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Example of a six-dimensional LCK solvmanifold

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Abstract: The purpose of this paper is to prove that there exists a lattice on a certain solvable Lie group and construct a six-dimensional locally conformal Kähler solvmanifold with non-parallel Lee form.

Keywords: solvmanifold; locally conformal Kähler manifold

MSC: Primary 53C55; Secondary 17B30

Introduction

Let G be a simply-connected solvable Lie group. A discrete co-compact subgroup Γ of G is said to be a *lattice* in G. We call a compact manifold $\Gamma\backslash G$ solvmanifold. It is well known that a nilpotent Lie group admits a lattice if and only if its Lie algebra has a basis whose structure constants are rational numbers [12].

In this paper, we consider the six-dimensional Lie group G given by

$$G = \left\{ \begin{pmatrix} \alpha_1^{t_1}\alpha_1'^{t_2} & 0 & 0 & 0 & x_1 \\ 0 & \alpha_2^{t_1}\alpha_2'^{t_2} & 0 & 0 & x_2 \\ 0 & 0 & \beta^{t_1}\beta'^{t_2} & 0 & z \\ 0 & 0 & 0 & \bar{\beta}^{t_1}\bar{\beta}'^{t_2} & \bar{z} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : t_i, x_i \in \mathbb{R}, z \in \mathbb{C} \right\},$$

where $\alpha_1, \alpha_2, \beta, \bar{\beta}$ are distinct roots of the polynomial $f_1(x) = x^4 - 2x^3 - 2x^2 + x + 1$, and $\alpha_i' = \alpha_i^{-1} + \alpha_i^{-2} (i = 1, 2)$, $\beta' = \beta^{-1} + \beta^{-2}$. Note that $\alpha_1', \alpha_2', \beta', \bar{\beta}'$ are the roots of the polynomial $f_2(x) = x^4 - 4x^3 + 4x^2 - 3x + 1$ (See section 1). Moreover, the Lie group G is solvable, because $[G, G]$ is abelian and $[G, [G, G]] \neq \{e\}$.

In section 1, we consider polynomials $f_1(x), f_2(x)$. They are eigenpolynomials of unimodular matrices, and we see that these unimodular matrices are abelian. Thus, we construct a lattice Γ in G:

Main Theorem . *The solvable Lie group G admits a lattice Γ.*

For a lattice in a solvable Lie group, Yamada [16] gave necessary and sufficient conditions for the existence of a lattice with respect to a basis of the Lie algebra with some properties by unimodular matrices (cf. [4]). The Lie algebra of the solvable Lie group G in Main Theorem satisfies these conditions (See Remark 2.3).

Inoue surface is a quotient of $\mathbb{H}\times\mathbb{C}$, where $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, and have a structure of a solvmanifold. The solvable Lie group G in Main Theorem is given by $\mathbb{H}^2 \times \mathbb{C}$, that is, the solvmanifold $\Gamma\backslash G$ is an extension of Inoue surface (See Remark 2.2).

Let (M, g, J) be a $2n$-dimensional compact Hermitian manifold. We denote by Ω the fundamental 2-form, that is, the 2-form defined by $\Omega(X, Y) = g(X, JY)$. A Hermitian manifold (M, g, J) is said to be a *locally conformal Kähler (LCK) manifold* if there exists a closed 1-form ω such that $d\Omega = \omega \wedge \Omega$. The closed 1-form ω is called *Lee form*. Note that if $\omega = df$, then $(M, e^{-f}g, J)$ is Kähler. A LCK manifold (M, g, J) is said to be *a*

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Vaisman manifold if the Lee form ω is parallel with respect to the Levi-Civita connection of g. LCK metrics are nowadays widely studied, and are connected to several topics, see e.g., [6], [1], [3], [5], [10], [9].

Oeljeklaus-Toma [8] constructed examples of LCK manifolds with non-parallel Lee form (*OT-manifold of type* $(s, 1)$) by using the number fields. Since their very definition, OT-manifolds appeared to be interesting from many points of view. For instance, they provide a counterexample to a conjecture of Vaisman [8, Remark 2.8], they provide the first example of non-trivial LCK rank [11], and they do not admits any small deformation [2]. Note that OT-manifolds of type $(1, 1)$ are Inoue surfaces (cf. [14]). Kasuya [7] showed that OT-manifolds are in fact solvmanifolds, and proved that OT-manifold of type $(s, 1)$ has no Vaisman structures. We see that the solvmanifold $\Gamma\backslash G$ is OT-manifold of type $(2, 1)$. From Main theorem, we can construct a lattice Γ on G by the method that is different from Oeljeklaus-Toma's one.

