Research Article

Martin Stynes and José Luis Gracia

Boundary Layers in a Two-Point Boundary Value Problem with a Caputo Fractional Derivative

Abstract: A two-point boundary value problem is considered on the interval $[0, 1]$, where the leading term in the differential operator is a Caputo fractional derivative of order δ with $1 < \delta < 2$. Writing u for the solution of the problem, it is known that typically $u''(x)$ blows up as $x \to 0$. A numerical example demonstrates the possibility of a further phenomenon that imposes difficulties on numerical methods: u may exhibit a boundary layer at $x = 1$ when δ is near 1. The conditions on the data of the problem under which this layer appears are investigated by first solving the constant-coefficient case using Laplace transforms, determining precisely when a layer is present in this special case, then using this information to enlighten our examination of the general variable-coefficient case (in particular, in the construction of a barrier function for u). This analysis proves that usually no boundary layer can occur in the solution u at $x = 0$, and that the quantity $M = \max_{x \in [0,1]} b(x)$, where b is the coefficient of the first-order term in the differential operator, is critical: when $M < 1$, no boundary layer is present when δ is near 1, but when $M \geq 1$ then a boundary layer at $x = 1$ is possible. Numerical results illustrate the sharpness of most of our results.

Keywords: Fractional Differential Equation, Caputo Fractional Derivative, Boundary Value Problem, Boundary Layer

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Martin Stynes: Applied Mathematics Division, Beijing Computational Science Research Center, P. R. China; and Department of Mathematics, National University of Ireland, Cork, Ireland, e-mail: m.stynes@ucc.ie
José Luis Gracia: Department of Applied Mathematics and Institute of Mathematics and Applications, University of Zaragoza, Spain, e-mail: jlgracia@unizar.es

1 Introduction

Boundary value problems whose differential operators involve fractional derivatives are of great interest, as these non-classical derivatives can model some physical processes where integer-order derivatives are unsuitable; see [6, 8] for an extensive list of recent applications and mathematical developments in this area. Thus the precise behaviour of solutions to fractional-derivative boundary value problems is of fundamental importance.

Let $\delta \in (1, 2)$. Let $g \in C^1[0, 1]$ with g' absolutely continuous on $[0, 1]$. Then the Caputo fractional derivative $D_*^\delta g$ associated with the point $x = 0$ is defined by

$$D_*^\delta g(x) := \frac{1}{\Gamma(2 - \delta)} \int_{t=0}^{x} (x - t)^{1-\delta} g''(t) \, dt \quad \text{for } 0 < x \leq 1;$$

see [4, 10, 15]. The Riemann–Liouville fractional derivative D_{RL}^δ of order δ associated with the point $x = 0$ is defined by

$$D_{RL}^\delta g(x) := \frac{d^2}{dx^2} \left[\frac{1}{\Gamma(2 - \delta)} \int_{t=0}^{x} (x - t)^{1-\delta} g(t) \, dt \right] \quad \text{for } 0 < x \leq 1;$$

see [4]. These fractional derivatives are related by the formula (see [4, Lemma 3.4])

$$D_*^\delta g(x) = D_{RL}^\delta g(x) - \frac{g(0)}{\Gamma(1 - \delta)} x^{-\delta} - \frac{g'(0)}{\Gamma(2 - \delta)} x^{1-\delta}. \tag{1.1}$$

In the present paper we shall consider the two-point boundary value problem

$$Lu(x) := -D_*^\delta u(x) + b(x)u'(x) + c(x)u(x) = f(x) \quad \text{for } x \in (0, 1), \tag{1.2a}$$

$$u(0) - \alpha_0 u'(0) = \gamma_0, \quad u(1) + \alpha_1 u'(1) = \gamma_1, \tag{1.2b}$$

where $1 < \delta < 2$. The constants $\alpha_0, \alpha_1, \gamma_0, \gamma_1$ and the functions b, c and f are given. We assume that $b, c, f \in C^1[0, 1]$ with $c \geq 0$ in $[0, 1]$. We assume also that

$$\alpha_1 \geq 0 \quad \text{and} \quad \alpha_0 \geq \frac{1}{\delta - 1}. \tag{1.3}$$

The conditions on c, α_0 and α_1 guarantee that (1.2) satisfies a comparison/maximum principle; see Theorem 3.1 below.

The problem (1.2) models superdiffusion of particle motion when convection is present; see the discussion and references in [6, Section 1]. It is a member of the general class of boundary value problems that is analysed in [10]. It is also discussed in [1]. Numerical methods for its solution are presented in [5, 7, 15] and their references.

Existence and uniqueness of a classical solution to (1.2) is established in [15]. It is proved that $u \in C^1[0, 1] \cap C^2(0, 1]$, and for some constants $\tilde{C}_i$ one has the sharp bounds

$$|u^{(i)}(x)| \leq \begin{cases} \tilde{C}_i & \text{if } i = 0, 1, \\ \tilde{C}_2 x^{\delta-i} & \text{if } i = 2, \end{cases} \quad \text{for all } x \in (0, 1]. \tag{1.4}$$

Thus $u''(x)$ may blow up at the interval endpoint $x = 0$.

These results tell us a lot about the nature of the solution u, but in one respect they are seriously deficient: all constants $\tilde{C}_i$ that appear above depend on the parameter δ, but they can be extremely large when δ is near 1, because in certain cases (as we shall see) the solution u develops a *boundary layer* at $x = 1$ (i.e., $|u'(1)|$ becomes very large) when δ is near 1. It is well known that, when computing numerical solution of problems with integer-order differential operators, such layers can cause a deterioration in accuracy [13]. This is also the case for the fractional-derivative problem (1.2): see [5, 6, 15], where computed solutions of (1.2) become less accurate when δ is near 1. This loss of accuracy appears in only some numerical examples in these papers, and no explanation is given there, but it is in fact confined to problems whose solutions exhibit a boundary layer at $x = 1$. In the present paper we shall cast light on when such layers appear in solutions of (1.2).

For a concrete example exhibiting a boundary layer, consider (1.2) with $b \equiv 1.9$, $c \equiv 0$ and $f \equiv 1$. Take $\alpha_0 = 1/(\delta - 1)$, $\alpha_1 = 0$, $\gamma_0 = 0.4$ and $\gamma_1 = 1.7$. For constant-coefficient problems like this, an explicit formula for $u(x)$ will be derived in Section 2 below. Using this formula, we plot the solution for the values $\delta = 1.6, 1.4, 1.2, 1.1$ in Figure 1. It is clear from this figure that a boundary layer at $x = 1$ develops in the solution when δ is near 1.

Figure 1 of [12] also shows a boundary layer developing as the order of the fractional derivative approaches a certain limiting value, though the boundary value problem under discussion there is not the same as (1.2).

Furthermore, there is currently great interest in the construction of numerical methods for problems with variable order of fractional derivative (see, e.g., [3, 14]); this implies that one should pay close attention to how the solution of (1.2) depends on the value of the parameter δ.

For these two reasons (loss of accuracy in numerical solution only for certain values of δ; design of numerical methods for variable δ) our aim in the present paper is to investigate in detail how u behaves as a function of δ (as well as a function of x).

The structure and main results of the paper are as follows. In Section 2 Laplace transforms are used to derive an explicit formula for the solution u of (1.2) in the special case when b is a nonzero constant and $c \equiv 0$. From this formula we deduce that, when δ is near 1, a boundary layer in u at $x = 1$ can appear only if $b \geq 1$, while u never has a boundary layer at $x = 0$ even though $\max_{[0,1]}|u(x)|$ blows up as $\delta \to 1^+$ if $b \geq 1$. In Section 3 the general case of (1.2) is considered and a comparison/maximum principle is used, with some guidance from Section 2, to explore how u depends on δ. In this general case we find that a boundary layer in u at $x = 1$ when δ is near 1 is possible only when $\max_{[0,1]} b(x) \geq 1$. Note here that it is the maximum of b, not of $|b|$, that is the significant quantity. It is shown also that a boundary layer at $x = 0$ (when δ is near 1) is possible only if $\max_{[0,1]} b(x) > 1$ and $\min_{[0,1]} b(x) \leq 0$ and $\min_{[0,1]} c(x) = 0$.
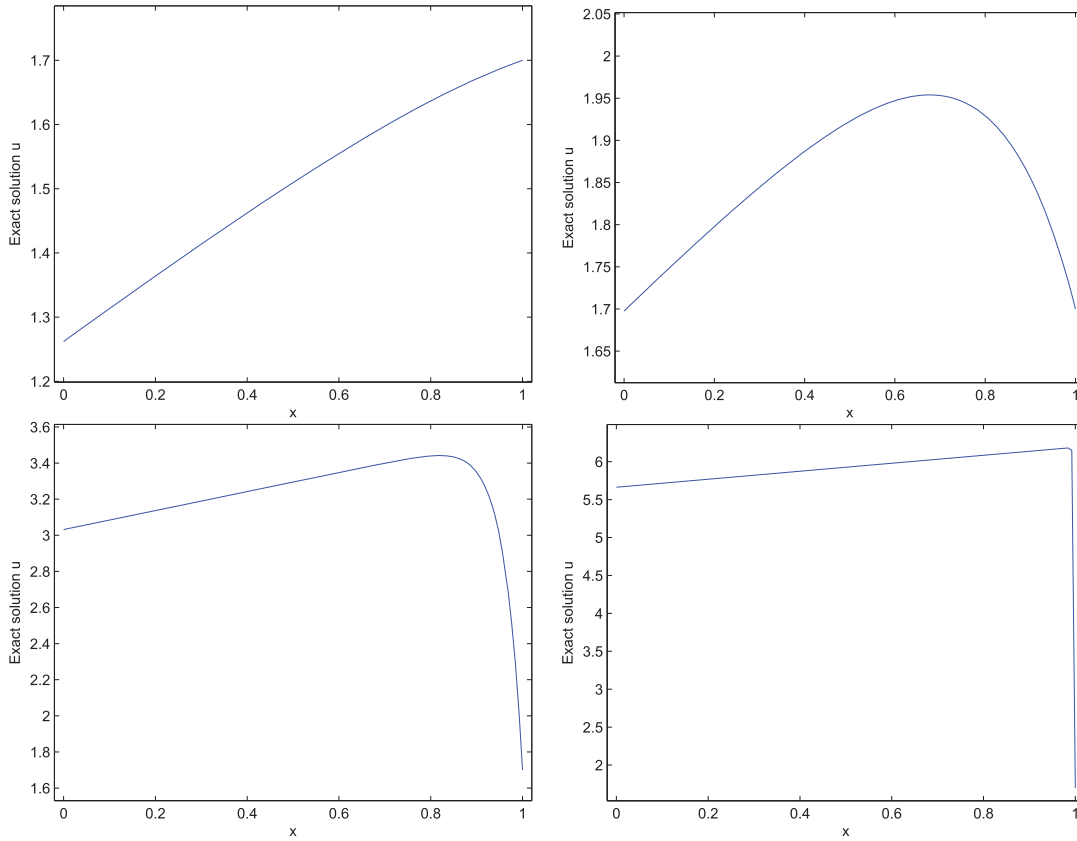
Figure 1. Exact solution for $b \equiv 1.9$, $c \equiv 0$, $f \equiv 1$, $\alpha_0 = 1/(\delta - 1)$, $\alpha_1 = 0$, $\gamma_0 = 0.4$ and $\gamma_1 = 1.7$, with $\delta = 1.6$ (1st row, left), $\delta = 1.4$ (1st row, right), $\delta = 1.2$ (2nd row, left) and $\delta = 1.1$ (2nd row, right), showing development of a boundary layer when δ is near 1.

