DE GRUYTER OPEN Complex Manifolds 2017; 4: 73–83

Complex Manifolds Open Access

Research Article

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Some relations between Hodge numbers and invariant complex structures on compact nilmanifolds

DOI 10.1515/coma-2017-0006
Received March 10, 2017; accepted May 25, 2017.

Abstract: Let N be a simply connected real nilpotent Lie group, $\mathfrak{n}$ its Lie algebra, and Γ a lattice in N. If a left-invariant complex structure on N is Γ-rational, then $H^{s,t}_{\bar\partial}(\Gamma\backslash N) \cong H^{s,t}_{\bar\partial}(\mathfrak{n}^{\mathbb{C}})$ for each s, t. We can construct different left-invariant complex structures on one nilpotent Lie group by using the complexification and the scalar restriction. We investigate relationships to Hodge numbers of associated compact complex nilmanifolds.

Keywords: Nilmanifold, Dolbeault cohomology group, Complex structure

MSC: 53C30, 22E25

1 Introduction

Invariant complex structures on compact nilmanifolds have been studied by many mathematicians (e.g.[6],[4]). However, it seems that relations between two distinct complex structures on one nilmanifold has not studied enough. In the previous papers [7–9], we investigated relations between invariant complex structures and Hodge numbers of compact nilmanifolds.

Let N be a real nilpotent Lie group defined by

$$N = \left\{ \left. \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right| z_i \in \mathbb{C} \right\}.$$

Consider the following two global complex coordinate systems of N:

$$\varphi_1 : \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (z_1, z_2, z_3) \in \mathbb{C}^3, \quad \varphi_2 : \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (\bar z_1, z_2, z_3) \in \mathbb{C}^3.$$

Then, for each global coordinate system, the group structure of N can be written as

$$(z_1, z_2, z_3) \cdot (w_1, w_2, w_3) = (z_1 + w_1, z_2 + w_2, z_3 + z_1 w_2 + w_3),$$
$$(z_1, z_2, z_3) \cdot (w_1, w_2, w_3) = (z_1 + w_1, z_2 + w_2, z_3 + \bar z_1 w_2 + w_3),$$

respectively. Put $\mathcal{S}_1 = \{(N, \varphi_1)\}$, $\mathcal{S}_2 = \{(N, \varphi_2)\}$. Then, complex manifolds $N_1 = (N, \mathcal{S}_1)$ and $N_2 = (N, \mathcal{S}_2)$ are isomorphic as real Lie groups, however, N_1 and N_2 are not isomorphic as real Lie groups with left-invariant complex structures. Then, $\Gamma\backslash N_1$ and $\Gamma\backslash N_2$ are not biholomorphic as complex manifolds, where Γ is a lattice in N.

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Moreover, we have seen that the Hodge numbers satisfy $h^{s,t}(\Gamma\backslash N_1) = h^{t,s}(\Gamma\backslash N_2)$ for each s, t ([7]). We have considered a generalization of the pair of compact complex nilmanifolds, and compared Hodge numbers of pairs of complex nilmanifolds.

Let $(\mathfrak{h}, J)$ be a Lie algebra with a complex structure, and $\mathfrak{h}_J^{\pm}$ the vector spaces of the $\pm\sqrt{-1}$ eigenvectors of the complex structure J, respectively. We denote by $H^{*,*}_{\bar{\partial}_J}(\mathfrak{h}^{\mathbb{C}})$ the cohomology ring of the differential bigraded algebra $\bigwedge^{*,*}(\mathfrak{h}^{\mathbb{C}})^*$, associated to $\mathfrak{h}^{\mathbb{C}}$ with respect to the operator $\bar{\partial}_J$ in the canonical decomposition $d = \partial_J + \bar{\partial}_J$ on $\bigwedge^{*,*}(\mathfrak{h}^{\mathbb{C}})^*$. Let $\mathfrak{g}$ be a real Lie algebra, and $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$ a decomposition such that $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{b}$ is an ideal of $\mathfrak{g}$. Then, we can construct complex structures J and $\tilde{J}$ on $_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ from the decomposition (for the details of J and $\tilde{J}$, see Section 3), where $\mathfrak{g}^{\mathbb{C}}$ is the complexification of $\mathfrak{g}$, and $_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ is the real Lie algebra obtained from $\mathfrak{g}^{\mathbb{C}}$ by the scalar restriction. We denote $\dim H^{s,t}_{\bar{\partial}_J}(_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}))$ by $h^{s,t}(\mathfrak{g}_J)$. Let N be a simply connected real nilpotent Lie group, Γ a lattice in N. If a left-invariant complex structure J on N is a Γ-rational complex structure, then $H^{s,t}_{\bar{\partial}}(\Gamma\backslash N) \cong H^{s,t}_{\bar{\partial}}(\mathfrak{n}^{\mathbb{C}})$ for each s, t ([1]). Thus, results on $H^{s,t}_{\bar{\partial}}(\mathfrak{n}^{\mathbb{C}})$ of the nilpotent Lie algebra with rational complex structures yield results on $H^{s,t}_{\bar{\partial}}(\Gamma\backslash N)$ of a compact nilmanifold with invariant rational complex structures.

In this paper, we consider a construction of nilpotent Lie algebras $\mathfrak{n}$ with a decomposition $\mathfrak{n} = \mathfrak{a} \ltimes \mathfrak{b}$ by root systems (Section 4). We also consider a modification of the construction of a complex structure $\tilde{J}$. Indeed, if $\mathfrak{a}$ has a good decomposition, we can consider other complex structure J' on $_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ which is a generalization of $\tilde{J}$ (Section 3). As an application, we obtain a compact nilmanifold with invariant complex structures such that Hodge numbers are quite different for each complex structure (see Example 7.3). In the previous paper [9], we have several results in the case where $\mathfrak{b}$ is abelian, e.g., if $\mathfrak{b}$ is abelian, then

$$\sum_{s+t=r} h^{s,t}(\mathfrak{g}_J) = \sum_{s+t=r} h^{s,t}(\mathfrak{g}_{\tilde{J}})$$

for each r. Thus, we have an interest in the case where $\mathfrak{b}$ is non-abelian. For example, we have the following:

Theorem 1.1.

$$\sum_{s+t=r} h^{s,t}(\mathfrak{g}_J) - \sum_{s+t=r} h^{s,t}(\mathfrak{g}_{\tilde{J}}) = \sum_{s+t=r} (\dim H^s(\mathfrak{g} \times \mathbb{R}^q) - \dim H^s(\mathfrak{g}_0 \times \mathfrak{b})) \cdot \binom{p}{t},$$

where $\mathfrak{g}_0$ is a Lie algebra which is constructed from a decomposition $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$, $p = \dim \mathfrak{a}$, and $q = \dim \mathfrak{b}$. In particular,

$$\sum_{s+t=1} h^{s,t}(\mathfrak{g}_J) - \sum_{s+t=1} h^{s,t}(\mathfrak{g}_{\tilde{J}}) = \dim([\mathfrak{a}, \mathfrak{b}] \cap [\mathfrak{b}, \mathfrak{b}]) \geq 0.$$

If $\mathfrak{b}$ is abelian, then $\mathfrak{g} \times \mathbb{R}^q \cong \mathfrak{g}_0 \times \mathfrak{b}$. Thus, this theorem is a generalization of a result in the previous paper [9]. However, it is not always natural to consider relations of $h^{s,t}(\mathfrak{g}_J)$ and $h^{s,t}(\mathfrak{g}_{\tilde{J}})$ for investigating dualities. For the details, see Theorem 6.3, which is one of main results.

