then, (2.1) is an equivalent formulation of

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma. \end{aligned} \tag{2.4}$$

The solution of (2.1) is the normal derivative $\phi = \partial_n u \in \mathcal{H} := H^{-1/2}(\Gamma)$ of the solution $u \in H^1(\Omega)$ of (2.4). The operator $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is an elliptic and symmetric isomorphism (see, e.g., [27–29]). It thus provides a scalar product defined by $\langle\langle\phi,\psi\rangle\rangle := \langle V\phi,\psi\rangle_{L^2(\Gamma)}$. This scalar product induces an equivalent energy norm on $H^{-1/2}(\Gamma)$, which will be denoted by $|||\psi||| := \langle\langle\psi,\psi\rangle\rangle^{1/2}$. For some Γ dependent constant $C_{\text{norm}} > 0$, it holds

$$C_{\text{norm}}^{-1}|||\psi||| \leqslant \|\psi\|_{H^{-1/2}(\Gamma)} \leqslant C_{\text{norm}}|||\psi||| \quad \text{for all } \psi \in H^{-1/2}(\Gamma). \tag{2.5}$$

Whereas $g \in H^{1/2}(\Gamma)$ is sufficient to guarantee the solvability of (2.1), the weighted-residual error estimator η_ℓ (see (2.9) below) needs the given boundary data to satisfy $g \in H^1(\Gamma)$. The usual adaptive algorithm of the type (1.1) reads as follows:

Algorithm 2.1. ***Input:*** *Initial partition* $\mathcal{T}_0$*, adaptivity parameter* $0 < \theta < 1$*, counter* $\ell := 0$.

(i) *Compute discrete solution* Φ_ℓ *corresponding to* $\mathcal{T}_\ell$.

(ii) *Compute refinement indicators* $\eta_\ell(T)$ *for all* $T \in \mathcal{T}_\ell$.

(iii) *Determine set* $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ *of minimal cardinality such that Dörfler marking*

$$\theta \sum_{T\in\mathcal{T}_\ell} \eta_\ell(T)^2 \leqslant \sum_{T\in\mathcal{M}_\ell} \eta_\ell(T)^2. \tag{2.6}$$

is satisfied.

(iv) *Refine (at least) marked elements* $T \in \mathcal{T}_\ell$ *to obtain new partition* $\mathcal{T}_{\ell+1}$.

(v) *Increase counter* $\ell \mapsto \ell + 1$ *and iterate.*

Output: *Discrete solutions* Φ_ℓ *and error estimators* $\eta_\ell := (\sum_{T\in\mathcal{T}_\ell}\eta_\ell(T)^2)^{1/2}$ *for* $\ell \geqslant 0$.

This section provides an overview on this work and its main results. We start with a discussion of the concrete realization of the modules which compose the adaptive algorithm (Algorithm 2.1).

2.2. Algorithm 2.1, Step (i): `solve`

Let $\mathcal{T}_\ell$ denote a partition of the boundary Γ into affine line segments. As usual, we denote the L^2-scalar product on the boundary Γ by $\langle\cdot,\cdot\rangle_{L^2(\Gamma)}$ and extend it to the duality brackets of $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$ by continuity. The lowest-order conforming Galerkin discretization of the continuous model problem (2.1) reads: Find $\Phi_\ell\in\mathcal{P}^0(\mathcal{T}_\ell)$ such that

$$\langle\!\langle\Phi_\ell,\Psi_\ell\rangle\!\rangle = \langle(K+\tfrac{1}{2})g,\Psi_\ell\rangle_{L^2(\Gamma)}\quad\text{for all }\Psi_\ell\in\mathcal{P}^0(\mathcal{T}_\ell), \tag{2.7}$$

where we use the polynomial spaces $\mathcal{P}^p([0,1]):=\{v\in C^\infty([0,1]):\frac{\partial^{p+1}}{\partial s^{p+1}}v=0\}$ to define

$$\mathcal{P}^p(\mathcal{T}_\ell):=\big\{v\in L^2(\Gamma): v\circ F_T\in\mathcal{P}^p([0,1])\text{ for all }T\in\mathcal{T}_\ell\big\}.$$

Here, $F_T:[0,1]\to T$ is an affine transformation which maps the unit interval onto the element $T\in\mathcal{T}_\ell$. As in the continuous setting, it follows that (2.7) allows for a unique solution. For simplicity, we assume that the module `solve` computes the exact discrete solution. However, it is possible to include an approximate solver into our analysis. As an immediate consequence of the Galerkin orthogonality $\langle\!\langle\phi-\Phi_\ell,\Psi_\ell\rangle\!\rangle=0$ for all $\Psi_\ell\in\mathcal{P}^0(\mathcal{T}_\ell)$, we get the best approximation property, also known as Céa's lemma,

$$|\!|\!|\phi-\Phi_\ell|\!|\!| = \min_{\Psi_\ell\in\mathcal{P}^0(\mathcal{T}_\ell)}|\!|\!|\phi-\Psi_\ell|\!|\!|. \tag{2.8}$$

2.3. Algorithm 2.1, Step (ii): `estimate`

We recall the definition of the residual-based error estimator η_ℓ which dates back to the seminal work [14] for 2D and has been extended to 3D in [12]. The local contributions of η_ℓ are defined by

$$\eta_\ell(T):=\operatorname{diam}(T)^{1/2}\|\tfrac{\partial}{\partial s}(V\Phi_\ell-f)\|_{L^2(T)}\quad\text{for all }T\in\mathcal{T}_\ell. \tag{2.9}$$

Here, $\frac{\partial}{\partial s}$ denotes the arclength derivative along Γ. We define the local mesh-width function $h_\ell\in L^\infty(\Gamma)$ by $h_\ell|_T:=\operatorname{diam}(T)$, where $\operatorname{diam}(T)$ denotes the Euclidean length of an element $T\in\mathcal{T}_\ell$. Now, there holds reliability (cf. [11, Theorem 1])

$$C_{\rm rel}^{-1}|\!|\!|\phi-\Phi_\ell|\!|\!|\leqslant\eta_\ell:=\Big(\sum_{T\in\mathcal{T}_\ell}\eta_\ell(T)^2\Big)^{1/2}=\|h_\ell^{1/2}\tfrac{\partial}{\partial s}(V\Phi_\ell-f)\|_{L^2(\Gamma)} \tag{2.10}$$

for all $\ell\in\mathbb{N}$, where $C_{\rm rel}>0$ depends only on Γ and the K-mesh constant $\kappa(\mathcal{T}_\ell)$ (see (2.11) below). Note that the assumption $g\in H^1(\Gamma)$, and the mapping properties of V and K guarantee that $f=(K+\frac{1}{2})g\in H^1(\Gamma)$ as well as $V\Phi_\ell\in H^1(\Gamma)$ (cf. [27–29]). Therefore, the estimator η_ℓ is well defined.

2.4. Algorithm 2.1, Step (iv): `refine`

For a given set $\mathcal{M}_\ell\subset\mathcal{T}_\ell$ of marked elements, we refine $\mathcal{T}_\ell$ such that at least all marked elements $T\in\mathcal{M}_\ell$ are bisected into two sons of half length and such that the K-mesh constant

$$\kappa(\mathcal{T}_\ell):=\max\big\{h_\ell|_T/h_\ell|_{T'}:T,T'\in\mathcal{T}_\ell\text{ with }T\cap T'\neq\emptyset\big\} \tag{2.11}$$

remains uniformly bounded in the sense of

$$\sup_{\ell\in\mathbb{N}} \kappa(\mathcal{T}_\ell) < \infty. \tag{2.12}$$

The following algorithm proposed and used in [19, 20, 25] guarantees (2.12), as stated in Theorem 2.3 below.

Algorithm 2.2. ***Input:*** *Partition $\mathcal{T}_\ell$, marked elements $\mathcal{M}_\ell^{(0)} := \mathcal{M}_\ell$, counter $i := 0$.*

(i) *Define $\mathcal{U}^{(i)} := \bigcup_{T\in\mathcal{M}_\ell^{(i)}}\{T' \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell^{(i)}$ neighbor of $T : h_\ell|_{T'} > \kappa(\mathcal{T}_0)\, h_\ell|_T\}$.*

(ii) *If $\mathcal{U}^{(i)} \neq \emptyset$, define $\mathcal{M}_\ell^{(i+1)} := \mathcal{M}_\ell^{(i)} \cup \mathcal{U}^{(i)}$, increase counter $i \mapsto i+1$, and goto* (i).

(iii) *Otherwise, bisect all marked elements $T \in \mathcal{M}_\ell^{(i)}$ to obtain $\mathcal{T}_{\ell+1}$.*

Output: *Refined boundary partition $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$ as well as sets of refined elements $\mathcal{M}_\ell^{(i)} = \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1} \supseteq \mathcal{M}_\ell$.*

A detailed analysis of this algorithm is given in Section 3, while its essential properties are stated in Theorem 2.3.

2.5. Function Spaces Involved

For $\nu \in (0, 5/2]$, we define $H^\nu(\Gamma)$ as the trace space, i.e.,

$$H^\nu(\Gamma) := \big\{v|_\Gamma : v \in H^{\nu+1/2}(\Omega)\big\}$$

equipped with the norm

$$\|w\|_{H^\nu(\Gamma)} := \inf\big\{\|v\|_{H^{\nu+1/2}(\Omega)} : w = v|_\Gamma,\ v \in H^{\nu+1/2}(\Gamma)\big\}.$$

This definition is equivalent to the classical definition of $H^\nu(\Gamma)$ as a Sobolev space on the 1D Lipschitz-manifold Γ for $\nu \in (0, 1]$. In particular, one may use the *Sobolev–Slobodeckij* norm

$$|w|_{H^\nu(\Gamma)} := \int_\Gamma \int_\Gamma \frac{|w(x) - w(y)|^2}{|x-y|^{1+2\nu}}\, dx\, dy$$

to define an equivalent norm $(\|\cdot\|_{L^2(\Gamma)}^2 + |\cdot|_{H^\nu(\Gamma)}^2)^{1/2}$ on $H^\nu(\Gamma)$, for $\nu \in (0, 1)$. For $\nu = 1$, $H^1(\Gamma)$ is equipped with the equivalent norm

$$\|w\|_{H^1(\Gamma)}^2 := \|w\|_{L^2(\Gamma)}^2 + \|\tfrac{\partial}{\partial s} w\|_{L^2(\Gamma)}^2 \quad \text{for all } w \in H^1(\Gamma),$$

where $\frac{\partial}{\partial s}$ denotes the arclength derivative along Γ.

Finally, for $\nu \in (0, 1)$, we may equivalently define $H^\nu(\Gamma)$ as the real interpolation space of $L^2(\Gamma)$ and $H^1(\Gamma)$ (cf. [8]). All mentioned definitions of $H^\nu(\Gamma)$ are – at least for $\nu \in (0, 1)$ – equivalent. The norm equivalency constants, however, depend on the boundary Γ.

2.6. Main Results

The first result of this work states that the 1D mesh-refinement algorithm (Algorithm 2.2) is optimal. To that end, let $\mathbb{T}$ denote the set of all locally refined meshes $\widetilde{\mathcal{T}}_\ell$, which can be obtained from the initial partition $\mathcal{T}_0$ by Algorithm 2.2, i.e., $\widetilde{\mathcal{T}}_\ell$ is obtained inductively by $\widetilde{\mathcal{T}}_{j+1} = \texttt{refine}(\widetilde{\mathcal{T}}_j, \widetilde{\mathcal{M}}_j)$ for $j = 0, \dots, \ell-1$, with $\widetilde{\mathcal{T}}_0 = \mathcal{T}_0$, and arbitrary $\ell \in \mathbb{N}$ as well as arbitrary marked elements $\widetilde{\mathcal{M}}_j \subseteq \widetilde{\mathcal{T}}_j$.

Theorem 2.3. *Algorithm 2.2 has the following properties:*

(i) *For all meshes $\mathcal{T} \in \mathbb{T}$, it holds that*

$$\kappa(\mathcal{T}) \leqslant 2\kappa(\mathcal{T}_0). \tag{2.13}$$

(ii) *Given two meshes $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$, there exists a coarsest common refinement $\mathcal{T} \oplus \mathcal{T}' \in \mathbb{T}$ such that*

$$\#(\mathcal{T} \oplus \mathcal{T}') \leqslant \#\mathcal{T} + \#\mathcal{T}' - \#\mathcal{T}_0. \tag{2.14}$$

(iii) *The additional refinements which guarantee* (2.13) *do not lead to substantially more refined elements, i.e.,*

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \leqslant C_{\mathrm{mesh}} \sum_{j=0}^{\ell-1} \#\mathcal{M}_j \tag{2.15}$$

for some ℓ-independent constant $C_{\mathrm{mesh}} > 0$ and sets of marked elements $\mathcal{M}_j$.

The second theorem is the mathematical heart of this work and states efficiency of the weighted-residual error estimator η_ℓ on locally refined meshes up to terms of higher order.

Theorem 2.4 (Efficiency of η_ℓ)**.** *Let the given boundary data satisfy $g \in H^{s_{\mathrm{reg}}}(\Gamma)$ for some $s_{\mathrm{reg}} > 2$. Let ϕ denote the solution of* (2.1). *Then, for $\mathcal{T}_\ell \in \mathbb{T}$ the error estimator η_ℓ is efficient in the following sense:*

$$C_{\mathrm{eff}}^{-1} \eta_\ell \leqslant \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell. \tag{2.16}$$

Here, $C_{\mathrm{eff}} > 0$ depends only on Γ and $\kappa(\mathcal{T}_\ell)$. The higher-order term hot_ℓ *is given in detail in Definition 4.7 below. For all $\varepsilon > 0$, it satisfies*

$$\mathrm{hot}_\ell = \Big(\sum_{T \in \mathcal{T}_\ell} \mathrm{hot}_\ell(T)^2\Big)^{1/2} \quad \text{and} \quad \mathrm{hot}_\ell(T) \leqslant C_{\mathrm{hot}} (h_\ell|_T)^{\min\{s_{\mathrm{reg}}, 5/2\} - 1/2 - \varepsilon}, \tag{2.17}$$

where $C_{\mathrm{hot}} > 0$ depends only on Γ, $\kappa(\mathcal{T}_\ell)$, $s_{\mathrm{reg}} > 2$, and $\varepsilon > 0$.

