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Research Article

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Regularization of closed positive currents and intersection theory

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Abstract: We prove the existence of a closed regularization of the integration current associated to an effective analytic cycle, with a bounded negative part. By means of the King formula, we are reduced to regularize a closed differential form with L^1_{loc} coefficients, which by extension has a test value on any positive current with the same support as the cycle. As a consequence, the restriction of a closed positive current to a closed analytic submanifold is well defined as a closed positive current. Lastly, given a closed smooth differential (q', q')-form on a closed analytic submanifold, we prove the existence of a closed (q', q')-current having a restriction equal to that differential form. After blowing up we deal with the case of a hypersurface and then the extension current is obtained as a solution of a linear differential equation of order 1.

Keywords: Chern class, Green operator, MacPherson graph construction, Modification, Positive current, Residue current

MSC: 14C17, 32C30, 32J25

1 Introduction

Let (X, ω) be a compact Kähler manifold of dimension n and let T be a closed positive current in X of bidegree (q, q). We prove (see Proposition 9.2) the following result.

Proposition. *There are a constant $C \geq 0$ and a sequence T_l of $\mathcal{C}^\infty$ closed differential (q, q)-forms in X, weakly converging in X to T and satisfying $T_l \geq -C\omega^q$ in X for any l.*

This theorem is a version in any codimension q of Demailly's regularization theorem (see [11]) known only for $q = 1$. Recall the statement of Demailly's theorem in its most precise form.

Let u be a $\mathcal{C}^\infty$ positive differential $(1, 1)$-form on X such that $\frac{i}{2\pi}\Theta_{TX} + u \otimes \mathrm{id}_{TX} \geq 0$ in the sense of Griffiths in X, where $\Theta_{TX} \in \mathcal{C}^\infty_{1,1}(X, T^*X \otimes TX)$ is the Chern curvature form. For T of bidegree $(1, 1)$, we can write $T = \lim_l T_l$ weakly in X with (T_l) a sequence of $\mathcal{C}^\infty$ closed differential $(1, 1)$-forms in X such that $T_l \geq -\lambda_l u$ where (λ_l) is a decreasing sequence of continuous functions in X satisfying $\lambda_l(x) \to \nu(T, x)$ for every $x \in X$, with $\nu(T, x)$ the Lelong number of T at x. In particular, $\lambda_l u$ converges to 0 weakly in X.

Recall also the regularization theorem of Dinh-Sibony (see [13]) which applies to any q but only claims the existence of a closed regularization with a negative part that is bounded in the L^1_{loc} sense.

This theorem asserts precisely the existence of a constant C dependent of ω such that all T in X can be written $T = \lim_l (T_l^+ - T_l^-)$ weakly in X, where T_l^+, T_l^- are some closed positive differential (q, q)-forms of class $\mathcal{C}^\infty$

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on X satisfying $\int_X \|T_l^+\| \leq C\int_X \|T\|$ and $\int_X \|T_l^-\| \leq C\int_X \|T\|$. Here, we denote by $\|T\|$ the mass measure of T. A canonical decomposition $T = T^+ - T^-$ with T^+, T^- closed positive is then obtained.

We prove by a direct method (see Propositions 2.2 and 5.1) the following result, similar to that of Dinh-Sibony.

Proposition. *There is a sequence (T_l) of $\mathcal{C}^\infty$ differential (q,q)-forms closed in X weakly converging in X to T and such that $T_l \geq -u_l$ with (u_l) a sequence of $\mathcal{C}^\infty$ differential (q,q)-forms in X converging to 0 for the $\mathrm{L}^1_{\mathrm{loc}}$ topology in X.*

The existence of the sequence T_l in the first above Proposition is obtained here by a biduality argument and T_l has no explicit construction.

On the other hand we give an explicit construction of a closed regularization in the sense of Dinh-Sibony of the integration current associated with an effective algebraic cycle $\sum_j m_j Z_j$, using a Green form of this cycle (see [22, 24]). The Green form is obtained from a locally free projective resolution of the ideal sheaves, following Bismut-Bost-Gillet-Soulé (see [4, 6]), but here without any assumption of smoothness and without any hypothesis of compatibility between the Hermitian metrics.

Specifically we assume X projective and we consider Z an analytic subset of codimension q of X which is the locus of zeroes of a holomorphic section s of a Hermitian holomorphic vector bundle E above X. To calculate a Green form of the codimension q cycle $\sum_j m_j Z_j$ associated to s, we denote by $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaves generated by the components of s and we suppose given a projective resolution

$$0 \longrightarrow \mathcal{O}(F_n) \xrightarrow{g_n} \mathcal{O}(F_{n-1}) \xrightarrow{g_{n-1}} \dots \xrightarrow{g_2} \mathcal{O}(F_1) \xrightarrow{g_1} \mathcal{O}(E^*) \xrightarrow{g_0} \mathcal{I} \longrightarrow 0$$

with holomorphic vector bundles F_i defined in X, equipped with Hermitian metrics $h_{0,i}$ of class $\mathcal{C}^\infty$ in X.

We get explicitly a differential form Γ with coefficients of class $\mathcal{C}^\infty$ in $X \smallsetminus Z$ and $\mathrm{L}^1_{\mathrm{loc}}$ in X such that in the sense of currents on X, we have

$$\sum_j m_j[Z_j] - \mathrm{dd^c}\Gamma = c_q(\Theta_E) + (-1)^{q-1} \times \text{component of bidegree } (q,q) \text{ of } \prod_{1\leq i\leq n} c(\Theta_{F_i})^{(-1)^{i-1}}, \tag{1}$$

with $c_q(\Theta_E)$ the q^{th} Chern form of the curvature form Θ_E of (E,h) and with $c(\Theta_{F_i}) = \sum_k c_k(\Theta_{F_i})$ the total Chern form of Θ_{F_i}, for Hermitian metrics h and h_i naturally derived from that of E and $h_{0,i}$.

We proceed here directly from the formula of King

$$\sum_j m_j[Z_j] = (\mathrm{dd^c}\log\|s\|)^q - \mathbb{1}_{X\smallsetminus Z}(\mathrm{dd^c}\log\|s\|)^q.$$

Applying the Bott-Chern calculations of transgression forms for the Chern classes involved in the exact sequence $0 \longrightarrow \mathbb{C}s \longrightarrow E \longrightarrow E/\mathbb{C}s \longrightarrow 0$ in $X \smallsetminus Z$, we first make explicit a differential form ψ_1 with coefficients of class $\mathcal{C}^\infty$ in $X \smallsetminus Z$ and $\mathrm{L}^1_{\mathrm{loc}}$ in X, such that in X in the sense of currents, we have

$$\sum_j m_j[Z_j] = c_q(\Theta_E) - c_q(\Theta_{E/\mathbb{C}s}) + \mathrm{dd^c}\psi_1.$$

A generalization of the Poincaré-Lelong formula was also obtained by Andersson (see [1, 2]).

To express $c_q(\Theta_{E/\mathbb{C}s})$ we again argue on $X \smallsetminus Z$ by breaking the resolution in short exact sequences of holomorphic vector bundles. The local integrability of Γ is obtained as a result of the construction of MacPherson of the graph (see [3]), which gives the existence on the blow up of X along Z of an exact complex of holomorphic vector bundles G_i, extending the one given on $X \smallsetminus Z$.

In conclusion and as an application, we obtain (see Proposition 8.3) the following result.

Proposition. *There is an explicit sequence of differential forms $\tilde{W}_l$ of class $\mathcal{C}^\infty$, closed in X, converging weakly in X to $\sum_j m_j[Z_j]$ and satisfying $\tilde{W}_l \geq -\tilde{U}_l$ where $\tilde{U}_l$ is of class $\mathcal{C}^\infty$ in X and converges to 0 for the $\mathrm{L}^1_{\mathrm{loc}}$ topology.*

When Z is smooth, it is further shown that every differential (q',q')-form of class $\mathcal{C}^\infty$ that is closed in Z extends as a closed (q',q')-current in X. This allows for any manifold X to extend formula (1) by setting

$$[Z] = \theta_q + \mathrm{dd^c}\psi_1 \tag{2}$$

with θ_q a closed (q,q)-current satisfying $\{\theta_{q|Z}\} = c_q(N_X Z)$ and with ψ_1 a differential $(q-1,q-1)$-form explicitly calculated (see Proposition 7.1).