2 Preliminaries

In this section, we recall results of Dolbeault cohomology groups of compact complex nilmanifolds.

Let H be a real Lie group, and $\mathfrak{h}$ its Lie algebra. We denote by $H^*(\mathfrak{h}) = H^*(\mathfrak{h}, \mathbb{R})$ the cohomology of the complex $\bigwedge^*(\mathfrak{h}^*)$ of left-invariant differential forms on the Lie group H. A left-invariant almost complex structure on H can be identified with a linear mapping $J : \mathfrak{h} \longrightarrow \mathfrak{h}$ such that $J^2 = -\mathrm{id}$. Such a structure determines in the usual way the subspace

$$\bigwedge\nolimits_J^{1,0} = \{\alpha - \sqrt{-1}J\alpha : \alpha \in \mathfrak{h}^*\}$$

of $\mathfrak{h}^* \otimes \mathbb{C}$ of left-invariant $(1,0)$-forms, its conjugate $\bigwedge_J^{0,1}$, and more general subspaces $\bigwedge_J^{p,q}$ of $\bigwedge^{p+q}\mathfrak{h}^* \otimes \mathbb{C}$. The almost complex structure J is said to be *integrable* if

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0$$

for all $X, Y \in \mathfrak{h}$. We shall refer to a pair $(\mathfrak{h}, J)$ consisting of a Lie algebra and an integrable almost complex structure simply as a *Lie algebra with a complex structure*. The equation $N_J(X, Y) = 0$ holds for all $X, Y \in \mathfrak{h}$ if and only if $d(\bigwedge_J^{1,0}) \subset \bigwedge_J^{2,0} \oplus \bigwedge_J^{1,1}$ (cf.[6, pp.313–pp.314]). If there exist no possibility of confusion, we omit the subscript J in $h^{s,t}(\mathfrak{h}_J)$ and $H^{s,t}_{\bar\partial_J}(\mathfrak{h}^{\mathbb{C}})$.

Let N be a simply connected real nilpotent Lie group, and $\mathfrak{n}$ be the Lie algebra of N. It is well known that there exists a lattice in N if and only if there exists a rational Lie subalgebera $\mathfrak{n}_{\mathbb{Q}}$ such that $\mathfrak{n} \cong \mathfrak{n}_{\mathbb{Q}} \otimes \mathbb{R}$. Let Γ be a lattice in N, and $\mathfrak{n}_{\mathbb{Q}}$ the $\mathbb{Q}$-span of $\exp^{-1}(\Gamma)$ in $\mathfrak{n}$, where $\exp : \mathfrak{n} \longrightarrow N$ is the exponential map (cf.[3, Chapter 2]). Then, a complex structure J on $\mathfrak{n}$ is said to be *Ă-rational* if $J(\mathfrak{n}_{\mathbb{Q}}) \subset \mathfrak{n}_{\mathbb{Q}}$ ([1]). Conversely, let $\mathfrak{n}_{\mathbb{Q}}$ be a rational Lie subalgebra such that $\mathfrak{n} \cong \mathfrak{n}_{\mathbb{Q}} \otimes \mathbb{R}$, and J a complex structure on $\mathfrak{n}$ such that $J(\mathfrak{n}_{\mathbb{Q}}) \subset \mathfrak{n}_{\mathbb{Q}}$. Moreover, let $\mathcal{L}$ be a lattice in $\mathfrak{n}$ contained in $\mathfrak{n}_{\mathbb{Q}}$. Then, the group Γ generated by $\exp \mathcal{L}$ is a lattice in N (cf.[3, Chapter 2]). Moreover, J is Γ-rational. We say a complex structure J on a Lie algebra $\mathfrak{h}$ is *rational* if $J(\mathfrak{h}_{\mathbb{Q}}) \subset \mathfrak{h}_{\mathbb{Q}}$ for some rational Lie subalgebra $\mathfrak{h}_{\mathbb{Q}}$ such that $\mathfrak{h} \cong \mathfrak{h}_{\mathbb{Q}} \otimes \mathbb{R}$.

Theorem 2.1 (Console-Fino[1]). *Let N be a simply connected nilpotent Lie group, and Γ a lattice in N. If J is a Γ-rational complex structure on $\mathfrak{n}$, then,*

$$H^{s,t}_{\bar\partial}(\Gamma\backslash N) \cong H^{s,t}_{\bar\partial}(\mathfrak{n}^{\mathbb{C}})$$

for each s, t.

Theorem 2.2 (Console-Fino[1]). *For any small deformation of a Γ-rational complex structure, the isomorphism*

$$H^{s,t}_{\bar\partial}(\Gamma\backslash N) \cong H^{s,t}_{\bar\partial}(\mathfrak{n}^{\mathbb{C}})$$

holds for each s, t.

Theorem 2.3 (Sakane[5]). *Let N be a simply connected complex nilpotent Lie group, and Γ a lattice in N. Then,*

$$H^{s,t}_{\bar\partial}(\Gamma\backslash N) \cong H^{0,t}_{\bar\partial}(\mathfrak{n}^-) \otimes \bigwedge^s(\mathfrak{n}^+)^* \cong H^t(\mathfrak{n}^-) \otimes \bigwedge^s(\mathfrak{n}^+)^*$$

for each s, t.

Thus, results on $H^{s,t}_{\bar\partial}(\mathfrak{n}^{\mathbb{C}})$ of a nilpotent Lie algebra with good complex structures yield results on $H^{s,t}_{\bar\partial}(\Gamma\backslash N)$ of a compact nilmanifold with invariant complex structures.

3 Complex structures

In this section, we construct complex structures on ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$, where $\mathfrak{g}$ is a Lie algebra.