Following the lines of [22] and re-interpreting their results (see Section 5 below), we are able to prove the optimal rate of convergence for the estimator: We define

$$\mathbb{T}_N := \big\{\mathcal{T}_\star \in \mathbb{T} : \#\mathcal{T}_\star - \#\mathcal{T}_0 \leqslant N\big\}$$

and

$$(\phi, g) \in \mathbb{A}_s^\eta \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \|(\phi, g)\|_{\mathbb{A}_s^\eta} := \sup_{N \in \mathbb{N}} \inf_{\mathcal{T}_\star \in \mathbb{T}_N} (N^s \eta_\star) < \infty, \tag{2.18}$$

where $\eta_\star$ is the weighted-residual error estimator for the mesh $\mathcal{T}_\star \in \mathbb{T}$. Using the efficiency estimate in Theorem 2.4, we may finally characterize the approximation class $\mathbb{A}_s^\eta$ in terms of the Galerkin error only. To that end, we introduce

$$\phi \in \mathbb{A}_s \quad \overset{\text{def.}}{\iff} \quad \|\phi\|_{\mathbb{A}_s} := \sup_{N\in\mathbb{N}} \inf_{\mathcal{T}_*\in\mathbb{T}_N} \inf_{\Psi_*\in\mathcal{P}^0(\mathcal{T}_*)} |||\phi - \Psi_*|||N^s < \infty. \tag{2.19}$$

Precisely, this quasi-optimality is characterized in the following theorem by means of the adaptive algorithm.

Theorem 2.5. *For arbitrary adaptivity parameter* $0 < \theta < 1$*, Algorithm 2.1 guarantees the existence of* $0 < \gamma, \kappa < 1$*, such that*

$$\Delta_{\ell+1} \leqslant \kappa\Delta_\ell, \quad \textit{where } \Delta_\ell := |||\phi - \Phi_\ell|||^2 + \gamma\eta_\ell^2 \textit{ and } \ell \geqslant 0. \tag{2.20}$$

In particular, this proves linear convergence of the Galerkin error to zero. Moreover, let $s > 0$ *and suppose that* $0 < \theta < 1$ *is sufficiently small. Then, Algorithm 2.1 is optimal in the sense of*

$$(\phi, g) \in \mathbb{A}_s^\eta \quad \iff \quad \eta_\ell \leqslant C_1 \,(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \quad \textit{for all } \ell \in \mathbb{N}. \tag{2.21}$$

Finally, provided that $g \in H^{s_{\rm reg}}(\Gamma)$ *for some* $s_{\rm reg} > 2$ *and* $0 < s < \min\{s_{\rm reg}, 5/2\} - 1/2$*, Algorithm 2.1 is even optimal in the sense of*

$$\phi \in \mathbb{A}_s \quad \iff \quad |||\phi - \Phi_\ell||| \leqslant C_2 \,(\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \quad \textit{for all } \ell \in \mathbb{N}. \tag{2.22}$$

The constants $C_1, C_2 > 0$ *depend only on* Γ *as well as* $\|(\phi, g)\|_{\mathbb{A}_s^\eta}$ *and* $\|\phi\|_{\mathbb{A}_s}$*, respectively.*

The novelty of this theorem lies in the second statement (2.22). It proves that the weighted-residual error estimator η_ℓ is in fact the optimal choice to steer the adaptive algorithm in the sense that asymptotically no other estimator can perform better. The generically optimal rate of convergence of lowest-order BEM for Symm's integral equation is $s = 3/2$ (see [29, Theorem 4.1.54]). Therefore, (2.22) states that if there is a sequence of meshes which reveals order $s = 3/2$, Algorithm 2.1 will produce a (maybe different) sequence of meshes such that the corresponding energy norm error converges with the same or even better rate. The first statement (2.20) as well as the quasi-optimality result (2.21) are proved for the 3D case in [22]. The latter result (2.21) states that the adaptive algorithm is optimal in the sense that the only quantity that is seen by the algorithm – the estimator – converges with optimal order. We recite (2.20)–(2.21) only for convenience of the reader. In Section 5, we work out the differences which occur in the proof of (2.20)–(2.21) due to the present 2D situation.

3. Proof of Theorem 2.3

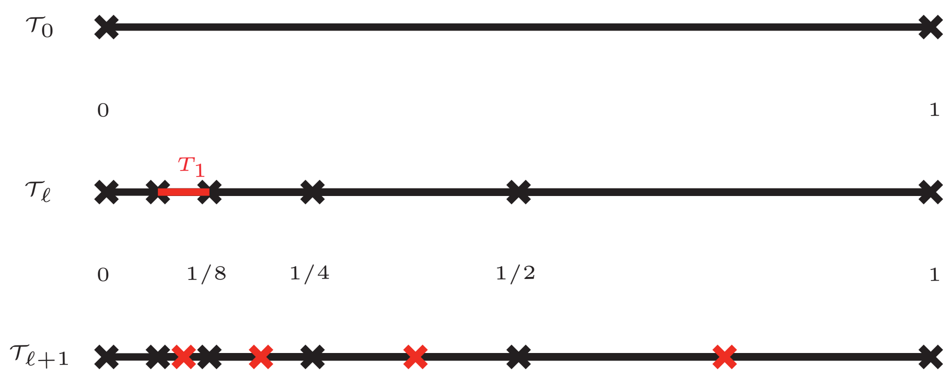
In this section, we aim to prove optimality (2.13)–(2.15) of the local mesh-refinement strategy in Algorithm 2.2. Suppose that $\mathcal{T}_0 = \{T_1, \dots, T_N\}$ is a given initial partition of Γ into affine boundary segments T_j and that a sequence of meshes $\mathcal{T}_\ell$ is obtained inductively by local refinement, where

$$\mathcal{T}_{\ell+1} = \texttt{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell) \tag{3.1}$$

is generated from $\mathcal{T}_\ell$ by refinement of (at least) certain marked elements $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$. Here, refinement of an element $T \in \mathcal{M}_\ell$ means that T is bisected into two elements $T_1, T_2 \in \mathcal{T}_{\ell+1}$ of half length, i.e., there holds $h_{\ell+1}|_T = \frac{1}{2}\, h_\ell|_T$.

Remark 3.1. Clearly, the boundedness estimate (2.13) cannot be improved in general. For instance, let $\mathcal{T}_0$ be a uniform partition with $\#\mathcal{T}_0 > 1$ and $\#\mathcal{M}_0 = 1$. Provided that the obtained partition satisfies $\#\mathcal{T}_1 < 2\,\#\mathcal{T}_0$, i.e., the local refinement does not lead to a uniform refinement, there holds $\kappa(\mathcal{T}_0) = 1$, whereas $\kappa(\mathcal{T}_1) = 2$.

Remark 3.2. Since the refined elements $\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ are bisected into two sons, it holds that $\#\mathcal{M}_\ell \leqslant \#(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}) = \#\mathcal{T}_{\ell+1} - \#\mathcal{T}_\ell$. Under (2.13), the converse inequality $\#\mathcal{T}_{\ell+1} - \#\mathcal{T}_\ell \lesssim \#\mathcal{M}_\ell$ cannot hold in general as the following elementary example proves: Let $\mathcal{T}_0$ denote the partition of [0, 1] depicted below



Obviously, the mesh-ratio is $\kappa(\mathcal{T}_0) = 1$. Repeated marking of the leftmost elements of $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_{\ell-1}$ generates the mesh $\mathcal{T}_\ell$ with $\kappa(\mathcal{T}_\ell) = 2$ and $\#\mathcal{T}_\ell = \ell$. Marking the highlighted element $T_1 \in \mathcal{T}_\ell$ results in the mesh $\mathcal{T}_{\ell+1} := \texttt{refine}(\mathcal{T}_\ell, \{T_1\})$, where $\ell - 1$ elements are refined to ensure $\kappa(\mathcal{T}_{\ell+1}) = 2$. Consequently, the number of additional refinements can be arbitrarily large.

Before tackling the original problem, we introduce a level-based mesh-refinement strategy.

3.1. Level-Based Mesh-Refinement

To imitate the analytical techniques developed in [9,30], we introduce the level of an element by induction: For $T \in \mathcal{T}_0$, let $\texttt{level}(T) := 0$. If $T \in \mathcal{T}_\ell$ is bisected into two sons $T_1, T_2 \in \mathcal{T}_{\ell+1}$, we define $\texttt{level}(T_1) := \texttt{level}(T_2) := \texttt{level}(T) + 1$. Instead of Algorithm 2.2, we consider the following level-based variant:

Algorithm 3.3. ***Input:*** *Partition $\mathcal{T}_\ell$, marked elements $\mathcal{M}_\ell^{(0)} := \mathcal{M}_\ell$, counter $i := 0$.*

(i) *Define $\mathcal{U}^{(i)} := \bigcup_{T \in \mathcal{M}_\ell^{(i)}} \{T' \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell^{(i)}$ neighbor of $T : \texttt{level}(T') < \texttt{level}(T)\}$.*

(ii) *If $\mathcal{U}^{(i)} \neq \emptyset$, define $\mathcal{M}_\ell^{(i+1)} := \mathcal{M}_\ell^{(i)} \cup \mathcal{U}^{(i)}$, increase counter $i \mapsto i+1$, and goto* (i).

(iii) *Otherwise, bisect all marked elements $T \in \mathcal{M}_\ell^{(i)}$ to obtain $\mathcal{T}_{\ell+1}$.*

Output: *Refined boundary partition $\mathcal{T}_{\ell+1}$ as well as $\mathcal{M}_\ell^{(i)} = \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$.*

Note that Algorithm 3.3 is well-defined in the sense that it terminates for some counter $0 \leqslant i \leqslant \#\mathcal{T}_\ell - 1$. Moreover, the definition of $\mathcal{U}^{(i)}$ in step (i) of Algorithm 3.3 guarantees

$$|\texttt{level}(T) - \texttt{level}(T')| \leqslant 1 \quad \text{for all } T, T' \in \mathcal{T}_\ell \text{ with } T \cap T' \neq \emptyset. \tag{3.2}$$

With this at hand, one can use the techniques from [9,30] to prove (2.15). Moreover, (2.13) and (2.14) follow from direct calculations. For details, we refer to the extended preprint [3] where the following proposition is proved.

Proposition 3.4. *The mesh-refinement strategy in Algorithm 3.3 satisfies the optimality properties* (2.13)–(2.15). □

3.2. κ-Based Mesh-Refinement

In this section, we use the level-based algorithm to prove that the mesh-refinement of Algorithm 2.2 also satisfies (2.13)–(2.15). The advantage of this is that there is no need to compute or store the level function. We now turn to the proof of Theorem 2.3. First, we prove the uniform boundedness (2.13) of the K-mesh constant.

Proof of Theorem 2.3 (i). Let $T, T' \in \mathcal{T}_{\ell+1}$ be neighbors, i.e., $T \neq T'$ and $T \cap T' \neq \emptyset$. Consequently, the fathers $\widehat{T}, \widehat{T}' \in \mathcal{T}_\ell$ of T and T' either coincide or are neighbors as well. We aim to provide an upper bound for the quotient $h_{\ell+1}|_{T'}/h_{\ell+1}|_T$. In case of $\widehat{T} = \widehat{T}'$, there holds $h_{\ell+1}|_T = h_{\ell+1}|_{T'}$. Therefore, we may assume that $\widehat{T} \neq \widehat{T}'$. We now consider four cases:

(a) If $\widehat{T}, \widehat{T}'$ are both not refined, there holds $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}$ and $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}$.

(b) If $\widehat{T}, \widehat{T}'$ are both refined, there holds $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}/2$ and $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}/2$.

(c) If $\widehat{T}'$ is refined and $\widehat{T}$ is not, there holds $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}/2$ and $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}$.

(d) If $\widehat{T}'$ is not refined and $\widehat{T}$ is refined, there holds $h_{\ell+1}|_{T'} = h_\ell|_{\widehat{T}'}$ and $h_{\ell+1}|_T = h_\ell|_{\widehat{T}}/2$. Moreover, Algorithm 2.2 implies $h_\ell|_{\widehat{T}'} \leqslant \kappa(\mathcal{T}_0) h_\ell|_{\widehat{T}}$.

In the cases (a)–(c), we thus observe

$$h_{\ell+1}|_{T'}/h_{\ell+1}|_T \leqslant h_\ell|_{\widehat{T}'}/h_\ell|_{\widehat{T}} \leqslant \kappa(\mathcal{T}_\ell).$$

In case (d), there holds

$$h_{\ell+1}|_{T'}/h_{\ell+1}|_T = 2\, h_\ell|_{\widehat{T}'}/h_\ell|_{\widehat{T}} \leqslant 2\,\kappa(\mathcal{T}_0).$$

Altogether, this proves

$$\frac{h_{\ell+1}|_{T'}}{h_{\ell+1}|_T} \leqslant \max\{\kappa(\mathcal{T}_\ell), 2\,\kappa(\mathcal{T}_0)\} \quad \text{for all neighboring elements } T, T' \in \mathcal{T}_{\ell+1},$$

whence $\kappa(\mathcal{T}_{\ell+1}) \leqslant \max\{\kappa(\mathcal{T}_\ell), 2\,\kappa(\mathcal{T}_0)\}$. By induction, we conclude $\kappa(\mathcal{T}_{\ell+1}) \leqslant 2\,\kappa(\mathcal{T}_0)$. □

Next, the overlay estimate (2.14) will be proved.