2 Closed regularization with negative part converging weakly towards 0

Let (X,ω) be a compact Kähler manifold of dimension n and let T be a closed positive current in X of bidegree (q,q). We give here a closed regularization of T obtained using the Green operator of (X,ω) (see [5, 16, 23]).

Proposition 2.1. *There is a linear operator*

$$N : \{\text{current of bidegree } (q,q) \text{ in } X\} \to \{\text{current of bidegree } (q,q) \text{ in } X\},$$

that is continuous for the weak topology and such that $\theta - N(\theta)$ is closed for every current θ of bidegree (q,q) in X and $N(\theta) = 0$ if θ is closed. Also, if θ is of class $\mathcal{C}^\infty$ in X then $N(\theta)$ is also of class $\mathcal{C}^\infty$.

Proof. We can write $\theta = h + \overline{\partial} A(\theta) + A(\overline{\partial}\theta)$ with harmonic h,

$$A = \overline{\partial}^* G = G\overline{\partial}^*$$

where G is the Green operator. With Δ the $\overline{\partial}$-Laplacian associated with ω, then $I - \Delta G$ is the orthogonal projection on the space of harmonic differential (q,q)-forms on X. Moreover $\partial G = G\partial$ and $\overline{\partial} G = G\overline{\partial}$. See [16, 23] for the construction of G and for the continuity of G for the weak topology of currents and the $\mathrm{L}^1_{\mathrm{loc}}$ topology.

We set $\alpha = A(\theta)$, then we can write $\alpha = h' + \partial A'(\alpha) + A'(\partial\alpha)$ with harmonic h',

$$A' = \partial^* G = G\partial^*.$$

Since $\overline{\partial}$ commutes with ∂^*, then $\overline{\partial}$ commutes with A' and we have $\theta = h + \overline{\partial}\partial A'(\alpha) + N(\theta)$ with $N(\theta) = A'(\overline{\partial}\partial\alpha) + A(\overline{\partial}\theta)$. Then $\theta - N(\theta) = h + \overline{\partial}\partial A'(A(\theta))$ is closed in X.

More we have $\partial\overline{\partial}\alpha = \partial\overline{\partial} A(\theta) = \partial\theta - \partial A(\overline{\partial}\theta) = \partial\theta - A(\partial\overline{\partial}\theta)$ since ∂ commutes with A. Thus $N(\theta) = A'(A(\partial\overline{\partial}\theta)) - A'(\partial\theta) + A(\overline{\partial}\theta)$ satisfies $N(\theta) = 0$ if θ is closed. □

Proposition 2.2. *There is a sequence (T_l) of closed $\mathcal{C}^\infty$ differential (q,q)-forms on X that weakly converge to T in X and satisfy $T_l \geq -u_l$, where (u_l) is a sequence of $\mathcal{C}^\infty$ differential (q,q)-forms on X that weakly converge to* 0.

Proof. (U_α) is a finite covering of X with open sets of coordinate maps and (λ_α) is a $\mathcal{C}^\infty$ partition of the unit subordinate to (U_α). With $\chi_\epsilon \geq 0$ of class $\mathcal{C}^\infty$ approximating the Dirac mass at 0 in $\mathbb{C}^n$, we set

$$\theta_\epsilon = \sum_\alpha \lambda_\alpha \big((T_{|U_\alpha}) * \chi_\epsilon\big)$$

which is a positive differential form of class $\mathcal{C}^\infty$ in X weakly converging in X to T. So $\theta_\epsilon - N(\theta_\epsilon)$ is of class $\mathcal{C}^\infty$ closed in X, weakly converges in X to $T - N(T) = T$ and is $\geq -N(\theta_\epsilon)$ with $N(\theta_\epsilon)$ weakly converging in X to $N(T) = 0$.

Note that $N(\theta_\epsilon)$ is smooth but not necessarily positive nor closed. □

3 Closed regularization constructed from the formula of King

We will give a closed regularization of the current of integration built using an approximation formula of [18].

Let Z be an analytical subset of codimension q of X, let $\mathcal{I} \subset \mathcal{O}_X$ be coherent ideal subsheaves such that $\operatorname{supp}(\mathcal{O}_X/\mathcal{I}) = Z$, let (Z_j) be the family of irreducible analytic components of Z of codimension exactly q and let $m_j \in \mathbb{N}^*$ be the generic multiplicity of $\mathcal{I}$ along Z_j. So

$$\sum_j m_j[Z_j] = \lim_{\epsilon\to 0^+} \frac{(q+1)\epsilon}{2^q(\rho+\epsilon)} \big(\mathrm{dd^c}\log(\rho+\epsilon)\big)^q$$

with a function ρ $\mathcal{C}^\infty$ in X always ≥ 0 such that $\rho^{-1}(0) = Z$ built using a method of [9] to be reminded now.

Let (U_α) be an open finite covering of X and let $f_\alpha : U_\alpha \to \mathbb{C}^{N_\alpha}$ be a holomorphic map satisfying: for every $x \in U_\alpha$, the germs in x of the components of f_α generate the ideal $\mathcal{I}_x$ and therefore $f_\alpha^{-1}(0) = Z \cap U_\alpha$. We denote by $H_\alpha(x)$ a positive definite Hermitian $N_\alpha \times N_\alpha$ matrix dependent in the $\mathcal{C}^\infty$ manner of $x \in U_\alpha$. We set

$$\psi_\alpha = {}^t f_\alpha H_\alpha \overline{f}_\alpha$$

which is a function $\mathcal{C}^\infty$ in U_α always ≥ 0 with $\psi_\alpha^{-1}(0) = Z \cap U_\alpha$. With (λ_α) a $\mathcal{C}^\infty$ partition of the unit subordinate to (U_α), taking

$$\rho = \sum_\alpha \lambda_\alpha^2 \psi_\alpha,$$

there is a constant $C \geq 0$ such that $\mathrm{dd^c}\log(\rho+\epsilon) + C\omega \geq 0$ for all $\epsilon > 0$.

Proposition 3.1. *We have the approximation formula of the integration current*

$$\sum_j m_j[Z_j] = \lim_{\epsilon\to 0^+} \frac{(q+1)\epsilon}{2^q(\rho+\epsilon)} \big(\mathrm{dd^c}\log(\rho+\epsilon) + C\omega\big)^q.$$

Proof. Using the binomial theorem. Just look for k with $1 \leq k \leq q$, we have

$$\frac{\epsilon}{\rho+\epsilon}\big(\mathrm{dd^c}\log(\rho+\epsilon)\big)^{q-k} \wedge \omega^k \to 0$$

weakly in X when $\epsilon \to 0^+$. By the extension theorem of Skoda-El Mir, the Monge-Ampère operator $(\mathrm{dd^c}\log\rho)^{q-k}$ is a differential form with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in X. So $\frac{\rho}{\rho+\epsilon}(\mathrm{dd^c}\log(\rho+\epsilon))^{q-k}$ weakly converges in X to $(\mathrm{dd^c}\log\rho)^{q-k}$ when $\epsilon \to 0^+$. □

We set $V_\epsilon = \frac{q+1}{2^q}(\mathrm{dd^c}\log(\rho+\epsilon) + C\omega)^q$ so

$$\sum_j m_j[Z_j] = \lim_{\epsilon\to 0^+} \frac{\epsilon}{\rho+\epsilon} V_\epsilon$$

with $V_\epsilon \geq 0$ of class $\mathcal{C}^\infty$. We apply the Proposition 2.1 taking $\theta = \frac{\epsilon}{\rho+\epsilon}V_\epsilon$. We set $U_\epsilon = N(\theta)$ and $W_\epsilon = \theta - N(\theta)$, so W_ϵ is a differential form of class $\mathcal{C}^\infty$, closed in X. When $\epsilon \to 0^+$, θ weakly converges in X to $\sum_j m_j[Z_j]$ which is closed and therefore U_ϵ weakly converges in X to 0. Then W_ϵ weakly converges in X to $\sum_j m_j[Z_j]$. More $W_\epsilon \geq -U_\epsilon$.