We consider the following Lie algebra $\mathfrak{g}$ over $\mathbb{R}$:

$$\mathfrak{g} = (\mathfrak{t} \ltimes \mathfrak{k}) \ltimes \mathfrak{b},$$

where $\mathfrak{a} = \mathfrak{t} \ltimes \mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{t}$ is abelian, $\mathfrak{k}$ is an ideal of $\mathfrak{a}$, and $\mathfrak{b}$ is an ideal of $\mathfrak{g}$. Take basis of the Lie subalgebras $\mathfrak{t}$, $\mathfrak{k}$ and $\mathfrak{b}$:

$$\mathfrak{t} = \mathrm{span}_{\mathbb{R}}\{U^1_1, \ldots, U^1_{p_1}\}, \mathfrak{k} = \mathrm{span}_{\mathbb{R}}\{U^1_{p_1+1}, \ldots, U^1_{p_1+p_2}\}, \mathfrak{b} = \mathrm{span}_{\mathbb{R}}\{V^1_1, \ldots, V^1_q\}.$$

Consider the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Since $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$, the real Lie algebra ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ has the following basis:

$$\{U^1_1, \ldots, U^1_p, V^1_1, \ldots, V^1_q, U^2_1, \ldots, U^2_p, V^2_1, \ldots, V^2_q\},$$

where $U^2_i = \sqrt{-1}U^1_i$, $V^2_j = \sqrt{-1}V^1_j$, and $p = p_1 + p_2$. Let

$$\{\alpha^1_1, \ldots, \alpha^1_p, \beta^1_1, \ldots, \beta^1_q, \alpha^2_1, \ldots, \alpha^2_p, \beta^2_1, \ldots, \beta^2_q\}$$

be the dual basis of

$$\{U_1^1,\ldots,U_p^1,V_1^1,\ldots,V_q^1,U_1^2,\ldots,U_p^2,V_1^2,\ldots,V_q^2\}.$$

Put

$$\lambda_i=\alpha_i^1+\sqrt{-1}\alpha_i^2,\ \mu_j=\beta_j^1+\sqrt{-1}\beta_j^2$$

for each i,j. Then,

$$d\lambda_k=-\sum_{i,j}C_{ij}^k\lambda_i\wedge\lambda_j, d\mu_t=-\sum_{i,s}D_{is}^t\lambda_i\wedge\mu_s-\sum_{s,h}E_{sh}^t\mu_s\wedge\mu_h$$

for each k,t, where $C_{ij}^k, D_{is}^t, E_{sh}^t\in\mathbb{R}$.

Let J be the complex structure on ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ defined by

$$JU_i^1=U_i^2\ (JU_i^2=-U_i^1),\ JV_j^1=V_j^2\ (JV_j^2=-V_j^1)$$

for each i,j. Note that $({}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}),J)$ is a complex Lie algebra. Let us consider a basis of ${}_{\mathbb{R}}(\mathfrak{t}^{\mathbb{C}})$:

$${}_{\mathbb{R}}(\mathfrak{t}^{\mathbb{C}})=\mathrm{span}\{X_1,\ldots,X_{p_1},Y_1,\ldots,Y_{p_1}\}.$$

Then, we consider other almost complex structure J' on ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ defined by

$$J'X_i=Y_i\ (J'Y_i=-X_i),\ J'U_{p_1+j}^1=-U_{p_1+j}^2\ (J'U_{p_1+j}^2=U_{p_1+j}^1),$$
$$J'V_k^1=V_k^2\ (J'V_k^2=-V_k^1)$$

for each i,j,k. Let ${}_{\mathbb{R}}(G^{\mathbb{C}})$ be the simply connected real Lie group corresponding to ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Then, we have the following theorem.

Theorem 3.1. *J' is integrable on ${}_{\mathbb{R}}(G^{\mathbb{C}})$.*

Proof. Note that $\mu_t,\bar\lambda_{p_1+j}\in\bigwedge_{J'}^{1,0}$ for each t,j, and

$$d\mu_t=-\sum_{i,s}D_{is}^t\lambda_i\wedge\mu_s-\sum_{s,h}E_{sh}^t\mu_s\wedge\mu_h\in\bigwedge_J^{2,0}.$$

Since $\lambda_i\in\bigwedge_{J'}^{1,0}\oplus\bigwedge_{J'}^{0,1}$ for $i=1,\ldots,p$, we have $d\mu_t\in\bigwedge_{J'}^{2,0}\oplus\bigwedge_{J'}^{1,1}$. Since

$$d\bar\lambda_{p_1+j}=-\sum_{k,h}C_{p_1+k,p_1+h}^{p_1+j}\bar\lambda_{p_1+k}\wedge\bar\lambda_{p_1+h}-\sum_{\substack{i=1,\ldots,p_1\\h=1,\ldots,p_2}}C_{i,p_1+h}^{p_1+j}\bar\lambda_i\wedge\bar\lambda_{p_1+h}$$

and $\bar\lambda_i\in\bigwedge_{J'}^{1,0}\oplus\bigwedge_{J'}^{0,1}$ for each $i=1,\ldots,p_1$, we have $d\bar\lambda_{p_1+j}\in\bigwedge_{J'}^{2,0}\oplus\bigwedge_{J'}^{1,1}$. Therefore, J' is integrable on ${}_{\mathbb{R}}(G^{\mathbb{C}})$. □

In the case of $\mathfrak{a}=\mathfrak{k}$, we use the symbol $\tilde J$ instead of J'. We have also proved that if J is a rational complex structure, then $\tilde J$ is also rational. By considering a parametrized basis of ${}_{\mathbb{R}}(\mathfrak{t}^{\mathbb{C}})$, we get a parametrized complex structure on ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$.

Example 3.2. *Let us consider the case of $\mathfrak{a}=\mathfrak{t}$. Put $e^{\theta J}=\cos\theta\cdot\mathrm{id}+\sin\theta\cdot J$ for $\theta\in\mathbb{R}$. Note that*

$$\begin{aligned}{}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})&=\mathrm{span}_{\mathbb{R}}\{U_1^1,\ldots,U_p^1,U_1^2,\ldots,U_p^2,V_1^1,\ldots,V_q^1,V_1^2,\ldots,V_q^2\}\\&=\mathrm{span}_{\mathbb{R}}\{U_1^1,\ldots,U_p^1,e^{\theta_1J}U_1^2,\ldots,e^{\theta_pJ}U_p^2,V_1^1,\ldots,V_q^1,V_1^2,\ldots,V_q^2\}\end{aligned}$$

for $\vartheta=(\theta_1,\ldots,\theta_p)\in\mathbb{R}^p$, where $0\le\theta_i\le\pi,\theta_i\ne\frac{\pi}{2}$. Consider a complex structure J_ϑ on ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ for $\vartheta=(\theta_1,\ldots,\theta_p)\in\mathbb{R}^p$ defined by

$$J_\vartheta U_i^1=e^{\theta_iJ}U_i^2\ (J_\vartheta(e^{\theta_iJ}U_i^2)=-U_i^1),\ J_\vartheta V_j^1=V_j^2\ (J_\vartheta V_j^2=-V_j^1)$$

for each i,j. Then, J_ϑ is integrable by Theorem 3.1.