Proof of Theorem 2.3 (ii). We aim to prove even a little bit more, i.e., for meshes $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$, there holds $\mathcal{T} \oplus \mathcal{T}' \in \mathbb{T}$ and

$$\begin{aligned} \mathcal{T} \oplus \mathcal{T}' = \mathcal{T}_\oplus := &\big\{T \in \mathcal{T} : \text{exists } T' \in \mathcal{T}' \text{ with } T \subseteq T'\big\} \\ &\cup \big\{T' \in \mathcal{T}' : \text{exists } T \in \mathcal{T} \text{ with } T' \subseteq T\big\}. \end{aligned} \tag{3.3}$$

If the characterization of $\mathcal{T} \oplus \mathcal{T}'$ above holds true, the estimate in (2.14) is fulfilled trivially. First, we show that $\mathcal{T}_\oplus$ as defined in (3.3) is a refinement of $\mathcal{T}$ and $\mathcal{T}'$. Assume it exists $T \in \mathcal{T}$ with $T \notin \mathcal{T}_\oplus$. Then, for all $T' \in \mathcal{T}'$, it holds $T \not\subseteq T'$. Because, the refinement rule generates a binary refinement tree, this implicates $T' \subseteq T$ or $|T \cap T'| = 0$ for all $T' \in \mathcal{T}'$. Therefore, we have $T'_1, \dots, T'_k \in \mathcal{T}'$ with

$$T = \bigcup_{i=1}^{k} T'_i.$$

By definition of $\mathcal{T}_\oplus$, $T'_i \in \mathcal{T}_\oplus$ for all $i = 1, \dots, k$ and therefore $\mathcal{T}_\oplus$ is a refinement of $\mathcal{T}$. The same argumentation for $\mathcal{T}'$ yields that $\mathcal{T}_\oplus$ is a refinement of $\mathcal{T}'$. Obviously, $\mathcal{T}_\oplus$ is the coarsest common refinement of $\mathcal{T}$ and $\mathcal{T}'$. Next, we aim to show $\kappa(T_\oplus) \leqslant 2\kappa(\mathcal{T}_0)$. We argue by contradiction. Therefore, assume neighbors $T, T' \in \mathcal{T}_\oplus$ with $\operatorname{diam}(T)/\operatorname{diam}(T') > \max\{\kappa(\mathcal{T}), \kappa(\mathcal{T}')\}$. By definition of the K-mesh constant κ, we obtain $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$. The definition of $\mathcal{T}_\oplus$ thus gives an element $\widehat{T}' \in \mathcal{T}'$ with $T \subset \widehat{T}'$. Now, we obtain the contradiction

$$\max\{\kappa(\mathcal{T}), \kappa(\mathcal{T}')\} < \frac{\operatorname{diam}(T)}{\operatorname{diam}(T')} \leqslant \frac{\operatorname{diam}(\widehat{T}')}{\operatorname{diam}(T')} \leqslant \kappa(\mathcal{T}'),$$

where we used that T and $\widehat{T}'$ are neighbors in $\mathcal{T}'$ or coincide. This shows $\kappa(T_\oplus) \leqslant 2\kappa(\mathcal{T}_0)$. Consequently, we may generate $\mathcal{T}_\oplus$ by iterative refinement of $\mathcal{T}_0 := \mathcal{T}$,

$$\mathcal{T}_{i+1} := \texttt{refine}(\mathcal{T}_i, \mathcal{T}_i \setminus \mathcal{T}_\oplus)$$

for all $i \geqslant 0$ with $\mathcal{T}_i \setminus \mathcal{T}_\oplus \neq \emptyset$. This yields $\mathcal{T}_\oplus \in \mathbb{T}$ and therefore $\mathcal{T}_\oplus = \mathcal{T} \oplus \mathcal{T}'$, which concludes the proof. □

We note that, by definition, Algorithm 2.2 provides the coarsest refinement $\mathcal{T}_{\ell+1}$ of a partition $\mathcal{T}_\ell$ with $\kappa(\mathcal{T}_\ell) \leqslant 2\,\kappa(\mathcal{T}_0)$ such that all elements $T \in \mathcal{M}_\ell$ are refined and that there holds $\kappa(\mathcal{T}_{\ell+1}) \leqslant 2\,\kappa(\mathcal{T}_0)$. The proof of (2.15) will be achieved by comparison of Algorithm 2.2 with Algorithm 3.3. More precisely, the optimality (2.15) for the κ-based mesh-refinement is obtained via the estimate for the level-based mesh-refinement from the previous section.

Proof of Theorem 2.3 (iii). Let $\widetilde{\texttt{refine}}$ denote the level-based mesh-refinement from Section 3.1. By induction, we now define an additional sequence of partitions by

$$\widetilde{\mathcal{T}}_{\ell+1} := \widetilde{\texttt{refine}}(\widetilde{\mathcal{T}}_\ell, \widetilde{\mathcal{M}}_\ell) \quad \text{with} \quad \widetilde{\mathcal{M}}_\ell := \mathcal{M}_\ell \cap \widetilde{\mathcal{T}}_\ell,$$

where $\widetilde{\mathcal{T}}_0 := \mathcal{T}_0$ and $\widetilde{\mathcal{M}}_0 := \mathcal{M}_0$. In the following, we prove that the partitions $\mathcal{T}_\ell$ generated by Algorithm 2.2 are coarser than the partitions $\widetilde{\mathcal{T}}_\ell$ generated by Algorithm 3.3 in the sense that each element $T \in \mathcal{T}_\ell$ is the union of elements from $\widetilde{\mathcal{T}}_\ell$, i.e.,

$$\forall \ell \in \mathbb{N}_0 \;\; \forall T \in \mathcal{T}_\ell \;\; \exists \mathcal{V}_\ell \subseteq \widetilde{\mathcal{T}}_\ell \qquad T = \bigcup_{\widetilde{T} \in \mathcal{V}_\ell} \widetilde{T}. \tag{3.4}$$

This implies $\#\mathcal{T}_\ell \leqslant \#\widetilde{\mathcal{T}}_\ell$. Moreover, there holds $\#\widetilde{\mathcal{M}}_\ell \leqslant \#\mathcal{M}_\ell$ by definition of the set $\widetilde{\mathcal{M}}_\ell$. Using the optimality (2.15) of the level-based refinement, we therefore infer optimality of the κ-based refinement

$$\#\mathcal{T}_\ell - \#\mathcal{T}_0 \leqslant \#\widetilde{\mathcal{T}}_\ell - \#\widetilde{\mathcal{T}}_0 \lesssim \sum_{j=0}^{\ell-1} \#\widetilde{\mathcal{M}}_j \leqslant \sum_{j=0}^{\ell-1} \#\mathcal{M}_j.$$

Here, the symbol $\lesssim$ suppresses the constant $C_{\rm mesh}$ from (2.15). Altogether, it thus only remains to verify (3.4).

This is done by induction on $\ell \in \mathbb{N}_0$: The case $\ell = 0$ follows by definition $\mathcal{T}_0 = \widetilde{\mathcal{T}}_0$. Now, suppose that (3.4) holds for $\mathcal{T}_\ell$ and $\widetilde{\mathcal{T}}_\ell$ and consider an arbitrary element $T \in \mathcal{T}_{\ell+1}$. We have to distinguish three cases:

Case 1: Let $T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}$. By the induction hypothesis, there is some $\mathcal{V} \subseteq \widetilde{\mathcal{T}}_\ell$ such that

$$T = \bigcup_{\widetilde{T} \in \mathcal{V}} \widetilde{T}.$$

For any $\widetilde{T} \in \mathcal{V}$, there holds either $\widetilde{T} \in \widetilde{\mathcal{T}}_{\ell+1}$ or $\widetilde{T} = \widetilde{T}' \cup \widetilde{T}''$ for some $\widetilde{T}', \widetilde{T}'' \in \widetilde{\mathcal{T}}_{\ell+1}$. Consequently, this implies

$$T = \bigcup_{\widetilde{T} \in \widetilde{\mathcal{V}}} \widetilde{T} \quad \text{with} \quad \widetilde{\mathcal{V}} := \big\{ \widetilde{T}' \in \widetilde{\mathcal{T}}_{\ell+1} : \text{exists } \widetilde{T} \in \mathcal{V} \text{ with } \widetilde{T}' \subseteq \widetilde{T} \big\}.$$

Case 2: Let $T \in \mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell$, fix the unique $\widehat{T} \in \mathcal{T}_\ell$ with $T \subsetneqq \widehat{T}$, and assume that $\widehat{T} \in \mathcal{T}_\ell \backslash \widetilde{\mathcal{T}}_\ell$. By the induction hypothesis, there is some $\mathcal{V} \subseteq \widetilde{\mathcal{T}}_\ell$ such that

$$\widehat{T} = \bigcup_{\widetilde{T} \in \mathcal{V}} \widetilde{T}.$$

Moreover, $\widehat{T} \in \mathcal{T}_\ell \backslash \widetilde{\mathcal{T}}_\ell$ implies $\mathcal{V} \subseteq \widetilde{\mathcal{T}}_{\ell+1}$. Now, recall that bisection leads to a binary refinement tree. Consequently, the two sons of $\widehat{T}$ have an analogous representation. In particular, this implies

$$T = \bigcup_{\widetilde{T} \in \widetilde{\mathcal{V}}} \widetilde{T} \quad \text{with} \quad \widetilde{\mathcal{V}} := \big\{ \widetilde{T} \in \mathcal{V} : \widetilde{T} \subseteq T \big\} \subseteq \widetilde{\mathcal{T}}_{\ell+1}.$$

Case 3: Let $T \in \mathcal{T}_{\ell+1} \backslash \mathcal{T}_\ell$, fix the unique $\widehat{T} \in \mathcal{T}_\ell$ with $T \subsetneqq \widehat{T}$, and assume that $\widehat{T} \in \mathcal{T}_\ell \cap \widetilde{\mathcal{T}}_\ell$. In particular, $\widehat{T}$ is refined by the κ-based mesh-refinement from Algorithm 2.2. We now aim to show that $\widehat{T}$ will be marked for refinement by the level-based mesh-refinement from Algorithm 3.3 as well. To that end, we again consider the three possible cases:

Case 3.1: We note that $\widehat{T} \in \mathcal{M}_\ell$ implies $\widehat{T} \in \widetilde{\mathcal{M}}_\ell$ due to $\widehat{T} \in \mathcal{T}_\ell \cap \widetilde{\mathcal{T}}_\ell$. Therefore, we obtain $T \in \widetilde{\mathcal{T}}_{\ell+1}$.

Case 3.2: Assume that $\widehat{T} \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell$ has a marked neighbor $\widehat{T}' \in \mathcal{M}_\ell$ which leads to the additional marking of $\widehat{T}$, i.e., $h_\ell|_{\widehat{T}} > \kappa(\mathcal{T}_0) h_\ell|_{\widehat{T}'}$. Let $\widehat{T}_0, \widehat{T}_0' \in \mathcal{T}_0$ be the – not necessarily distinct – unique elements with $\widehat{T} \subseteq \widehat{T}_0$ and $\widehat{T}' \subseteq \widehat{T}_0'$. By definition of $\kappa(\mathcal{T}_0)$, there holds $h_0|_{\widehat{T}_0} \leqslant \kappa(\mathcal{T}_0) h_0|_{\widehat{T}_0'}$. From the definition of the level-function, we infer

$$h_\ell|_{\widehat{T}} = 2^{-\,\mathtt{level}(\widehat{T})} h_0|_{\widehat{T}_0} \quad \text{and} \quad h_\ell|_{\widehat{T}'} = 2^{-\,\mathtt{level}(\widehat{T}')} h_0|_{\widehat{T}_0'}.$$

Combining these relations, we obtain

$$2^{-\,\mathtt{level}(\widehat{T})} h_0|_{\widehat{T}_0} = h_\ell|_{\widehat{T}} > \kappa(\mathcal{T}_0) h_\ell|_{\widehat{T}'} = \kappa(\mathcal{T}_0) 2^{-\,\mathtt{level}(\widehat{T}')} h_0|_{\widehat{T}_0'}$$

and end up with

$$\kappa(\mathcal{T}_0) \geqslant \frac{h_0|_{\widehat{T}_0}}{h_0|_{\widehat{T}_0'}} > \kappa(\mathcal{T}_0) 2^{\mathtt{level}(\widehat{T}) - \mathtt{level}(\widehat{T}')}$$

and hence $\texttt{level}(\widehat{T}') > \texttt{level}(\widehat{T})$. According to the induction hypothesis for $\widehat{T}' \in \mathcal{T}_\ell$ and the level-estimate (3.2), we infer that $\widehat{T}' \in \widetilde{\mathcal{T}}_\ell$. Consequently, $\widehat{T}' \in \mathcal{M}_\ell$ implies $\widehat{T}' \in \widetilde{\mathcal{M}}_\ell$ according to our first observation. Now, $\widehat{T}' \in \widetilde{\mathcal{M}}_\ell$ and $\texttt{level}(\widehat{T}') > \texttt{level}(\widehat{T})$ enforces refinement of $\widehat{T}$ by the level-based Algorithm 3.3. This and $\widehat{T} \in \widetilde{\mathcal{T}}_\ell$ imply $T \in \widetilde{\mathcal{T}}_{\ell+1}$.