Notation. We use the standard notation $C^k(I)$ to denote the space of real-valued functions whose derivatives up to order k are continuous on an interval I, and write $C(I)$ for $C^0(I)$. For each $g \in C[0, 1]$, set $\|g\|_\infty = \max_{x \in [0,1]} |g(x)|$.

In several inequalities C denotes a generic positive constant that depends on the data $b, c, f, \gamma_0, \gamma_1, \alpha_1$ of the boundary value problem (1.2) but is *independent of δ and x*; note that C can take different values in different places. A subscripted C (e.g., C_1) denotes a fixed positive constant that can depend on all the data of the boundary value problem (1.2) except δ and x.

2 The Case Where b is Constant and $c \equiv 0$

In Section 2 we consider the special case of problem (1.2) where $c \equiv 0$, and b is constant with $b \neq 0$. Our results could be extended to the case where b and c are arbitrary constants, but when $c \neq 0$ the details become much more complex; see Remark 2.1. We shall use Laplace transforms to derive an explicit formula for the solution u of (1.2) in terms of Mittag-Leffler functions.

Our examination of this special case gives useful and penetrating insights into the properties of the solution u of (1.2). Furthermore, the precise form of the solution that we find in Section 2.2 is very helpful when constructing a barrier function in Section 3 to analyse the solution of (1.2) when b and c are no longer constants.

2.1 General Right-Hand Side f

Extend the domain of f to $[0, \infty)$ in such a way that the extension (which we also call f) is smooth and has support in $[0, 2]$. To solve (1.2), we treat it as an initial-value problem on $[0, \infty)$ with the initial condition $u(0) - \alpha_0 u'(0) = \gamma_0$ from (1.2b), and apply the standard Laplace transform operator $\mathcal{L}$, defined by $\mathcal{L}v(s) = \int_{t=0}^{\infty} e^{-st} v(t)\, dt$. The second boundary condition $u(1) + \alpha_1 u'(1) = \gamma_1$ of (1.2b) will be invoked later.

Define the two-parameter Mittag-Leffler function by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{for } \alpha, \beta, z \in \mathbb{R} \text{ with } \alpha > 0. \tag{2.1}$$

We shall need the properties [11, (1.80), (1.82)] that for constant α, β, γ and λ one has

$$\mathcal{L}\{x^{\beta-1} E_{\alpha,\beta}(\pm \lambda x^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda} \tag{2.2}$$

and

$$D_{\mathrm{RL}}^\gamma \left(x^{\beta-1} E_{\alpha,\beta}(\lambda x^\alpha)\right) = x^{\beta-\gamma-1} E_{\alpha,\beta-\gamma}(\lambda x^\alpha). \tag{2.3}$$

Note that when $\gamma = 1$ then $D_{\mathrm{RL}}^\gamma \equiv d/dx$ (see [4, p. 27]); thus (2.3) implies that

$$\frac{d}{dx}\left(x^{\beta-1} E_{\alpha,\beta}(\lambda x^\alpha)\right) = x^{\beta-2} E_{\alpha,\beta-1}(\lambda x^\alpha). \tag{2.4}$$

Applying $\mathcal{L}$ to (1.2a) and observing that $\mathcal{L}\{D_*^\delta u\} = s^{\delta-2}[s^2 \mathcal{L}\{u\} - su(0) - u'(0)]$ from [11, (2.253)], we obtain

$$(-s^\delta + bs)\mathcal{L}\{u\} + s^{\delta-1}u(0) + s^{\delta-2}u'(0) - bu(0) = \mathcal{L}\{f\}.$$

Hence

$$\mathcal{L}\{u\} = -\frac{\mathcal{L}\{f\}}{s(s^{\delta-1} - b)} + \frac{u(0)}{s} + \frac{s^{\delta-3}u'(0)}{s^{\delta-1} - b} = -\frac{\mathcal{L}\{f\}}{s(s^{\delta-1} - b)} + \frac{\gamma_0 + \alpha_0 u'(0)}{s} + \frac{s^{\delta-3}u'(0)}{s^{\delta-1} - b} \tag{2.5}$$

using the boundary condition (1.2b) at $x = 0$.

To find the inverse Laplace transform of $-\mathcal{L}\{f\}/[s(s^{\delta-1} - b)]$, we imitate [4, p. 135]. Using the integration theorem for Laplace transforms,

$$\frac{\mathcal{L}\{f\}}{s(s^{\delta-1} - b)} = \frac{\mathcal{L}\{\int_{r=0}^{x} f(r)dr\}}{s^{\delta-1} - b}. \tag{2.6}$$

By (2.4) one has

$$\frac{d}{dx} E_{\delta-1,1}(bx^{\delta-1}) = x^{-1} E_{\delta-1,0}(bx^{\delta-1}), \tag{2.7}$$

where the first term in the Mittag-Leffler series for $E_{\delta-1,0}(bx^{\delta-1})$ vanishes since $\Gamma(0) = \infty$. But (2.2) yields

$$\mathcal{L}\{E_{\delta-1,1}(bx^{\delta-1})\} = \frac{s^{\delta-2}}{s^{\delta-1} - b}$$

so the differentiation theorem for Laplace transforms gives

$$\mathcal{L}\{x^{-1} E_{\delta-1,0}(bx^{\delta-1})\} = s\mathcal{L}\{E_{\delta-1,1}(bx^{\delta-1})\} - E_{\delta-1,1}(0) = \frac{s^{\delta-1}}{s^{\delta-1} - b} - 1 = \frac{b}{s^{\delta-1} - b}.$$

Consequently, from (2.6) we have

$$\frac{\mathcal{L}\{f\}}{s(s^{\delta-1} - b)} = \frac{1}{b} \mathcal{L}\{x^{-1} E_{\delta-1,0}(bx^{\delta-1})\} \mathcal{L}\left\{ \int_{r=0}^{x} f(r)dr \right\}.$$

Now the convolution theorem for Laplace transforms yields

$$\mathcal{L}^{-1}\left\{ \frac{\mathcal{L}\{f\}}{s(s^{\delta-1} - b)} \right\}(x) = \frac{1}{b} \int_{t=0}^{x} t^{-1} E_{\delta-1,0}(bt^{\delta-1}) \left[\int_{r=0}^{x-t} f(r)\, dr \right] dt. \tag{2.8}$$

Taking the inverse transform of (2.5) and invoking (2.8) and (2.2), we get

$$u(x) = \gamma_0 + \alpha_0 u'(0) + u'(0)x E_{\delta-1,2}(bx^{\delta-1}) - \frac{1}{b}\int\limits_{t=0}^{x} t^{-1}E_{\delta-1,0}(bt^{\delta-1})\left[\int\limits_{r=0}^{x-t} f(r)\,dr\right]dt. \tag{2.9}$$

By virtue of (2.4) one can differentiate (2.9) to obtain

$$u'(x) = u'(0)E_{\delta-1,1}(bx^{\delta-1}) - \frac{1}{b}\int\limits_{t=0}^{x} t^{-1}E_{\delta-1,0}(bt^{\delta-1})f(x-t)\,dt. \tag{2.10}$$

Consequently, imposing the boundary condition $u(1) + \alpha_1 u'(1) = \gamma_1$ of (1.2b), one has

$$\gamma_1 = \gamma_0 + \alpha_0 u'(0) + u'(0)E_{\delta-1,2}(b) - \frac{1}{b}\int\limits_{t=0}^{1} t^{-1}E_{\delta-1,0}(bt^{\delta-1})\left[\int\limits_{r=0}^{1-t} f(r)\,dr\right]dt$$

$$+ \alpha_1\left\{u'(0)E_{\delta-1,1}(b) - \frac{1}{b}\int\limits_{t=0}^{1} t^{-1}E_{\delta-1,0}(bt^{\delta-1})f(1-t)\,dt\right\}$$

whence

$$u'(0) = \frac{\gamma_1 - \gamma_0 + \frac{1}{b}\int_{t=0}^{1} t^{-1}E_{\delta-1,0}(bt^{\delta-1})\left[\int_{r=0}^{1-t} f(r)\,dr + \alpha_1 f(1-t)\right]dt}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)}. \tag{2.11}$$

Substitution of (2.11) into (2.9) and (2.10) yields explicit formulas for $u(x)$ and $u'(x)$.

2.2 Constant Right-Hand Side f

In this section we simplify the results of Section 2.1 by taking f to be constant so that we can then investigate in detail the solution u. Furthermore, the formulas of Section 2.2 will be of great help in the construction of a barrier function in Section 3 to analyse the structure and behaviour of u in the case of variable b, c and f.

