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Boundary asymptotics of the relative Bergman kernel metric for hyperelliptic curves

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Abstract: We survey variations of the Bergman kernel and their asymptotic behaviors at degeneration. For a Legendre family of elliptic curves, the curvature form of the relative Bergman kernel metric is equal to the Poincaré metric on $\mathbb{C} \setminus \{0, 1\}$. The cases of other elliptic curves are either the same or trivial. Two proofs depending on elliptic functions' special properties and Abelian differentials' Taylor expansions are discussed, respectively. For a holomorphic family of hyperelliptic nodal or cuspidal curves and their Jacobians, we announce our results on the Bergman kernel asymptotics near various singularities. For genus-two curves particularly, asymptotic formulas with precise coefficients involving the complex structure information are written down explicitly.

Keywords: variation of Bergman kernel; degeneration of hyperelliptic curve; node; cusp

MSC: Primary 32A25; Secondary 32G15, 32U05, 30F45

1 Introduction: old and new results

This paper is a survey on the Bergman kernel, a reproducing kernel (determined by the complex structure) of the space of L^2 holomorphic top-degree forms on a connected complex manifold, with a focus on its variation (in particular its asymptotic behaviors) at degeneration. For a compact manifold X, let L be a positive line bundle equipped with a Hermitian metric h and let $\{s_1, \dots, s_N\}$ be an orthonormal basis of $H^0(X, L)$. Then the Bergman kernel for L over X (which does not depend on choices of local coordinates) is defined as

$$B := \sum_{j=1}^{N} |s_j|_h^2. \tag{1.1}$$

The Bergman kernel plays big roles in the study of several complex variables and complex geometry. As the complex structure deforms, for pseudoconvex domains the variation of the Bergman kernels was initially studied by Maitani & Yamaguchi [20] and generalized to higher dimensional cases by Berndtsson [1]. For general results on arbitrary dimensional Stein manifolds and complex projective algebraic manifolds, see Berndtsson [2], Tsuji [26], Berndtsson & Păun [6] and Păun & Takayama [25], etc. These important results indicating semi-positivity properties of the relative canonical bundles recently turn out to have close relations with the Ohsawa-Takegoshi L^2 extension theorem [8, 21], space of Kähler metrics [3], etc. For simplicity (for each fixed $l = 1, 2, \dots$), let us consider the one-dimensional case, namely a holomorphic family of Riemann surfaces $X_\lambda^{(l)}$ parametrized by one complex variable λ. The Bergman kernel on $X_\lambda^{(l)}$ can be written as $B_\lambda^{(l)} = k_\lambda^{(l)}(z) dz \wedge d\bar{z}$, in some local coordinate z. Then, the above log-plurisubharmonic variation results imply that

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$\log k_\lambda^{(l)}(z)$ is plurisubharmonic in (λ, z) and particularly guarantee the following semi-positivity:

$$L_{\lambda,z}^{(l)} := \sqrt{-1}\,\partial_\lambda\bar{\partial}_\lambda \log k_\lambda^{(l)}(z) \geq 0, \tag{1.2}$$

whenever $X_\lambda^{(l)}$ is smooth. (The last inequality is a restricted version in the sense that the transversal direction is looked at, and the strict positivity of $L_{\lambda,z}^{(l)}$ relates to the hyperellipticity and Weierstrass points, cf. [4, 19].) Now suppose some $X_{\lambda_0}^{(l)}$ is a singular algebraic curve, and thus a possibly interesting question is to study $\log k_\lambda^{(l)}(z)$ or $L_{\lambda,z}^{(l)}$ and their asymptotic behaviors as λ approaches λ_0. Near the degenerate boundary, the limiting case is not fully understood. In general, there are at least three approaches: elliptic functions, Taylor expansions and pinching coordinates.

1.1 Elliptic curves with nodes or cusps

The so-called Legendre family of elliptic curves $X_\lambda^{(1)} := \{y^2 = x(x-1)(x-\lambda)\}$ gives a general description of genus one compact Riemann surfaces, whose moduli space is $\mathbb{C} \setminus \{0, 1\}$. Also, $X_\lambda^{(1)}$ degenerates to a singular curve with a node as λ tends to 0, 1 or ∞. Using the Weierstrass-$\wp$ function's coordinate parameterization and the elliptic modular lambda function's Taylor expansion, the author in [9] showed that $L_{\lambda,z}^{(1)}$ is strictly positive everywhere inside the moduli space and thus defines a Kähler metric which blows up and has hyperbolic growth as $\lambda \to 0$. In comparison, the Poincaré hyperbolic metric $\omega_{\mathbb{D}^*}$ on the punctured unit disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ has exactly the same asymptotic behavior near the origin. However, an explicit four-term asymptotic expansion formula as $\lambda \to 0$, which gives more geometrical intuitions, shows that $L_{\lambda,z}^{(1)}$ and $\omega_{\mathbb{D}^*}$ are indeed not the same [10].