1 Preliminary

In this section, we consider the roots of the polynomials $f_1(x) = x^4 - 2x^3 - 2x^2 + x + 1$ and $f_2(x) = x^4 - 4x^3 + 4x^2 - 3x + 1$.

We get

Lemma 1.1. *On* $(-\infty, \frac{1}{\sqrt{2}}]$, $f_1(x) = x^4 - 2x^3 - 2x^2 + x + 1 > 0$.

Proof. We see that

$$f_1(x) = x^4 - 2x^2(x + 1) + x + 1 = x^4 + (x + 1)(1 - 2x^2).$$

It follows that if $x \leq -1$ or $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$, then $f_1(x) > 0$. Moreover, if $-1 < x < -\frac{1}{\sqrt{2}}$, then $0 < x + 1 < 1$ and $1 - 2x^2 < 0$. Thus, we get $(x + 1)(1 - 2x^2) > 1 - 2x^2$. It follows that

$$f_1(x) = x^4 + (x + 1)(1 - 2x^2) > x^4 + 1 - 2x^2 = (x^2 - 1)^2 > 0.$$

Therefore we get our claim. □

By Descartes' rule of signs, we easily see that the polynomials $f_1(x)$ has only two positive roots $\alpha_1, \alpha_2 (\alpha_1 < \alpha_2)$. We have a statement about α_1, α_2 in more details:

Lemma 1.2. *On* $(\frac{1}{\sqrt{2}}, 1)$, $f_1(x) = x^4 - 2x^3 - 2x^2 + x + 1$ *has only one root* α_1.

Proof. We see that

$$f_1'(x) = 4x^3 - 6x^2 - 4x + 1 = 2(2x + 1)x(x - 2) + 1.$$

Since $\frac{1}{\sqrt{2}} < x < 1$, we get $x - 2 < -1$. It follows that

$$\begin{aligned} f_1'(x) &= 2(2x + 1)x(x - 2) + 1 \\ &< 2(2x + 1)x(-1) + 1 < 2(\tfrac{2}{\sqrt{2}} + 1)\tfrac{1}{\sqrt{2}}(-1) + 1 < 0. \end{aligned}$$

Then we see that $f_1(x)$ is monotone decreasing. Moreover, we get $f_1(\frac{1}{\sqrt{2}}) > 0$ and $f_1(1) < 0$. Thus, we have our claim. □

Lemma 1.3. *On* $(1, \infty)$, $f_1(x) = x^4 - 2x^3 - 2x^2 + x + 1$ *has only one root* α_2.

Proof. For $1 < x \leq 2$, we see that

$$f_1(x) = x^3(x - 2) - (2x + 1)(x - 1) < x^3(x - 2) \leq 0.$$

Moreover, for $2 < x$, since

$$f_1'(x) = 4x^3 - 6x^2 - 4x + 1 = 2(2x + 1)x(x - 2) + 1 > 0,$$

we see that $f_1(x)$ is monotone increasing. Thus, we have our claim. □

Remark 1.4. Since $f_1(\frac{5}{2}) < 0$ and $f_1(3) > 0$, we see that $\frac{5}{2} < \alpha_2 < 3$.

Therefore, we have

Proposition 1.5. *The polynomial $f_1(x) = x^4 - 2x^3 - 2x^2 + x + 1$ has only two real roots. Moreover, they are positive.*

Next, we consider the roots of the polynomials $f_2(x) = x^4 - 4x^3 + 4x^2 - 3x + 1$.