First, taking f constant in (2.11) and recalling (2.7), we have

$$\begin{aligned} u'(0) &= \frac{\gamma_1 - \gamma_0 + \frac{f}{b}\int_{t=0}^{1}[\alpha_1 + (1-t)]\frac{d}{dt}E_{\delta-1,1}(bt^{\delta-1})\,dt}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)} \\ &= \frac{\gamma_1 - \gamma_0 + \frac{f}{b}\left[\alpha_1 E_{\delta-1,1}(b) - \alpha_1 - 1 + \int_{t=0}^{1}E_{\delta-1,1}(bt^{\delta-1})\,dt\right]}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)} \\ &= \frac{\gamma_1 - \gamma_0 + \frac{f}{b}\left[\alpha_1 E_{\delta-1,1}(b) - \alpha_1 - 1 + E_{\delta-1,2}(b)\right]}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)}, \end{aligned} \tag{2.12}$$

where we integrated by parts then invoked (2.4) with $\beta = 2$. Substituting (2.12) into (2.9) gives us $u(x)$; this formula can be written in a variety of ways, the most compact of which seems to be

$$u(x) = \gamma_0 + \frac{(\alpha_0 + x)f}{b} + \frac{\left[\alpha_0 + x E_{\delta-1,2}(bx^{\delta-1})\right]\left[\gamma_1 - \gamma_0 - (1 + \alpha_0 + \alpha_1)f/b\right]}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)}. \tag{2.13}$$

The correctness of this formula can be verified by substitution into (1.2), using (2.4) and, by (1.1) and (2.3),

$$D_*^{\delta}\left(x E_{\delta-1,2}(bx^{\delta-1})\right) = x^{1-\delta}E_{\delta-1,2-\delta}(bx^{\delta-1}) - \frac{x^{1-\delta}}{\Gamma(2-\delta)}$$

$$= x^{1-\delta}\sum_{k=1}^{\infty}\frac{(bx^{\delta-1})^k}{\Gamma(k(\delta-1) + 2 - \delta)} = bE_{\delta-1,1}(bx^{\delta-1}).$$

Remark 2.1. In the case of constant b, c, f with $c \neq 0$, one can use the above Laplace transform technique to obtain a closed-form representation of the solution by imitating [9, (35)]. For example, to invert $[s(s^\delta - bs - c)]^{-1}$, write

$$\frac{1}{s(s^\delta - bs - c)} = \frac{1}{s(s^\delta - c)(1 - \frac{bs}{s^\delta - c})} = \frac{1}{s(s^\delta - c)} \sum_{r=0}^{\infty} \left(\frac{bs}{s^\delta - c}\right)^r = \sum_{r=0}^{\infty} \frac{b^r s^{r-1}}{(s^\delta - c)^{r+1}}$$

then appeal to [11, (1.80)]:

$$\mathcal{L}\{x^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\pm \lambda x^\alpha)\} = \frac{k! \, s^{\alpha - \beta}}{(s^\alpha \mp \lambda)^{k+1}},$$

where $E_{\alpha,\beta}^{(k)}(y) = (d^k/dy^k) E_{\alpha,\beta}(y)$. This will yield an infinite sum of derivatives of Mittag-Leffler functions, which is very complicated. As our main aim in Section 2 is to gain insight into the solution of the general problem (1.2), we do not consider $c \neq 0$ any further.

Remark 2.2. Differentiating (2.13) twice by invoking (2.4), one gets

$$u''(x) = \left[\frac{\gamma_1 - \gamma_0 - (1 + \alpha_0 + \alpha_1)f/b}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)} \right] \frac{1}{x} E_{\delta-1,0}(bx^{\delta-1}). \tag{2.14}$$

When $\gamma_1 - \gamma_0 \neq (1 + \alpha_0 + \alpha_1)f/b$, on recalling the definition (2.1) and $\Gamma(0) = \infty$ it follows that as $x \to 0$ one has $u''(x) \sim \tilde{C}x^{\delta-2}$. This coincides with the bound on $|u''(x)|$ that is derived in [15, Corollary 3.5] for the general case of variable b, c and f. Furthermore, the next term in the series expansion of the Mittag-Leffler function in (2.14) is $\tilde{C}x^{-1}(x^{\delta-1})^2 = \tilde{C}x^{2\delta-3}$ which matches [15, Example 3.7], where it was shown that the derivative bounds (1.4) are sharp.

Remark 2.3. The coefficient of f in the formula (2.13) for $u(x)$ is

$$\frac{(\alpha_0 + x)[\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)] - [\alpha_0 + x E_{\delta-1,2}(bx^{\delta-1})](1 + \alpha_0 + \alpha_1)}{b[\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)]}$$

$$= \frac{1}{b[\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)]} \Big\{ \alpha_0 [E_{\delta-1,2}(b) - 1 - x E_{\delta-1,2}(bx^{\delta-1}) + x]$$

$$+ x E_{\delta-1,2}(b) - x E_{\delta-1,2}(bx^{\delta-1}) + \alpha_1 [(\alpha_0 + x) E_{\delta-1,1}(b) - \alpha_0 - x E_{\delta-1,2}(bx^{\delta-1})] \Big\}$$

$$= \frac{1}{1 + \alpha_0 + b E_{\delta-1,\delta+1}(b) + \alpha_1 [1 + b E_{\delta-1,\delta}(b)]} \Big\{ \alpha_0 [E_{\delta-1,\delta+1}(b) - x^\delta E_{\delta-1,\delta+1}(bx^{\delta-1})]$$

$$+ x E_{\delta-1,\delta+1}(b) - x^\delta E_{\delta-1,\delta+1}(bx^{\delta-1}) + \alpha_1 [(\alpha_0 + x) E_{\delta-1,\delta}(b) - x^\delta E_{\delta-1,\delta+1}(bx^{\delta-1})] \Big\}, \tag{2.15}$$

where we used the elementary identity

$$E_{\delta-1,1+i}(z) = z E_{\delta-1,\delta+i}(z) + 1 \quad \text{for } i = 0, 1.$$

The formula (2.15) is the inspiration for our choice of barrier function in Theorem 3.10.

We turn now to our main interest: investigating how the solution u changes as δ approaches 1, and in particular determining when boundary layers appear in u. Observe first that, since $\alpha_0 > 1$ by condition (1.3), it follows from (2.12) that $|u'(0)| \leq C$ for some constant C, i.e., there is never a boundary layer in u at $x = 0$.

In the subsections that follow, we show that when δ is near 1, the magnitudes of $\|u\|_\infty$ and $|u'(1)|$ depend strongly on whether $b < 1$ or $b \geq 1$.

Differentiating (2.13) gives

$$u'(1) = \frac{f}{b} \cdot \frac{\alpha_0[1 - E_{\delta-1,1}(b)] + E_{\delta-1,2}(b) - E_{\delta-1,1}(b)}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)} + \frac{(\gamma_1 - \gamma_0) E_{\delta-1,1}(b)}{\alpha_0 + E_{\delta-1,2}(b) + \alpha_1 E_{\delta-1,1}(b)}. \tag{2.16}$$
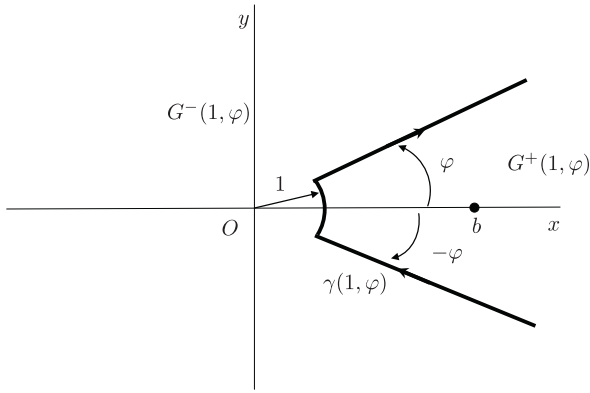
Figure 2. Contour $\gamma(1, \varphi)$ of equation (2.18).

2.2.1 Case $b > 1$

Assume in Section 2.2.1 that $b > 1$ and $f \neq 0$. Suppose that $1/(\delta - 1) = K$ for some positive integer K. Then for $i = 1, 2$ we have

$$E_{\delta-1,i}(b) = \sum_{k=0}^{\infty} \frac{b^k}{\Gamma((\delta - 1)k + i)} \geq \frac{b^K}{\Gamma(1 + i)} \geq \frac{1}{2} b^{1/(\delta-1)}. \tag{2.17}$$

It follows that $\lim_{\delta\to 1^+} E_{\delta-1,i}(b) = \infty$ and $E_{\delta-1,i}(b) \gg 1/(\delta - 1)$ as $\delta \to 1^+$. Consequently, when $\alpha_0 = 1/(\delta - 1)$, by virtue of (2.12) one has $\lim_{\delta\to 1^+} u'(0) = f/b$ and hence, recalling (1.2b),

$$\lim_{\delta\to 1^+} u(0) = \lim_{\delta\to 1^+} \left[\gamma_0 + \alpha_0 u'(0)\right] = \begin{cases} \infty & \text{if } f > 0, \\ -\infty & \text{if } f < 0. \end{cases}$$

Thus when $b > 1$ and $f \neq 0$ we do *not* have $\|u\|_\infty \leq C$ independently of δ. Nevertheless, recalling from above that $\lim_{\delta\to 1^+} u'(0) = f/b$, there is no boundary layer at $x = 0$.