If each $(\cos\theta_i,\sin\theta_i)$ is a rational point on the unit circle, and $\{U_1^1,\ldots,U_p^1,V_1^1,\ldots,V_q^1\}$ is a basis of $\mathfrak{g}$ such that the structure constants are rational, then $\{U_1^1,\ldots,U_p^1,e^{\theta_1J}U_1^2,\ldots,e^{\theta_pJ}U_p^2,V_1^1,\ldots,V_q^1,V_1^2,\ldots,V_q^2\}$ is a basis of ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ such that the structure constants are rational. Then, J_ϑ is a rational complex structure on ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ by definition.

4 Construction of nilpotent Lie algebras with a decomposition

In this section, we construct nilpotent Lie algebras $\mathfrak{n}$ with a decomposition $\mathfrak{n} = \mathfrak{a} \ltimes \mathfrak{b}$ by root systems.

Let G be a compact semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{h}$ a maximal abelian subalgebra. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of root system Δ. We denote by Δ^+ the set of all positive root relative to Π. Let Π_0, Π_1 be subsets of Π. We put $[\Pi_i] = \Delta \cap \{\Pi_i\}_{\mathbb{Z}}$ for each $i = 1, 2$. Consider the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

Then,

$$\mathfrak{n}^{\mathbb{C}} = \sum_{\alpha \in \Delta^+ - [\Pi_0]} \mathfrak{g}^{\mathbb{C}}_{\alpha}$$

is a nilpotent Lie algebra. Indeed, $\mathfrak{n}^{\mathbb{C}}$ is the nilradical of

$$\mathfrak{u} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in \Delta^+ \cup [\Pi_0]} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

We put $\Delta_{\mathfrak{m}} = \Delta^+ - [\Pi_0]$. Take a normal basis $E_\alpha \in \mathfrak{g}^{\mathbb{C}}_{\alpha}$ ($\alpha \in \Delta$). Then, structure constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = N_{-\alpha,-\beta} \in \mathbb{R}$, where $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ if $\alpha, \beta, \alpha+\beta \in \Delta$. Let $\mathfrak{g}_\alpha = \mathbb{R}E_\alpha$, and $\mathfrak{n} = \sum_{\alpha \in \Delta_{\mathfrak{m}}} \mathfrak{g}_\alpha$. Put

$$\mathfrak{b} = \sum_{\Delta_{\mathfrak{m}} - [\Pi_1]} \mathfrak{g}_\alpha, \mathfrak{a} = \sum_{\Delta_{\mathfrak{m}} \cap [\Pi_1]} \mathfrak{g}_\alpha.$$

Theorem 4.1. *$\mathfrak{a}$ is a subalgebra of $\mathfrak{n}$, and $\mathfrak{b}$ is an ideal of $\mathfrak{n}$.*

Proof. Let $\Pi_1 = \{\alpha_{j_1}, \dots, \alpha_{j_k}\}$. By properties of $\Delta_{\mathfrak{m}} \cap [\Pi_1]$ and $\Delta_{\mathfrak{m}} - [\Pi_1]$, $\mathfrak{a}$, $\mathfrak{b}$ are subalgebras of $\mathfrak{n}$. Let $\beta \in \Delta_{\mathfrak{m}} - [\Pi_1]$, $\alpha \in \Delta_{\mathfrak{m}} \cap [\Pi_1]$. Assume that $\alpha + \beta \in \Delta_{\mathfrak{m}}$. Then, $\alpha + \beta \notin \Delta_{\mathfrak{m}} \cap [\Pi_1]$ because $\alpha + \beta$ can not be written as a linear combination of $\alpha_{j_1}, \dots, \alpha_{j_k}$. Since $(\Delta_{\mathfrak{m}} - [\Pi_1]) \cup (\Delta_{\mathfrak{m}} \cap [\Pi_1]) = \Delta_{\mathfrak{m}}$, we have $\alpha + \beta \in \Delta_{\mathfrak{m}} - [\Pi_1]$. Hence, $\mathfrak{b}$ is an ideal of $\mathfrak{n}$. □

Example 4.2. *Let $G = SU(l+1)$, and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ a basis of the root system of type A_l. Moreover, let $\Pi_0 = \{\alpha_2, \dots, \alpha_{l-1}\}$ and $\Pi_1 = \{\alpha_1, \dots, \alpha_k\}$. Then, $\mathfrak{n}$ is a $(2l-1)$-dimensional Heisenberg Lie algebra*

$$\mathfrak{n} = span\{X_1, \dots, X_{l-1}, Z, Y_1, \dots, Y_{l-1}\}$$

where $[X_i, Y_i] = Z$ for each i, and

$$\mathfrak{a} = \mathfrak{a}_k = span\{X_1, \dots, X_k\}, \mathfrak{b} = \mathfrak{b}_k = span\{X_{k+1}, \dots, X_{l-1}, Z, Y_1, \dots, Y_{l-1}\}.$$

5 Hodge numbers

In this section, we see relations between $h^{s,t}(\mathfrak{g}_J)$ and $h^{s,t}(\mathfrak{g}_{\tilde{J}})$.

We shall now consider the case of $\mathfrak{a} = \mathfrak{k}$. We assume that $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b}$, C^k_{ij}, D^t_{is}, and E^t_{sh} as in section 3. Then, note that

$$[U^1_i, U^1_j] = \sum_{k=1}^{p} C^k_{ij} U^1_k,\ [U^1_i, V^1_s] = \sum_{t=1}^{q} D^t_{is} V^1_t,\ [V^1_s, V^1_t] = \sum_{h=1}^{q} E^h_{st} V^1_h$$

for each i, j, s and t.

Let $\mathfrak{g}_0$ be the Lie algebra defined by

$$\mathfrak{g}_0 = \mathrm{span}\{U_1, \dots, U_p, V_1, \dots, V_q\}$$

which satisfies

$$[U_i, U_j] = \sum_{k=1}^{p} C_{ij}^k U_k, \quad [U_i, V_s] = \sum_{t=1}^{q} D_{is}^t V_t \quad (i, j = 1, \dots, p, s = 1, \dots, q)$$

with other brackets vanishing.

In previous papers [8, 9], we have

Theorem 5.1 ([8])**.** *For each r,*

$$h^{0,r}(\mathfrak{g}_{\bar{J}}) = \dim H^r(\mathfrak{a} \times \mathfrak{b}).$$

Corollary 5.2 ([9])**.**

$$h^{1,0}(\mathfrak{g}_J) - h^{0,1}(\mathfrak{g}_{\bar{J}}) = \dim[\mathfrak{a}, \mathfrak{a}] + \dim[\mathfrak{b}, \mathfrak{b}].$$

Theorem 5.3 ([8])**.** *For each r,*

$$\sum_{s+t=r} h^{s,t}(\mathfrak{g}_{\bar{J}}) = \dim H^r(\mathfrak{g}_0 \times \mathfrak{b} \times \mathbb{R}^{\dim \mathfrak{a}}).$$

Then, we have the following.