We have

$$W_\epsilon = h_\epsilon + \overline{\partial}\partial A'\big(A(\frac{\epsilon}{\rho+\epsilon}V_\epsilon)\big)$$

with harmonic h_ϵ convergent when $\epsilon \to 0^+$ in the space of differential forms of class $\mathcal{C}^\infty$ of bidegree (q,q) in X.

As $\frac{\epsilon}{\rho+\epsilon}V_\epsilon$ is positive and bounded by mass, by extracting a sequence, we can assume $\frac{\epsilon}{\rho+\epsilon}V_\epsilon$ weakly converging to a positive current, thus a current with measure coefficients. Because of the orders of singularity in G and $A = \overline{\partial}^* G$, then $G(\frac{\epsilon}{\rho+\epsilon}V_\epsilon)$ converges for the $\mathrm{L}^1_{\mathrm{loc}}$ topology and $A(\frac{\epsilon}{\rho+\epsilon}V_\epsilon)$ too. This implies that $A'(A(\frac{\epsilon}{\rho+\epsilon}V_\epsilon))$ converges for the $\mathrm{L}^1_{\mathrm{loc}}$ topology.

Finally

$$U_\epsilon = N(\frac{\epsilon}{\rho+\epsilon}V_\epsilon) = N(V_\epsilon - \frac{\rho}{\rho+\epsilon}V_\epsilon) = -N(\frac{\rho}{\rho+\epsilon}V_\epsilon)$$

since V_ϵ is closed and therefore $N(V_\epsilon) = 0$. But $\frac{\rho}{\rho+\epsilon}V_\epsilon$ converges for the $\mathrm{L}^1_{\mathrm{loc}}$ topology since $\rho(\frac{1}{2}\mathrm{dd^c}\log\rho)^q = \rho(\sum_j m_j[Z_j] + (\frac{1}{2}\mathrm{dd^c}\log\rho)^q_{|X\setminus Z})$ and $\rho_{|Z} = 0$.

Moreover $\partial(\frac{\rho}{\rho+\epsilon}V_\epsilon) = \{\frac{\partial\rho}{\rho+\epsilon} - \frac{\rho\partial\rho}{(\rho+\epsilon)^2}\} \wedge V_\epsilon$ converges for the $\mathrm{L}^1_{\mathrm{loc}}$ topology. In effect $\frac{\partial\rho}{\rho+\epsilon} \wedge V_\epsilon$ converges for the $\mathrm{L}^1_{\mathrm{loc}}$ topology since

$$\partial\rho \wedge (\frac{1}{2}\mathrm{dd^c}\log\rho)^q = \partial\rho \wedge \big(\sum_j m_j[Z_j] + (\frac{1}{2}\mathrm{dd^c}\log\rho)^q_{|X\setminus Z}\big) = \partial\rho \wedge (\frac{1}{2}\mathrm{dd^c}\log\rho)^q_{|X\setminus Z}$$

and since $\frac{\partial\rho}{\rho} \wedge (\frac{1}{2}\mathrm{dd^c}\log\rho)^q_{|X\setminus Z} = \partial\log\rho \wedge (\frac{1}{2}\mathrm{dd^c}\log\rho)^q_{|X\setminus Z}$ is with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in X, as being the direct image by the blow up μ of X of center Z of a differential form with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in $\tilde{X}$. Also $\frac{\rho\partial\rho}{(\rho+\epsilon)^2} \wedge V_\epsilon$ converges for the $\mathrm{L}^1_{\mathrm{loc}}$ topology.

In the same manner $\partial\overline{\partial}(\frac{\rho}{\rho+\epsilon}V_\epsilon) = \{\frac{\partial\overline{\partial}\rho}{\rho+\epsilon} - \frac{2\partial\rho\wedge\overline{\partial}\rho}{(\rho+\epsilon)^2} - \frac{\rho\partial\overline{\partial}\rho}{(\rho+\epsilon)^2} + \frac{2\rho\partial\rho\wedge\overline{\partial}\rho}{(\rho+\epsilon)^3}\} \wedge V_\epsilon$ converges for the $\mathrm{L}^1_{\mathrm{loc}}$ topology, thanks to $\partial\overline{\partial}\log\rho = \frac{\partial\overline{\partial}\rho}{\rho} - \partial\log\rho \wedge \overline{\partial}\log\rho$.

Consequently $-U_\epsilon = N(\frac{\rho}{\rho+\epsilon}V_\epsilon) = A'A(\partial\overline{\partial}(\frac{\rho}{\rho+\epsilon}V_\epsilon)) - A'(\partial(\frac{\rho}{\rho+\epsilon}V_\epsilon)) + A(\overline{\partial}(\frac{\rho}{\rho+\epsilon}V_\epsilon))$ converges for the $\mathrm{L}^1_{\mathrm{loc}}$ topology to 0.

Note the expression

$$h_\epsilon = \frac{\epsilon}{\rho+\epsilon}V_\epsilon - \Delta G(\frac{\epsilon}{\rho+\epsilon}V_\epsilon)$$

for the orthogonal projection of $\theta = \frac{\epsilon}{\rho+\epsilon}V_\epsilon$ on the space of harmonic differential (q,q)-forms and note that cohomologically $\sum_j m_j\{Z_j\} = \lim_{\epsilon\to 0}\{h_\epsilon\}$.

4 Restriction of a closed positive current to a closed complex submanifold

Suppose Z is a closed complex submanifold of codimension q in X and T is a closed positive current in X of bidimension (p', p'). To define the restriction $T_{|Z}$, we have to define the intersection $T \wedge [Z]$ under the formula $i_*(T_{|Z}) = T \wedge [Z]$ where $i : Z \to X$ is the canonical injection.

Proposition 4.1. *With the notation of Proposition 3.1, there is a sequence $\epsilon_l \to 0^+$ such that the positive not necessarily closed currents*

$$T \wedge \frac{(q+1)\epsilon_l}{2^q(\rho+\epsilon_l)}\big(\mathrm{dd^c}\log(\rho+\epsilon_l) + C\omega\big)^q$$

converge towards a closed positive current which is $T \wedge [Z]$.

Proof. We use that

$$T \wedge \frac{(q+1)\epsilon}{2^q(\rho+\epsilon)}\big(\mathrm{dd^c}\log(\rho+\epsilon) + C\omega\big)^q \leq T \wedge \frac{q+1}{2^q}\big(\mathrm{dd^c}\log(\rho+\epsilon) + C\omega\big)^q$$

and the total mass $\int_X T \wedge \frac{q+1}{2^q}(\mathrm{dd^c}\log(\rho+\epsilon) + C\omega)^q \wedge \omega^{p'-q} = \int_X \{T\}\frac{q+1}{2^q}C^q\{\omega\}^{p'}$ is constant.

To see that $T \wedge [Z]$ is closed, we use that

$$\frac{\epsilon}{(\rho+\epsilon)^2}\mathrm{d}\rho \wedge \big(\mathrm{dd^c}\log(\rho+\epsilon)\big)^j \to 0$$

when $\epsilon \to 0$ for $0 \leq j \leq q$. In effect $\lim_{\epsilon\to 0^+} T \wedge \frac{\epsilon}{\rho+\epsilon}(\mathrm{dd^c}\log(\rho+\epsilon))^j$ has support in Z and is written i_*R. Thus $\lim_{\epsilon\to 0^+} T \wedge \frac{\epsilon}{(\rho+\epsilon)^2}\mathrm{d}\rho \wedge (\mathrm{dd^c}\log(\rho+\epsilon))^j = \lim_{\epsilon\to 0^+}(i_*R \wedge \frac{\mathrm{d}\rho}{\rho+\epsilon}) = 0$ since $\rho_{|Z} = 0$. Also a restriction $T_{|Z}$ can be expressed by means of μ^*T which is closed when T is closed. □

A priori the limit $T \wedge [Z]$ in the Proposition 4.1 depends on the sequence $\epsilon_l \to 0^+$.