Case 3.3: For any element $\widehat{T} \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell$ which is refined by Algorithm 2.2, we find a marked element $\widehat{T}^{(0)} \in \mathcal{M}_\ell$ and a chain of elements $\widehat{T}^{(1)}, \dots, \widehat{T}^{(i)} \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell$ such that

$$\kappa(\mathcal{T}_0)\, h_\ell|_{\widehat{T}^{(j-1)}} < h_\ell|_{\widehat{T}^{(j)}} \quad \text{for } j = 1, \dots, i \text{ and } \widehat{T}^{(i)} = \widehat{T}.$$

In particular, all these elements will be refined by call of Algorithm 2.2. Proceeding as in the previous case, we see that there holds $\widehat{T}^{(j)} \in \widetilde{\mathcal{T}}_\ell$ for all $j = 0, \dots, i$ as well as $\widehat{T}^{(0)} \in \widetilde{\mathcal{M}}_\ell$ and that all these elements will be refined by the level-based mesh-refinement as well. As above, we thus obtain $T \in \widetilde{\mathcal{T}}_{\ell+1}$. □

4. Proof of Theorem 2.4

A key ingredient for the proof of Theorem 2.4 is the following inverse estimate from [2, Theorem 1], which is also found in [22, Theorem 3.1] for the case of discrete functions $\psi \in \mathcal{P}^0(\mathcal{T}_\ell)$.

Lemma 4.1. *Let $\mathcal{T}_\ell \in \mathbb{T}$ denote a mesh with corresponding mesh-width function h_ℓ. Then, for $\psi \in L^2(\Gamma)$, it holds*

$$C_{\mathrm{V}}^{-1} \|h_\ell^{1/2} \tfrac{\partial}{\partial s} V\psi\|_{L^2(\Gamma)} \leqslant \|\psi\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}\psi\|_{L^2(\Gamma)}, \tag{4.1}$$

where the constant $C_{\mathrm{V}} > 0$ depends only on Γ and $\kappa(\mathcal{T}_\ell)$. □

We want to use the statement of Lemma 4.1 with $\psi := \phi - \Phi_\ell$, which gives us

$$\eta_\ell = \|h_\ell^{1/2} \tfrac{\partial}{\partial s} V(\phi - \Phi_\ell)\|_{L^2(\Gamma)} \lesssim \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}. \tag{4.2}$$

With the L^2-orthogonal projection $\Pi_\ell : L^2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_\ell)$, we see

$$\begin{aligned}
\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} &\leqslant \|h_\ell^{1/2}(1 - \Pi_\ell)\phi\|_{L^2(\Gamma)} + \|h_\ell^{1/2}(\Pi_\ell\phi - \Phi_\ell)\|_{L^2(\Gamma)} \\
&\lesssim \|h_\ell^{1/2}(1 - \Pi_\ell)\phi\|_{L^2(\Gamma)} + \|\Pi_\ell\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \\
&\leqslant \|h_\ell^{1/2}(1 - \Pi_\ell)\phi\|_{L^2(\Gamma)} + \|(1 - \Pi_\ell)\phi\|_{H^{-1/2}(\Gamma)} + \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} \\
&\lesssim \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}(1 - \Pi_\ell)\phi\|_{L^2(\Gamma)},
\end{aligned}$$

where we used the local approximation property of Π_ℓ (cf. [13, Theorem 4.1]) as well as the inverse estimate from [26, Theorem 3.6]. The main task now is to bound the last term of the preceding estimate appropriately and to absorb it on the left-hand side. To formulate the next statement, we define

$$\mathcal{T}_{\ell,k} := \mathrm{unif}^{(k)}(\mathcal{T}_\ell) \tag{4.3}$$

as the mesh which is generated by bisecting all elements $T \in \mathcal{T}_\ell$ k-times. Moreover, $\mathrm{unif}^{(k)}(T)$ denotes the set of sons $T_i \in \mathrm{unif}^{(k)}(\mathcal{T}_\ell)$, $i = 1, \dots, 2^k$ of $T \in \mathcal{T}_\ell$. Furthermore, for any $\nu > 0$ the broken Sobolev space is defined by

$$H^\nu(\mathcal{T}_\ell) := \big\{ v \in L^2(\Gamma) : v|_T \in H^\nu(T) \text{ for all } T \in \mathcal{T}_\ell \big\}.$$

Proposition 4.2. *Let the given boundary data satisfy $g \in H^{s_{\rm reg}}(\Gamma)$ for some $s_{\rm reg} > 2$. We consider a mesh $\mathcal{T}_\ell \in \mathbb{T}$. Then, the unique solution of (2.1) can be decomposed as $\phi = \phi_0 + \phi_{\rm sing}$. The smooth part satisfies $\phi_0 \in H^{\nu_{\rm reg}-1-\varepsilon}(\mathcal{T}_\ell)$ for all $\varepsilon > 0$, where $\nu_{\rm reg} := \min\{s_{\rm reg}, 5/2\}$. The singular part fulfills $\phi_{\rm sing} \in L^2(\Gamma)$. Moreover, it exists $h_0 > 0$ such that for all $\mathcal{T}_\ell$ with mesh-width $\|h_\ell\|_{L^\infty(\Gamma)} < h_0$ and for all $\kappa > 0$, there exists $k \in \mathbb{N}$ such that*

$$\|h_\ell^{1/2}(1-\Pi_{\ell,k})\phi\|_{L^2(\Gamma)} \leqslant \kappa \|h_\ell^{1/2}(1-\Pi_\ell)\phi\|_{L^2(\Gamma)} + C_3\|h_\ell^{1/2}(1-\Pi_\ell^{(1)})\phi_0\|_{L^2(\Gamma)}, \tag{4.4}$$

where $\Pi_{\ell,k} : L^2(\Gamma) \to \mathcal{P}^0(\mathrm{unif}^{(k)}(\mathcal{T}_\ell))$ and $\Pi_\ell^{(1)} : L^2(\Gamma) \to \mathcal{P}^1(\mathcal{T}_\ell)$ denote the respective L^2-orthogonal projections. The constants $C_3 > 0, h_0 > 0$, and $k \in \mathbb{N}$ depend only on Γ and $\kappa(\mathcal{T}_\ell)$. The function ϕ_0 depends on $\mathcal{T}_\ell$ and $s_{\rm reg} > 2$, but for all $\varepsilon > 0$ the elementwise norm is bounded uniformly, i.e.,

$$\sum_{T\in\mathcal{T}_\ell} \|\phi_0\|^2_{H^{\nu_{\rm reg}-1-\varepsilon}(T)} \leqslant C_{\rm hot} < \infty \tag{4.5}$$

and $C_{\rm hot} > 0$ depends only on Γ, $\kappa(\mathcal{T}_\ell)$, $s_{\rm reg} > 2$, and $\varepsilon > 0$.

The proof of this proposition needs several preliminary lemmata and the definition of the space of singularity functions: Let $\beta_j \in (-1/2, 2]$, $j = 1, \dots, m$, with $\beta_j \neq \beta_i$ for $i \neq j$. Then, for an interval $T \subseteq \mathbb{R}$,

$$\begin{aligned}\mathcal{H}_{\rm sing}(T, (\beta_j)_{j=1}^m) := \mathrm{span}\big(&\{s \mapsto s^{\beta_j} : j = 1, \dots, m\}\\ &\cup \{s \mapsto s^{\beta_j}\log(s) : j = 1, \dots, m\}\big) \oplus \mathcal{P}^1(T)\end{aligned} \tag{4.6}$$

is called the singularity space for $(\beta_j)_{j=1}^m$.

Lemma 4.3. *Assume $h > 0$ and $r \geqslant h/\nu$ for some $\nu > 0$. Let $x_0, s_1, s_2 \in [r, r+h]$. For $|s_2 - x_0| \geqslant h/4$ and $\beta \in (-1/2, 2]$, it holds*

$$\frac{|\int_{s_2}^{s_1} t^{\beta-2}dt|}{|\int_{x_0}^{s_2} t^{\beta-2}dt|} \leqslant C_4, \tag{4.7}$$

where $C_4 = 4/(1+\nu)^{\beta-2} > 0$.

Proof. Because $\beta - 2 \leqslant 0$, we may estimate

$$\max_{t\in[r,r+h]} t^{\beta-2} = r^{\beta-2} \quad \text{and} \quad \min_{t\in[r,r+h]} t^{\beta-2} = (r+h)^{\beta-2} \geqslant (1+\nu)^{\beta-2} r^{\beta-2}.$$

With $|s_2 - x_0| \geqslant h/4$, we see

$$\frac{|\int_{s_2}^{s_1} t^{\beta-2}dt|}{|\int_{x_0}^{s_2} t^{\beta-2}dt|} \leqslant \frac{hr^{\beta-2}}{h/4(1+\nu)^{\beta-2}r^{\beta-2}} \leqslant \frac{4}{(1+\nu)^{\beta-2}}.$$

This concludes the proof. □

In the following, we will write $(\cdot)' = \frac{\partial}{\partial s}$ to abbreviate the notation.

Lemma 4.4. *Assume $h > 0$ and $r \geqslant h/\nu$ for some $\nu > 0$. Consider the interval $T := [r, r+h]$ and $\psi \in \mathcal{H}_{\rm sing}(T, (\beta_j)_{j=1}^m)$. Then, there exists $r_0 > 0$ such that, for $r < r_0$,*

$$\max_T |\psi'| \leqslant C_5 \min_{T'} |\psi'|, \tag{4.8}$$

where $T' := [r, r + h/4]$ or $T' := [r + 3h/4, r + h]$. The constants $C_5 > 0$ and $r_0 > 0$ depend only on $\nu > 0$ and $(\beta_j)_{j=1}^m \in (-1/2, 2]$.

Proof. The function ψ can be written as

$$\psi(s) = \sum_{j=1}^m a_j s^{\beta_j} + b_j s^{\beta_j} \log(s) + a_T, \tag{4.9}$$

with $a_T \in \mathcal{P}^1(T)$ and $a_j, b_j \in \mathbb{R}$. First, we observe that $(s \mapsto s^1) \in \mathcal{P}^1(T)$. Therefore, we assume $a_j = 0$ for $\beta_j = 1$. Note that the statement (4.8) is trivial if ψ' is constant. Due to the last observation, this happens only if all coefficients a_j, b_j are zero. Therefore, we may additionally assume that at least one coefficient a_j or b_j is non-zero. Let $|\psi'(x_0)| = \min_{x \in T} |\psi'(x)|$ for $x_0 \in T$. We use the minimality of $|\psi'(x_0)|$ to show that, for all $s \in T$, either one of the terms $\psi'(x_0)$ and $\int_{x_0}^s \psi''(t)dt$ is zero or that both terms must have the same sign. We argue by contradiction and assume $\psi'(x_0) \int_{x_0}^s \psi''(t)dt < 0$, i.e., both terms have opposite sign for some $s \in [r, r+h]$. We choose $x_1 \in [r, r+h]$ such that

$$\Big| \int_{x_0}^{x_1} \psi''(t)dt \Big| < |\psi'(x_0)| \quad \text{and} \quad \psi'(x_0) \int_{x_0}^{x_1} \psi''(t)dt < 0. \tag{4.10}$$

This is possible because $\int_{x_0}^{x_1} \psi''(t)dt \to 0$ for $x_1 \to x_0$. With (4.10), we obtain

$$|\psi'(x_1)| = \Big| \psi'(x_0) + \int_{x_0}^{x_1} \psi''(t)dt \Big| = |\psi'(x_0)| - \Big| \int_{x_0}^{x_1} \psi''(t)dt \Big| < |\psi'(x_0)|, \tag{4.11}$$

which is a contradiction to the minimality of $|\psi'(x_0)|$. We just proved

$$\psi'(x_0) \int_{x_0}^{s} \psi''(t)dt \geqslant 0 \quad \text{for all } s \in T,$$

i.e., both terms have the same sign or at least one of them is zero. With this result, we may write

$$|\psi'(s)| = \Big| \psi'(x_0) + \int_{x_0}^{s} \psi''(t)dt \Big| = |\psi'(x_0)| + \Big| \int_{x_0}^{s} \psi''(t)dt \Big| \tag{4.12}$$

for all $s \in T$.

Now, we fix the index j_0 with the smallest exponent $\beta_{j_0} \in (-1/2, 2]$ and $a_{j_0} \neq 0$ or $b_{j_0} \neq 0$ in (4.9). Note that we can explicitly compute ψ'', i.e.,

$$\psi''(s) = \sum_{j=1}^m a_j \beta_j (\beta_j - 1) s^{\beta_j - 2} + b_j s^{\beta_j - 2} \big(\beta_j (\beta_j - 1) \log(s) + 2\beta_j - 1 \big).$$

Now, we have to distinguish two cases:

Case 1: It holds that $b_{j_0} = 0$. Due to our assumptions, we have $\beta_{j_0} \neq 1$, since $a_{j_0} \neq 0$. Then, we choose $r_0 < 1/(1+\nu)$ sufficiently small such that for all $0 < s < r_0(1+\nu)$ holds

$$0 < \tfrac{1}{2} |a_{j_0} \beta_{j_0} (\beta_{j_0} - 1) s^{\beta_{j_0} - 2}| \leqslant |\psi(s)''| \leqslant 2 |a_{j_0} \beta_{j_0} (\beta_{j_0} - 1) s^{\beta_{j_0} - 2}|, \tag{4.13}$$

which is possible because $\beta_j - 2 \leqslant 0$ and the term with the smallest exponent dominates the function ψ''.

Case 2: It holds that $b_{j_0} \neq 0$. If $\beta_{j_0} \neq 1$, we choose $r_0 < 1/(1+\nu)$ sufficiently small such that for all $0 < s < r_0(1+\nu)$ holds

$$0 < \tfrac{1}{2}|b_{j_0}\beta_{j_0}(\beta_{j_0}-1)\log(s)s^{\beta_{j_0}-2}| \leqslant |\psi(s)''| \leqslant 2|b_{j_0}\beta_{j_0}(\beta_{j_0}-1)\log(s)s^{\beta_{j_0}-2}|, \tag{4.14}$$

which is possible because $\beta_j - 2 \leqslant 0$ and the term with the smallest exponent dominates the function ψ''. If $\beta_{j_0} = 1$, the log-term vanishes and we get

$$0 < \tfrac{1}{2}|b_{j_0}s^{-1}| \leqslant |\psi(s)''| \leqslant 2|b_{j_0}s^{-1}|, \tag{4.15}$$

i.e., case 1 with different constants. All arguments for case 1 in the proof below work analogously for this case.