Lemma 1.6. *If $f_1(\alpha) = 0$, then $f_2(\alpha^{-1} + \alpha^{-2}) = 0$.*

Proof. Let α be $f_1(\alpha) = 0$. By a straightforward computation, we find that

$$\begin{aligned} f_2(\alpha^{-1} + \alpha^{-2}) &= (\alpha^{-1} + \alpha^{-2})^4 - 4(\alpha^{-1} + \alpha^{-2})^3 + 4(\alpha^{-1} + \alpha^{-2})^2 - 3(\alpha^{-1} + \alpha^{-2}) + 1 \\ &= \alpha^{-8} + 4\alpha^{-7} + 2\alpha^{-6} - 8\alpha^{-5} - 7\alpha^{-4} + 4\alpha^{-3} + \alpha^{-2} - 3\alpha^{-1} + 1 \\ &= \alpha^{-8}(\alpha^8 - 3\alpha^7 + \alpha^6 + 4\alpha^5 - 7\alpha^4 - 8\alpha^3 + 2\alpha^2 + 4\alpha + 1) \\ &= \alpha^{-8}(\alpha^4 - \alpha^3 + \alpha^2 + 3\alpha + 1)(\alpha^4 - 2\alpha^3 - 2\alpha^2 + \alpha + 1). \end{aligned}$$

Since $f_1(\alpha) = \alpha^4 - 2\alpha^3 - 2\alpha^2 + \alpha + 1 = 0$, we have $f_2(\alpha^{-1} + \alpha^{-2}) = 0$. □

Let $\beta, \bar{\beta}$ be the complex roots of $f_1(x)$. From Lemma 1.6, $\beta^{-1} + \beta^{-2}, \bar{\beta}^{-1} + \bar{\beta}^{-2}$ are roots of $f_2(x)$. We see

Lemma 1.7. $\beta^{-1} + \beta^{-2} \neq \bar{\beta}^{-1} + \bar{\beta}^{-2}$.

Proof. Since $f_1(\beta) = f_1(\bar{\beta}) = 0$, we see that

$$\begin{aligned} \beta^4 - 2\beta^3 - 2\beta^2 + \beta + 1 &= 0 \\ \beta + 1 &= -\beta^4 + 2\beta^3 + 2\beta^2 \\ \beta^{-1} + \beta^{-2} &= -\beta^2 + 2\beta + 2 \end{aligned}$$

and $\bar{\beta}^{-1} + \bar{\beta}^{-2} = -\bar{\beta}^2 + 2\bar{\beta} + 2$.

We assume that $\beta^{-1} + \beta^{-2} = \bar{\beta}^{-1} + \bar{\beta}^{-2}$. Then we get

$$\begin{aligned} -\beta^2 + 2\beta + 2 &= -\bar{\beta}^2 + 2\bar{\beta} + 2 \\ -(\beta - \bar{\beta})(\beta + \bar{\beta}) + 2(\beta - \bar{\beta}) &= 0 \\ -(\beta + \bar{\beta}) + 2 &= 0. \end{aligned}$$

It follows that $\mathrm{Re}\beta = 1$, that is, $|\beta| > 1$. On the other hand, we see that $\alpha_1\alpha_2\beta\bar{\beta} = 1$ because $\alpha_1, \alpha_2, \beta, \bar{\beta}$ are roots of $f_1(x)$. Since $\frac{1}{\sqrt{2}} < \alpha_1 < 1$ and $\frac{5}{2} < \alpha_2 < 3$, we get $\alpha_1\alpha_2 > \frac{5}{2\sqrt{2}} > 1$, that is, $|\beta| < 1$. This is a contradiction. □

Therefore, we have

Proposition 1.8. *The polynomial $f_2(x) = x^4 - 4x^3 + 4x^2 - 3x + 1$ has only two real roots $\alpha_i' = \alpha_i^{-1} + \alpha_i^{-2} (i = 1, 2)$. Moreover, they are positive.*

Remark 1.9. Since $f_1(\alpha_1) = 0$, we see that

$$\alpha_1' = \alpha_1^{-1} + \alpha_1^{-2} \text{ and } \alpha_1' = -\alpha_1^2 + 2\alpha_1 + 2 = -(\alpha_1 - 1)^2 + 3.$$

Since $\frac{1}{\sqrt{2}} < \alpha_1 < 1$, we have $2 < \alpha_1' < 3$.

Since $\frac{5}{2} < \alpha_2 < 3$, we get $\frac{1}{3} < \alpha_2^{-1} < \frac{2}{5}$ and $\frac{1}{9} < \alpha_2^{-2} < \frac{4}{25}$. It follows that $\frac{4}{9} < \alpha_2' < \frac{14}{25}$.