Theorem 1.1. *For $\lambda \in \mathbb{C} \setminus \{0, 1\}$, in the local coordinate z write $B_\lambda^{(1)} = k_\lambda^{(1)}(z)dz \wedge d\bar{z}$. Then, as $\lambda \to 0$,*

$$L_{\lambda,z}^{(1)} = \frac{\sqrt{-1}d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}\left(1 + 2\frac{\log 16}{\log|\lambda|} + 3\left(\frac{\log 16}{\log|\lambda|}\right)^2 + 4\left(\frac{\log 16}{\log|\lambda|}\right)^3 + O\left(\frac{1}{(\log|\lambda|)^4}\right)\right).$$

It turns out that the subleading terms in the asymptotic expansion contain more "logarithmic" information, slowing down the growth order at infinity. As we can see, even though the 2nd and 4th terms tend to $-\infty$, the left hand side which is mainly affected by the leading term, still tends to $+\infty$. It is expected that each term should have certain geometrical interpretations. In general, this result is much related to the variation of Hodge structures, especially Schmid's famous Nilpotent Orbit Theorem (see [16, 23]). Then, how about the cases of other families of elliptic curves (degenerating to a singular one with a node or a cusp) where the special elliptic function method could not apply? The answer is that we can determine accurately not only the leading term but also the subleading terms by a method based on the Taylor expansions of Abelian differentials (see [12, 14]). Using this alternative approach, we deal with another nodal family of elliptic curves $X_\lambda^{(2)} := \left\{y^2 = (x-1)(x^2-\lambda)\right\}$, and the result is the same as our previous one for $X_\lambda^{(1)}$.

Theorem 1.2. *For $\lambda \in \mathbb{C} \setminus \{0, 1\}$, in the local coordinate z write $B_\lambda^{(2)} = k_\lambda^{(2)}(z)dz \wedge d\bar{z}$. Then, as $\lambda \to 0$,*

$$L_{\lambda,z}^{(2)} \sim \frac{\sqrt{-1}d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}.$$

Here the symbol " $\sim$ " means that the (element-wise non-zero) ratio of both sides (possibly $g \times g$ matrices) tends to 1 as $\lambda \to 0$. In other words, it represents the leading term asymptotics. For the cuspidal degeneration, the type of singularities surely determines various boundary behaviors: either trivial with a constant period or reducible to the Legendre family case, depending on the complex structure. Here we deal with two families of degenerate Riemann surfaces, namely $X_\lambda^{(3)} := \left\{y^2 = x(x^2-\lambda)\right\}$ and $X_\lambda^{(4)} := \left\{y^2 = x(x-\lambda)(x-\lambda^2)\right\}$, which have constant and non-constant periods, respectively.

Theorem 1.3. *For $\lambda \in \mathbb{C} \setminus \{0\}$, in the local coordinate z write $B_\lambda^{(3)} = k_\lambda^{(3)}(z)dz \wedge d\bar{z}$. Then, as $\lambda \to 0$,*

$$L_{\lambda,z}^{(3)} \equiv 0.$$

Theorem 1.4. *For $\lambda \in \mathbb{C} \setminus \{0\}$, in the local coordinate z write $B_\lambda^{(4)} = k_\lambda^{(4)}(z)dz \wedge d\bar{z}$. Then, as $\lambda \to 0$,*

$$L_{\lambda,z}^{(4)} \sim \frac{\sqrt{-1}d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}.$$

For the elliptic curve $X_\lambda^{(4)}$ with a non-constant period, a possible explanation for the appearance of hyperbolic growth might be that we can change coordinates and reduce the family $X_\lambda^{(4)}$ to $X_\lambda^{(1)}$. This interesting connection between the cusp and node cases in some way strengthens the importance of a Legendre family.

1.2 Hyperelliptic and general curves with nodes

For a family of genus two curves $X_\lambda^{(5)} := \{y^2 = x(x-\lambda)(x-1)(x-a)(x-b)\}$ degenerating to a singular one with a non-separating node, where a, b, λ are distinct numbers in $\mathbb{C} \setminus \{0, 1\}$ satisfying $1 < |a| < |b|$, we determine the precise coefficient as follows.