In either case, we see that for $r < r_0$, we get $r + h \leqslant r(1+\nu) \leqslant r_0(1+\nu)$. Therefore, $s \in T$ satisfies $s \leqslant r_0(1+\nu)$, and we get with (4.13)–(4.14) that ψ'' has no zero on T. Using this and (4.12), we get for $s_1, s_2 \in T$, $s_2 \neq x_0$,

$$\begin{aligned}\frac{|\psi'(s_1)|}{|\psi'(s_2)|} &= \frac{|\psi'(x_0)| + |\int_{x_0}^{s_1}\psi''(t)dt|}{|\psi'(x_0)| + |\int_{x_0}^{s_2}\psi''(t)dt|} \leqslant \frac{|\psi'(x_0)| + |\int_{x_0}^{s_2}\psi''(t)dt| + |\int_{s_2}^{s_1}\psi''(t)dt|}{|\psi'(x_0)| + |\int_{x_0}^{s_2}\psi''(t)dt|} \\ &\leqslant 1 + \frac{|\int_{s_2}^{s_1}\psi''dt|}{|\psi'(x_0)| + |\int_{x_0}^{s_2}\psi''dt|} \leqslant 1 + \frac{|\int_{s_2}^{s_1}\psi''dt|}{|\int_{x_0}^{s_2}\psi''dt|}.\end{aligned} \tag{4.16}$$

Again we use (4.13) and (4.14), to estimate

$$\frac{|\psi'(s_1)|}{|\psi'(s_2)|} \leqslant 1 + \frac{2|a_{j_0}\beta_{j_0}(\beta_{j_0}-1)\int_{s_2}^{s_1}t^{\beta_{j_0}-2}dt|}{\frac{1}{2}|a_{j_0}\beta_{j_0}(\beta_{j_0}-1)\int_{x_0}^{s_2}t^{\beta_{j_0}-2}dt|} = 1 + 4\frac{|\int_{s_2}^{s_1}t^{\beta_{j_0}-2}dt|}{|\int_{x_0}^{s_2}t^{\beta_{j_0}-2}dt|}, \tag{4.17}$$

for case 1, and by use of $r + h \leqslant r_0(1+\nu) < 1$,

$$\frac{|\psi'(s_1)|}{|\psi'(s_2)|} \leqslant 1 + \frac{2|b_{j_0}\beta_{j_0}(1-\beta_{j_0})\int_{s_2}^{s_1}t^{\beta_{j_0}-2}\log(t)|dt}{\frac{1}{2}|b_{j_0}\beta_{j_0}(1-\beta_{j_0})\int_{x_0}^{s_2}t^{\beta_{j_0}-2}\log(t)dt|} \leqslant 1 + 4\frac{|\int_{s_2}^{s_1}t^{\beta_j-2}dt|}{|\int_{x_0}^{s_2}t^{\beta_j-2}dt|}\frac{|\log(r)|}{|\log(r+h)|} \tag{4.18}$$

for case 2. If we restrict ourselves to $|s_2 - x_0| \geqslant h/4$, all assumptions of Lemma 4.3 are satisfied, and we get for case 1

$$\frac{|\psi'(s_1)|}{|\psi'(s_2)|} \leqslant 1 + 4C_4 \tag{4.19}$$

by help of equation (4.17). For case 2, we additionally have to bound $|\log(r)/\log(r+h)| \leqslant C_6$ in (4.18) by

$$C_6 = \sup_{0<r<r_0}\frac{|\log(r)|}{|\log(r+h)|} \leqslant \sup_{0<r<r_0}\frac{|\log(r)|}{|\log(r) + \log(1+\nu)|} < \infty,$$

where we used $r + h \leqslant r(1+\nu) \leqslant r_0(1+\nu) < 1$. With the definition

$$T' := \begin{cases}[r, r+h/4] & \text{for } x_0 \in [r+h/2, r+h], \\ [r+3h/4, r+h] & \text{for } x_0 \in [r, r+h/2)\end{cases}$$

and $s_2 \in T'$, we ensure $|s_2 - x_0| \geqslant h/4$. Plugging everything together, we use (4.19) to prove the statement (4.8). □

Lemma 4.5. *Assume $h > 0$ and $r \geqslant h/\nu$ for some $\nu > 0$. Consider the interval $T := [r, r+h]$ and $\psi \in \mathcal{H}_{\rm sing}(T, (\beta_j)_{j=1}^m)$. Then, there exists $r_0 > 0$ such that for $r < r_0$, it holds*

$$\|(1-\Pi_k)\psi\|_{L^2(T)}^2 \leqslant C_7 2^{-2k} \|(1-\Pi)\psi\|_{L^2(T)}^2 \quad \text{for all } k \in \mathbb{N}, \tag{4.20}$$

where the constants $C_7 > 0$ and $r_0 > 0$ depend only on $(\beta_j)_{j=1}^m \in (-1/2, 2]$ and $\nu > 0$. Here, $\Pi_k : L^2(T) \to \mathcal{P}^0(\mathrm{unif}^{(k)}(T))$ and $\Pi : L^2(T) \to \mathcal{P}^0(T)$ denote the L^2-orthogonal projections.

Proof. The statement is trivial for constant ψ, i.e., we may assume $\psi'(s) \neq 0$ for at least one $s \in T$. For $r < r_0$, Lemma 4.4 proves

$$0 < \max_T |\psi'| \leqslant C_5 \min_{T'} |\psi'|. \tag{4.21}$$

Next, we use that $(1-\Pi_k)\psi$ has a zero s_{T_i} on each $T_i \in \mathrm{unif}^{(k)}(T), i = 1, \ldots, 2^k$. Therefore

$$|(1-\Pi_k)\psi(s)| = \Big|\int_{s_{T_i}}^s ((1-\Pi_k)\psi)' dt\Big| = \Big|\int_{s_{T_i}}^s \psi' dt\Big| \leqslant h_{T_i} \max_{T_i} |\psi'|$$

for $s \in T_i$. With (4.21) and $h_{T_i} = h_{T_1} = 2^{-k}h$, we conclude

$$\begin{aligned}\|(1-\Pi_k)\psi\|_{L^2(T)}^2 &\lesssim \sum_{i=1}^{2^k} h_{T_i}^3 \max_{T_i} |\psi'|^2 \leqslant h_{T_1}^3 2^k \max_T |\psi'|^2 \lesssim h_{T_1}^3 2^k \min_{T'} |\psi'|^2 \\ &= 2^{-2k} h^3 \min_{T'} |\psi'|^2. \end{aligned} \tag{4.22}$$

Now, we calculate for $s_0, s \in T'$

$$h^3 \min_{T'} |\psi'| \simeq \int_{T'} \Big(\int_{s_0}^s \min_{T'} |\psi'| dt\Big)^2 ds \leqslant \int_{T'} \Big|\int_{s_0}^s \psi' dt\Big|^2 ds, \tag{4.23}$$

where we used $4|T'| = |T| = h$ and the fact that ψ' doesn't change sign on T' because of (4.21). To bound the last term in the estimate above, we introduce the L^2-orthogonal projection $\Pi' : L^2(T') \to \mathcal{P}^0(T')$. Let $s_0 \in T'$ denote the zero of $(1-\Pi')\psi$ and note that $((1-\Pi')\psi)' = \psi'$ on T'. With this and the estimates (4.22) and (4.23), we end up with

$$\begin{aligned}\|(1-\Pi_k)\psi\|_{L^2(T)}^2 &\lesssim 2^{-2k} \int_{T'} \Big|\int_{s_0}^s ((1-\Pi')\psi)' dt\Big|^2 ds = 2^{-2k} \|(1-\Pi')\psi\|_{L^2(T')}^2 \\ &\leqslant 2^{-2k} \|(1-\Pi)\psi\|_{L^2(T')}^2 \leqslant 2^{-2k} \|(1-\Pi)\psi\|_{L^2(T)}^2, \end{aligned} \tag{4.24}$$

due to the best-approximation property of Π' on T'. This proves the assertion. □

Lemma 4.6. *Assume $h > 0$. Consider the interval $T := [0, h]$ and $\psi \in \mathcal{H}_{\rm sing}(T, (\beta_j)_{j=1}^m)$. Then, there holds*

$$\|(1-\Pi_k)\psi\|_{L^2(T)}^2 \leqslant C_8 2^{-2\varepsilon k} \|(1-\Pi)\psi\|_{L^2(T)}^2 \quad \text{for all } k \in \mathbb{N}, \tag{4.25}$$

where the constant $C_8 > 0$ and $\varepsilon > 0$ depend only on $(\beta_j)_{j=1}^m \in (-1/2, 2]$. Here, $\Pi_k : L^2(T) \to \mathcal{P}^0(\mathrm{unif}^{(k)}(T))$ and $\Pi : L^2(T) \to \mathcal{P}^0(T)$ denote the L^2-orthogonal projections.

Proof. For $\varepsilon = (\min_{j=1,\dots,m}\beta_j + 1/2)/2$, we consider $\mu \in \mathcal{H}_{\rm sing}(T,(\beta_j)_{j=1}^m) \subset H^\varepsilon(T)$. We define the fractional Sobolev norms by interpolation. Recall that all definitions of the fractional Sobolev norms are equivalent on the whole space $H^\varepsilon(\Gamma)$. But as the constants depend on the domain, we get some elementwise properties like the Poincaré inequality (cf. [8])

$$\|(1-\Pi)v\|_{L^2(T)} \lesssim \|h^\varepsilon v\|_{H^\varepsilon(T)} \quad \text{for all } v \in H^\varepsilon(T)$$

more easily if we choose the definition by interpolation. Let $\widehat{\mu}(s) := \mu(hs)$. First we prove that $\widehat{\mu}$ belongs to a finite dimensional space:

$$\widehat{\mu}(s) = \sum_{j=1}^m a_j h^{\beta_j} s^{\beta_j} + b_j\big(h^{\beta_j}s^{\beta_j}\log(s) + h^{\beta_j}\log(h)s^{\beta_j}\big) + a_T(hs) \in \mathcal{H}_{\rm sing}([0,1],(\beta_j)_{j=1}^m),$$

where $\dim\mathcal{H}_{\rm sing}([0,1],(\beta_j)_{j=1}^m) \leqslant 2m+2$. With this and standard scaling arguments, one obtains

$$\|\mu\|^2_{H^\varepsilon(T)} \lesssim h^{1-2\varepsilon}\|\widehat{\mu}\|^2_{H^\varepsilon([0,1])} \lesssim h^{1-2\varepsilon}\|\widehat{\mu}\|^2_{L^2([0,1])} \lesssim h^{-2\varepsilon}\|\mu\|^2_{L^2(T)}, \tag{4.26}$$

where the second estimate holds because of norm equivalence on finite dimensional spaces. By use of (4.26) with $\mu = (1-\Pi)\psi$, we conclude

$$\begin{aligned}\|(1-\Pi_k)\psi\|^2_{L^2(T)} &= \sum_{i=1}^{2^k}\|(1-\Pi_k)(1-\Pi)\psi\|^2_{L^2(T_i)} \lesssim h_{T_1}^{2\varepsilon}\sum_{i=1}^{2^k}\|(1-\Pi)\psi\|^2_{H^\varepsilon(T_i)}\\ &\lesssim h_{T_1}^{2\varepsilon}\|(1-\Pi)\psi\|^2_{H^\varepsilon(T)} \lesssim (h_{T_1}/h)^{2\varepsilon}\|(1-\Pi)\psi\|^2_{L^2(T)}\\ &\lesssim 2^{-2\varepsilon k}\|(1-\Pi)\psi\|^2_{L^2(T)},\end{aligned} \tag{4.27}$$

where we used the Poincaré inequality for fractional Sobolev norms and the fact that $\sum_{i=1}^{2^k}\|w\|^2_{H^\varepsilon(T_i)} \lesssim \|w\|^2_{H^\varepsilon(T)}$ for all $w \in H^\varepsilon(T)$ (see [8]). □

Now, we are ready to prove Proposition 4.2.