5 Closed regularization with negative part converging to 0 for the $\mathrm{L}^1_{\mathrm{loc}}$ topology

We now use the writing

$$T = \Theta + \mathrm{dd^c} S$$

with Θ a $\mathcal{C}^\infty$ differential (q,q)-form closed in X and S a $(q-1,q-1)$-current in X. This is obtained from Proposition 2.1 which gives that $T - \overline{\partial}\partial A'(A(T))$ is of class $\mathcal{C}^\infty$ since T is closed.

Proposition 5.1. *There is a sequence (T_l) of $\mathcal{C}^\infty$ differential (q,q)-forms closed in X weakly converging in X to T and such that $T_l \geq -u_l$ with (u_l) a sequence of $\mathcal{C}^\infty$ differential (q,q)-forms in X converging to 0 for the $\mathrm{L}^1_{\mathrm{loc}}$ topology in X.*

Proof. Since T is positive thus with measure coefficients, $A'(A(T))$ as S is with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients. But $\overline{\partial}A'(A(T)) = A'(\overline{\partial}A(T)) = A'(T-h)$ since $\overline{\partial}T = 0$. As a consequence $\overline{\partial}S$ is with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients and since S is real, ∂S too.

In other words, one can choose S with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients, with ∂S and $\overline{\partial}S$ with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients. We set

$$\tilde{T}_\epsilon = \Theta + \mathrm{dd^c}\Big(\sum_\alpha \lambda_\alpha\big((S_{|U_\alpha}) * \chi_\epsilon\big)\Big)$$

which is a differential (q,q)-form of class $\mathcal{C}^\infty$ closed in X weakly converging in X to T when $\epsilon \to 0^+$. We have

$$\tilde{T}_\epsilon = \Theta + \sum_\alpha \big\{\lambda_\alpha\big((\mathrm{dd^c}S_{|U_\alpha}) * \chi_\epsilon\big) + \mathrm{d}\lambda_\alpha \wedge \big((\mathrm{d^c}S_{|U_\alpha}) * \chi_\epsilon\big) + \big((\mathrm{d}S_{|U_\alpha}) * \chi_\epsilon\big) \wedge \mathrm{d^c}\lambda_\alpha + (\mathrm{dd^c}\lambda_\alpha) \wedge \big((S_{|U_\alpha}) * \chi_\epsilon\big)\big\}$$

then using that $\mathrm{dd^c}S = T - \Theta$ we have $\tilde{T}_\epsilon = \sum_\alpha \lambda_\alpha((T_{|U_\alpha}) * \chi_\epsilon) - \tilde{u}_\epsilon$ with

$$-\tilde{u}_\epsilon = \Theta - \sum_\alpha \lambda_\alpha\big((\Theta_{|U_\alpha}) * \chi_\epsilon\big) + \sum_\alpha \big\{\mathrm{d}\lambda_\alpha \wedge \big((\mathrm{d^c}S_{|U_\alpha}) * \chi_\epsilon\big) + \big((\mathrm{d}S_{|U_\alpha}) * \chi_\epsilon\big) \wedge \mathrm{d^c}\lambda_\alpha + (\mathrm{dd^c}\lambda_\alpha) \wedge \big((S_{|U_\alpha}) * \chi_\epsilon\big)\big\}.$$

We have $\tilde{T}_\epsilon \geq -\tilde{u}_\epsilon$ with $\tilde{u}_\epsilon$ which converges for the $\mathrm{L}^1_{\mathrm{loc}}$ topology necessarily to 0. □

Note that $\tilde{u}_\epsilon$ is smooth but not necessarily positive nor closed.

Since $\tilde{u}_\epsilon \to 0$ for the $\mathrm{L}^1_{\mathrm{loc}}$ topology when $\epsilon \to 0^+$, there are $\epsilon_l \to 0^+$ and $g \geq 0$ and $\mathrm{L}^1_{\mathrm{loc}}$ in X such that $\|\tilde{u}_{\epsilon_l}\| \leq g$ for all l (see the theorem of Fischer-Riesz in [12], Proposition 13.11.4 (ii)). The regularization obtained is therefore of the same type of the Dinh-Sibony regularization (see [13]).

6 Closed extension of a closed current defined on a closed complex submanifold

Suppose Z is a closed complex submanifold of codimension q in X and θ is a $\mathcal{C}^\infty$ differential form closed in Z of bidegree (q',q').

Let $\mu : \tilde{X} \to X$ be the blow up of center Z and let T be a closed current of bidegree (q',q') in X. We define the restriction of T to Z by

$$T_{|Z} = (\mu_{|H})_*\{(\mu^*T)_{|H} \wedge c_1(\Theta_{\mathcal{O}_{P(N_X Z)}(1)})^{q-1}\}$$

by denoting by $H \subset \tilde{X}$ the exceptional divisor.

We are looking for T such that

$$T_{|Z} = \theta$$

and it will be so when $(\mu^*T)_{|H} = \alpha$ with $\alpha = (\mu_{|H})^*\theta$ which is closed in H. We will set $T = \mu_* S$ with S a current closed in $\tilde{X}$ satisfying $S_{|H} = \alpha$. This equality returns to

$$S \wedge [H] = j_*\alpha$$

with $j : H \to \tilde{X}$ the canonical injection.

Proposition 6.1. *With $\sigma \in H^0(\tilde{X}, \mathcal{O}(H))$ satisfying $\sigma^{-1}(0) = H$, we can write $[H] = \gamma \mathrm{D}\sigma \wedge \mathrm{D}\sigma^*$ with γ a measure in $\tilde{X}$ with support H and $j_*\alpha = \beta\gamma \wedge \mathrm{D}\sigma \wedge \mathrm{D}\sigma^*$ with β a $\mathcal{C}^\infty$ differential form of bidegree (q', q') in $\tilde{X}$.*

Proof. Locally $\sigma = z_n e$ where e is a local frame of $\mathcal{O}(H)$ and

$$[H] = C\delta(z_n)\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n.$$

But $\mathrm{D}\sigma = \mathrm{d}z_n \otimes e + z_n \mathrm{D}e$ and $\mathrm{D}\sigma^* = \mathrm{d}\overline{z}_n \otimes e^* + \overline{z}_n \mathrm{D}e^*$ and therefore $\mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n = (\mathrm{d}z_n \otimes e) \wedge (\mathrm{d}\overline{z}_n \otimes e^*) = (\mathrm{D}\sigma - z_n \mathrm{D}e) \wedge (\mathrm{D}\sigma^* - \overline{z}_n \mathrm{D}e^*) = \mathrm{D}\sigma \wedge \mathrm{D}\sigma^*$ modulo terms containing z_n or $\overline{z}_n$.