Theorem 1.5. *In the local coordinate $z = \sqrt{x}$ on $X_\lambda^{(5)}$, write its Bergman kernel as $B_\lambda^{(5)} = k_\lambda^{(5)}(z)dz \wedge d\bar{z}$. Then, as $\lambda \to 0$ for $0 \neq |z| < \sqrt{\frac{c_1}{|c_2|}}$, it holds that*

$$\partial\bar{\partial}\log k_\lambda^{(5)}(z) \sim \left|\frac{1}{z^2} - \frac{c_2}{c_1}\right|^2 \frac{c_1 d\lambda \wedge d\bar{\lambda}}{2|\lambda|^2(-\log|\lambda|)^3} + \frac{(c_2 - c_1 z^{-2})\bar{z}^{-3} d\lambda \wedge d\bar{z}}{\lambda(-\log|\lambda|)^2} + \frac{(c_2 - c_1\bar{z}^{-2})z^{-3} dz \wedge d\bar{\lambda}}{\bar{\lambda}(-\log|\lambda|)^2} + \frac{4c_1 dz \wedge d\bar{z}}{|z|^6(-\log|\lambda|)},$$

where $c_2 = \operatorname{Im}\left\{\int_a^b \frac{\sqrt{ab}dx}{x\sqrt{(x-1)(x-a)(x-b)}}\right\}$ and $c_1 := \operatorname{Im}\tau_{a,b}$, such that $\tau_{a,b}$ is the period of the elliptic curve $\{y^2 = (x-1)(x-a)(x-b)\}$.

For general curves near both separating and nonseparating nodes, Habermann & Jost [18] obtained by using the pinching-coordinate method the asymptotic results for the Bergman kernel and its induced L^2 metric on the Teichmuller space. The leading term in our results here can be interpreted as special cases of Proposition 3.2 (ii) in [18] for $m = 1$. For the asymptotics of the period matrices, see [15, 27]. Without caring precise coefficients, our results can serve as an alternative proof to these facts. Nevertheless, our results are based on a different method and has an advantage that the precise coefficients can be explicitly written down, especially if one wants to know how the given complex structures relate to these coefficients, which usually indicate the geometry of base varieties and their singularities. By embedding each $X_\lambda^{(5)}$ to its Jacobian, we observe that the curvature form of the relative Bergman kernel metric on their Jacobians in the transversal direction has hyperbolic growth again (see (4.2) in Chapter 4). This can be regarded as a higher dimensional generalization of the leading term in Theorem 1.1.

1.3 Hyperelliptic curves with cusps: Case I

Let $p(x)$ be a polynomial of degree at least 2 with roots of distinct absolute values different from $|\lambda|$ and 0. For a family of hyperelliptic curves $X_\lambda^{(6)} := \{y^2 = x(x^2 - \lambda) \cdot p(x)\}$, degenerating to a singular one with a cusp as $\lambda \to 0$, we announce our result on asymptotic behaviors of the Bergman kernel as follows (see [13] for the details of proofs).

Theorem 1.6. *In the local coordinate $z = \sqrt{x}$ on $X_\lambda^{(6)}$, write its Bergman kernel as $B_\lambda^{(6)} = k_\lambda^{(6)}(z)dz \wedge d\bar{z}$. Then, as $\lambda \to 0$ for small $|z| \neq 0$, it holds that*

$$\log k_\lambda^{(6)}(z) = \log\frac{4 + O(z^4)}{|z^4 \cdot p(z^2)|} + \frac{\operatorname{Re}\left(\sum_{j=2}^g z^{2(j-1)}\right)}{1 + O(z^4)} \cdot O(\lambda^{\frac{1}{4}}).$$

Notice that both the first and second terms above are harmonic with respect to λ, which vanishes under the $\partial_\lambda \bar{\partial}_\lambda$ operator. Also, it seems that the Jacobian varieties of $X_\lambda^{(6)}$ remain being manifolds (do not degenerate), as $\lambda \to 0$.

1.4 Hyperelliptic curves with cusps: Case II

For a family of genus two curves $X_\lambda^{(7)} := \left\{y^2 = x(x-\lambda)(x-\lambda^2)(x-a)(x-b)\right\}$, where a, b, λ are distinct complex numbers in $\mathbb{C} \setminus \{0\}$ satisfying $|a| < |b|$, we determine precise coefficients as follows.