To discuss $\lim_{\delta\to 1^+} u'(1)$, we use more sophisticated machinery. Set $\sigma = \delta - 1$ for brevity. Let $\gamma(1, \varphi)$ be the complex-plane contour of Figure 2 where we choose $\varphi = 2\sigma\pi/3$. This contour divides the complex plane into two regions, which we denote by $G^-(1, \varphi)$ and $G^+(1, \varphi)$. The real number $b > 1$ lies in the region $G^+(1, \varphi)$, so by [11, Theorem 1.1], for arbitrary but fixed $n > 0$ we have

$$E_{\sigma,n}(b) = \frac{1}{\sigma} b^{(1-n)/\sigma} \exp(b^{1/\sigma}) + \frac{1}{2\pi\sigma\iota} \int_{\gamma(1,\varphi)} \frac{\zeta^{(1-n)/\sigma} \exp(\zeta^{1/\sigma})}{\zeta - b} \, d\zeta, \tag{2.18}$$

where ι is the imaginary unit.

Our interest lies in what happens when $\delta \to 1^+$ (i.e., $\sigma \to 0^+$, so $\varphi \to 0^+$) with $b > 1$ fixed. For $\zeta \in \gamma(1, \varphi)$ and φ sufficiently small we have

$$|\zeta - b| \geq \min\{b - 1, \, b\sin\varphi\} = b\sin\varphi \geq b\varphi/2 = b\sigma\pi/3.$$

Consequently (cf. [11, p. 33]), for some positive constants C one obtains

$$\left| \frac{1}{2\pi\sigma\iota} \int_{\gamma(1,\varphi)} \frac{\zeta^{(1-n)/\sigma} \exp(\zeta^{1/\sigma})}{\zeta - b} \, d\zeta \right| \leq \frac{C}{\sigma^2} \int_{\gamma(1,\varphi)} |\zeta^{(1-n)/\sigma}| \exp(|\zeta|^{1/\sigma} \cos(2\pi/3)) \, d\zeta \leq \frac{C}{\sigma^2}$$

as $\sigma \to 0^+$, where the contour integral is bounded since $\cos(2\pi/3) < 0$. Thus (2.18) implies that for $b > 1$ one has

$$E_{\delta-1,n}(b) = \frac{1}{\delta - 1} b^{(1-n)/(\delta-1)} \exp(b^{1/(\delta-1)}) + O\left(\frac{1}{(\delta - 1)^2}\right) \quad \text{as } \delta \to 1^+. \tag{2.19}$$

It follows from (2.19) that $E_{\delta-1,1}(b)/E_{\delta-1,2}(b) \approx b^{1/(\delta-1)}$ when δ is near 1. Consequently, when δ is near 1, if $\alpha_1 > 0$ then (2.16) and (2.17) imply that

$$|u'(1)| \approx C \min\{\alpha_0, E_{\delta-1,1}(b)\}, \tag{2.20}$$

while if $\alpha_1 = 0$, then (2.16) and (2.17) imply that

$$|u'(1)| \approx \frac{C\alpha_0 E_{\delta-1,1}(b)}{\alpha_0 + E_{\delta-1,2}(b)} \approx \frac{C\alpha_0 b^{1/(\delta-1)} E_{\delta-1,2}(b)}{\alpha_0 + E_{\delta-1,2}(b)} \approx Cb^{1/(\delta-1)} \min\{\alpha_0, E_{\delta-1,2}(b)\}. \tag{2.21}$$

Thus when $b > 1$ there is a boundary layer at $x = 1$, and it is much stronger when $\alpha_1 = 0$.

2.2.2 Case $b = 1$

Assume in Section 2.2.2 that $b = 1$ and $f \neq 0$. Then (2.13) yields

$$
\begin{aligned}
u(0) &= \gamma_0 + \alpha_0 f + \frac{\alpha_0[\gamma_1 - \gamma_0 - (1 + \alpha_0 + \alpha_1)f]}{\alpha_0 + E_{\delta-1,2}(1) + \alpha_1 E_{\delta-1,1}(1)} \\
&= \gamma_0 + \alpha_0\left\{\frac{\gamma_1 - \gamma_0 + f[E_{\delta-1,2}(1) + \alpha_1 E_{\delta-1,1}(1) - 1 - \alpha_1]}{\alpha_0 + E_{\delta-1,2}(1) + \alpha_1 E_{\delta-1,1}(1)}\right\}.
\end{aligned} \tag{2.22}
$$

Suppose that $1/(\delta - 1)$ is an integer. Then

$$
\begin{aligned}
E_{\delta-1,2}(1) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma((\delta-1)k+2)} \geq \sum_{k=0}^{\frac{1}{\delta-1}-1} \frac{1}{\Gamma(3)} + \sum_{k=\frac{1}{\delta-1}}^{\frac{2}{\delta-1}-1} \frac{1}{\Gamma(4)} + \sum_{k=\frac{2}{\delta-1}}^{\frac{3}{\delta-1}-1} \frac{1}{\Gamma(5)} + \cdots \\
&= \frac{1}{\delta-1}\left[\frac{1}{\Gamma(3)} + \frac{1}{\Gamma(4)} + \frac{1}{\Gamma(5)} + \cdots\right] = \frac{e-2}{\delta-1}.
\end{aligned} \tag{2.23}
$$

Likewise, one has

$$
\begin{aligned}
E_{\delta-1,2}(1) &\leq \sum_{k=0}^{\frac{1}{\delta-1}-1} \frac{1}{\Gamma(2)} + \sum_{k=\frac{1}{\delta-1}}^{\frac{2}{\delta-1}-1} \frac{1}{\Gamma(3)} + \sum_{k=\frac{2}{\delta-1}}^{\frac{3}{\delta-1}-1} \frac{1}{\Gamma(4)} + \cdots \\
&= \frac{1}{\delta-1}\left[\frac{1}{\Gamma(2)} + \frac{1}{\Gamma(3)} + \frac{1}{\Gamma(4)} + \cdots\right] = \frac{e-1}{\delta-1}.
\end{aligned} \tag{2.24}
$$

One can show similarly that

$$\frac{e-1}{\delta-1} \leq E_{\delta-1,1}(1) \leq \frac{e}{\delta-1}. \tag{2.25}$$

Invoking (2.23)–(2.25) in (2.22), for $f > 0$ and δ sufficiently close to 1 one obtains

$$u(0) \geq \gamma_0 + \alpha_0\left\{\frac{\gamma_1 - \gamma_0 + f\left[\frac{e-2+\alpha_1(e-1)}{\delta-1} - 1 - \alpha_1\right]}{\alpha_0 + \frac{e-1+\alpha_1 e}{\delta-1}}\right\} = \gamma_0 + \frac{\gamma_1 - \gamma_0 + f\left[\frac{e-2+\alpha_1(e-1)}{\delta-1} - 1 - \alpha_1\right]}{1 + \frac{e-1+\alpha_1 e}{\alpha_0(\delta-1)}}.$$

But $\alpha_0 \geq 1/(\delta - 1)$, so it follows that as $\delta \to 1^+$ with $1/(\delta - 1)$ an integer, then $u(0) \to \infty$ if $f > 0$. Otherwise, if $f < 0$ then $u(0) \to -\infty$ as $\delta \to 1^+$. Thus when $b = 1$ and $f \neq 0$ we do *not* have $\|u\|_\infty \leq C$ independently of δ.

Furthermore, the condition (1.3) compared with (2.24) and (2.25) shows that $\alpha_0 \geq \frac{1}{3}E_{\delta-1,i}(1)$ for $i = 1, 2$ when δ is near 1. Consequently, (2.16) implies that $|u'(1)| \approx CE_{\delta-1,1}(1) \approx C/(\delta - 1)$ when δ is near 1; thus u exhibits a boundary layer at $x = 1$.

2.2.3 Case $0 \leq b < 1$

Assume in Section 2.2.3 that $0 \leq b < 1$. Then

$$E_{\delta-1,2}(b) = \sum_{k=0}^{\infty} \frac{b^k}{\Gamma((\delta-1)k+2)} \leq \sum_{k=0}^{\infty} b^k = \frac{1}{1-b}. \tag{2.26}$$

The estimate (2.26) is qualitatively sharp when δ is near 1 because for $0 \le k \le \lfloor 1/(\delta - 1)\rfloor$, where $\lfloor n\rfloor$ denotes the greatest integer less than or equal to n, one has

$$\frac{1}{\Gamma((\delta - 1)k + 2)} \ge \frac{1}{\Gamma(3)} = \frac{1}{2}$$

so

$$E_{\delta-1,2}(b) \ge \sum_{k=0}^{\lfloor 1/(\delta-1)\rfloor} \frac{b^k}{2} = \frac{1 - b^{1+\lfloor 1/(\delta-1)\rfloor}}{2(1 - b)} \ge \frac{1}{4(1 - b)} \quad \text{for } \delta \text{ sufficiently close to 1.} \tag{2.27}$$

Similarly to (2.26) and (2.27), one has

$$\frac{1}{2(1 - b)} \le E_{\delta-1,1}(b) \le \frac{1}{\theta(1 - b)}, \tag{2.28}$$

where $\theta = \min\{\Gamma(x) : 1 \le x \le 2\}$.

From (2.26)–(2.28) and (2.12), since $\alpha_0 \ge 1/(\delta - 1)$ one obtains $|u'(0)| \le C/\alpha_0$ for some constant C, so $u'(0) \to 0$ as $\delta \to 1^+$. Furthermore, (2.13) and (2.16) now yield

$$\|u\|_\infty + |u'(1)| \le C \quad \text{for } 0 \le b < 1$$

for some constant $C = C(b)$. Thus no boundary layers are present in u when $0 \le b < 1$.

Remark 2.4. The analysis above of the cases $b > 1$, $b = 1$ and $0 \le b < 1$ shows that, as a function of δ, the nature of the solution u undergoes a fundamental change when b moves from $b \ge 1$ to $b < 1$. This is shown graphically in Figure 3, where u changes dramatically as b moves from 0.9 to 1.1 (as well as the evident changes in shape in these graphs, note the changes in scale of the y axis).

The regime $b < 0$ is not discussed separately in Section 2 since we are able to handle it easily in Section 3 when we discuss the general variable-coefficient case; see Theorem 3.5.

3 Boundary Layers in the Variable-Coefficient Problem

In numerical solutions of (1.2) computed by the finite difference method of [15], when δ is near 1 we have observed boundary layers at $x = 1$ in certain examples but we have never observed a layer at $x = 0$. The results of Section 3 will extend those of Section 2, prove in most cases that a boundary layer cannot occur at $x = 0$, and provide substantial information about when boundary layers can appear at $x = 1$ when δ is near 1.