Theorem 5.4.

$$\sum_{s+t=r} h^{s,t}(\mathfrak{g}_J) - \sum_{s+t=r} h^{s,t}(\mathfrak{g}_{\bar{J}}) = \sum_{s+t=r} (\dim H^s(\mathfrak{g} \times \mathbb{R}^q) - \dim H^s(\mathfrak{g}_0 \times \mathfrak{b})) \cdot \binom{p}{t}$$

where $p = \dim \mathfrak{a}$, and $q = \dim \mathfrak{b}$.

Proof. Considering the case where $\mathfrak{a} = \{0\}$ and $\mathfrak{b} = \mathfrak{g}$, we have

$$\sum_{s+t=r} h^{s,t}(\mathfrak{g}_J) = \dim_{\mathbb{R}} H^r(\mathbb{R}^{\dim \mathfrak{g}} \times \mathfrak{g} \times \{0\})$$

by Theorem 5.3. Then, we have

$$\begin{aligned} &\sum_{s+t=r} h^{s,t}(\mathfrak{g}_J) - \sum_{s+t=r} h^{s,t}(\mathfrak{g}_{\bar{J}}) \\ &= \dim H^r((\mathfrak{g} \times \mathbb{R}^q) \times \mathbb{R}^p) - \dim H^r((\mathfrak{g}_0 \times \mathfrak{b}) \times \mathbb{R}^p) \\ &= \sum_{s+t=r} \dim H^s(\mathfrak{g} \times \mathbb{R}^q) \cdot \dim H^t(\mathbb{R}^p) - \sum_{s+t=r} \dim H^s(\mathfrak{g}_0 \times \mathfrak{b}) \cdot \dim H^t(\mathbb{R}^p) \\ &= \sum_{s+t=r} (\dim H^s(\mathfrak{g} \times \mathbb{R}^q) - \dim H^s(\mathfrak{g}_0 \times \mathfrak{b})) \cdot \dim H^t(\mathbb{R}^p) \end{aligned}$$

by Theorem 5.3. □

Corollary 5.5.

$$\sum_{s+t=1} h^{s,t}(\mathfrak{g}_J) - \sum_{s+t=1} h^{s,t}(\mathfrak{g}_{\bar{J}}) = \dim([\mathfrak{a}, \mathfrak{b}] \cap [\mathfrak{b}, \mathfrak{b}]) \geq 0.$$

Proof. By Theorem 5.4, we see

$$\sum_{s+t=1} h^{s,t}(\mathfrak{g}_J) - \sum_{s+t=1} h^{s,t}(\mathfrak{g}_{\bar{J}}) = \dim H^1(\mathfrak{g} \times \mathbb{R}^q) - \dim H^1(\mathfrak{g}_0 \times \mathfrak{b}),$$

where $q = \dim \mathfrak{b}$. Then,

$$\dim H^1(\mathfrak{g} \times \mathbb{R}^q) = (\dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}]) + \dim \mathfrak{b}.$$

On the other hand,

$$\begin{aligned}\dim H^1(\mathfrak{g}_0 \times \mathfrak{b}) &= (\dim \mathfrak{g} - \dim[\mathfrak{g}_0, \mathfrak{g}_0]) + (\dim \mathfrak{b} - \dim[\mathfrak{b}, \mathfrak{b}]) \\ &= \dim \mathfrak{g} - \dim[\mathfrak{a}, \mathfrak{a}] - \dim[\mathfrak{a}, \mathfrak{b}] + \dim \mathfrak{b} - \dim[\mathfrak{b}, \mathfrak{b}].\end{aligned}$$

Since $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{a}, \mathfrak{a}] \oplus ([\mathfrak{a}, \mathfrak{b}] + [\mathfrak{b}, \mathfrak{b}])$,

$$\dim[\mathfrak{g}, \mathfrak{g}] = \dim[\mathfrak{a}, \mathfrak{a}] + \dim[\mathfrak{a}, \mathfrak{b}] + \dim[\mathfrak{b}, \mathfrak{b}] - \dim([\mathfrak{a}, \mathfrak{b}] \cap [\mathfrak{b}, \mathfrak{b}]).$$

Thus,

$$\sum_{s+t=1} h^{s,t}(\mathfrak{g}_J) - \sum_{s+t=1} h^{s,t}(\mathfrak{g}_{\tilde{J}}) = \dim([\mathfrak{a}, \mathfrak{b}] \cap [\mathfrak{b}, \mathfrak{b}]).$$ □

Corollary 5.6. *If $h^{1,0}(\mathfrak{g}_J) = h^{0,1}(\mathfrak{g}_{\tilde{J}})$, then $h^{0,1}(\mathfrak{g}_J) = h^{1,0}(\mathfrak{g}_{\tilde{J}})$.*

Proof. By Corollary 5.2, $\mathfrak{a}$ and $\mathfrak{b}$ is abelian. Thus, we have

$$h^{1,0}(\mathfrak{g}_J) + h^{0,1}(\mathfrak{g}_J) = h^{1,0}(\mathfrak{g}_{\tilde{J}}) + h^{0,1}(\mathfrak{g}_{\tilde{J}}).$$

Thus, $h^{0,1}(\mathfrak{g}_J) = h^{1,0}(\mathfrak{g}_{\tilde{J}})$. □

6 Dualities of Hodge numbers of compact nilmanifolds of type A_l

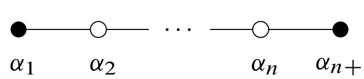
In the previous paper [8], we have the following theorem:

Theorem 6.1. *Let $H_{\mathbb{R}}(n)$ be a $(2n+1)$-dimensional real Heisenberg Lie group, and $\mathfrak{h}_{\mathbb{R}}(n)$ its Lie algebra. Let ${}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}})$ be the simply connected nilpotent Lie group corresponding to ${}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}(n)^{\mathbb{C}})$. Then, there exist a lattice Γ and left-invariant complex structures $\tilde{J}_0, \ldots, \tilde{J}_n$ on ${}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}})$ which satisfy*
(1) *If $k \neq h$, then $(\Gamma \backslash {}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_k)$ and $(\Gamma \backslash {}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_h)$ are not biholomorphic.*
(2) *$\sum_{s+t=r} h^{s,t}(\Gamma \backslash H(n;k)) = \sum_{s+t=r} h^{s,t}(\Gamma \backslash H(n;n-k))$ for each r, where $\Gamma \backslash H(n;h) = (\Gamma \backslash {}_{\mathbb{R}}(H_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_h)$ for each h.*

In this section, we generalize second result.