For the other formula, α is $\mathcal{C}^\infty$ closed in H and there is β a differential form $\mathcal{C}^\infty$ in $\tilde{X}$ not necessarily closed satisfying $\alpha = \beta_{|H}$. So $j_*\alpha = j_*(\beta_{|H}) = \beta \wedge [H]$. □

The aim is therefore of finding S closed in $\tilde{X}$ satisfying

$$j_*\alpha = S \wedge [H] \Leftrightarrow \beta\gamma \wedge \mathrm{D}\sigma \wedge \mathrm{D}\sigma^* = S\gamma \wedge \mathrm{D}\sigma \wedge \mathrm{D}\sigma^*.$$

But we know that $\mathrm{d}\beta_{|H} = 0 \Leftrightarrow \mathrm{d}\beta = \sigma \otimes U + V \otimes \sigma^* + \mathrm{D}\sigma \otimes W + T \otimes \mathrm{D}\sigma^*$ for some U, V, W, T. We will set

$$S = \beta + \sigma \otimes u + v \otimes \sigma^* + \mathrm{D}\sigma \otimes w + t \otimes \mathrm{D}\sigma^*$$

with u, v, w, t to determine so that $\mathrm{d}S = 0$. This is equivalent to

$$\begin{aligned}
-(\sigma \otimes U + V \otimes \sigma^* + \mathrm{D}\sigma \otimes W + T \otimes \mathrm{D}\sigma^*) &= \mathrm{d}(\sigma \otimes u + v \otimes \sigma^* + \mathrm{D}\sigma \otimes w + t \otimes \mathrm{D}\sigma^*)\\
&= \mathrm{D}\sigma \otimes u + \sigma \otimes \mathrm{D}u + \mathrm{D}^2\sigma \otimes w - \mathrm{D}\sigma \otimes \mathrm{D}w + \mathrm{D}v \otimes \sigma^* + v \otimes \mathrm{D}\sigma^* + \mathrm{D}t \otimes \mathrm{D}\sigma^* - t \otimes \mathrm{D}^2\sigma^*\\
&= \sigma \otimes (\mathrm{D}u + \Theta w) + (\mathrm{D}v + \Theta t) \otimes \sigma^* + \mathrm{D}\sigma \otimes (u - \mathrm{D}w) + (v + \mathrm{D}t) \otimes \mathrm{D}\sigma^*
\end{aligned}$$

with Θ the $(1, 1)$-form of curvature of the connection D.

Thus u, v, w, t must satisfy the linear differential equation of order 1

$$\sigma \otimes (\mathrm{D}u + \Theta w + U) + \mathrm{D}\sigma \otimes (u - \mathrm{D}w + W) + (\mathrm{D}v + \Theta t + V) \otimes \sigma^* + (v + \mathrm{D}t + T) \otimes \mathrm{D}\sigma^* = 0. \quad (3)$$

Suppose θ and β real, so $\sigma \otimes U + V \otimes \sigma^* + \mathrm{D}\sigma \otimes W + T \otimes \mathrm{D}\sigma^* = U^* \otimes \sigma^* + \sigma \otimes V^* + W^* \otimes \mathrm{D}\sigma^* + \mathrm{D}\sigma \otimes T^*$. Then we can assume that $U = V^*$ and $W = T^*$ and a solution S will be real if $u = v^*$ and $w = -t^*$.

The differential of $\sigma \otimes (\mathrm{D}u + \Theta w + U) + \mathrm{D}\sigma \otimes (u - \mathrm{D}w + W)$ is

$$\mathrm{D}\sigma \otimes (\mathrm{D}u + \Theta w + U) + \sigma \otimes (\mathrm{D}^2u + \mathrm{D}\Theta w + \mathrm{D}U) + \mathrm{D}^2\sigma \otimes (u - \mathrm{D}w + W) - \mathrm{D}\sigma \otimes (\mathrm{D}u - \mathrm{D}^2w + \mathrm{D}W)$$

$$= \sigma \otimes (\mathrm{D}U + \Theta W) + \mathrm{D}\sigma \otimes (U - \mathrm{D}W) + \mathrm{D}\sigma \otimes \Theta w + \mathrm{D}\sigma \otimes \overline{\Theta} w + \sigma \otimes \overline{\Theta} u + \Theta\sigma \otimes u + \sigma \otimes \mathrm{D}\Theta w - \Theta\sigma \otimes \mathrm{D}w$$

so it's $\sigma \otimes (\mathrm{D}U + \Theta W) + \mathrm{D}\sigma \otimes (U - \mathrm{D}W)$ using that $\Theta + \overline{\Theta} = 0$ and $\mathrm{D}(\Theta w) = \Theta \mathrm{D}w$.

Moreover

$$(\mathrm{D}v + \Theta t + V) \otimes \sigma^* + (v + \mathrm{D}t + T) \otimes \mathrm{D}\sigma^* = (\mathrm{D}u^* + \overline{\Theta} w^* + U^*) \otimes \sigma^* + (u^* - \mathrm{D}w^* + W^*) \otimes \mathrm{D}\sigma^*$$

is the adjoint of $\sigma \otimes (\mathrm{D}u + \Theta w + U) + \mathrm{D}\sigma \otimes (u - \mathrm{D}w + W)$ and its differential is the adjoint of $\sigma \otimes (\mathrm{D}U + \Theta W) + \mathrm{D}\sigma \otimes (U - \mathrm{D}W)$ which is

$$(\mathrm{D}U^* + \overline{\Theta} W^*) \otimes \sigma^* - (U^* - \mathrm{D}W^*) \otimes \mathrm{D}\sigma^* = (\mathrm{D}V - \Theta T) \otimes \sigma^* + (\mathrm{D}T - V) \otimes \mathrm{D}\sigma^*.$$

The existence of a solution to the equation (3) therefore causes the necessary condition

$$\sigma \otimes (\mathrm{D}U + \Theta W) + \mathrm{D}\sigma \otimes (U - \mathrm{D}W) + (\mathrm{D}V - \Theta T) \otimes \sigma^* + (\mathrm{D}T - V) \otimes \mathrm{D}\sigma^* = 0. \quad (4)$$

But we know that $\sigma \otimes U + V \otimes \sigma^* + \mathrm{D}\sigma \otimes W + T \otimes \mathrm{D}\sigma^*$ is exact thus closed. Thus it is known that

$$\begin{aligned}
0 &= \mathrm{D}\sigma \otimes U + \sigma \otimes \mathrm{D}U + \mathrm{D}^2\sigma \otimes W - \mathrm{D}\sigma \otimes \mathrm{D}W + \mathrm{D}V \otimes \sigma^* - V \otimes \mathrm{D}\sigma^* + \mathrm{D}T \otimes \mathrm{D}\sigma^* + T \otimes \mathrm{D}^2\sigma^*\\
&= \sigma \otimes (\mathrm{D}U + \Theta W) + (\mathrm{D}V - \Theta T) \otimes \sigma^* + \mathrm{D}\sigma \otimes (U - \mathrm{D}W) + (\mathrm{D}T - V) \otimes \mathrm{D}\sigma^*
\end{aligned}$$

i.e. that equation (4) is satisfied. S is first constructed in $\tilde{X} \setminus H$ ie in $\sigma \neq 0$ and then extends as a closed current in $\tilde{X}$.

Proposition 6.2. *There is a current T closed in X of bidegree (q', q') such that $T_{|Z} = \theta$.*

7 Formula of King

The restriction $\pi : H = P(N_X Z) \to Z$ of the blow up μ in the exceptional divisor H is a fibration and $\mu_{|H} = i \circ \pi$ where $i : Z \to X$ is the canonical injection. For $y \in \tilde{X}$ the differential $\mathrm{d}\mu(y) : T_y\tilde{X} \to T_{\mu(y)}X$ satisfies $\mathrm{d}\mu(y)(T_y H) \subset T_{\mu(y)}Z$ if $y \in H$. Hence there is an induced map $(N_{\tilde{X}}H)_y \to (N_X Z)_x$ if $x = \mu(y)$, whose image is $\mathcal{O}_{P(N_X Z)}(-1)_y$. So $N_{\tilde{X}}H = \mathcal{O}_{P(N_X Z)}(-1) = \mathcal{O}_H(-1)$ then $\mathcal{O}(H)_{|H} = N_{\tilde{X}}H = \mathcal{O}_H(-1)$.