The key tool in the analysis presented in Section 3 is the following comparison principle, which will be used several times.

Theorem 3.1 ([15, Theorem 2.1]). *Let* $z \in C^1[0, 1] \cap C^2(0, 1]$. *Assume that* $|z''(x)| \le Kx^{-\theta}$ *for* $0 < x \le 1$, *where* $\theta \in (0, 1)$ *and* K *are constants that are independent of* x. *Let* $b, c \in C[0, 1]$ *with* $c(x) \ge 0$ *for all* $x \in (0, 1)$. *Assume that* z *satisfies the inequalities*

$$-D_*^\delta z + bz' + cz \ge 0 \quad \text{on } (0, 1),$$
$$z(0) - \alpha_0 z'(0) \ge 0 \quad \text{and} \quad z(1) + \alpha_1 z'(1) \ge 0,$$

where α_0 *and* α_1 *satisfy* (1.3). *Then* $z \ge 0$ *on* $[0, 1]$.

The bounds (1.4) show that the solution u of (1.2) satisfies the regularity hypotheses imposed on z in Theorem 3.1. We shall apply Theorem 3.1 to $B_i \pm u$ for various barrier functions B_i, concluding that $B_i \pm u \ge 0$, i.e., that $|u| \le B_i$ on $[0, 1]$.

We begin with a general result that provides a useful relationship between $u'(1)$ and $u'(0)$.

For any function $\phi \in C[0, 1]$, set $\|\phi\|_{[0,x]} = \max_{0 \le t \le x}|\phi(t)|$ for $0 \le x \le 1$.

Lemma 3.2. *There exists a constant C such that*

$$|u'(x)| \le |u'(0)| + Cx^{\delta-1}\big[1 + \|u\|_{[0,x]} + |u'(0)|\big]\, E_{\delta-1,1}\big(\|b\|_{[0,x]}\, x^{\delta-1}\big) \quad \text{for } 0 \le x \le 1.$$
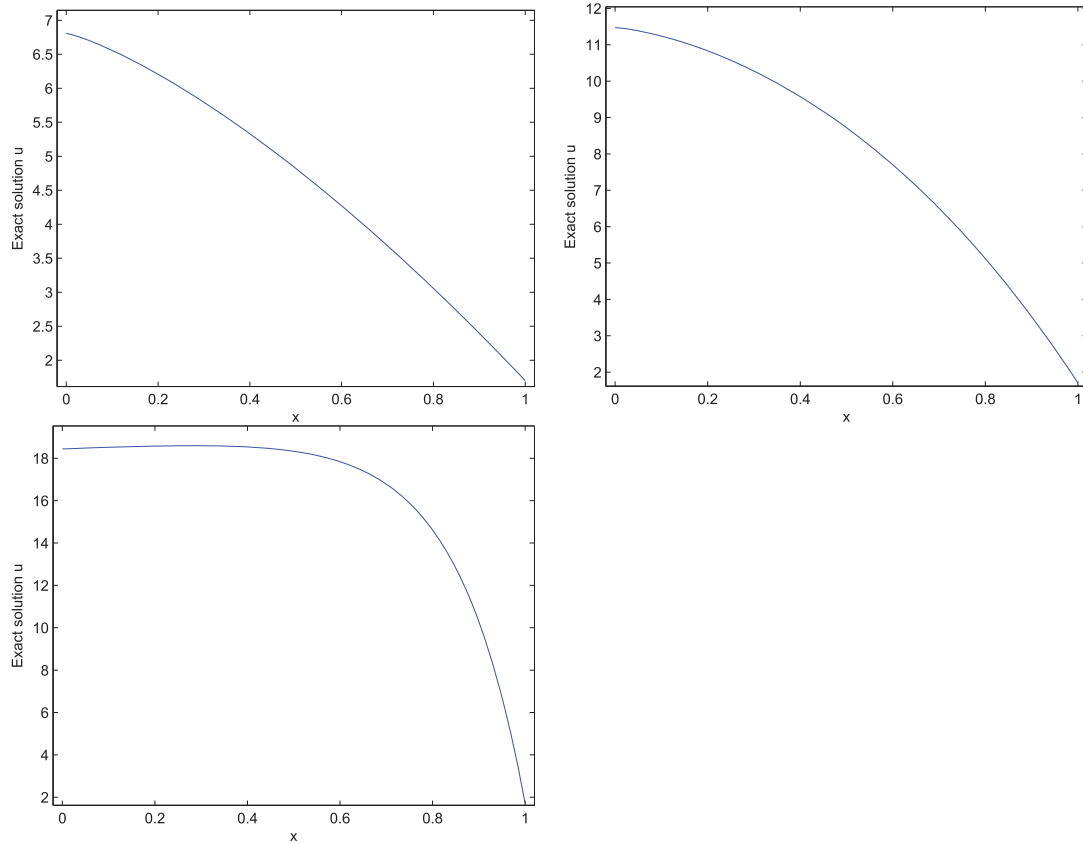
Figure 3. Exact solution for $\delta = 1.05$, $c \equiv 0$, $f \equiv 1$, $\alpha_0 = 1/(\delta - 1)$, $\alpha_1 = 0$, $\gamma_0 = 0.4$, $\gamma_1 = 1.7$, and $b = 0.9$ (top left figure), $b = 1.0$ (top right figure), $b = 1.1$ (bottom figure), showing effect on solution u of moving from $b < 1$ to $b > 1$.

Proof. Set $y(x) = u'(x) - u'(0)$ for $0 \le x \le 1$. On multiplying (1.2a) by $(x - t)^{\delta-2}/\Gamma(\delta - 1)$ then integrating from $t = 0$ to $t = x$, after some manipulation of the fractional derivative term one obtains [7] a weakly singular Volterra integral equation of the second kind in the unknown y: for $0 < x \le 1$,

$$y(x) - \frac{1}{\Gamma(\delta - 1)} \int_{t=0}^{x} (x - t)^{\delta-2} b(t) y(t) \, dt = \frac{1}{\Gamma(\delta - 1)} \int_{t=0}^{x} (x - t)^{\delta-2} \{ b(t) u'(0) + c(t) u(t) - f(t) \} \, dt$$

$$:= g(x), \quad \text{say.} \tag{3.1}$$

From [2, p. 343] the solution of (3.1) can be expressed as

$$y(x) = g(x) + \int_{t=0}^{x} R_\alpha(x, t) g(t) \, dt \quad \text{for } 0 \le x \le 1, \tag{3.2}$$

where

$$R_\alpha(x, t) := (x - t)^{\delta-2} \sum_{n=1}^{\infty} (x - t)^{(n-1)(\delta-1)} \Phi_n(x, t; \delta) \tag{3.3}$$

and the Φ_n are defined iteratively by

$$\Phi_1(x, t; \delta) := b(t)/\Gamma(\delta - 1),$$

$$\Phi_n(x, t; \delta) := \frac{1}{\Gamma(\delta - 1)} \int_{z=0}^{1} (1 - z)^{\delta-2} z^{(n-1)(\delta-1)-1} b(t + (x - t)z) \Phi_{n-1}(t + (x - t)z, t; \delta) \, dz \quad \text{for } n = 1, 2, \ldots$$

An inductive argument using Euler's beta function shows easily (cf. [2, Lemma 6.1.3]) that

$$|\Phi_n(x,t;\delta)| \le \frac{\|b\|_{[0,x]}^n}{\Gamma(n(\delta-1))} \quad \text{for } n = 1, 2, \dots \text{ and } 0 \le t \le x \le 1. \tag{3.4}$$

Consequently, the infinite series in (3.3) is uniformly convergent for $0 \le t \le x \le 1$. Hence we can move the summation sign of (3.3) outside the integral in (3.2). Again appealing to (3.4), from (3.2) we get

$$
\begin{aligned}
|y(x)| &\le |g(x)| + \|g\|_{[0,x]} \sum_{n=1}^{\infty} \frac{\|b\|_{[0,x]}^n}{\Gamma(n(\delta-1))} \int_{t=0}^{x} (x-t)^{(n-1)(\delta-1)+\delta-2} \, dt \\
&= |g(x)| + \|g\|_{[0,x]} \sum_{n=1}^{\infty} \frac{\|b\|_{[0,x]}^n x^{n(\delta-1)}}{n(\delta-1)\Gamma(n(\delta-1))} \\
&= |g(x)| + \|g\|_{[0,x]} \sum_{n=1}^{\infty} \frac{(\|b\|_{[0,x]} x^{\delta-1})^n}{\Gamma(n(\delta-1)+1)} \\
&\le \|g\|_{[0,x]} \, E_{\delta-1,1}\big(\|b\|_{[0,x]} \, x^{\delta-1}\big)
\end{aligned}
\tag{3.5}
$$

by the definition (2.1) of the Mittag-Leffler function.

Recalling the definition of g in (3.1), it follows that

$$
\begin{aligned}
\|g\|_{[0,x]} &\le C\big(1 + \|u\|_{[0,x]} + |u'(0)|\big) \frac{1}{\Gamma(\delta-1)} \int_{t=0}^{x} (x-t)^{\delta-2} \, dt \\
&= C\big(1 + \|u\|_{[0,x]} + |u'(0)|\big) \frac{x^{\delta-1}}{\Gamma(\delta)} \\
&\le C\big(1 + \|u\|_{[0,x]} + |u'(0)|\big) x^{\delta-1}
\end{aligned}
\tag{3.6}
$$

as $\inf_{\delta \in (1,2)} \Gamma(\delta) > 0$. Combining (3.5), (3.6) and $u'(x) = u'(0) + y(x)$ yields the required result. $\qquad\square$

Inequality (3.5) is sharp if g and b are positive constants; see [2, Theorem 6.1.1].

First, we show that in the special case where $c > 0$ on $[0,1]$, the solution u is uniformly bounded for $\delta \in (1,2)$ and no boundary layer appears at $x = 0$ when δ is near 1; there is also no boundary layer at $x = 1$ if $\alpha_1 > 0$.