We have

Proposition 1.10. $\begin{vmatrix} \log\alpha_1 & \log\alpha_2 \\ \log\alpha'_1 & \log\alpha'_2 \end{vmatrix} \neq 0$

Proof. From Remark 1.9, we see that

$$\log\tfrac{1}{2} < \log\tfrac{1}{\sqrt{2}} < \log\alpha_1 < 0, \quad \log\tfrac{5}{2} < \log\alpha_2 < \log 3,$$
$$\log 2 < \log\alpha'_1 < \log 3, \quad \log\tfrac{4}{9} < \log\alpha'_2 < \log\tfrac{14}{25}.$$

It follows that

$$\log\alpha_1\log\alpha'_2 - \log\alpha_2\log\alpha'_1 < \log\tfrac{1}{2}\log\tfrac{4}{9} - \log\tfrac{5}{2}\log 2 = \log 2(\log\tfrac{9}{4} - \log\tfrac{5}{2}) = \log 2\log\tfrac{9}{10} < 0.$$

Therefore, we have our claim. □

2 Proof of Main Theorem

In this section, we prove Main Theorem and give some remarks. We use same notation introduced in section 1.

We consider unimodular matrices as follows:

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}, B_2 = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We easily check that $f_i(x)$ is the characteristic polynomial of $B_i (i = 1, 2)$. From section 1, B_1 has real positive eigenvalues α_1, α_2 and complex eigenvalues $\beta, \bar{\beta}$, and B_2 has real positive eigenvalues $\alpha'_1 = \alpha_1^{-1} + \alpha_1^{-2}, \alpha'_2 = \alpha_2^{-1} + \alpha_2^{-2}$ and complex eigenvalues $\beta' = \beta^{-1} + \beta^{-2}, \bar{\beta}' = \bar{\beta}^{-1} + \bar{\beta}^{-2}$. Moreover, by a straightforward computation, we have $B_1B_2 = B_2B_1$.

Since $B_1B_2 = B_2B_1$, they are simultaneously diagonalizable. Indeed, put $\mathbf{a}_i = (1, \alpha_i, \alpha_i^2, \alpha_i^3) (i = 1, 2)$, $\mathbf{b} = (1, \beta, \beta^2, \beta^3)$. Then, we easily see that $\mathbf{a}_i, \mathbf{b}$ are left eigenvectors of α_i, β for B_1, and left eigenvectors of α'_i, β' for B_2, respectively. Then, there exists ${}^tP = \begin{pmatrix} {}^t\mathbf{a}_1 & {}^t\mathbf{a}_2 & {}^t\mathbf{b} & {}^t\bar{\mathbf{b}} \end{pmatrix} \in GL(4, \mathbb{C})$ such that

$$PB_1P^{-1} = \mathrm{diag}(\alpha_1, \alpha_2, \beta, \bar{\beta}), \; PB_2P^{-1} = \mathrm{diag}(\alpha'_1, \alpha'_2, \beta', \bar{\beta}').$$

Let φ be a diffeomorphism from $\mathbb{R}^6$ to G given by

$$\varphi((t_i, x_i, y_i)) = \left(\begin{array}{ccccc} \alpha_1^{t_1}\alpha_1'^{t_2} & 0 & 0 & 0 & \\ 0 & \alpha_2^{t_1}\alpha_2'^{t_2} & 0 & 0 & P\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \\ 0 & 0 & \beta^{t_1}\beta'^{t_2} & 0 & \\ 0 & 0 & 0 & \bar{\beta}^{t_1}\bar{\beta}'^{t_2} & \\ 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Put $\Gamma = \varphi(\mathbb{Z}^6)$. It is Γ is discrete and $\varphi(U)$ is compact, where $U = \{(t_i, x_i, y_i) \in \mathbb{R}^7; 0 \leq t_i, x_i, y_i \leq 1\}$. Moreover, since

$$\begin{aligned} \mathrm{diag}(\alpha_1^{s_1}\alpha_1'^{s_2}, \alpha_2^{s_1}\alpha_2'^{s_2}, \beta^{s_1}\beta'^{s_2}, \bar{\beta}^{s_1}\bar{\beta}'^{s_2}) &= \mathrm{diag}(\alpha_1^{s_1}, \alpha_2^{s_1}, \beta^{s_1}, \bar{\beta}^{s_1})\mathrm{diag}(\alpha_1'^{s_2}, \alpha_2'^{s_2}, \beta'^{s_2}, \bar{\beta}'^{s_2}) \\ &= PB_1^{s_1}B_2^{s_2}P^{-1} \end{aligned}$$

for $s_1, s_2 \in \mathbb{Z}$, we have

$$\Gamma = \left\{ \begin{pmatrix} PB_1^{s_1}B_2^{s_2}P^{-1} & P\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} \\ 0 \;\; 0 \;\; 0 \;\; 0 & 1 \end{pmatrix} : s_i, u_i, v_i \in \mathbb{Z} \right\}.$$