Proof of Proposition 4.2. According to [17, Theorem 4.8], the solution ϕ has the form

$$\phi(x) = \widetilde{\phi}_0(x) + \phi_{\rm sing} := \widetilde{\phi}_0(x) + \sum_{j=1}^m \chi_j(x)\phi_j(|x-c_j|) \quad \text{for all } x \in \Gamma, \tag{4.28}$$

where $m \in \mathbb{N}$ is the number of corners c_j of Γ and $\widetilde{\phi}_0 \in H^{\nu_{\rm reg}-1-\varepsilon}(\mathcal{T}_0)$ for all $\varepsilon > 0$. The singularity functions ϕ_j satisfy

$$\phi_j(s) = \sum_{i=1}^M a_{i,j}s^{\beta_{i,j}} + b_{i,j}s^{\beta_{i,j}}\log(s) \in \mathcal{H}_{\rm sing}\big([0,\infty],\big((\beta_{i,j})_{i=1}^M\big)_{j=1}^m\big), \tag{4.29}$$

where the exponents $\beta_{i,j} > -1/2$ are determined by the inner angle α_j in c_j through $\beta_{i,j}+1 = k_i\pi/\alpha_j$ for some non-negative integer $k_i \in \mathbb{N}$. Moreover χ_j is a smooth cutoff function with $c_i \notin \operatorname{supp}(\chi_j)$ for all $i \neq j$. For each χ_j, it exists a neighborhood $U_j \subset \Gamma$ of c_j such that $\chi_j \equiv 1$ in U_j. We choose $h_0 > 0$ sufficiently small so that the ball $B_{h_0}(c_j) \cap \Gamma \subset U_j$ for all $j = 1,\dots,m$. Additionally, we observe that for $\beta_{i,j} > 2$ the corresponding term in (4.29) is smoother than $\widetilde{\phi}_0$. Thus, it is sufficient to consider $\beta_{i,j} \in (-1/2, 2]$. Of course, we want to exploit Lemma 4.5 and Lemma 4.6. We prove estimate (4.4) elementwise, i.e.,

$$\|h_\ell^{1/2}(1-\Pi_{\ell,k})\phi\|_{L^2(\Gamma)}^2 = \sum_{T\in\mathcal{T}_\ell} h_\ell|_T \|(1-\Pi_{\ell,k})\phi\|_{L^2(T)}^2. \tag{4.30}$$

By use of an affine transformation, we can treat each element that appears in the sum as an interval on the real axis, i.e., we identify the corner c_j with zero and $T = [r, r+h]$ for some $r \geqslant 0$, $h = h_\ell|_T$. If $r > 0$, there exists at least one element T' with $T' \cap T \neq \emptyset$, which is located between the corner c_j and T. Mesh regularity thus gives

$$r \geqslant h_\ell|_{T'} \geqslant \frac{h_\ell|_T}{\kappa(\mathcal{T}_\ell)}. \tag{4.31}$$

Now, we consider equation (4.30) and distinguish three cases:

(i) If $T = [0, h]$, the assumption on the mesh-width shows $h < h_0$ and therefore

$$\|(1-\Pi_{\ell,k})\phi\|_{L^2(T)}^2 \lesssim \|(1-\Pi_{\ell,k})(\phi_j + a_T)\|_{L^2(T)}^2 + \|\widetilde{\phi}_0 - a_T\|_{L^2(T)}^2.$$

We choose $a_T = (\Pi_\ell^{(1)}\widetilde{\phi}_0)|_T \in \mathcal{P}^1(T)$ and apply Lemma 4.6 to estimate the first term

$$\begin{aligned}\|(1-\Pi_{\ell,k})\phi\|_{L^2(T)}^2 &\lesssim 2^{-2\varepsilon k}\|(1-\Pi_\ell)(\phi_j + a_T)\|_{L^2(T)}^2 + \|(1-\Pi_\ell^{(1)})\widetilde{\phi}_0\|_{L^2(T)}^2 \\ &\lesssim 2^{-2\varepsilon k}\|(1-\Pi_\ell)\phi\|_{L^2(T)}^2 + \|(1-\Pi_\ell^{(1)})\widetilde{\phi}_0\|_{L^2(T)}^2. \end{aligned} \tag{4.32}$$

(ii) If $T = [r, r+h]$ with $r + h < h_0$ and additionally $r < r_0$ with the constant $r_0 > 0$ from Lemma 4.5, we obtain

$$\begin{aligned}\|(1-\Pi_{\ell,k})\phi\|_{L^2(T)}^2 &\lesssim 2^{-2k}\|(1-\Pi_\ell)(\phi_j + a_T)\|_{L^2(T)}^2 + \|(1-\Pi_\ell^{(1)})\widetilde{\phi}_0\|_{L^2(T)}^2 \\ &\lesssim 2^{-2k}\|(1-\Pi_\ell)\phi\|_{L^2(T)}^2 + \|(1-\Pi_\ell^{(1)})\widetilde{\phi}_0\|_{L^2(T)}^2 \end{aligned} \tag{4.33}$$

by use of Lemma 4.5.

(iii) If $T = [r, r+h]$ with $r \geqslant r_0$ or $r + h \geqslant h_0$, we obtain by use of mesh regularity $r\kappa(\mathcal{T}_\ell) \geqslant h$ that $r \geqslant \min\{r_0, h_0/(1+\kappa(\mathcal{T}_\ell))\} > 0$. Therefore, $\phi_{\rm sing}|_T$ is smooth and $\phi|_T \in H^{\nu_{\rm reg}-1-\varepsilon}(T)$. We apply Lemma 4.5 with $\psi = a_T := (\Pi_\ell^{(1)}\phi)|_T$ to see

$$\begin{aligned}\|(1-\Pi_{\ell,k})\phi\|_{L^2(T)}^2 &\lesssim 2^{-2k}\|(1-\Pi_\ell)a_T\|_{L^2(T)}^2 + \|(1-\Pi_\ell^{(1)})\phi\|_{L^2(T)}^2 \\ &\lesssim 2^{-2k}\|(1-\Pi_\ell)\phi\|_{L^2(T)}^2 + \|(1-\Pi_\ell^{(1)})\phi\|_{L^2(T)}^2. \end{aligned} \tag{4.34}$$

Finally, we define ϕ_0 elementwise by

$$\phi_0|_T := \begin{cases} \widetilde{\phi}_0|_T & \text{for cases (i) and (ii),} \\ \phi|_T & \text{for case (iii),} \end{cases}$$

and obtain $\phi_0 \in H^{\nu_{\rm reg}-1-\varepsilon}(\mathcal{T}_\ell)$ for all $\varepsilon > 0$. Choosing $k \in \mathbb{N}$ sufficiently large in the estimates above, we insert (4.32)–(4.34) in (4.30) to prove the assertion. □

With this result, we may prove the first estimate (2.16) of Theorem 2.4.

Proof of Theorem 2.4. Due to (4.2), it remains to estimate the term $\|h_\ell^{1/2}(\phi-\Phi_\ell)\|_{L^2(\Gamma)}$. Let $\Pi_{\ell,k} : L^2(\Gamma) \to \mathcal{P}^0(\mathrm{unif}^{(k)}(\mathcal{T}_\ell))$ denote the L^2-orthogonal projection. First, note that due to the approximation properties of $\Pi_{\ell,k}$ (cf. [13, Theorem 4.1]), it holds

$$\begin{aligned}\|\Pi_{\ell,k}\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} &\leqslant \|(1-\Pi_{\ell,k})\phi\|_{H^{-1/2}(\Gamma)} + \|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)}\\ &\leqslant C\|h_{\ell,k}^{1/2}(1-\Pi_{\ell,k})\phi\|_{L^2(\Gamma)} + \|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)}\\ &= C2^{-k/2}\|h_\ell^{1/2}(1-\Pi_{\ell,k})\phi\|_{L^2(\Gamma)} + \|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)} \qquad (4.35)\end{aligned}$$

for all $k \in \mathbb{N}$. Here, the constant $C > 0$ stems from the inverse estimate in [26, Theorem 3.6] and is independent of $\ell, k \in \mathbb{N}$. Consequently, we may estimate

$$\begin{aligned}\|h_\ell^{1/2}(\phi-\Phi_\ell)\|_{L^2(\Gamma)} &\leqslant \|h_\ell^{1/2}(\phi-\Pi_{\ell,k_1}\phi)\|_{L^2(\Gamma)} + \|h_\ell^{1/2}(\Pi_{\ell,k_1}\phi-\Phi_\ell)\|_{L^2(\Gamma)}\\ &\leqslant \|h_\ell^{1/2}(\phi-\Pi_{\ell,k_1}\phi)\|_{L^2(\Gamma)} + C\|h_\ell/h_{\ell,k_1}\|_{L^\infty(\Gamma)}^{1/2}\|\Pi_{\ell,k_1}\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)}\\ &\leqslant (1+C)\|h_\ell^{1/2}(\phi-\Pi_{\ell,k_1}\phi)\|_{L^2(\Gamma)} + C2^{k_1/2}\|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)}, \qquad (4.36)\end{aligned}$$

where $C > 0$ again stems from the inverse estimate in [26, Theorem 3.6]. With $h_0 > 0$ from Proposition 4.2, we choose k_1 sufficiently large such that $\|h_{\ell,k_1}\|_{L^\infty(\Gamma)} < h_0$ for all $\ell \in \mathbb{N}$. For $k_2 \in \mathbb{N}$, we get

$$\begin{aligned}\|h_\ell^{1/2}(1-\Pi_{\ell,k_1})\phi\|_{L^2(\Gamma)} &\leqslant \|h_\ell^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} + \|h_\ell^{1/2}(1-\Pi_{\ell,k_1})\Pi_{\ell,k_1+k_2}\phi\|_{L^2(\Gamma)}\\ &\leqslant \|h_\ell^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} + \|h_\ell^{1/2}\Pi_{\ell,k_1+k_2}(\phi-\Phi_\ell)\|_{L^2(\Gamma)}\\ &\leqslant \|h_\ell^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} + C2^{(k_1+k_2)/2}\|\Pi_{\ell,k_1+k_2}(\phi-\Phi_\ell)\|_{H^{-1/2}(\Gamma)}\\ &\leqslant (1+C)\|h_\ell^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} + C2^{(k_1+k_2)/2}\|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)}, \qquad (4.37)\end{aligned}$$

where we applied the inverse inequality from [26, Theorem 3.6] as well as (4.35). Given $\kappa > 0$, Proposition 4.2 now provides $k_2 \in \mathbb{N}$ such that

$$\|h_{\ell,k_1}^{1/2}(1-\Pi_{\ell,k_1+k_2})\phi\|_{L^2(\Gamma)} \leqslant \kappa\|h_{\ell,k_1}^{1/2}(1-\Pi_{\ell,k_1})\phi\|_{L^2(\Gamma)} + C_3\|h_{\ell,k_1}^{1/2}(1-\Pi_{\ell,k_1}^{(1)})\phi_0\|_{L^2(\Gamma)}. \quad (4.38)$$

Plugging (4.38) into (4.37) and rearranging the terms, we get

$$(1-(1+C)\kappa)\|h_\ell^{1/2}(1-\Pi_{\ell,k_1})\phi\|_{L^2(\Gamma)} \lesssim \|\phi-\Phi_\ell\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}(1-\Pi_{\ell,k_1}^{(1)})\phi_0\|_{L^2(\Gamma)}.$$

For $\kappa > 0$ sufficiently small, combine the estimate above with (4.36) to prove the assertion. Note that $\kappa > 0$ determines $k_2 \in \mathbb{N}$ as well as h_0 determines $k_1 \in \mathbb{N}$. Therefore, the hidden constants in the estimate above are fixed uniformly. □

Definition 4.7. Let the given boundary data satisfy $g \in H^{s_{\mathrm{reg}}}(\Gamma)$ for some $s_{\mathrm{reg}} > 2$. With $\nu_{\mathrm{reg}} := \min\{s_{\mathrm{reg}}, 5/2\}$, we define the higher-order term hot_ℓ by

$$\mathrm{hot}_\ell := \|h_\ell^{1/2}(1-\Pi_{\ell,k}^{(1)})\phi_0\|_{L^2(\Gamma)} \quad \text{with} \quad \mathrm{hot}_\ell(T) := \|h_\ell^{1/2}(1-\Pi_{\ell,k}^{(1)})\phi_0\|_{L^2(T)}$$

for all $T \in \mathcal{T}_\ell$. Here, $k = k_1 \in \mathbb{N}$ as in the proof of Theorem 2.4 depends only on Γ. As stated in Proposition 4.2, the function $\phi_0 \in H^{\nu_{\mathrm{reg}}-1-\varepsilon}(\mathcal{T}_{\ell,k})$ for all $\varepsilon > 0$ depends on $\mathcal{T}_\ell$ and $s_{\mathrm{reg}} > 2$, but the piecewise norm is uniformly bounded, i.e.,

$$\sum_{T\in\mathcal{T}_{\ell,k}} \|\phi_0\|_{H^{\nu_{\mathrm{reg}}-1-\varepsilon}(T)}^2 \leqslant C_{\mathrm{hot}} < \infty, \qquad (4.39)$$

where $C_{\text{hot}} > 0$ depends only on Γ, $\kappa(\mathcal{T}_\ell)$, $s_{\text{reg}} > 2$, and $\varepsilon > 0$. Therefore, the Poincaré inequality for fractional Sobolev norms yields

$$\text{hot}_\ell(T)^2 = \sum_{T_i \in \text{unif}^{(k)}(T)} \|h_\ell^{1/2}(1 - \Pi_{\ell,k}^{(1)})\phi_0\|_{L^2(T_i)}^2 \lesssim (h_\ell|_T)^{2(\nu_{\text{reg}}-1/2-\varepsilon)} C_{\text{hot}}^2, \tag{4.40}$$

for all $\varepsilon > 0$. Note that $\nu_{\text{reg}} - 1/2 - \varepsilon > 3/2$ for $\varepsilon > 0$ sufficiently small. Considering the generic rate of convergence $\mathcal{O}(h^{3/2})$ of lowest-order BEM for Symm's integral equation and smooth solutions (cf. [29, Theorem 4.1.54]), we confirm that hot_ℓ is indeed a term of higher order.

5. Proof of Theorem 2.5

Proof of Theorem 2.5, equations (2.20)–(2.21). We argue analogously to the proofs of [22, Theorem 4.1] (convergence result (2.20)) and [22, Theorem 5.1] (quasi-optimality result (2.21)). Therein, the 3D-based proofs rely on the uniform shape regularity of the meshes $\mathcal{T}_\ell$ generated by newest vertex bisection (NVB) as well as the fact that NVB satisfies the properties (2.14) and (2.15). In the present situation, the uniform shape regularity corresponds to uniform boundedness of the K-mesh constant. The necessary optimality properties (2.13)–(2.15) are provided by Theorem 2.3. Finally, we stress that in [22] the approximation class $\mathbb{A}_s^\eta$ is characterized by the total error $|||\phi - \Phi_\ell|||^2 + \text{osc}_\ell^2$, where $\text{osc}_\ell := \|h_\ell^{1/2}\nabla_\Gamma(1 - J_\ell)(f - V\Phi_\ell)\|_{L^2(\Gamma)}$ denote the so-called data oscillations which include the Scott–Zhang projection $J_\ell : L^2(\Gamma) \to \mathcal{S}^1(\mathcal{T}_\ell)$. However, according to [22, Proposition 5.2], the total error is equivalent to the error estimator η_ℓ, so that our definition of $\mathbb{A}_s^\eta$ is in fact, equivalent. Now, the results of [22] hold accordingly. □

Everything what remains to do, is to characterize the approximation class $\mathbb{A}_s$ in terms of the Galerkin error.