If $c \ge \underline{c} > 0$ for some constant $\underline{c}$, set

$$C_1 = \max\left\{ |\gamma_0|, |\gamma_1|, \frac{\|f\|_\infty}{\underline{c}} \right\}.$$

Theorem 3.3. *Assume that $c \ge \underline{c} > 0$ for some constant $\underline{c}$. Then*
(i) $\|u\|_\infty \le C_1$,
(ii) $|u'(0)| \le \frac{C_1 + |\gamma_0|}{\alpha_0} \le (\delta-1)(C_1 + |\gamma_0|)$,
(iii) $|u'(1)| \le \frac{C_1 + |\gamma_1|}{\alpha_1}$ *if $\alpha_1 > 0$,*
(iv) $|u'(1)| \le C E_{\delta-1,1}(\|b\|_\infty)$ *for some constant C if $\alpha_1 = 0$.*

Proof. (i) Define the constant barrier function $B_1(x) \equiv C_1$. Then $B_1 \pm u \ge 0$ by Theorem 3.1, so (i) is valid.

(ii) By (1.2b) we have $u(0) - \alpha_0 u'(0) = \gamma_0$, so (1.3) and (i) yield

$$|u'(0)| = \left| \frac{u(0) - \gamma_0}{\alpha_0} \right| \le \frac{\|u\|_\infty + |\gamma_0|}{\alpha_0} \le \frac{C_1 + |\gamma_0|}{\alpha_0} \le (\delta-1)(C_1 + |\gamma_0|).$$

(iii) Use the boundary condition $u(1) + \alpha_1 u'(1) = \gamma_1$ and (i) to derive (iii).

(iv) This follows immediately from Lemma 3.2, (i) and (ii). $\qquad\square$

Theorem 3.3 shows that when c is strictly positive on $[0,1]$ there is no boundary layer at $x = 0$ and, if one also has $\alpha_1 > 0$, there is no boundary layer at $x = 1$. But when $\alpha_1 = 0$ one may have a layer at $x = 1$, as the next example demonstrates.
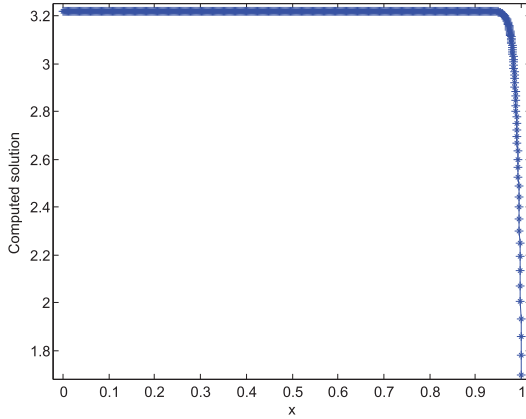
Figure 4. Example showing boundary layer at $x = 1$ is possible when $c > 0$ and $\alpha_1 = 0$.

Example 3.4. Consider our boundary value problem (1.2) with $\delta = 1.01$, $\alpha_0 = 1/(\delta - 1)$, $\alpha_1 = 0$, $\gamma_0 = 0.4$, $\gamma_1 = 1.7$ and constant functions $b(x) \equiv 1.1$, $c(x) \equiv 1$ and $f(x) \equiv 3.25$. In Figure 4 we show the solution (values at the mesh points, joined by a piecewise linear curve) computed by the finite difference scheme of [15] on a uniform mesh of width $1/2048$. A boundary layer at $x = 1$ is clearly visible.

Set

$$M = \max_{x \in [0,1]} b(x).$$

We observe that M can have any sign. In Theorems 3.5, 3.8 and 3.10 we shall consider the regimes $M < 0$, $0 \le M \le 1$ and $M > 1$ respectively and in each case we shall derive bounds on $\|u\|_\infty$, $|u'(0)|$ and $|u'(1)|$.

The case where b is strictly negative is addressed first. For $M < 0$, set

$$C_2 = \max\left\{|\gamma_0|, |\gamma_1| - \frac{(1 + \alpha_1)\|f\|_\infty}{M}\right\} \quad \text{and} \quad C_3 = \max\left\{|\gamma_0| + C_2, \frac{\|f\|_\infty + C_2 \|c\|_\infty}{-M}\right\}.$$

Theorem 3.5. *Assume that* $M < 0$. *Then*

(i) $\|u\|_\infty \le C_2$,

(ii) $|u'(0)| \le (\delta - 1)(C_2 + |\gamma_0|)$,

(iii) $|u'(1)| \le C_3$.

Proof. (i) Set $B_2(x) = C_2 + x\|f\|_\infty/M$ for $0 \le x \le 1$. Then $B_2 \ge 0$ so

$$LB_2(x) = b(x)B_2'(x) + c(x)B_2(x) \ge \frac{b(x)\|f\|_\infty}{M} \ge \|f\|_\infty,$$

$$B_2(0) - \alpha_0 B_2'(0) = C_2 - \frac{\alpha_0\|f\|_\infty}{M} \ge |\gamma_0| \quad \text{and} \quad B_2(1) + \alpha_1 B_2'(1) = C_2 + \frac{(1 + \alpha_1)\|f\|_\infty}{M} \ge |\gamma_1|$$

by definition of C_2. It follows from (1.2) and Theorem 3.1 that $B_2 \pm u \ge 0$, i.e., $|u(x)| \le B_2(x)$ for $0 \le x \le 1$. As $B_2(x) \le C_2$, we have proved (i).

(ii) To obtain (ii), use the boundary condition $u(0) - \alpha_0 u'(0) = \gamma_0$, (1.3) and (i).

(iii) Set $w(x) = u(x) - u(1)$. Then by (1.2) and (i) we have

$$|Lw(x)| = |f(x) - c(x)u(1)| \le \|f\|_\infty + C_2\|c\|_\infty,$$

$$|w(0) - \alpha_0 w'(0)| = |\gamma_0 - u(1)| \le |\gamma_0| + C_2, \quad w(1) = 0.$$

Set $v(x) = C_3(1 - x)$. Then

$$Lv(x) = -C_3 b(x) + C_3(1 - x)c(x) \ge -C_3 M,$$

$$v(0) - \alpha_0 v'(0) = C_3(1 + \alpha_0) \ge C_3, \quad v(1) = 0.$$

It follows from the definition of C_3 and Theorem 3.1 that $v \pm w \geq 0$, i.e., $|w(x)| \leq v(x)$ for all x. Hence

$$|u'(1)| = \lim_{x \to 1^-} \frac{|u(1) - u(x)|}{1 - x} \leq \lim_{x \to 1^-} \frac{v(x)}{1 - x} = C_3,$$

which completes the proof of (iii). □

In the proof of Theorem 3.5 (iii) we did not use Lemma 3.2 since it will yield only a crude bound on $|u'(1)|$ when $|M| \geq 1$.

Thus when $M < 0$, the solution u of (1.2) is bounded independently of δ and has no boundary layers.

For $0 < M \leq 1$ set

$$\sigma_0(M, \delta) = \min\left\{ \frac{1.13}{1 - M}, \frac{\delta[0.13 + \exp(M^{1/(\delta-1)})]}{\delta - 1} \right\}, \quad \sigma_1(M, \delta) = \min\left\{ \frac{1}{1 - M}, \frac{\delta[\exp(M^{1/(\delta-1)}) - 1]}{(\delta - 1)M^{1/(\delta-1)}} \right\}.$$

In these definitions, if $M = 1$ then each σ_i equals the second term in $\{\dots\}$, while if $M \to 1^-$ with δ fixed, then the first term blows up so the second term gives the minimum, and if $\delta \to 1^+$ with $M \in (0, 1)$ fixed, then by L'Hôpital's rule the second term is approximately $\delta/(\delta - 1)$ so the first term gives the minimum.

Lemma 3.6. *For $0 \leq M \leq 1$ one has*

$$0 < E_{\delta-1,\delta}(M) \leq \sigma_0(M, \delta) \quad \text{and} \quad 0 < E_{\delta-1,\delta+1}(M) \leq \sigma_1(M, \delta).$$

Proof. For $0 \leq M < 1$ we have $0 < E_{\delta-1,\delta+1}(M) \leq 1/(1 - M)$ by the argument used to prove (2.26).

For $0 \leq M \leq 1$, letting $\lceil r \rceil$ denote the smallest integer satisfying $\lceil r \rceil \geq r$, we also have the alternative bound

$$E_{\delta-1,\delta+1}(M) \leq \sum_{k=0}^{\lceil \frac{1}{\delta-1} \rceil - 1} \frac{1}{\Gamma(2)} + \sum_{k=\lceil \frac{1}{\delta-1} \rceil}^{\lceil \frac{2}{\delta-1} \rceil - 1} \frac{M^{1/(\delta-1)}}{\Gamma(3)} + \sum_{k=\lceil \frac{2}{\delta-1} \rceil}^{\lceil \frac{3}{\delta-1} \rceil - 1} \frac{M^{2/(\delta-1)}}{\Gamma(4)} + \cdots$$

$$\leq \left(\frac{1}{\delta - 1} + 1 \right) \cdot \frac{1}{M^{1/(\delta-1)}} \left[M^{1/(\delta-1)} + \frac{M^{2/(\delta-1)}}{2!} + \frac{M^{3/(\delta-1)}}{3!} + \cdots \right]$$

$$= \frac{\delta}{(\delta - 1) M^{1/(\delta-1)}} \left[\exp(M^{1/(\delta-1)}) - 1 \right].$$

The arguments for $E_{\delta-1,\delta}(M)$ are similar; observe that $1/\Gamma(\delta) < 1.13$ since $\min_{1 < \delta < 2} \Gamma(\delta) \approx 0.885603$. □

One could derive similar and sharper estimates for $E_{\delta-1,\delta}(M)$ and $E_{\delta-1,\delta+1}(M)$ by imitating the sophisticated analysis of [11, Theorem 1.1], but our approach is simpler and adequate for our purposes.