Then we see that Γ is a subgroup of G. Thus, Γ is a lattice of G, and we have proved the following theorem:

Theorem 2.1. *The solvable Lie group G admits a lattice Γ.*

We give some remarks:

Remark 2.2. (cf. [13]) By Proposition 1.10, we can construct the solvable Lie group G from $\mathbb{H}^2 \times \mathbb{C}$, where $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Since $\begin{vmatrix} \log \alpha_1 & \log \alpha_2 \\ \log \alpha'_1 & \log \alpha'_2 \end{vmatrix} \neq 0$, we get $\mathbb{R}^2 = \operatorname{span}\{(\log \alpha_1, \log \alpha'_1), (\log \alpha_2, \log \alpha'_2)\}$. Then we see that

$$\begin{aligned} \mathbb{H}^2 \times \mathbb{C} &= \{(x_1 + \sqrt{-1}e^{t_1 \log \alpha_1 + t_2 \log \alpha'_1}, x_2 + \sqrt{-1}e^{t_1 \log \alpha_2 + t_2 \log \alpha'_2}, z) : t_i, x_i \in \mathbb{R}, z \in \mathbb{C}\} \\ &= \{(x_1 + \sqrt{-1}\alpha_1^{t_1}\alpha_1'^{t_2}, x_2 + \sqrt{-1}\alpha_2^{t_1}\alpha_2'^{t_2}, z) : t_i, x_i \in \mathbb{R}, z \in \mathbb{C}\}. \end{aligned}$$

Thus, we define a group structure on $\mathbb{H}^2 \times \mathbb{C}$ as follows:

$$\begin{aligned} &(x_1 + \sqrt{-1}\alpha_1^{t_1}\alpha_1'^{t_2}, x_2 + \sqrt{-1}\alpha_2^{t_1}\alpha_2'^{t_2}, z) \cdot (x'_1 + \sqrt{-1}\alpha_1^{t'_1}\alpha_1'^{t'_2}, x'_2 + \sqrt{-1}\alpha_2^{t'_1}\alpha_2'^{t'_2}, z') \\ &= (\alpha_1^{t_1}\alpha_1'^{t_2}x'_1 + x_1 + \sqrt{-1}\alpha_1^{t_1+t'_1}\alpha_1'^{t_2+t'_2}, \alpha_2^{t_1}\alpha_2'^{t_2}x'_2 + x_2 + \sqrt{-1}\alpha_2^{t_1+t'_1}\alpha_2'^{t_2+t'_2}, \beta^{t_1}\beta'^{t_2}z' + z). \end{aligned}$$

Let f be a map from $(\mathbb{H}^2 \times \mathbb{C}, \cdot)$ to G given by

$$f((x_1 + \sqrt{-1}\alpha_1^{t_1}\alpha_1'^{t_2}, x_2 + \sqrt{-1}\alpha_2^{t_1}\alpha_2'^{t_2}, z)) = \begin{pmatrix} \alpha_1^{t_1}\alpha_1'^{t_2} & 0 & 0 & 0 & x_1 \\ 0 & \alpha_2^{t_1}\alpha_2'^{t_2} & 0 & 0 & x_2 \\ 0 & 0 & \beta^{t_1}\beta'^{t_2} & 0 & z \\ 0 & 0 & 0 & \bar{\beta}^{t_1}\bar{\beta}'^{t_2} & \bar{z} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We easily see that f is an isomorphism, that is, $(\mathbb{H}^2 \times \mathbb{C}, \cdot)$ can be expressed by the solvable Lie group G.