Proposition 5.1. *Let the given boundary data satisfy $g \in H^{s_{\text{reg}}}(\Gamma)$ for some $s_{\text{reg}} > 2$. Then, it holds equivalency*

$$(\phi, g) \in \mathbb{A}_s^\eta \quad \Longleftrightarrow \quad \phi \in \mathbb{A}_s$$

for all $0 < s < \min\{s_{\text{reg}}, 5/2\} - 1/2$.

Proof. First, we assume $(\phi, g) \in \mathbb{A}_s^\eta$. Then the reliability estimate (2.10) proves

$$\|\phi\|_{\mathbb{A}_s} \leqslant C_{\text{rel}} \|(\phi, g)\|_{\mathbb{A}_s^\eta} < \infty,$$

i.e., $\phi \in \mathbb{A}_s$.

Second, we assume $\phi \in \mathbb{A}_s$ for some $0 < s < \min\{s_{\text{reg}}, 5/2\} - 1/2$. The definition of the approximation class $\mathbb{A}_s$ guarantees a mesh $\mathcal{T}_{N/2} \in \mathbb{T}$ with

$$\#\mathcal{T}_{N/2} - \#\mathcal{T}_0 \leqslant N/2 \quad \text{and} \quad \inf_{\Psi_{N/2} \in \mathcal{P}^0(\mathcal{T}_{N/2})} |||\phi - \Psi_{N/2}|||(N/2)^s \leqslant \|\phi\|_{\mathbb{A}_s}.$$

Because of the Céa lemma (2.8), we get

$$|||\phi - \Phi_{N/2}|||(N/2)^s \leqslant \|\phi\|_{\mathbb{A}_s}.$$

For $N > 4\#\mathcal{T}_0$, we construct a quasi-uniform mesh $\mathcal{T}_u \in \mathbb{T}$ by splitting each element $T \in \mathcal{T}_0$ uniformly in exactly $k = \lfloor N/(2\#\mathcal{T}_0)\rfloor$ parts. Then, it holds

$$\#\mathcal{T}_u - \#\mathcal{T}_0 = (k-1)\#\mathcal{T}_0 \leqslant \frac{N}{2\#\mathcal{T}_0}\#\mathcal{T}_0 = N/2.$$

We define the overlay $\mathcal{T}_+ := \mathcal{T}_{N/2} \oplus \mathcal{T}_u$. The mesh $\mathcal{T}_u$ has at least

$$k\#\mathcal{T}_0 \geqslant \Big(\frac{N}{2\#\mathcal{T}_0} - 1\Big)\#\mathcal{T}_0 = N/2 - \#\mathcal{T}_0 \geqslant N/4$$

elements. Therefore and by (4.40), it holds that $\mathrm{hot}_+ \leqslant 4^{s_\star} C_{\mathrm{hot}} N^{-s_\star}$. Here, the ε-dependent constant $s_\star$ is defined as $s_\star := \min\{s_{\mathrm{reg}}, 5/2\} - 1/2 - \varepsilon$ for all $\varepsilon > 0$ and $C_{\mathrm{hot}} > 0$ depends on $\varepsilon > 0$. Note that it is sufficient to choose $\varepsilon > 0$ such that $s < s_\star$. With the Céa lemma (2.8), we get

$$|||\phi - \Phi_+||| \leqslant |||\phi - \Phi_{N/2}|||.$$

With this, we then obtain

$$\begin{aligned}(|||\phi - \Phi_+|||^2 + \mathrm{hot}_+^2)N^{2s} &\leqslant |||\phi - \Phi_{N/2}|||^2(\tfrac{N}{2})^{2s}\, 2^{2s} + 4^{s_\star} C_{\mathrm{hot}} N^{2s-2s_\star}\\ &\leqslant 4^s\|\phi\|_{\mathbb{A}_s}^2 + 4^{s_\star} C_{\mathrm{hot}} < \infty.\end{aligned}$$

Efficiency of η_ℓ now gives

$$\eta_+^2 \lesssim |||\phi - \Phi_+|||^2 + \mathrm{hot}_+^2 \tag{5.1}$$

and therefore

$$\eta_+^2 N^{2s} \lesssim (|||\phi - \Phi_+|||^2 + \mathrm{hot}_+^2)N^{2s} \lesssim 4^s\|\phi\|_{\mathbb{A}_s}^2 + 4^{s_\star} C_{\mathrm{hot}}. \tag{5.2}$$

The overlay estimate (2.14) finally yields $\#\mathcal{T}_+ - \#\mathcal{T}_0 \leqslant \#\mathcal{T}_{N/2} + \#\mathcal{T}_u - 2\#\mathcal{T}_0 \leqslant N$. This proves $(\phi, g) \in \mathbb{A}_s^\eta$. □

Proof of Theorem 2.5, equation (2.22). For $0 < s < \min\{s_{\mathrm{reg}}, 5/2\} - 1/2$, Proposition 5.1 states

$$\phi \in \mathbb{A}_s \quad\Longleftrightarrow\quad (\phi, g) \in \mathbb{A}_s.$$

By use of Theorem 2.5 (2.21) for $0 < \theta < 1$ sufficiently small, this is equivalent to

$$\phi \in \mathbb{A}_s \quad\Longleftrightarrow\quad \eta_\ell \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \text{ for all } \ell \in \mathbb{N}.$$

Finally, with reliability (2.10), we immediately see

$$\phi \in \mathbb{A}_s \quad\Longrightarrow\quad |||\phi - \Phi_\ell||| \lesssim (\#\mathcal{T}_\ell - \#\mathcal{T}_0)^{-s} \text{ for all } \ell \in \mathbb{N}.$$

The converse implication is trivial. □

6. Numerical Examples

We consider the model problem (2.1) with several example data g. Programming was done with the MATLAB-BEM library `Hilbert` [1]. To deal with the integral operator on the right-hand side of (2.1), we replace the exact boundary data g by its nodal interpoland $G_\ell := I_\ell g \in \mathcal{S}^1(\mathcal{T}_\ell)$. Analogously to, e.g., [5], this introduces an additional approximation error which can be bounded above by

$$\mathrm{osc}_\ell := \|h_\ell^{1/2}\tfrac{\partial}{\partial s}(g - G_\ell)\|_{L^2(\Gamma)}.$$

Additionally, we plot the following quantities with respect to the number of elements:

Instead of the energy norm error $|||\phi - \Phi_\ell|||$ which can hardly be computed analytically, we plot the following reliable error bound:

$$|||\phi - \Phi_\ell||| \lesssim \mathrm{err}_\ell + \mathrm{osc}_\ell, \quad \text{with } \mathrm{err}_\ell := \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}.$$

The integral is computed via Gauss–Legendre quadrature. Note that under the regularity assumptions of Theorem 2.4, we obtain that err_ℓ is up to terms of higher order and oscillation terms even a lower bound for the energy norm error, i.e.,

$$\mathrm{err}_\ell \lesssim |||\phi - \Phi_\ell||| + \mathrm{osc}_\ell + \mathrm{hot}_\ell$$

for all $\ell \in \mathbb{N}$.

We plot the error indicator $\eta_\ell = \|h_\ell^{1/2}\frac{\partial}{\partial s}(V\Phi_\ell - (K+\frac{1}{2})G_\ell))\|_{L^2(\Gamma)}$. The functions $(V\Phi_\ell)(x)$ and $(KG_\ell)(x)$ are computed analytically, which is possible, since $\Phi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ and $G_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ are discrete functions. The L^2-norm is then computed by an adapted numerical quadrature.

An important quantity in our analysis is the term of higher order hot_ℓ from Definition 4.7. Even if we prescribe the solution ϕ, we do not know ϕ_0 in general. Therefore, we aim to visualize the behavior of hot_ℓ as follows:

$$\widetilde{\mathrm{hot}}_\ell := \|h_\ell^{1/2}(1 - \Pi_\ell^{(1)})\phi\|_{L^2(\Gamma_{\mathrm{reg}})},$$

where $\Gamma_{\mathrm{reg}} = \Gamma \setminus \bigcup_{j=1}^m B_\delta(c_j)$. Here, $\delta > 0$ is small compared to the size of the domain (for the depicted domain sizes in Figure 1, we choose $\delta = 0.01$) and c_j, $j = 1, \ldots, m$ denote the corners of the boundary, i.e., the generic singularities of ϕ. From the expansion (4.28), we know that $\phi|_{\Gamma_{\mathrm{reg}}}$ has the same regularity as $\phi_0|_{\Gamma_{\mathrm{reg}}}$. Therefore, $\widetilde{\mathrm{hot}}_\ell$ should give a good representation of hot_ℓ.

To compare the adaptive approach presented in Algorithm 2.1 versus the uniform mesh-refinement, we want to consider the computational times:

- The time t_{unif} to compute the solution $\Phi^{(\ell)}$ of the uniform approach is the time needed to perform ℓ uniform refinements of the initial mesh $\mathcal{T}_0$, plus the time needed to build and solve the linear system corresponding to $\mathcal{T}^{(\ell)}$. Obviously, the second contribution is vastly dominant.

- The time t_{adap} to compute the solution Φ_ℓ of the adaptive approach in Algorithm 2.1 is the time to build and solve the system corresponding to the mesh $\mathcal{T}_\ell$ plus the time needed to compute all the previous solutions, to compute the error estimators, to discretize the data g, and to mark and refine the meshes.

Although this definition seems to favor the uniform approach, we think that it provides a fair comparison between those strategies. Throughout, all the occurring linear systems were solved directly with the MATLAB backslash operator. In all experiments, the adaptivity parameter in Algorithm 2.1 is chosen as

$$\theta = 1/2.$$

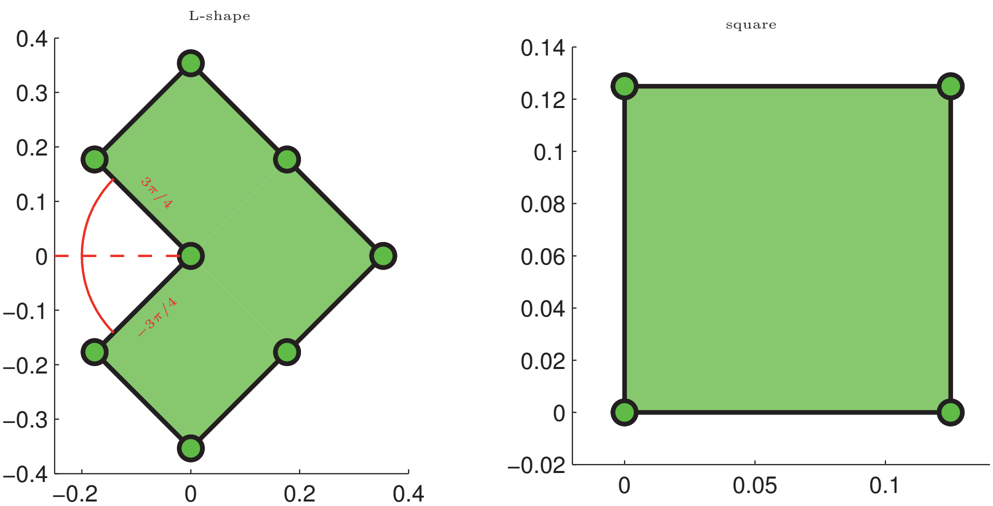


Figure 1. Different domains Ω with initial partitions of the boundary $\mathcal{T}_0$.

6.1. Experiment on L-Shape with Singular Solution

Here, Γ is the boundary of the L-shaped domain Ω in Figure 1 (left). We prescribe the solution u of

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma \end{aligned} \tag{6.1}$$

as $u(x,y) := r^{2/3}\cos(2\alpha/3)$ with polar coordinates (r,α) with respect to $(0,0) \in \mathbb{R}^2$. It is easy to check that $u|_\Gamma = g$ is smooth and therefore meets the regularity assumptions of Theorem 2.4. We compute the data and solution thereof. Figure 2 shows that the error and the error estimator converge with optimal order $\mathcal{O}(N^{-3/2})$ on adaptively generated meshes. The term $\widetilde{\text{hot}}_\ell$ converges with even higher order, which underlines that the error estimator is efficient. Recall that $u \in H^{1+2/3-\varepsilon}(\Omega)$ for all $\varepsilon > 0$ has a generic singularity in the reentrant corner. Therefore, uniform refinement leads to a suboptimal rate of convergence $\mathcal{O}(N^{-2/3})$. We see that despite the computational overhead which comes with adaptive refinement, this strategy is superior to uniform refinement after only a few iterations.

6.2. Experiment on Square with Smooth Solution

Here, Γ is the boundary of the square Ω in Figure 1 (right). We prescribe the smooth solution u of (6.1) as $u(x,y) := \sinh(2\pi x)\cos(2\pi y)$. Figure 3 shows the results of the experiment. Note that for a smooth solution, uniform mesh-refinement is asymptotically the best strategy to approximate the solution. This can be easily confirmed with results from a priori analysis. Nevertheless, Figure 3 shows that adaptive mesh-refinement does not need significantly more computational time to reach the same accuracy.