With $q = \operatorname{codim}_X Z$ we have

$$[Z] = \mu_*\big([H] \wedge c_1(\Theta_{\mathcal{O}_H(1)})^{q-1}\big)$$

with $[H] = c_1(\Theta_{\mathcal{O}(H)}) + \mathrm{dd^c}\log\|\sigma\|$ where $\sigma \in H^0(\tilde{X}, \mathcal{O}(H))$ satisfies $\sigma^{-1}(0) = H$. So $-c_1(\Theta_{\mathcal{O}(H)}) = \mathrm{dd^c}\log\|\sigma\|$ in $\tilde{X} \setminus H$ then

$$\begin{aligned}[Z] &= \mu_*\big([H] \wedge c_1(\Theta_{\mathcal{O}(-H)})^{q-1}\big) = \mu_*\big([H] \wedge \big(-c_1(\Theta_{\mathcal{O}(H)})\big)^{q-1}\big)\\ &= \mathrm{dd^c}\mu_*\big(\log\|\sigma\|(\mathrm{dd^c}\log\|\sigma\|)^{q-1}\big) - \mu_*\big((\mathrm{dd^c}\log\|\sigma\|)^q\big)\\ &= \mathrm{dd^c}\big(\mu_*(\log\|\sigma\|)\big(\mathrm{dd^c}\mu_*(\log\|\sigma\|)\big)^{q-1}\big) - \big(\mathrm{dd^c}\mu_*(\log\|\sigma\|)\big)^q_{|X\setminus Z}\end{aligned}$$

hence the formula of King

$$[Z] = \mathrm{dd^c}\big(\log\|\mu_*\sigma\|(\mathrm{dd^c}\log\|\mu_*\sigma\|)^{q-1}\big) - (\mathrm{dd^c}\log\|\mu_*\sigma\|)^q_{|X\setminus Z}$$

with

$$\mu_*\sigma = (\mu_{|\tilde{X}\setminus H})^{-1*}\sigma \in H^0\big(X \setminus Z, (\mu_{|\tilde{X}\setminus H})^{-1*}\big(\mathcal{O}(H)_{|\tilde{X}\setminus H}\big)\big).$$

But $\mathcal{O}(-H) = \mu^*(\mathcal{I}_Z)$ and then $\mu_*(\mathcal{O}(H)) = \mu_*\mu^*(\mathcal{I}_Z^\vee)$ extends the holomorphic vector bundle $(\mu_{|\tilde{X}\setminus H})^{-1*}(\mathcal{O}(H)_{|\tilde{X}\setminus H})$ defined in $X \setminus Z$.

We will now express the term $-(\mathrm{dd^c}\log\|\mu_*\sigma\|)^q_{|X\setminus Z}$ considering the exact sequence

$$0 \to \mathcal{O}_H(-1) \to \pi^* N_X Z \to Q \to 0$$

above H and with Q the quotient vector bundle. As Q has rank $q-1$, we have $c_q(Q) = 0$. But $c(Q) = \pi^* c(N_X Z) \wedge c(\mathcal{O}_H(-1))^{-1} = \sum_j \pi^* c_j(N_X Z) \wedge \sum_k (-c_1(\mathcal{O}_H(-1)))^k$ and therefore $c_q(Q) = \sum_{0\le k\le q} \pi^* c_{q-k}(N_X Z) \wedge c_1(\mathcal{O}_H(1))^k$ ie

$$0 = c_1\big(\mathcal{O}_H(1)\big)^q + \sum_{1\le k\le q-1} \pi^* c_{q-k}(N_X Z) \wedge c_1\big(\mathcal{O}_H(1)\big)^k + \pi^* c_q(N_X Z).$$

Since $\mathcal{O}_H(1) = \mathcal{O}(-H)_{|H}$, with $\xi = -c_1(\mathcal{O}(H))$ it comes

$$0 = \xi^q_{|H} + \sum_{1\le k\le q-1} \pi^* c_{q-k}(N_X Z) \wedge \xi^k_{|H} + \pi^* c_q(N_X Z).$$

For $0 \le k \le q-1$, using Proposition 6.2, we can write

$$c_{q-k}(N_X Z) = \{\theta_{q-k|Z}\} \tag{5}$$

where θ_{q-k} is a closed current in X of bidegree $(q-k, q-k)$. Since $i \circ \pi = \mu \circ j$ we have $\pi^* c_{q-k}(N_X Z) = \{\pi^* i^* \theta_{q-k}\} = \{j^*\mu^*\theta_{q-k}\} = \{(\mu^*\theta_{q-k})_{|H}\}$. So

$$0 = \xi^q_{|H} + \sum_{1\le k\le q-1} \big\{\big(\mu^*\theta_{q-k} \wedge \big(-c_1(\Theta_{\mathcal{O}(H)})\big)^k\big)_{|H}\big\} + \{(\mu^*\theta_q)_{|H}\}$$

then

$$\xi^q + \sum_{1\le k\le q-1} \big\{\mu^*\theta_{q-k} \wedge \big(-c_1(\Theta_{\mathcal{O}(H)})\big)^k\big\} + \{\mu^*\theta_q\} = \{\mu^*\varphi\}$$

with φ a closed current in X of bidegree (q,q) which satisfies $\{\varphi_{|Z}\} = 0$. Replacing θ_q by $\theta_q - \varphi$, we can assume

$$\xi^q + \sum_{1\le k\le q-1} \big\{\mu^*\theta_{q-k} \wedge \big(-c_1(\Theta_{\mathcal{O}(H)})\big)^k\big\} + \{\mu^*\theta_q\} = 0.$$

With $\alpha = (-c_1(\Theta_{\mathcal{O}(H)}))^q + \sum_{1\le k\le q-1} \mu^*\theta_{q-k} \wedge (-c_1(\Theta_{\mathcal{O}(H)}))^k + \mu^*\theta_q$, then there is w such that $\alpha = \mathrm{dd^c} w$ in $\tilde{X}$.

Let m be an integer such that $\mu_*(\|\sigma\|^m w)$ is $\mathcal{C}^\infty$ in X and 0 on Z, so

$$\mu_* w = \frac{1}{\|\mu_*\sigma\|^m}\big(\mu_*\sigma \otimes \mu_* a + \mu_* b \otimes \mu_*\sigma^* + \mathrm{D}(\mu_*\sigma) \otimes \mu_* c + \mu_* d \otimes \mathrm{D}(\mu_*\sigma^*)\big).$$

Since $-c_1(\Theta_{\mathcal{O}(H)}) = \mathrm{dd^c}\log\|\sigma\|$ in $\tilde{X} \setminus H$, it finally comes the relation

$$\begin{aligned}(\mathrm{dd^c}\log\|\mu_*\sigma\|)^q_{|X\setminus Z} = &- \sum_{1\le k\le q-1} \theta_{q-k} \wedge (\mathrm{dd^c}\log\|\mu_*\sigma\|)^k - \theta_q \\ &+ \mathrm{dd^c}\big\{\|\mu_*\sigma\|^{-m}\big(\mu_*\sigma \otimes \mu_* a + \mu_* b \otimes \mu_*\sigma^* + \mathrm{D}(\mu_*\sigma) \otimes \mu_* c + \mu_* d \otimes \mathrm{D}(\mu_*\sigma^*)\big)\big\}.\end{aligned}$$

Proposition 7.1. *With θ_{q-k} the currents closed in X defined by* (5)*, we can write*

$$\begin{aligned}[Z] = (\mathrm{dd^c}\log\|\mu_*\sigma\|)^q &+ \sum_{1\le k\le q-1} \theta_{q-k} \wedge (\mathrm{dd^c}\log\|\mu_*\sigma\|)^k + \theta_q \\ &- \mathrm{dd^c}\big\{\|\mu_*\sigma\|^{-m}\big(\mu_*\sigma \otimes \mu_* a + \mu_* b \otimes \mu_*\sigma^* + \mathrm{D}(\mu_*\sigma) \otimes \mu_* c + \mu_* d \otimes \mathrm{D}(\mu_*\sigma^*)\big)\big\}\end{aligned}$$

with differential forms a, c with values in $\mathcal{O}(-H)$ and b, d with values in $\mathcal{O}(H)$.