Theorem 1.7. *In the local coordinate $z = \sqrt{x}$ on $X_\lambda^{(7)}$, write its Bergman kernel as $B_\lambda^{(7)} = k_\lambda^{(7)}(z) dz \wedge d\bar{z}$. Then, as $\lambda \to 0$ for small $|z| \neq 0$, it holds that*

$$\partial\bar{\partial} \log k_\lambda^{(7)}(z) \sim \frac{c}{|z|^4} \left\{ \frac{d\lambda \wedge d\bar{\lambda}}{2|\lambda|^2(-\log|\lambda|)^3} - \frac{d\lambda \wedge d\bar{z}}{\bar{z}\lambda(-\log|\lambda|)^2} - \frac{dz \wedge d\bar{\lambda}}{z\bar{\lambda}(-\log|\lambda|)^2} + \frac{4dz \wedge d\bar{z}}{|z|^2(-\log|\lambda|)} \right\},$$

where $c := \pi \operatorname{Im}\left\{\tau_{a,b}\right\} > 0$, such that $\tau_{a,b}$ is the period (scalar) of the elliptic curve $\{y^2 = x(x-a)(x-b)\}$.

For a family of hyperelliptic curves $X_\lambda^{(8)} := \left\{y^2 = x(x-\lambda^2)(x-\lambda) \cdot p(x)\right\}$, where $p(x)$ is a polynomial of degree at least 2 with distinct roots different from λ and 0, the result is more or less the same. We find that the curvature form transversally induces an incomplete metric on the parameter space. For the Jacobian varieties of $X_\lambda^{(8)}$, as $\lambda \to 0$, hyperbolic growth appears again! (See the details in Chapter 6 and [13].)

2 Backgrounds: Bergman kernel's variation and period matrix

As the complex structure changes, the variation of the Bergman kernels was initially studied by Maitani & Yamaguchi [20], who started from the Bergman-Schiffer formula and the Hopf lemma (regarding the Green function as a defining function), and proved the plurisubharmonicity results concerning the Robin constants and logarithms of the Bergman kernels by using differential geometrical computations.

Theorem 2.1 (Maitani-Yamaguchi). *Let Ω be a pseudoconvex domain in $\mathbb{C}_z \times \mathbb{C}_t$ with a smooth boundary. Let $B_t(z)$ be the Bergman kernel function of $\Omega \cap (\mathbb{C}_z \times \{t\})$. Then $\log B_t(z)$ is a plurisubharmonic function on Ω.*

After that, generalizations of [20] to higher dimensional cases were made by Berndtsson [1] using L^2 methods. Moreover, Guan & Zhou [17, 22] provided an alternative proof of the log-plurisubharmonic variation of Bergman kernels by using the Ohsawa-Takegoshi L^2 extension theorem with optimal constant, regarded as a sub-mean-value property of the fiber-wise Bergman kernels. Conversely, Berndtsson & Lempert [5] showed that the log-plurisubharmonic variation of Bergman kernels can give a proof of rather general versions of the Ohsawa-Takegoshi theorem.

Theorem 2.2 (Berndtsson). *Let D be a pseudoconvex domain in $\mathbb{C}_z^n \times \mathbb{C}_t^k$, and let Φ be a plurisubharmonic function on D. For each t, set $D_t := D \cap \left(\mathbb{C}_z^n \times \{t\}\right)$ and $\Phi_t := \Phi|_{D_t}$. Let $B_t(z)$ be the Bergman kernel of the space $A^2(D_t, \Phi_t) := \left\{ f \in \mathcal{O}(D_t) | \int_{D_t} e^{-\Phi_t} |f|^2 < +\infty \right\}$. Then $\log B_t(z)$ is a plurisubharmonic function on D.*

An equivalent definition of the Bergman kernel on compact Riemann surfaces can be given by the Riemann period matrix. As a natural generalization of the genus-two case, a hyperelliptic compact Riemann surface of genus $g \geq 2$ may be written as $X_\lambda := \{y^2 = P_\lambda(x)\}$, with $P_\lambda(x)$ being a polynomial of degree> 4 having distinct roots λ, a_j, such that $\lambda \in \mathbb{C} \setminus \{0, \cup_j a_j\}$. There exists a globally defined basis $\omega_1 := dx/y$, $\omega_2 := x dx/y$, ... , $\omega_g := x^{g-1} dx/y$ for the Hilbert space of L^2 holomorphic 1-forms. To get the Bergman kernel (the sum of orthogonal base wedging their conjugates), one needs to ortho-normalize the bases by L^2 inner products via the Gram-Schmidt process. A hyperelliptic curve can be obtained as a double covering of the Riemann sphere, cutting itself at the intervals $[0, \lambda]$, $[a_1, a_2]$, ... We get two types of cycles δ_i and γ_j which forms a homologous basis of the curve, and their intersection number is $\delta_i \cdot \gamma_j = 1$, for $i = j$. Each δ_i is a circle containing the interval $[0, \lambda]$, $[a_1, a_2]$, ..., respectively, and each γ_j switches from one sheet to another. Define two $g \times g$ matrices $A_{i,j} = \int_{\delta_j} \omega_i$ (invertible) and $B_{i,j} = \int_{\gamma_j} \omega_i$. Then, a new matrix $Z := A^{-1}B$ is symmetric and has a positive imaginary part, i.e., $\operatorname{Im} Z > 0$, due to the Hodge-Riemann bilinear relation and the Stokes formula. Thus, the Bergman kernel is equivalently defined as