Define

$$\phi_r(x) = x^\delta E_{\delta-1,\delta+1}(rx^{\delta-1}) \quad \text{for } r \in \mathbb{R} \text{ and } 0 \leq x \leq 1. \tag{3.7}$$

In this notation, the function $\phi_b(x)$ appears prominently in the formula (2.15) of Remark 2.3. For each $r \in \mathbb{R}$, one has $\phi'_r(x) = x^{\delta-1} E_{\delta-1,\delta}(rx^{\delta-1})$ by (2.4) so $\phi_r(0) = \phi'_r(0) = 0$. Hence, using (1.1) and (2.3), we get

$$-D_*^\delta \phi_r(x) + r\phi'_r(x) = -E_{\delta-1,1}(rx^{\delta-1}) + rx^{\delta-1} E_{\delta-1,\delta}(rx^{\delta-1})$$

$$= -\sum_{k=0}^\infty \frac{(rx^{\delta-1})^k}{\Gamma(k(\delta - 1) + 1)} + \sum_{m=0}^\infty \frac{(rx^{\delta-1})^{m+1}}{\Gamma(m(\delta - 1) + \delta)} = -1, \tag{3.8}$$

as $\Gamma(m(\delta - 1) + \delta) = \Gamma((m + 1)(\delta - 1) + 1)$ so the infinite series now cancel each other except for the $k = 0$ term.

When $r \geq 0$ it is easy to see that $\phi'_r(x) > 0$ for $0 < x < 1$. Hence for $r \geq 0$ one has $0 \leq \phi_r(x) \leq \phi_r(1)$ for $0 \leq x \leq 1$.

The barrier functions B_3 and B_4 of Lemma 3.7 and Theorem 3.10 will make use of the property (3.8) with $r = M$.

Lemma 3.7. *Assume that $M \geq 0$. Set*

$$B_3(x) = \max\{|\gamma_0|, |\gamma_1|\} + \|f\|_\infty [\phi_M(1) - \phi_M(x) + \alpha_1 \phi'_M(1)] \quad \text{for } 0 \leq x \leq 1.$$

Then $|u(x)| \leq B_3(x)$ for $x \in [0, 1]$ and

$$\|u\|_\infty \leq \|B_3\|_\infty = B_3(0) = \max\{|\gamma_0|, |\gamma_1|\} + \|f\|_\infty [\phi_M(1) + \alpha_1 \phi'_M(1)]. \tag{3.9}$$

Proof. The discussion preceding the lemma of the properties of ϕ_r implies that $B_3(x) \geq 0$ for $0 \leq x \leq 1$. By (3.8), $\phi_M'(x) > 0$ on $(0, 1)$ and the definition of M, we have

$$LB_3(x) = -D_*^\delta B_3(x) + MB_3'(x) + [b(x) - M]B_3'(x) + c(x)B_3(x)$$
$$= \|f\|_\infty + [M - b(x)]\phi_M'(x)\|f\|_\infty + c(x)B_3(x) \geq \|f\|_\infty$$

and

$$B_3(0) - \alpha_0 B_3'(0) = \max\{|\gamma_0|, |\gamma_1|\} + \|f\|_\infty[\phi_M(1) + \alpha_1\phi_M'(1)] \geq |\gamma_0|,$$
$$B_3(1) + \alpha_1 B_3'(1) = \max\{|\gamma_0|, |\gamma_1|\} \geq |\gamma_1|.$$

It follows from Theorem 3.1 that B_3 is a barrier function for $\pm u$, i.e., $|u(x)| \leq B_3(x)$ on $[0, 1]$. Then (3.9) is immediate from the properties of ϕ_M. □

Theorem 3.8. *Assume that* $0 \leq M \leq 1$. *Then*
(i) $\|u\|_\infty \leq \max\{|\gamma_0|, |\gamma_1|\} + \|f\|_\infty[\sigma_1(M, \delta) + \alpha_1\sigma_0(M, \delta)]$,
(ii) $|u'(0)| \leq (\delta - 1)[|\gamma_0| + \max\{|\gamma_0|, |\gamma_1|\}] + \delta[(e - 1) + \alpha_1(e + 0.13)]\|f\|_\infty$,
(iii) $|u'(1)| \leq \begin{cases} (|\gamma_1| + \|u\|_\infty)/\alpha_1 & \text{if } \alpha_1 > 0, \\ \sigma_0(M, \delta)\max\{\|f\|_\infty + \|c\|_\infty|\gamma_1|, 2[|\gamma_0| + |\gamma_1|]\} & \text{if } \alpha_1 = 0. \end{cases}$

Proof. (i) Combine Lemmas 3.6 and 3.7.

(ii) The boundary condition (1.2b) and the condition (1.3) yield $|u'(0)| \leq (\delta - 1)(|\gamma_0| + \|u\|_\infty)$. Now invoke part (i) and observe that $\sigma_0(M, \delta) \leq \delta(0.13 + e)/(\delta - 1)$ and $\sigma_1(M, \delta) \leq \delta(e - 1)/(\delta - 1)$.

(iii) If $\alpha_1 > 0$, then the result follows from the boundary condition (1.2b). Thus, assume that $\alpha_1 = 0$. Set $w_1(x) = u(x) - u(1) = u(x) - \gamma_1$. Then by (1.2) we have

$$|Lw_1(x)| = |f(x) - c(x)\gamma_1| \leq \|f\|_\infty + \|c\|_\infty|\gamma_1|,$$
$$|w_1(0) - \alpha_0 w_1'(0)| = |\gamma_0 - \gamma_1| \leq |\gamma_0| + |\gamma_1|, \quad w_1(1) = 0.$$

Set $v_1(x) = K[\phi_M(1) - \phi_M(x)]$, where ϕ_M is defined in (3.7) and

$$K = \max\{\|f\|_\infty + \|c\|_\infty|\gamma_1|, [|\gamma_0| + |\gamma_1|]/\phi_M(1)\}.$$

Then $v_1 \geq 0$ and $v_1' \leq 0$, so by (3.8) we get

$$Lv_1(x) = -D_*^\delta v_1(x) + Mv_1'(x) + [b(x) - M]v_1'(x) + c(x)v_1(x) \geq K,$$
$$v_1(0) - \alpha_0 v_1'(0) = K[\phi_M(1) - \phi_M(0) + \alpha_0\phi_M'(0)] = K\phi_M(1), \quad v_1(1) = 0.$$

It follows from the definition of K and Theorem 3.1 that $v_1 \pm w_1 \geq 0$, i.e., $|w_1(x)| \leq v_1(x)$ for all x. Hence

$$|u'(1)| = \lim_{x \to 1^-} \frac{|u(1) - u(x)|}{1 - x} \leq \lim_{x \to 1^-} \frac{v_1(x)}{1 - x} = K\phi_M'(1). \tag{3.10}$$

But $0 < \phi_M'(1) < \sigma_0(M, \delta)$ by Lemma 3.6, and $\phi_M(1) = E_{\delta-1,\delta+1}(M) \geq 1/2$ on taking the first term in the Mittag-Leffler series. Combining these inequalities with (3.10) and the definition of K completes the proof. □

If instead of $0 \leq M \leq 1$ one has the stronger hypothesis that $\|b\|_\infty \leq 1$, then a bound similar to that for the case $\alpha_1 = 0$ in Theorem 3.8 (iii) can be derived quickly using Lemma 3.2, Theorem 3.8 (i), (ii) and (2.25).

For $0 \leq M < 1$, Theorem 3.8 (i) bounds $\|u\|_\infty$ independently of δ. Theorem 3.8 (ii) shows that for $0 \leq M \leq 1$, when δ is near 1 there is no boundary layer in u at $x = 0$. The situation in Theorem 3.8 (iii) is more complicated; to clarify it, we give now a simpler but slightly less sharp corollary of this part of the theorem.

Corollary 3.9. *Assume that* $0 \leq M \leq 1$. *Then for some constant C we have*

$$|u'(1)| \leq \begin{cases} C/(1 - M) & \text{if } 0 \leq M < 1, \\ C/(\delta - 1) & \text{if } M = 1. \end{cases}$$

Proof. Substitute the bound of Theorem 3.8 (i) into Theorem 3.8 (iii) and recall the definitions of $\sigma_1(M, \delta)$ and $\sigma_0(M, \delta)$. □

Theorem 3.8 (ii) shows that as $\delta \to 1^+$ there is no boundary layer at $x = 0$ when $0 \leq M \leq 1$. Corollary 3.9 shows that as $\delta \to 1^+$, no boundary layer is present at $x = 1$ when $0 \leq M < 1$, but when $M = 1$ a boundary layer is possible there and the discussion in Section 2.2.2 shows that then the estimate $|u'(1)| \leq C/(\delta - 1)$ of Corollary 3.9 is sharp in general.

In Theorem 3.8 and Corollary 3.9 we state no result for the case $M > 1$, despite the fact that our arguments can be extended to this case, because the bound provided by Lemma 3.7 is then excessively large.

The next result gives a satisfactory bound on $\|u\|_\infty$ when $M > 1$ under the extra hypothesis that $b > 0$ on $[0, 1]$. In the definition of B_4 below, the expression multiplying $\|f\|_\infty$ imitates closely the coefficient of f that we reported in (2.15).

Theorem 3.10. *Assume that $M > 1$ and $b \geq \underline{b} > 0$ for some constant $\underline{b}$. For $0 \leq x \leq 1$, set*

$$B_4(x) = \max\{|\gamma_0|, |\gamma_1|\} + \frac{\|f\|_\infty \{\alpha_0[\phi_M(1) - \phi_M(x)] + x\phi_M(1) - \phi_M(x) + \alpha_1[(\alpha_0 + x)\phi'_M(1) - \phi_M(x)]\}}{1 + \alpha_0 + \underline{b}\phi_M(1) + \alpha_1[1 + \underline{b}\phi'_M(1)]}.$$

Then $|u(x)| \leq B_4(x)$ for $x \in [0, 1]$ and for some constant C one has

(i) $\|u\|_\infty \leq \|B_4\|_\infty \leq C \min\{\alpha_0, \phi_M(1) + \alpha_1\phi'_M(1)\}$,

(ii) $|u'(0)| \leq C$,

(iii) $|u'(1)| \leq C \min\{\alpha_0, \phi_M(1) + \alpha_1\phi'_M(1)\}$ *if $\alpha_1 > 0$*,

(iv) $|u'(1)| \leq CM^{1/(\delta-1)} \min\{\alpha_0, \phi_M(1)\}$ *if $\alpha_1 = 0$*.