Remark 2.3. The solvable Lie algebra $\mathfrak{g}$ of G satisfies the necessary and sufficient conditions for lattices in [16]. The solvable Lie algebra $\mathfrak{g}$ is given by

$$\begin{aligned} &\mathfrak{g} = \operatorname{span}\{T_1, T_2, X_1, X_2, Z, \bar{Z}\} \\ &[T_1, X_i] = \log \alpha_i X_i, \quad [T_1, Z] = \log \beta Z, \quad [T_1, \bar{Z}] = \log \bar{\beta}\bar{Z} \\ &[T_2, X_i] = \log \alpha'_i X_i, \quad [T_2, Z] = \log \beta' Z, \quad [T_2, \bar{Z}] = \log \bar{\beta}'\bar{Z}, \end{aligned}.$$

We put $(\tilde{X}_1, \tilde{X}_2, \tilde{Y}_1, \tilde{Y}_2) = (X_1, X_2, Z, \bar{Z})P$. Then B_i is the matrix presentation of $\exp \operatorname{ad}(T_i)$ with respect to $\{\tilde{X}_1, \tilde{X}_2, \tilde{Y}_1, \tilde{Y}_2\}$ for each i.

Remark 2.4. [7] Let $\mathfrak{g}$ be the solvable Lie algebra of G. Then $\mathfrak{g}^*$ has a basis $\{\lambda_i, \mu_i, \upsilon, \bar{\upsilon}\}$ such that

$$\begin{aligned} &d\lambda_1 = 0, d\lambda_2 = 0, \\ &d\mu_1 = -\{(\log \alpha_1)\lambda_1 + (\log \alpha'_1)\lambda_2\} \wedge \mu_1,\ d\mu_2 = -\{(\log \alpha_2)\lambda_1 + (\log \alpha'_2)\lambda_2\} \wedge \mu_2, \\ &d\upsilon = -\{(\log \beta)\lambda_1 + (\log \beta')\lambda_2\} \wedge \upsilon,\ d\bar{\upsilon} = -\{(\log \bar{\beta})\lambda_1 + (\log \bar{\beta}')\lambda_2\} \wedge \bar{\upsilon}. \end{aligned}$$

Since $\begin{vmatrix} \log \alpha_1 & \log \alpha_2 \\ \log \alpha'_1 & \log \alpha'_2 \end{vmatrix} \neq 0$ and $\alpha_1\alpha_2\beta\bar{\beta} = \alpha'_1\alpha'_2\beta'\bar{\beta}' = 1$, we have a basis $\{\varphi_i, \mu_i, \nu_i\}$ of $\mathfrak{g}^*$ as follows:

$$\begin{aligned} &d\varphi_1 = 0, d\varphi_2 = 0, \\ &d\mu_1 = -\varphi_1 \wedge \mu_1,\ d\mu_2 = -\varphi_2 \wedge \mu_2, \\ &d\nu_1 = \tfrac{1}{2}(\varphi_1 + \varphi_2) \wedge \nu_1 + (c_1\varphi_1 + c_2\varphi_2) \wedge \nu_2, \\ &d\nu_2 = \tfrac{1}{2}(\varphi_1 + \varphi_2) \wedge \nu_2 - (c_1\varphi_1 + c_2\varphi_2) \wedge \nu_1, \end{aligned}$$

where $c_1^2 + c_2^2 \neq 0$. The solvmanifold $\Gamma\backslash G$ has a left-invariant complex structure J such that $w_i = \varphi_i + \sqrt{-1}\mu_i$ for $i = 1, 2$ and $w_3 = \nu_1 + \sqrt{-1}\nu_2$ are $(1, 0)$-form, and has a LCK structure with the fundamental 2-form Ω:

$$\Omega = -2(\varphi_1 \wedge \mu_1 + \varphi_2 \wedge \mu_2) - (\varphi_1 \wedge \mu_2 + \varphi_2 \wedge \mu_1) - \nu_1 \wedge \nu_2.$$

Note that the Lee form $\varphi_1 + \varphi_2$ is not parallel.

Remark 2.5. (cf. [7]) The first Betti number $b_1 = \dim H^1_{DR}(\Gamma\backslash G) = \mathrm{rank}[\Gamma, \Gamma]\backslash\Gamma = 2$. Vaisman [15] proved that the first Betti number of a Vaisman manifold is odd. Thus, the solvmanifold $\Gamma\backslash G$ has a LCK structure, but it has no Vaisman structures.

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