$$B_\lambda := \sum_{i,j=1}^{g} ((\operatorname{Im} Z)^{-1})_{ij} \omega_i \wedge \overline{\omega_j}.$$

And after changing the coordinate, the coefficients of the Bergman kernel are the same up to a harmonic function (the determinant of the Jacobian) which is killed by the $\partial_\lambda \bar{\partial}_\lambda$ operator. In other words, $L_{\lambda,z}$ is well-defined.

3 Degenerations of the Bergman kernel on various elliptic curves

It is known that for an elliptic curve $E_\lambda := \{y^2 = p_\lambda(x)\}$, $p_\lambda(x)$ being a polynomial of x depending on λ of degree 3 or 4, there exists a globally defined holomorphic 1-form $\omega := dx/y$. After normalizing by the L^2 inner product $\omega_0 := C_\lambda^{-1/2}\omega$ will then contribute to the Bergman kernel on E_λ (for its canonical bundle), which is a $(1, 1)$-form locally written as $K_\lambda = C_\lambda^{-1}\omega \wedge \overline{\omega}$, where $C_\lambda := \frac{\sqrt{-1}}{2} \int_E \omega \wedge \overline{\omega} > 0$ is a positive real number depending only on λ. Since $\partial_\lambda \bar{\partial}_\lambda \log C_\lambda^{-1} = \partial_\lambda \bar{\partial}_\lambda \log k_\lambda(\cdot)$, for any local coefficient (function) $k_\lambda(\cdot)$, only the term C_λ matters for the curvature form.

3.1 A Legendre family of elliptic curves

In particular, if $p_\lambda(x) := x(x-1)(x-\lambda)$, we get a Legendre family of elliptic curves X_λ. For small $\lambda \to 0$, a double covering of the Riemann sphere can be made by cutting itself from 0 to λ, and from 1 to ∞. Then, we get two cycles δ and γ forming a homologous basis of the elliptic curve, and containing $\{0, \lambda\}$ and $\{\lambda, 1\}$, respectively. The bilinear relation implies that $C_\lambda = \operatorname{Im}\left(\int_\gamma \omega \cdot \int_\delta \overline{\omega}\right)$. On the one hand, we observe that $L_{\lambda,z}$ is the Poincaré hyperbolic metric of $\mathbb{C} \setminus \{0, 1\}$, whose curvature is identically equal to "−4". This result seems to suggest a connection between the Bergman kernel's variation and the moduli space's hyperbolic metric. On the other hand, a four-term expansion formula of the Poincaré hyperbolic metric of $\mathbb{C} \setminus \{0, 1\}$ can be obtained as a corollary. The leading term of the above expansion formula implies that near the origin $\omega_{0,1}$ is asymptotically similar to $\omega_{\mathbb{D}^*}$, and the negative second term seems to support that the latter is bigger. Actually, it always holds that $\omega_{0,1} \leq \omega_{\mathbb{D}^*}$, wherever inside $\mathbb{D}^*$ (cf. [24]).

Corollary 3.1. *Let $\omega_{0,1}$ denote the Poincaré hyperbolic metric of $\mathbb{C} \setminus \{0, 1\}$. Then, as $\lambda \to 0$,*

$$\omega_{0,1} = \frac{\sqrt{-1} d\lambda \otimes d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}\left(1 + 2\frac{\log 16}{\log|\lambda|} + 3\left(\frac{\log 16}{\log|\lambda|}\right)^2 + 4\left(\frac{\log 16}{\log|\lambda|}\right)^3 + O\left(\frac{1}{(\log|\lambda|)^4}\right)\right).$$

Near the moduli space boundary points 1 and ∞, we also achieved in [11] explicit asymptotic formulas of the relative Bergman kernel metrics and its curvature forms. The proof of the following Theorem 3.1 is mainly due to the elliptic modular lambda function's special properties (in particular its behavior under the composition with inverse or translation mappings), which are also used in [9].