Let Δ be the root system of A_l, and $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ a basis of the root system of type A_l. Put $S = \{\gamma \in \mathrm{Aut}(\Delta) \mid \gamma(\Pi) = \Pi\}$.

In the case of $l = n+1$ and $\Pi_0 = \{\alpha_2, \ldots, \alpha_n\}$, $\sigma(\Pi_0) = \Pi_0$ for $\sigma \in S$, i.e., the painted Dynkin diagram corresponding to Π_0 is symmetric:



We have seen that the decomposition $\mathfrak{h}_{\mathbb{R}}(n) = \mathfrak{a}_k + \mathfrak{b}_k$ can be constructed by the root system of A_{n+1} and subsets Π_0, $\Pi_1 = \{\alpha_1, \ldots, \alpha_k\}$ of Π in Example 4.2.

Recall that

$$\Delta^+ = \left\{ \sum_{h=i}^{j} \alpha_h \,\middle|\, 1 \le i \le j \le l \right\}.$$

Put $\Pi_{1,k} = \{\alpha_1, \ldots, \alpha_k\}$, and

$$\Delta_{1,k} = \left\{ \sum_{h=i}^{j} \alpha_h \,\middle|\, 1 \leq i \leq k, i \leq j \leq l \right\},$$
$$\Delta_{1,k}^{b} = \left\{ \sum_{h=i}^{j} \alpha_h \,\middle|\, 1 \leq i \leq l, i \leq j, k+1 \leq j \leq l \right\},$$
$$\Delta_1^{k} = \left\{ \sum_{h=i}^{j} \alpha_h \,\middle|\, k \leq i \leq j \leq l \right\}.$$

Then, we see

$$\Delta^+ = \Delta_{1,k+1} \cup \Delta_1^{k+2} = ([\Pi_{1,k}]_{\mathbb{Z}} \cap \Delta^+) \cup \Delta_{1,k}^{b},$$
$$\Delta_{1,k+1} \cap \Delta_1^{k+2} = ([\Pi_{1,k}] \cap \Delta^+) \cap \Delta_{1,k}^{b} = \emptyset.$$

Note that $\Delta_1^{k+2} \subset \Delta_1^{k+1} \subset \Delta_{1,k}^{b}$, and $\sharp\Delta_{1,k} = \sharp\Delta_{1,l-k}^{b}$. Moreover, note that if $\alpha \in [\Pi_{1,k}] \cap \Delta^+$, $\beta \in \Delta_1^{k+2}$, then $\alpha + \beta \notin \Delta^+$, and if $\alpha = \sum_{h=i}^{k} \alpha_h \in [\Pi_{1,k}] \cap \Delta^+$, $\beta = \sum_{h=k+1}^{j} \alpha_h \in \Delta_1^{k+1} \setminus \Delta_1^{k+2}$, then $\alpha + \beta \in \Delta^+$, where $1 \leq i \leq k$ and $k+1 \leq j \leq l$.

Let Π_0 be a subset of Π. Put

$$\mathfrak{b}_k = \sum_{\Delta_{\mathrm{m}} - [\Pi_{1,k}]} \mathfrak{g}_\alpha, \mathfrak{a}_k = \sum_{\Delta_{\mathrm{m}} \cap [\Pi_{1,k}]} \mathfrak{g}_\alpha,$$

and $\mathfrak{n} = \mathfrak{a}_k \ltimes \mathfrak{b}_k$. Similarly as in the case of $\mathfrak{g}_0$ in section 5, let $\mathfrak{n}_{0,k}$ be the nilpotent Lie algebra which is constructed from the nilpotent Lie algebra $\mathfrak{n}$ with the decomposition $\mathfrak{n} = \mathfrak{a}_k \ltimes \mathfrak{b}_k$. Then, we see

$$\mathfrak{n}_{0,k} = \sum_{\alpha \in \Delta_{\mathrm{m}} \cap \Delta_{1,k+1}} \mathfrak{g}_\alpha \times \mathbb{R}^{\sharp(\Delta_{\mathrm{m}} \cap \Delta_1^{k+2})},$$
$$\mathfrak{b}_k = \sum_{\alpha \in \Delta_{1,k}^{b} \cap \Delta_{\mathrm{m}}} \mathfrak{g}_\alpha,$$
$$\mathbb{R}^{\dim \mathfrak{a}_k} = \mathbb{R}^{\sharp(\Delta_{\mathrm{m}} \cap ([\Pi_{1,k}] \cap \Delta^+))}.$$

Then, we have the following:

Theorem 6.2. *If the painted Dynkin diagram corresponding to Π_0 is symmetric, then*

$$\mathfrak{n}_{0,k} \times \mathfrak{b}_k \times \mathbb{R}^{\dim \mathfrak{a}_k} \cong \mathfrak{n}_{0,l-k-1} \times \mathfrak{b}_{l-k-1} \times \mathbb{R}^{\dim \mathfrak{a}_{l-k-1}}$$

as Lie algebras.

Proof. Note that

$$\mathfrak{n}_{0,l-k-1} = \sum_{\alpha \in \Delta_{\mathrm{m}} \cap \Delta_{1,l-k}} \mathfrak{g}_\alpha \times \mathbb{R}^{\sharp(\Delta_{\mathrm{m}} \cap \Delta_1^{l-k+1})},$$
$$\mathfrak{b}_{l-k-1} = \sum_{\alpha \in \Delta_{1,l-k-1}^{b} \cap \Delta_{\mathrm{m}}} \mathfrak{g}_\alpha,$$
$$\mathbb{R}^{\dim \mathfrak{a}_{l-k-1}} = \mathbb{R}^{\sharp(\Delta_{\mathrm{m}} \cap ([\Pi_{1,l-k-1}] \cap \Delta^+))}.$$

Since the painted Dynkin diagram is symmetric, we have

$$\sum_{\alpha \in \Delta_{\mathrm{m}} \cap \Delta_{1,k+1}} \mathfrak{g}_\alpha \cong \sum_{\alpha \in \Delta_{\mathrm{m}} \cap \in \Delta_{1,l-k-1}^{b}} \mathfrak{g}_\alpha,$$
$$\sharp(\Delta_{\mathrm{m}} \cap \Delta_1^{k+2}) = \sharp(\Delta_{\mathrm{m}} \cap ([\Pi_{1,l-k-1}] \cap \Delta^+)),$$
$$\sharp(\Delta_{\mathrm{m}} \cap ([\Pi_{1,k}] \cap \Delta^+)) = \sharp(\Delta_{\mathrm{m}} \cap \Delta_1^{l-k+1}).$$

Thus, we obtain our claim. $\square$

Theorem 6.3. *Let Π_0 be a subset of Π such that the painted Dynkin diagram corresponding to Π_0 is symmetric. Let $\{W_\alpha\}_{\alpha\in\Delta_{\mathrm{m}}}$ is a basis of $\mathfrak{n} = \sum_{\alpha\in\Delta_{\mathrm{m}}} \mathfrak{g}_\alpha$ such that $W_\alpha \in \mathfrak{g}_\alpha$ for each α. Moreover, let $\tilde{J}_k$ be the complex structure on ${}_{\mathbb{R}}(\mathfrak{n}^{\mathbb{C}})$ corresponding to decomposition $\mathfrak{n} = \mathfrak{a}_k \ltimes \mathfrak{b}_k$ and a basis $\{W_\alpha\}_{\alpha\in\Delta_{\mathrm{m}}}$ for each $k = 0,\dots,l$. Then,*

$$\sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_k}) = \sum_{s+t=r} h^{s,t}(\mathfrak{n}_{\tilde{J}_{l-k-1}})$$

for each k and r.