Proof. The expression $x\phi_M(1) - \phi_M(x)$ vanishes at $x = 0, 1$ and its second-order derivative is negative by (2.4), so the expression is positive on $(0, 1)$. It is easy to see that $\phi'_M(1) > \phi_M(1)$ so we also have $x\phi'_M(1) - \phi_M(x) > 0$ on $(0, 1)$. Consequently, $B_4(x) \geq 0$ for $0 \leq x \leq 1$.

Now

$$B_4(0) - \alpha_0 B'_4(0) = \max\{|\gamma_0|, |\gamma_1|\} \geq |\gamma_0| \quad \text{and} \quad B_4(1) + \alpha_1 B'_4(1) = \max\{|\gamma_0|, |\gamma_1|\} \geq |\gamma_1|.$$

For $0 < x < 1$, by (3.8) one has

$$LB_4(x) = c(x)B_4(x) + \frac{\|f\|_\infty \{\alpha_0 + b(x)\phi_M(1) + 1 + \alpha_1[b(x)\phi'_M(1) + 1] + (M - b)\phi'_M(x)(1 + \alpha_0 + \alpha_1)\}}{1 + \alpha_0 + \underline{b}\phi_M(1) + \alpha_1[1 + \underline{b}\phi'_M(1)]} \geq \|f\|_\infty.$$

Thus B_4 is a barrier function for $\pm u$ by Theorem 3.1. Hence, recalling that $\alpha_0 \geq 1/(\delta - 1) > 1$, for some C we get

$$\|u\|_\infty \leq \|B_4\|_\infty \leq \max\{|\gamma_0|, |\gamma_1|\} + \frac{\|f\|_\infty \{\alpha_0\phi_M(1) + \phi_M(1) + \alpha_1(\alpha_0 + 1)\phi'_M(1)\}}{1 + \alpha_0 + \underline{b}\phi_M(1) + \alpha_1[1 + \underline{b}\phi'_M(1)]}$$

$$\leq C\left[1 + \frac{\alpha_0\phi_M(1) + \alpha_0\alpha_1\phi'_M(1)}{\alpha_0 + \phi_M(1) + \alpha_1\phi'_M(1)}\right] \leq C \min\{\alpha_0, \phi_M(1) + \alpha_1\phi'_M(1)\}.$$

This proves (i).

For (ii) and (iii), use the result of (i) and the boundary conditions (1.2b).

To prove (iv), define the function

$$\tilde{u}(x) = u(x) - (1 - x)\frac{\gamma_0 - \gamma_1}{1 + \alpha_0} - \gamma_1, \tag{3.11}$$

which is the solution of the problem

$$L\tilde{u}(x) = \tilde{f}(x) \quad \text{for } x \in (0, 1),$$

$$\tilde{u}(0) - \alpha_0 \tilde{u}'(0) = 0, \quad \tilde{u}(1) = 0,$$

with

$$\tilde{f}(x) = f(x) + [b(x) - (1 - x)c(x)]\frac{\gamma_0 - \gamma_1}{1 + \alpha_0} - \gamma_1 c(x).$$

	$\delta = 1.1$	$\delta = 1.01$	$\delta = 1.001$	$\delta = 1.0001$	$\delta = 1.00001$	$\delta = 1.000001$	$\delta = 1.0000001$
$(\delta - 1)\max_{0 \le j \le N}|u_j|$	5.8374	7.6438	7.6043	7.6004	7.6000	7.6000	7.6000
$(\delta-1)|u_N-u_{N-1}|/(x_N-x_{N-1})$	5.0045	7.5954	7.5795	7.5777	7.5775	7.5775	7.5775

Table 1. Verifying sharpness of Theorem 3.10 (i), (iii).

Applying the barrier function B_4 of (i) above to $\tilde{u}$ yields

$$|\tilde{u}(x)| \le \frac{\|\tilde{f}\|_\infty \{\alpha_0[\phi_M(1) - \phi_M(x)] + x\phi_M(1) - \phi_M(x)\}}{1 + \alpha_0 + \underline{b}\phi_M(1)} \quad \text{for } x \in [0, 1].$$

Hence

$$|\tilde{u}'(1)| = \lim_{x \to 1^-} \frac{|\tilde{u}(1) - \tilde{u}(x)|}{1 - x} = \lim_{x \to 1^-} \frac{|\tilde{u}(x)|}{1 - x} \le \frac{\|\tilde{f}\|_\infty[(\alpha_0 + 1)\phi_M'(1) + \phi_M(1)]}{1 + \alpha_0 + \underline{b}\phi_M(1)}.$$

Recalling the identity (2.25), we have

$$\phi_M(1) = E_{\delta-1,\delta+1}(M) = [E_{\delta-1,2}(M) - 1]/M, \quad \phi_M'(1) = E_{\delta-1,\delta}(M) = [E_{\delta-1,1}(M) - 1]/M$$

and (2.19) implies that $E_{\delta-1,1}(M)/E_{\delta-1,2}(M) \approx M^{1/(\delta-1)}$ as $\delta \to 1^+$. Consequently,

$$|\tilde{u}'(1)| \le \frac{C\alpha_0\,\phi_M'(1)}{\alpha_0 + \phi_M(1)} \le \frac{C\alpha_0\,M^{1/(\delta-1)}\phi_M(1)}{\alpha_0 + \phi_M(1)} \le CM^{1/(\delta-1)}\min\{\alpha_0, \phi_M(1)\}.$$

Part (iv) of the theorem follows from this estimate and (3.11). □

Thus, in the case $M > 1$ with $b > 0$, Theorem 3.10 (ii) shows that there is no boundary layer at $x = 0$ when δ is near 1, even though $\|u\|_\infty$ may be unbounded as $\delta \to 1^+$ (see Section 2.2.1).

The accurate approximations (2.20) and (2.21) show that Theorem 3.10 (iii), (iv) is sharp for constant $b > 1$.

The next example shows numerically that the bounds of Theorem 3.10 (i), (iii) are sharp for a variable-coefficient problem when $\alpha_0 = 1/(\delta - 1) < \phi_M(1)$.

Example 3.11. Consider the test problem

$$-D_*^\delta u(x) + (x + 0.2)u'(x) = 2(3 + x) \quad \text{for } x \in (0, 1), \tag{3.12a}$$

$$u(0) - \frac{1}{\delta - 1}u'(0) = 0.4, \quad u(1) + u'(1) = 1.7. \tag{3.12b}$$

The exact solution of this problem is unknown. Observe that $M = 1.2$ so $\phi_M(1) \gg 1/(\delta - 1) = \alpha_0$ by an inequality similar to (2.17). To check the bounds of Theorem 3.10 (i), (iii), we use the finite difference scheme of [15] to compute an approximate numerical solution $\{u_j\}_{j=0}^N$ of (3.12) on a uniform mesh $\{x_j := j/N\}_{j=0}^N$ with $N = 2048$ for various values of δ close to 1, then evaluate $(\delta - 1)\max_{0 \le j \le N}|u_j|$ and $(\delta - 1)|u_N - u_{N-1}|/(x_N - x_{N-1})$ to approximate $(\delta - 1)\|u\|_\infty$ and $(\delta - 1)|u'(1)|$.

As each row of Table 1 is approximately constant as $\delta \to 1^+$, the bounds of Theorem 3.10 (i), (iii) are sharp for this example.

4 Conclusions

We considered a two-point boundary value problem whose leading term is a Caputo fractional derivative of order δ with $1 < \delta < 2$. The dependence of the solution on the parameter δ has not previously been investigated analytically, despite a growing interest in the research literature in the numerical solution of problems with variable fractional derivatives. By considering first the special case of a constant-coefficient operator, for which the solution can be determined explicitly, we showed that when δ is near 1, the solution of the boundary value problem may exhibit a boundary layer at the endpoint $x = 1$ of the domain. Moving on to the general case of a variable-coefficient differential operator, we then determined conditions on the data of

the problem under which boundary layers at each endpoint ($x = 0, 1$) cannot occur. This analysis showed that a crucial parameter in the presence or absence of a boundary layer at $x = 1$ is the quantity $M := \max_{x \in [0,1]} b(x)$, where b is the coefficient of the first-order term in the differential operator.

In all cases considered, we showed that $|u'(0)| \leq C$, i.e., no boundary layer in u appears at $x = 0$ when δ is near 1. The only data regime where this bound is not guaranteed by our theory (Theorems 3.3 and 3.10) is when $\min_{[0,1]} c(x) = 0$ and $M > 1$ without the additional property that $b > 0$ on $[0, 1]$, but our numerical experience (using the finite difference method of [15]) is that in this case also no boundary layer appears at $x = 0$. At $x = 1$, our theory proves rigorously that when δ is near 1, no boundary layer appears in u if $M < 1$, but one can have such layers when $M \geq 1$. This agrees with our numerical experience.

This analysis of the solution u of (1.2) leads naturally to the question: can one construct a numerical method that will yield an accurate approximation of u when it has a boundary layer at $x = 1$?

Correction Statement (December 6, 2014): In the electronic version of this paper (published September 26, 2014), the statement and proof of Theorem 3.10 parts (i) and (iii) were incorrect owing to an error of the authors. The expression $\min\{\alpha_0, \phi_M(1)\}$ in parts (i) and (iii) of the original electronic version is now replaced by $\min\{\alpha_0, \phi_M(1) + \alpha_1 \phi'_M(1)\}$. Note that in Theorem 3.10 (iv), the same expression $\min\{\alpha_0, \phi_M(1)\}$ appears and should not be changed.

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