Theorem 3.1. *Under the same assumptions as in Theorem 1.1, it follows that*
(i) *as $\lambda \to 1$, $\log k_\lambda^{(1)}(z) \sim \log(-\log|\lambda - 1|)$,*
(ii) *and as $\lambda \to \infty$, $\log k_\lambda^{(1)}(z) \sim \log(\log|\lambda|)$.*

3.2 Other families of elliptic curves

For other families of elliptic curves degenerating to singular ones with a node or a cusp, we observe that it is either trivial with a constant period or reducible to the Legendre family case. The approach we use is based on Abelian differentials' Taylor expansions and elliptic curves' periods, which do not depend on special elliptic functions. Firstly, if $p(x) = (x-1)(x^2-\lambda)$, on the elliptic curve $X_\lambda^{(2)}$ $(\lambda \in \mathbb{C} \setminus \{0, 1\})$, δ is a big circle centered at the origin containing $-\sqrt{\lambda}$ and $\sqrt{\lambda}$, and γ contains $\sqrt{\lambda}$ and 1. Secondly, if $p(x) = x(x^2-\lambda)$, on the elliptic curve $X_\lambda^{(3)}$ $(\lambda \in \mathbb{C} \setminus \{0\})$, δ contains $-\sqrt{\lambda}$ and 0, and γ contains 0 and $\sqrt{\lambda}$. Thirdly, if $p(x) = x(x-\lambda)(x-\lambda^2)$, on the elliptic curve $X_\lambda^{(4)}$ $(\lambda \in \mathbb{C} \setminus \{0\})$, δ contains 0 and λ^2, and γ contains λ^2 and λ. Lastly, Hodge-Riemann bilinear relations hold for all these above cases. We finally remark that the importance of this alternative approach is that it really works for higher genus cases (see [13]), where properties of special elliptic functions could not be applied to and this becomes the main difficulty.

4 Bergman kernel near a node: genus-two and general curves

For a holomorphic family of genus-two curves, we obtain asymptotic formulas (with precise coefficients involving the complex structure information) of the relative Bergman kernel metric near a node. The curvature form transversally tends to an incomplete metric on the parameter space, and these results are different from the elliptic curves.

Lemma 4.1. *Let $Z(\lambda)$ denote the period matrix of $X_\lambda^{(5)}$. Then, as $\lambda \to 0$,*

$$\operatorname{Im} Z(\lambda) \sim \frac{1}{\pi} \begin{pmatrix} -\log|\lambda| & c_2 \\ c_2 & c_1 \end{pmatrix}.$$

Proof of Lemma 4.1. It holds that $A(\lambda) \sim \begin{pmatrix} \frac{-2\pi}{\sqrt{ab}} & 0 \\ 0 & C_{ab} \end{pmatrix}$, $B(\lambda) \sim \begin{pmatrix} \frac{-2\log\lambda}{\sqrt{-ab}} & C'_{ab} \\ \frac{-2}{\sqrt{-ab}} & C''_{ab} \end{pmatrix}$, where $C_{ab} = \int_1^a \frac{-2dx}{\sqrt{(x-1)(x-a)(x-b)}}$, $C'_{ab} = \int_a^b \frac{-2dx}{x\sqrt{(x-1)(x-a)(x-b)}}$ and $C''_{ab} = \int_a^b \frac{-2dx}{\sqrt{(x-1)(x-a)(x-b)}}$. □

Proof of Theorem 1.5. By Lemma 4.1, it holds that

$$\log k_\lambda^{(5)}(z) = \log\left(\frac{4\pi\cdot|z|^2}{c_1\cdot|(z^2-1)(z^2-a)(z^2-b)|}\right) + \left|\frac{1}{z^2} - \frac{c_2}{c_1}\right|^2 \cdot \frac{c_1}{-\log|\lambda|} + O\left(\frac{1}{(-\log|\lambda|)^2}\right), \tag{4.1}$$

as $\lambda \to 0$ for $0 \neq |z| < \sqrt{\frac{c_1}{|c_2|}}$. Since the first term in the last similarity relation doesn't depend on λ and vanishes under the $\partial_\lambda\bar{\partial}_\lambda$ operator, the curvature form in Theorem 1.5 then follows from (4.1). □