Proof. By theorems 5.3 and 6.2, we have our claim. □

7 Examples

In this section, we see examples of Theorem 6.3, and an example such that Hodge numbers are quite different for each complex structure.

Example 7.1. *Let $\mathfrak{n}(n)$ be the real Lie algebra defined by*

$$\mathrm{span}\{\mathrm{X_{ij}}\}_{1\le \mathrm{i}<\mathrm{j}\le \mathrm{n}}$$

satisfying $[X_{ij}, X_{kh}] = \delta_{jk} X_{ih}$ for each i, j, k and h. Let $N(n)$ be the simply connected nilpotent Lie group corresponding to $\mathfrak{n}(n)$. For example, in the case of $n = 5$, we see

$$N(5) = \left\{ \left. \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 1 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 1 & x_{34} & x_{35} \\ 0 & 0 & 0 & 1 & x_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| x_{ij} \in \mathbb{R} \right\}.$$

Consider the following Lie subalgebras of $\mathfrak{n}(n)$:

$$\begin{aligned} \mathfrak{u}_k &= span\{X_{ij}\}_{1\le i<j\le k} = \mathfrak{a}_{k-1} \\ \mathfrak{v}_k &= span\{X_{ij}\}_{1\le i<j,\, k+1\le j\le n} = \mathfrak{b}_{k-1} \end{aligned}$$

for each k. Then, $\mathfrak{v}_k$ is an ideal of $\mathfrak{n}(n)$, and $\mathfrak{n}(n) = \mathfrak{u}_k + \mathfrak{v}_k$. Hence, we have a rational complex structure $\tilde{J}_k$ corresponding to the decomposition $\mathfrak{n}(n) = \mathfrak{u}_k + \mathfrak{v}_k$. The nilpotent Lie algebra $\mathfrak{n}(n)$ with the decomposition corresponds to the case where $G = SU(n)$, $\Pi = \{\alpha_1,\dots,\alpha_{n-1}\}$, $\Pi_0 = \emptyset$, and $\Pi_1 = \{\alpha_1,\dots,\alpha_{k-1}\}$. Put $\mathfrak{n}(n;k) = ({}_{\mathbb{R}}(\mathfrak{n}(n)^{\mathbb{C}}), \tilde{J}_{k-1})$. By a straightforward computation, we can easily see

$$\dim([\mathfrak{u}_k, \mathfrak{v}_k] \cap [\mathfrak{v}_k, \mathfrak{v}_k]) = \begin{cases} (k-1)\cdot(n-k-1) & (1\le k\le n-1) \\ 0 & (k=n). \end{cases}$$

Thus, we have

$$\sum_{s+t=1} h^{s,t}(\mathfrak{n}(n;1)) - \sum_{s+t=1} h^{s,t}(\mathfrak{n}(n;k)) = \begin{cases} (k-1)(n-k-1) & (1\le k\le n-1) \\ 0 & (k=n). \end{cases}$$

By theorem 6.3, we have

$$\sum_{s+t=r} h^{s,t}(\mathfrak{n}(n;k)) = \sum_{s+t=r} h^{s,t}(\mathfrak{n}(n;n-k))$$

for each k, r. However, $h^{1,0}(\mathfrak{n}(4,1)) = 6, h^{0,1}(\mathfrak{n}(4,1)) = 3$, $h^{1,0}(\mathfrak{n}(4,3)) = 4, h^{0,1}(\mathfrak{n}(4,3)) = 5$ (see Example 4.2 in [9]). On the other hand, we have that $h^{s,t}(\mathfrak{h}(4;2))$ satisfy the following interesting relations:

$$\begin{aligned} h^{s,t}(\mathfrak{h}(4;2)) &= h^{s,0}(\mathfrak{h}(4;2))\cdot h^{0,t}(\mathfrak{h}(4;2)), \\ h^{s,0}(\mathfrak{h}(4;2)) &= h^{0,s}(\mathfrak{h}(4;2)), \\ h^{s,t}(\mathfrak{h}(4;2)) &= h^{t,s}(\mathfrak{h}(4;2)), \end{aligned}$$

where $h^{0,1} = h^{0,5} = 4$, $h^{0,2} = h^{0,4} = 9$, $h^{0,3} = 12$ *(cf. [8, pp.201]). Indeed, for example, we have that*

$$\begin{aligned} h^{1,t} &= h^{0,t} \cdot h^{1,0} - \dim\{[\bar{\lambda}_{12} \wedge \alpha] \in H^{0,t}_{\bar{\partial}}\} \cdot \dim \mathrm{span}\{[\mu_{23}], [\mu_{24}]\} \\ &\quad + \dim\{[\bar{\lambda}_{12} \wedge \alpha] \in H^{0,t}_{\bar{\partial}}\} \cdot \dim \mathrm{span}\{\mu_{13}, \mu_{14}\} \\ &= h^{0,t} \cdot h^{1,0}, \end{aligned}$$

where

$$\begin{cases} \bar{\partial}\lambda_{12} = \bar{\partial}\mu_{23} = \bar{\partial}\mu_{24} = \bar{\partial}\mu_{34} = 0, \\ \bar{\partial}\mu_{13} = -\bar{\lambda}_{12} \wedge \mu_{23}, \quad \bar{\partial}\mu_{14} = -\bar{\lambda}_{12} \wedge \mu_{24}, \\ \bar{\partial}\bar{\lambda}_{12} = \bar{\partial}\bar{\mu}_{23} = \bar{\partial}\bar{\mu}_{13} = \bar{\partial}\bar{\mu}_{34} = 0, \\ \bar{\partial}\bar{\mu}_{24} = -\bar{\mu}_{23} \wedge \bar{\mu}_{34}, \quad \bar{\partial}\bar{\mu}_{14} = -\bar{\mu}_{13} \wedge \bar{\mu}_{34}. \end{cases}$$

However, it is not alway true that $h^{s,0}(\mathfrak{n}_{\tilde{J}_{k_0}}) = h^{0,s}(\mathfrak{n}_{\tilde{J}_{k_0}})$ *for* k_0 *such that* $k_0 = l - k_0 - 1$.