8 Closed regularization obtained from a locally free resolution of the ideal sheaves

Suppose the analytic subset Z of X of codimension $q = n - p$ is written $Z = s^{-1}(0)$ with $s \in H^0(X, E)$ a nonzero holomorphic section of $E \to X$ a Hermitian holomorphic vector bundle of rank N. Let $\mathcal{I}$ be the image of the morphism $\mathcal{O}(E^*) \to \mathcal{O}_X$ of sheaves of $\mathcal{O}_X$-modules induced by s.

King's formula (see [17, 20]) expresses here that the differential form $(\log\|s\|)\big(\mathrm{dd^c}\log\|s\|\big)^{q-1}$ which is $\mathcal{C}^\infty$ in $X \smallsetminus Z$ is with locally integrable coefficients in X and that

$$\sum_j m_j [Z_j] = \mathrm{dd^c}\big((\log\|s\|)\big(\mathrm{dd^c}\log\|s\|\big)^{q-1}\big) - \mathbb{1}_{X\smallsetminus Z}(\mathrm{dd^c}\log\|s\|)^q$$

where (Z_j) still refers to the family of irreducible analytic components of Z of dimension p exactly and $m_j \in \mathbb{N}^*$ is the generic multiplicity of $\mathcal{I}$ along Z_j.

The current $\mathrm{dd^c}((\log\|s\|)\big(\mathrm{dd^c}\log\|s\|\big)^{q-1})$ is also denoted by $(\mathrm{dd^c}\log\|s\|)^q$ and can be obtained from a Monge-Ampère operator in the sense of [10, 14].

Now we will express the term $-\big(\mathrm{dd^c}\log\|s\|\big)^q_{|X\smallsetminus Z}$ following the usual method of Bott-Chern (see [7, 8]) that we will remember.

Consider an exact sequence $0 \longrightarrow L \longrightarrow E \longrightarrow Q \longrightarrow 0$ of holomorphic vector bundles on X with $\mathrm{rk}\, L = 1$. The $\mathcal{C}^\infty$ Hermitian metric on E induces metrics on L and Q. Denote by Θ_E, Θ_L and Θ_Q the $(1,1)$-forms of curvature of the Chern connections respectively on E, L and Q.

Denoting by $c(\Theta_E) = \sum_k c_k(\Theta_E)$ the total Chern form associated to Θ_E, we have to make explicit a solution φ of class $\mathcal{C}^\infty$ in X of the equation

$$c(\Theta_E) - c(\Theta_L)c(\Theta_Q) = -\mathrm{dd^c}\varphi$$

where we did not include here the sign $\wedge$. To define φ we use the following notations: $\mathrm{Hom}(E, E) = E \otimes E^*$ injects itself into the exterior algebra $\bigwedge(E \oplus E^*)$ and the total Chern of Θ_E is then written as $c(\Theta_E) = (I + \frac{i}{2\pi}\Theta_E)^N$ identifying $\bigwedge^N E \otimes \bigwedge^N E^*$ with $\mathbb{C}$ using I_E^N, so $c_k(\Theta_E) = \binom{N}{k} I_E^{N-k}(\frac{i}{2\pi}\Theta_E)^k$.

With v a holomorphic local frame of L, $v^* \in E^*$ the adjoint, we set $\sigma = \frac{vv^*}{\|v\|^2}$ and $\alpha = \frac{DvDv^*}{\|v\|^2}$ where D means the Chern connection on E. We can then take

$$\varphi = \frac{N}{2} \sum_{1\le j\le N-1} \frac{1}{j}\sigma\big\{(I_E + \frac{i}{2\pi}\Theta_E)^j - I_E^j\big\}(I_E + \frac{i}{2\pi}\Theta_E + \frac{i}{2\pi}\alpha)^{N-j-1}. \tag{6}$$

We have also $c(\Theta_Q) = c(\Theta_L)^{-1}c(\Theta_E) + \mathrm{dd^c}(c(\Theta_L)^{-1}\varphi)$ and since $c(\Theta_L) = 1 + c_1(\Theta_L)$, we have

$$-\big(-c_1(\Theta_L)\big)^q = c_q(\Theta_E) - c_q(\Theta_Q) + \sum_{1\le k\le q-1} \big(-c_1(\Theta_L)\big)^k \wedge c_{q-k}(\Theta_E) + \mathrm{dd^c}\psi_0$$

where $\psi_0 = \sum_{0\le k\le q-1} (-c_1(\Theta_L))^k \wedge \varphi_{q-k-1}$ with φ_{q-k-1} the component of bidegree $(q-k-1, q-k-1)$ of φ.

Let $\mu : \tilde{X} \to X$ be the blow up of X along $\mathcal{I}$ and let H be the exceptional divisor. We take L the line sub-bundle of μ^*E such that $\mu^*\mathcal{I} = \mathcal{O}(L^*)$ (see [21]).

Let us apply the above to the exact sequence

$$0 \longrightarrow L \longrightarrow \mu^*E \longrightarrow \mu^*E/L \longrightarrow 0$$

on $\tilde{X}$ and take the direct images by μ. Since $c_1(\Theta_L) = -\mathrm{dd^c}\log\|\mu^*s\|$ in $\tilde{X}\smallsetminus H$ we have

$$-(\mathrm{dd^c}\log\|s\|)^q_{|X\smallsetminus Z} = c_q(\Theta_E) - c_q(\Theta_{E/\mathbb{C}s}) + \sum_{1\le k\le q-1} (\mathrm{dd^c}\log\|s\|)^k \wedge c_{q-k}(\Theta_E) + \mathrm{dd^c}\psi \tag{7}$$

where $\psi = \sum_{0\le k\le q-1} \big(\mathrm{dd^c}\log\|s\|\big)^k \wedge \varphi_{q-k-1}$ and φ is given by (6) with $\sigma = \frac{ss^*}{\|s\|^2}$ and $\alpha = \frac{DsDs^*}{\|s\|^2}$. Being the direct image by the blow up μ of a differential form $\mathcal{C}^\infty$ in $\tilde{X}$, ψ as $c_q(\Theta_{E/\mathbb{C}s})$ is with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in X.

Then, since in $X\smallsetminus Z$

$$(\mathrm{dd^c}\log\|s\|)^k = \mathrm{dd^c}\Big\{-\frac{1}{2(k-1)}\Big(\frac{\mathrm{dd^c}\|s\|^2}{2\|s\|^2}\Big)^{k-1}\Big\} = \mathrm{dd^c}\big(\log\|s\|(\mathrm{dd^c}\log\|s\|)^{k-1}\big)$$

for $k \ge 2$, we have

$$-\big(\mathrm{dd^c}\log\|s\|\big)^q_{|X\smallsetminus Z} = c_q(\Theta_E) - c_q(\Theta_{E/\mathbb{C}s}) + \mathrm{dd^c}\psi'$$

with

$$\psi' = \sum_{1\le k\le q-1} \log\|s\|(\mathrm{dd^c}\log\|s\|)^{k-1} \wedge c_{q-k}(\Theta_E) + \psi.$$

Then the following result generalizes the Poincaré-Lelong formula, a generalization is also due to Andersson (see [1, 2]). The differential forms $c_q(\Theta_{E/\mathbb{C}s})$ and ψ' are $\mathcal{C}^\infty$ in $X\smallsetminus Z$, are with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in X and we have in X the equality between currents

$$\sum_j m_j[Z_j] = c_q(\Theta_E) - c_q(\Theta_{E/\mathbb{C}s}) + \mathrm{dd^c}\psi_1 \tag{8}$$

with $\psi_1 = \psi' + (\log\|s\|)(\mathrm{dd^c}\log\|s\|)^{q-1}$.

Assume now X projective, equipped with an ample line bundle B. There are then $k \in \mathbb{N}$ and sections $s_1, \dots, s_N \in H^0(X, B^{\otimes k})$ such that $Z = s_1^{-1}(0) \cap \dots \cap s_N^{-1}(0)$. We can then take $E = (B^{\otimes k})^{\oplus N}$ and $s = (s_1, \dots, s_N)$.