Notice that the coefficients in front of $(-\log|\lambda|)^{-1}$ is strictly positive, for small z, and $\log k_\lambda^{(5)}(z)$ has Lelong number zero at the origin. This result can be generalized to the case of a family of hyperelliptic curves $Y_\lambda := \{y^2 = x(x-\lambda)p(x)\}$, where $p(x)$ is a polynomial of degree at least 3 with distinct roots a_j and $\lambda \in \mathbb{C}\setminus\{0, \cup_j a_j\}$, which degenerates to a singular one $Y_0 := \{y^2 = x^2 p(x)\}$ with a non-separating node. We remark that the Bergman kernel on the Jacobian of Y_λ of genus g can be written as $(\det \operatorname{Im} Z(\lambda))^{-1} dW \wedge d\overline{W} =: K_\lambda(W) dW \wedge d\overline{W}$, in the local coordinate $W \in \mathbb{C}^g$ near the singularity. Thus, it holds that $\log K_\lambda(W) \sim -\log(-\log|\lambda|)$, as $\lambda \to 0$, where $\det \operatorname{Im} Z(\lambda) \to +\infty$. This does not depend on a_j, which determines the complex structure of Y_λ. Therefore, as $\lambda \to 0$ for $|W| \neq 0$ small, it follows that

$$\partial_\lambda\bar{\partial}_\lambda \log K_\lambda(W) \sim \frac{d\lambda \wedge d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2}. \tag{4.2}$$

For general curves, the advantage of a pinching coordinate is that it does not prescribe the complex structure and it works not only for non-separating but also for separating nodes. Comparing [18] and our result, there seems to be no essential difference for the Bergman kernels near non-separating nodes on general degenerate curves and hyperelliptic degenerate curves. However, there is a big difference in degeneration between genus-one and higher-genus curves, probably due to the Uniformization Theorem.

5 Bergman kernel near a cusp I: genus-two and hyperelliptic curves

For $X_\lambda^{(6)}$, we can prove the following lemma by analyzing the asymptotics of the matrices A and B respectively.

Lemma 5.1. *Let $Z(\lambda)$ denote the period matrix of $X_\lambda^{(6)}$. Then, as $\lambda \to 0$,*

$$\operatorname{Im} Z(\lambda) \sim \begin{pmatrix} 1 & \operatorname{Im}\left\{C^{(2)}\lambda^{\frac{1}{4}}\right\} & \dots & \operatorname{Im}\left\{C^{(g)}\lambda^{\frac{1}{4}}\right\} \\ \operatorname{Im}\left\{C^{(2)}\lambda^{\frac{1}{4}}\right\} & r_{2,2} & \dots & r_{2,g} \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{Im}\left\{C^{(g)}\lambda^{\frac{1}{4}}\right\} & r_{g,2} & \dots & r_{g,g} \end{pmatrix},$$

where $C^{(j)}$ and $r_{i,j}$ are constants depending on $p(x)$.

Proof of Lemma 5.1. It holds that

$$A(\lambda) \sim \begin{pmatrix} C_{(1,2)}^{(1)} \cdot \lambda^{-\frac{1}{4}} & 0 & \dots & \dots & \dots & 0 \\ C_{(1,2)}^{(2)} \cdot \lambda^{\frac{1}{4}} & \alpha_{22} & 0 & \dots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 0 & \vdots \\ \vdots & \alpha_{g-1,2} & \dots & \dots & \alpha_{g-1,g-1} & 0 \\ C_{(1,2)}^{(g)} \cdot \sqrt{\lambda}^{\frac{1}{2}+g-2} & \alpha_{g,2} & \dots & \dots & \dots & \alpha_{g,g} \end{pmatrix},$$

$$B(\lambda) \sim \begin{pmatrix} \sqrt{-1} \cdot C_{(1,2)}^{(1)} \cdot \lambda^{-\frac{1}{4}} & \beta_{12} & \dots & \beta_{1,g} \\ -\sqrt{-1} \cdot C_{(1,2)}^{(2)} \cdot \lambda^{\frac{1}{4}} & \beta_{22} & \dots & \beta_{2,g} \\ \vdots & \vdots & & \vdots \\ (-1)^{g-1}\sqrt{-1} \cdot C_{(1,2)}^{(g)} \cdot \sqrt{\lambda}^{\frac{1}{2}+g-2} & \beta_{g,2} & \dots & \beta_{g,g} \end{pmatrix},$$

where $C_{(1,2)}^{(j)} := -2\int_0^1 \frac{(u-1)^{j-1}du}{\sqrt{u(u-1)(u-2)}} \cdot \frac{1}{\sqrt{-a_1(-a_2)(-a_3)\dots}}$, and α_{ij} and β_{ij} $(i \geq j \geq 2)$ are constants depending on $p(x)$. □