Example 7.2. *Let* $H_{\mathbb{R}}(n)$ *be the* $(2n+1)$*-dimensional real Heisenberg group, and* $\mathfrak{h}_{\mathbb{R}}(n)$ *its Lie algebra. Then,* $\mathfrak{h}_{\mathbb{R}}(n)$ *has a basis* $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ *satisfying* $[X_i, Y_i] = Z$ $(i = 1, \ldots, n)$ *with other brackets vanishing. Consider the following Lie subalgebras of* $\mathfrak{h}_{\mathbb{R}}(n)$*:*

$$\begin{aligned} \mathfrak{a}_k &= span\{X_1, \ldots, X_k\} \\ \mathfrak{b}_k &= span\{X_{k+1}, \ldots, X_n, Y_1, \ldots, Y_n, Z\} \end{aligned}$$

for each $0 \le k \le n$*. Then,* $\mathfrak{h}_{\mathbb{R}}(n) = \mathfrak{a}_k + \mathfrak{b}_k$*. Hence, we have a rational complex structure* $\tilde{J}_k$ *corresponding to the decomposition* $\mathfrak{h}_{\mathbb{R}}(n) = \mathfrak{a}_k + \mathfrak{b}_k$*. We write* $\mathfrak{h}(n;k) = (\mathbb{R}(\mathfrak{h}_{\mathbb{R}}(n)^{\mathbb{C}}), \tilde{J}_k)$*. Since*

$$\dim([\mathfrak{a}_k, \mathfrak{b}_k] \cap [\mathfrak{b}_k, \mathfrak{b}_k]) = \begin{cases} 0 & (k = 0, n) \\ 1 & (k \neq 0, n), \end{cases}$$

we have

$$\sum_{s+t=1} h^{s,t}(\mathfrak{h}(n;0)) - \sum_{s+t=1} h^{s,t}(\mathfrak{h}(n;k)) = \begin{cases} 0 & (k = 0, n) \\ 1 & (k \neq 0, n). \end{cases}$$

Example 7.3. *Let* N *be the* 5*-dimensional complex Heisenberg group:*

$$N = H_{\mathbb{C}}(2) = \left\{ \left. \begin{pmatrix} 1 & z_1 & z_2 & z_5 \\ 0 & 1 & 0 & z_3 \\ 0 & 0 & 1 & z_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| z_1, z_2, z_3, z_4, z_5 \in \mathbb{C} \right\}.$$

Let us consider the bijective map $\phi_\theta : \mathbb{C} \longrightarrow \mathbb{C}$ *defined by*

$$\phi_\theta : z \mapsto \frac{1}{2}\left(z + \bar{z} + \sqrt{-1}\tan\theta(z - \bar{z}) + \frac{1}{\cos\theta}(z - \bar{z}) \right)$$

for $\theta \in [0, \pi] \setminus \{\frac{\pi}{2}\}$*. Then, we can easily see*

$$\phi_\theta\left(\frac{1}{2}(z + \bar{z}) + \frac{1}{2}e^{\sqrt{-1}\theta}(z - \bar{z}) \right) = z.$$

Consider the following global coordinate system:

$$\varphi_{\theta_1,\theta_2} : \begin{pmatrix} 1 & z_1 & z_2 & z_5 \\ 0 & 1 & 0 & z_3 \\ 0 & 0 & 1 & z_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto (\phi_{\theta_1}(z_1), \phi_{\theta_2}(z_2), z_3, z_4, z_5),$$

where $(\theta_1, \theta_2) \in ([0, \pi] \setminus \{\frac{\pi}{2}\}) \times ([0, \pi] \setminus \{\frac{\pi}{2}\})$. *We denote the complex structure corresponding to this global coordinate system by* $J_{(\theta_1, \theta_2)}$, *which corresponds the case of* $\mathfrak{a} = \mathfrak{t}$ *in Section* 3. *Then, we have the basis of the space of the left-invariant* $(1, 0)$*-forms, and the basis of the space of the left-invariant* $(0, 1)$*-forms which satisfy*

$$\begin{cases} \bar{\partial}\omega_1 = \bar{\partial}\omega_2 = \bar{\partial}\omega_3 = \bar{\partial}\omega_4 = 0, \\ \bar{\partial}\omega_5 = -\frac{1}{2}(1 - e^{\sqrt{-1}\theta_1})\bar{\omega}_1 \wedge \omega_3 - \frac{1}{2}(1 - e^{\sqrt{-1}\theta_2})\bar{\omega}_2 \wedge \omega_4, \\ \bar{\partial}\bar{\omega}_1 = \bar{\partial}\bar{\omega}_2 = \bar{\partial}\bar{\omega}_3 = \bar{\partial}\bar{\omega}_4 = 0, \\ \bar{\partial}\bar{\omega}_5 = -\frac{1}{2}(1 + e^{\sqrt{-1}\theta_1})\bar{\omega}_1 \wedge \bar{\omega}_3 - \frac{1}{2}(1 + e^{\sqrt{-1}\theta_2})\bar{\omega}_2 \wedge \bar{\omega}_4. \end{cases}$$

Then, we have

$$\begin{aligned} h^{1,0} &= 5, \quad h^{0,1} = 4, \quad (\theta_1 = \theta_2 = 0), \\ h^{1,0} &= 4, \quad h^{0,1} = 5, \quad (\theta_1 = \theta_2 = \pi), \\ h^{1,0} &= 4, \quad h^{0,1} = 4, \quad \textit{otherwise}, \end{aligned}$$

and

$$h^{2,0} = \begin{cases} 10 & (\theta_1 = \theta_2 = 0) \\ 7 & (\theta_1 = 0 \text{ and } \theta_2 \neq 0, \textit{or } \theta_1 \neq 0 \text{ and } \theta_2 = 0) \\ 6 & \textit{otherwise}, \end{cases}$$

$$h^{1,1} = \begin{cases} 19 & (\theta_1 = \theta_2 = \pi) \\ 20 & (\theta_1 = \theta_2 = 0) \\ 16 & (\theta_1 = 0 \text{ and } \theta_2 \neq 0, \textit{or } \theta_1 \neq 0 \text{ and } \theta_2 = 0) \\ 15 & \textit{otherwise}, \end{cases}$$

$$h^{0,2} = \begin{cases} 10 & (\theta_1 = \theta_2 = \pi) \\ 7 & (\theta_1 = \pi \text{ and } \theta_2 \neq \pi, \textit{or } \theta_1 \neq \pi \text{ and } \theta_2 = \pi) \\ 5 & \textit{otherwise}. \end{cases}$$

Acknowledgement: The author would like to express his deep appreciation to Professor Yusuke Sakane for valuable advice and encouragement during his preparation of the paper. This work was supported by JSPS KAKENHI Grant number JP16K05131.

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