6.3. Experiment on L-Shape with Singular Solution and Singular Data

Again Γ is the boundary of the L-shaped domain Ω in Figure 1 (left). We prescribe the solution u of (6.1) as $u(x,y) := v_{2/3}(x,y) + v_{7/8}(x-z_1, y-z_2)$, where $v_\delta(x,y) := r^\delta\cos(\delta\alpha)$ and $z = (z_1, z_2)$ is the uppermost corner of the L-shape in Figure 1. The solution ϕ has a generic singularity in the reentrant corner and in addition a singularity resulting from the singular

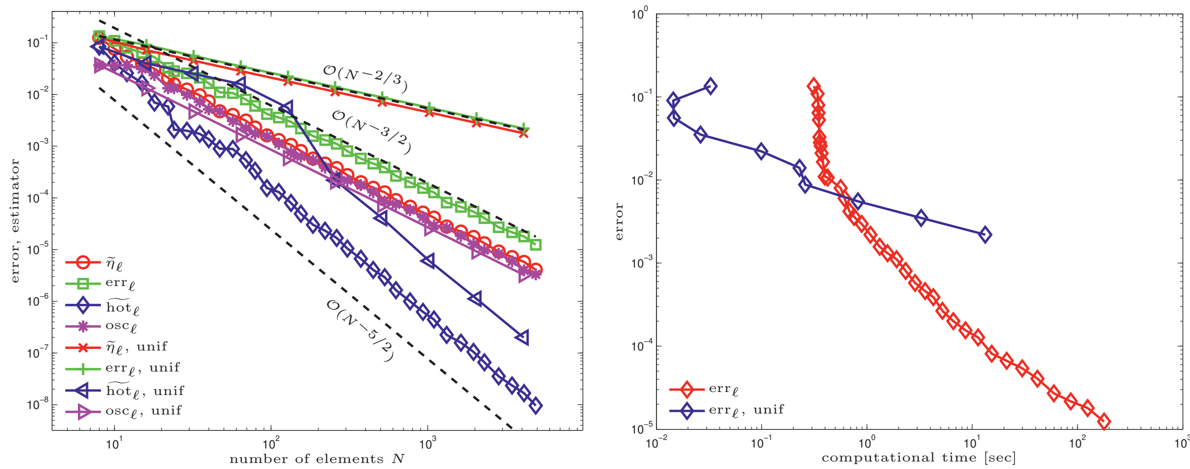


Figure 2. Experiment on L-shape with singular solution. We compare adaptive and uniform mesh-refinement in terms of the quantities err_ℓ, η_ℓ, $\widetilde{\text{hot}}_\ell$, and osc_ℓ plotted over the number of elements $N = \#\mathcal{T}_\ell$ (left). Additionally, err_ℓ is plotted versus the computational time in seconds (right).

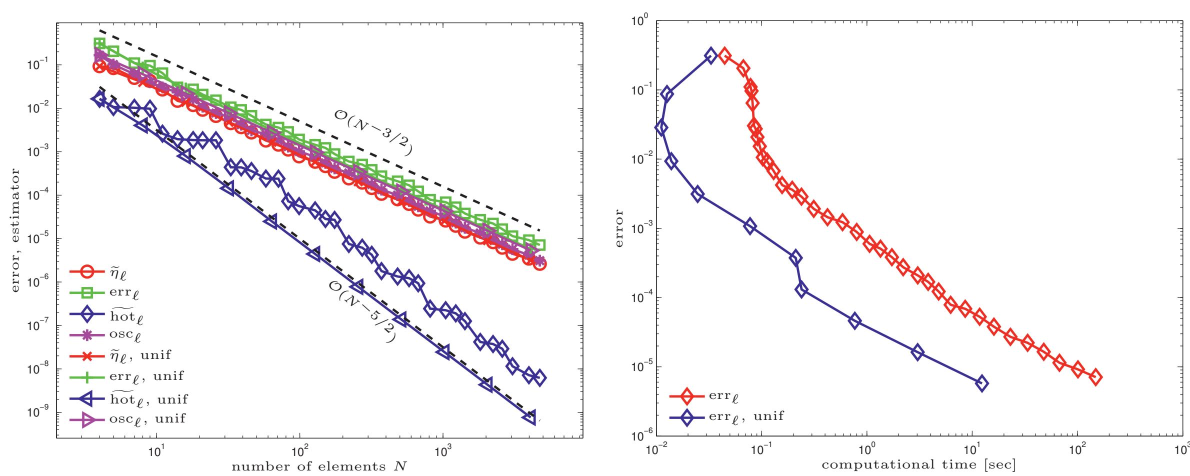


Figure 3. Experiment on square with smooth solution. We compare adaptive and uniform mesh-refinement in terms of the quantities err_ℓ, η_ℓ, $\widetilde{\text{hot}}_\ell$, and osc_ℓ plotted over the number of elements $N = \#\mathcal{T}_\ell$ (left). Additionally, err_ℓ is plotted versus the computational time in seconds (right).

data g. Note that $v_\delta \in H^{1+\delta-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. Therefore, $g \in H^{1/2+7/8-\varepsilon}(\Gamma) \not\subseteq H^{2+\varepsilon}(\Gamma)$ for all $\varepsilon > 0$. Hence, g does not meet the regularity assumptions of Theorem 2.4. Nevertheless, Figure 4 shows that the error bound err_ℓ and the error estimator behave perfectly in case of adaptive refinement. Even $\widetilde{\text{hot}}_\ell$ converges with higher order, which indicates $\text{err}_\ell \simeq |||\phi - \Phi_\ell|||$ for the computed steps. This shows that the regularity assumptions in Theorem 2.4 are not fully necessary. The error for uniform mesh-refinement converges with suboptimal rate $\mathcal{O}(N^{-2/3})$ and the data oscillations show suboptimal rate $\mathcal{O}(N^{-7/8})$, too.

A. Some Remarks on the Saturation Assumption

The saturation assumption for the boundary element method states that there exists $q \in (0,1)$ such that

$$|||\phi - \Phi_{\ell,1}||| \leqslant q|||\phi - \Phi_\ell||| \quad \text{for all } \ell \in \mathbb{N}, \tag{A.1}$$

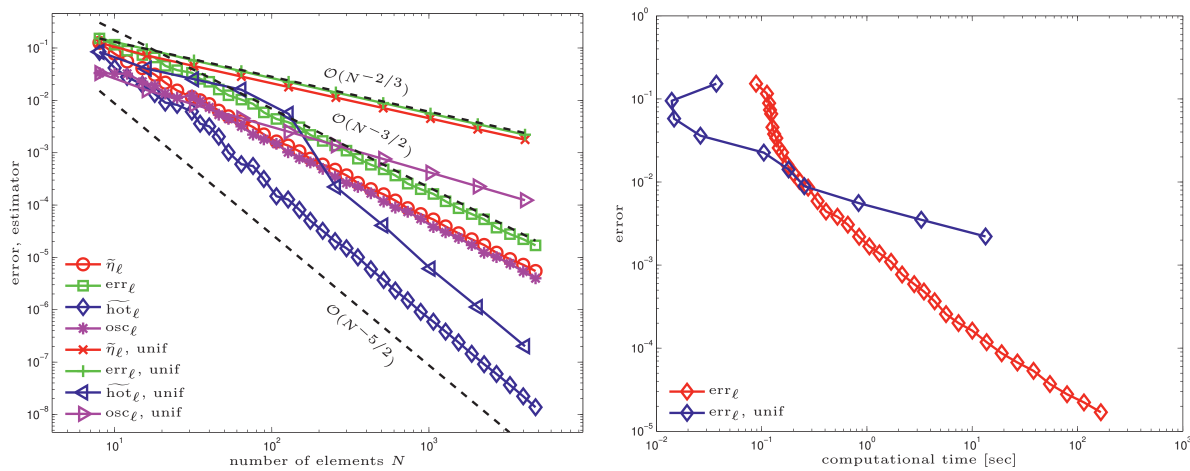


Figure 4. Experiment on L-shape with singular solution and singular data. We compare adaptive and uniform mesh-refinement in terms of the quantities err_ℓ, η_ℓ, $\widetilde{\mathrm{hot}}_\ell$, and osc_ℓ plotted over the number of elements $N = \#\mathcal{T}_\ell$ (left). Additionally, err_ℓ is plotted versus the computational time in seconds (right).

where $\Phi_{\ell,1}$ is the Galerkin solution with respect to the uniformly refined mesh $\mathcal{T}_{\ell,1} := \texttt{refine}(\mathcal{T}_\ell, \mathcal{T}_\ell)$. In terms of $(h - h/2)$ based error estimators as proposed in [6, 24, 25], the saturation assumption (A.1) is equivalent to the reliability of these error estimators. Therefore, it is of certain interest to confirm this assumption. Obviously, one can construct examples, for which assumption (A.1) fails to hold for an arbitrarily large number of steps by choosing $\phi \in \mathcal{P}^0(\mathcal{T}^{(n+1)})^\perp$, where $\mathcal{T}^{(n+1)} := \mathrm{unif}^{(n+1)}(\mathcal{T}_0)$. Then, there holds $\Phi_{\ell,1} = \Phi_\ell = 0$ for at least all meshes $\mathcal{T}_\ell$ with $\ell \leqslant n$. Up to data oscillation terms, (A.1) was proved for the finite element method and the Poisson problem [18], but still remains open for BEM. In this appendix, we attempt to prove a slightly weaker version of (A.1).

We assume the given boundary data to satisfy $g \in H^{s_{\mathrm{reg}}}(\Gamma)$ for some $s_{\mathrm{reg}} > 2$ throughout the whole section.

Lemma A.1. *Let $\mathcal{T}_\ell \in \mathbb{T}$ denote a mesh and let ϕ denote the solution of* (2.1)*. Then, it holds the following discrete efficiency estimate:*

$$C_9^{-1}\eta_\ell \leqslant |||\Phi_{\ell,k} - \Phi_\ell||| + \mathrm{hot}_\ell,$$

where $k \in \mathbb{N}$ and $C_9 \geqslant 1$ depend only on $\kappa(\mathcal{T}_\ell)$ and Γ. Here, $\Phi_{\ell,k}$ denotes the solution of (2.7) *with respect to the mesh $\mathcal{T}_{\ell,k} := \mathrm{unif}^{(k)}(\mathcal{T}_\ell)$.*

Proof. Recall the Céa lemma and norm equivalence (2.5) to see

$$\|\phi - \Phi_{\ell,k}\|_{H^{-1/2}(\Gamma)} \lesssim |||\phi - \Phi_{\ell,k}||| \leqslant |||(1 - \Pi_{\ell,k})\phi||| \lesssim \|(1 - \Pi_{\ell,k})\phi\|_{H^{-1/2}(\Gamma)}.$$

With the approximation properties of the L^2-projection (see [13, Theorem 4.1]), we conclude

$$\|\phi - \Phi_{\ell,k}\|_{H^{-1/2}(\Gamma)} \lesssim \|h_{\ell,k}^{1/2}(1 - \Pi_{\ell,k})\phi\|_{L^2(\Gamma)} \leqslant 2^{-k/2}\|h_\ell^{1/2}(1 - \Pi_\ell)\phi\|_{L^2(\Gamma)}. \tag{A.2}$$

Now, we argue as in the proof of Theorem 2.4 and conclude together with (A.2)

$$\begin{aligned}\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} &\lesssim \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell \\ &\leqslant \|\phi - \Phi_{\ell,k}\|_{H^{-1/2}(\Gamma)} + \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell \\ &\lesssim 2^{-k/2}\|h_\ell^{1/2}(1 - \Pi_\ell)\phi)\|_{L^2(\Gamma)} + \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell \\ &\lesssim 2^{-k/2}\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} + \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell.\end{aligned}$$

Hence, for $k \in \mathbb{N}$ sufficiently large, there holds

$$\|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} \lesssim \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell. \tag{A.3}$$

With (4.2) and the approximation properties of the Galerkin solution, we prove

$$\begin{aligned}\eta_\ell &\lesssim \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} \lesssim \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)} \\ &\lesssim \|\Phi_{\ell,k} - \Phi_\ell\|_{H^{-1/2}(\Gamma)} + \mathrm{hot}_\ell,\end{aligned}$$

where we inserted (A.3) to obtain the last estimate. Norm equivalence (2.5) proves the result. □

Now, we are able to prove the following result.

Proposition A.2 (weak saturation assumption)**.** *There exist constants $k \in \mathbb{N}$ and $0 < q < 1$ which depend only on $\kappa(\mathcal{T}_0)$ and Γ such that for all $\mathcal{T}_\ell \in \mathbb{T}$ with corresponding Galerkin solution Φ_ℓ, it holds*

$$|||\phi - \Phi_{\ell,k}|||^2 \leqslant q|||\phi - \Phi_\ell|||^2 + \mathrm{hot}_\ell^2.$$

Proof. We combine reliability (2.10), Lemma A.1, and the Galerkin orthogonality to see

$$\begin{aligned}|||\phi - \Phi_{\ell,k}|||^2 &= |||\phi - \Phi_\ell|||^2 - |||\Phi_{\ell,k} - \Phi_\ell|||^2 \leqslant |||\phi - \Phi_\ell|||^2 - \tfrac{1}{2}C_9^{-2}\eta_\ell^2 + \mathrm{hot}_\ell^2 \\ &\leqslant |||\phi - \Phi_\ell|||^2 - \tfrac{1}{2}C_9^{-2}C_{\mathrm{rel}}^{-2}|||\phi - \Phi_\ell|||^2 + \mathrm{hot}_\ell^2 \leqslant q|||\phi - \Phi_\ell|||^2 + \mathrm{hot}_\ell^2\end{aligned}$$

for $0 < q := 1 - \frac{1}{2}C_9^{-2}C_{\mathrm{rel}}^{-2} < 1$. Here, we used $C_{\mathrm{rel}}, C_9 \geqslant 1$ to guarantee $q > 0$. □

In contrast to (A.1), the result above needs a certain number of uniform refinements to achieve a contraction. This raises the question if one could construct examples in which one uniform refinement is actually not sufficient. This question is, however, beyond the scope and techniques of the present work and remains for future research.

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