Proofs of Theorems 1.6. By Lemma 5.1, as $\lambda \to 0$, it follows that

$$\begin{aligned} k_\lambda^{(6)}(z) &= \frac{4\sum_{i,j=1}^{g}(\operatorname{Im}^{-1} Z)_{ij} \cdot z^{2(i-1)} \cdot \overline{z}^{2(j-1)}}{|(z^4-\lambda)(z^2-a_1)(z^2-a_2)\dots|} \\ &= \frac{4}{|z^4(z^2-a_1)(z^2-a_2)\dots|}\left\{1 + O(z^4) + \operatorname{Re}\left(\sum_{j=2}^{g} z^{2(j-1)}\right) \cdot O(\lambda^{\frac{1}{4}})\right\} + O(\lambda^{\frac{1}{2}}), \end{aligned}$$

which proves Theorem 1.6 by taking the logarithm. □

We also remark that the Bergman kernel on the Jacobian of $X_\lambda^{(6)}$ can be written as $(\det \operatorname{Im} Z(\lambda))^{-1} dW \wedge d\overline{W} =: K_\lambda(W) dW \wedge d\overline{W}$, for $W \in \mathbb{C}^g$. Therefore, we know that $\log K_\lambda(W) = -\log\det\operatorname{Im} Z(\lambda) = C' + O(\lambda^{\frac{1}{2}})$, as $\lambda \to 0$. Here C' depends on a_j. So, it seems that the Jacobian of $X_\lambda^{(6)}$ remains non-degenerate, since $\det\operatorname{Im} Z(\lambda) \sim \exp(-C') < +\infty$.

6 Bergman kernel near a cusp II: genus-two and hyperelliptic curves

To prove Theorem 1.7, we will prove the following lemma by analyzing the asymptotics of period matrices.

Lemma 6.1. *Let $Z(\lambda)$ denote the period matrix of $X_\lambda^{(7)}$. Then as $\lambda \to 0$,*

$$\operatorname{Im} Z(\lambda) \sim \frac{1}{\pi} \begin{pmatrix} -\log|\lambda| & \operatorname{Im}\left\{c_2 \lambda^{\frac{1}{2}}\right\} \\ \operatorname{Im}\left\{c_2 \lambda^{\frac{1}{2}}\right\} & c_1 \end{pmatrix}. \tag{6.1}$$

Proof of Lemma 6.1. As $\lambda \to 0$, it holds that $A(\lambda) \sim \begin{pmatrix} \frac{2\pi}{\sqrt{ab}\sqrt{\lambda}} & 0 \\ C\lambda\sqrt{\lambda} & \tilde{C}_{ab} \end{pmatrix}$, $B(\lambda) \sim \begin{pmatrix} \frac{2\sqrt{-1}\log\lambda}{-\sqrt{ab}\sqrt{\lambda}} & \tilde{C}''_{a,b} \\ \frac{2}{\sqrt{ab}}\sqrt{-\lambda} & \tilde{C}'_{ab} \end{pmatrix}$, where $\tilde{C}_{a,b} := -2\int_0^a \frac{dx}{\sqrt{x(x-a)(x-b)}}$, $\tilde{C}''_{a,b} := -2\int_a^b \frac{dx}{x\sqrt{x(x-a)(x-b)}}$, and $\tilde{C}'_{a,b} := -2\int_a^b \frac{dx}{\sqrt{x(x-a)(x-b)}}$. □

Now, we could get the asymptotic result for the Bergman kernels by using the above lemma.

Proof of Theorem 1.7. Similar to the proof of Theorem 1.5, it follows by Lemma 6.1 that

$$k_\lambda^{(7)}(z) = \frac{4\sum_{i,j=1}^{2}(\operatorname{Im}^{-1} Z)_{ij} z^{2(2-i)} \cdot \bar{z}^{2(2-j)}}{|(z^2-\lambda)(z^2-\lambda^2)(z^2-a)(z^2-b)|} = \left\{\frac{1}{c_1} + \frac{1}{-\log|\lambda| \cdot |z|^4}\right\} \cdot \frac{4\pi}{\left|(z^2-a)(z^2-b)\right|} + O\left(\frac{1}{(-\log|\lambda|)^2}\right),$$

as $\lambda \to 0$. Taking the logarithm and computing $\partial\bar{\partial}$, we finish the proof. □

We remark that for the Jacobian varieties of $X_\lambda^{(7)}$ (more generally $X_\lambda^{(8)}$), as $\lambda \to 0$, hyperbolic growth appears again by (6.1) in Lemma 6.1.

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