Now calculate as Bismut-Bost-Gillet-Soulé (see [4, 6, 22]) a Green form of Z in X assuming given an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}(F_n) \xrightarrow{g_n} \mathcal{O}(F_{n-1}) \xrightarrow{g_{n-1}} \dots \xrightarrow{g_2} \mathcal{O}(F_1) \xrightarrow{g_1} \mathcal{O}(E^*) \xrightarrow{g_0} \mathcal{I} \longrightarrow 0$$

with each F_i a holomorphic vector bundle over X, with a Hermitian metric $h_{0,i}$ of class $\mathcal{C}^\infty$ in X. Denote by $h_{0,0}$ the Hermitian metric induced on E^* by that of E.

In $X \smallsetminus Z$, we have an exact sequence of holomorphic vector bundles

$$0 \longrightarrow F_n \xrightarrow{g_n} F_{n-1} \xrightarrow{g_{n-1}} \dots \xrightarrow{g_2} F_1 \xrightarrow{g_1} E^* \xrightarrow{g_0} \mathbb{C} \longrightarrow 0,$$

we can break into short exact sequences of holomorphic vector bundles

$$0 \longrightarrow K_0 \longrightarrow E^* \longrightarrow \mathbb{C} \longrightarrow 0,$$

$$0 \longrightarrow K_i \longrightarrow F_i \xrightarrow{g_i} K_{i-1} \longrightarrow 0$$

for $1 \le i \le n-2$,

$$0 \longrightarrow F_n \xrightarrow{g_n} F_{n-1} \xrightarrow{g_{n-1}} K_{n-2} \longrightarrow 0$$

with K_i defined in $X \smallsetminus Z$. We set $F_0 = E^*$, $F_{-1} = \mathbb{C}$. For $0 \le i \le n$, F_i is immersed in $F_i \oplus F_{i-1}$ identifying $(F_i)_x$ with the graph of $g_i(x)$ for $x \in X$ and we denote by h_i the metric induced in F_i.

We will now still apply the Bott-Chern calculations in the case of an exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow Q \longrightarrow 0$$

but this time without any assumption on rk L. We can then take

$$\varphi = N\sigma \int_0^1 \{(I_E + \frac{i}{2\pi}\Theta_t)^{N-1} - (I_E + \frac{i}{2\pi}\Theta_0)^{N-1}\}\frac{\mathrm{d}t}{t} \tag{9}$$

with

$$\Theta_t = \Theta_E + (1-t)(\mathrm{D}'\mathrm{d}''\sigma - \mathrm{d}''\mathrm{D}'\sigma) - (1-t)^2(\mathrm{d}''\sigma\mathrm{D}'\sigma + \mathrm{D}'\sigma\mathrm{d}''\sigma)$$

and $\sigma : E \to E$ the orthogonal projection onto L.

We can write in $X \smallsetminus Z$ the equalities

$$c(\Theta_{K_{i-1},q_{i-1}}) = c(\Theta_{K_i,h_i})^{-1}c(\Theta_{F_i})\big(1 + \mathrm{dd^c}\big(c(\Theta_{F_i})^{-1}\Phi_i\big)\big)$$

for $1 \le i \le n-2$ and

$$c(\Theta_{K_{n-2},q_{n-2}}) = c(\Theta_{F_n,h_{n-1}})^{-1}c(\Theta_{F_{n-1}})\big(1 + \mathrm{dd^c}\big(c(\Theta_{F_{n-1}})^{-1}\Phi_{n-1}\big)\big)$$

noting q_i the quotient metric on $K_i = F_{i+1}/K_{i+1}$ and Φ_i the differential form φ of (9) for the exact sequence $0 \longrightarrow K_i \longrightarrow F_i \longrightarrow K_{i-1} \longrightarrow 0$. Since more

$$c(\Theta_{K_i,h_i}) - c(\Theta_{K_i,q_i}) = -\mathrm{dd^c}\Lambda_i$$

where

$$\Lambda_i = \frac{\mathrm{rk}\ K_i}{2}\int_0^1 L_{K_i,t}(I_{K_i} + \frac{i}{2\pi}\Theta_{K_i,M_t})^{\mathrm{rk}\ K_i - 1}\mathrm{d}t \tag{10}$$

with M_t the Hermitian metric $(1-t)q_i + th_i$ on K_i and with $L_{K_i,t}$ the endomorphism of K_i for writing the Hermitian form $h_i - q_i$ with M_t (see [7], page 83), we have

$$\begin{aligned} c(\Theta_{K_0}) = &\prod_{1\le i\le n} c(\Theta_{F_i})^{(-1)^{i-1}} \prod_{1\le i\le n-1}\big(1 + \mathrm{dd^c}\big(c(\Theta_{F_i})^{-1}\Phi_i\big)\big)^{(-1)^{i-1}} \\ &\cdot \prod_{0\le i\le n-1}\big(1 + \mathrm{dd^c}\big(c(\Theta_{K_i,h_i})^{-1}\Lambda_i\big)\big)^{(-1)^{i-1}}, \end{aligned} \tag{11}$$

with $K_{n-1} = F_n$ and $c(\Theta_{K_{n-1},q_{n-1}}) = c(\Theta_{F_n})$, the metric q_{n-1} being obtained from the exact sequence $0 \to F_n \to K_{n-1} \to 0$ and with $c(\Theta_{K_0}) = c(\Theta_{K_0,h_0})$, the metric h_0 being obtained from the exact sequence $0 \to K_0 \to E^* \to \mathbb{C} \to 0$ which amounts to the exact sequence $0 \to \mathbb{C}s \to E \to E/\mathbb{C}s \to 0$.

We equip E with the Hermitian metric h dual of the Hermitian metric h_0 on $F_0 = E^*$ and $E/\mathbb{C}s$ with the Hermitian metric induced by h. So

$$c_q(\Theta_{K_0}) = (-1)^q c_q(\Theta_{E/\mathbb{C}s}, h)$$

and we get a differential $(q-1, q-1)$-form Ψ of class $\mathcal{C}^\infty$ in $X \smallsetminus Z$ such that in $X \smallsetminus Z$, we have

$$(-1)^q c_q(\Theta_{E/\mathbb{C}s}, h) - \mathrm{dd^c}\Psi = \text{component of bidegree } (q,q) \text{ of } \prod_{1\le i\le n} c(\Theta_{F_i})^{(-1)^{i-1}}. \tag{12}$$

To prove that Ψ is with $\mathrm{L}^1_{\mathrm{loc}}$ coefficients in X and that the equality holds in X in the sense of currents, MacPherson's construction using the graph (see [3] page 120, [15]) gives the following result.

Proposition 8.1. *Let still $\mu : \tilde{X} \to X$ be the blow up of X along Z with $H = \mu^{-1}(Z)$ the exceptional divisor. There is an exact sequence*

$$0 \longrightarrow G_n \xrightarrow{\gamma_n} G_{n-1} \xrightarrow{\gamma_{n-1}} \dots \xrightarrow{\gamma_2} G_1 \xrightarrow{\gamma_1} G_0 \xrightarrow{\gamma_0} \mathbb{C} \longrightarrow 0$$

of holomorphic vector bundles on $\tilde{X}$ satisfying for $0 \le i \le n$,
(i) G_i is a holomorphic vector subbundle of $\mu^* F_i \oplus \mu^* F_{i-1}$,
(ii) $G_{i|\tilde{X} \smallsetminus H}$ is the graph of $\mu^* F_i \xrightarrow{\mu^* g_i} \mu^* F_{i-1}$ and is isomorphic to $(\mu^* F_i)_{|\tilde{X} \smallsetminus H}$,
(iii) with $G_{-1} = \mathbb{C}$, the morphism $G_i \xrightarrow{\gamma_i} G_{i-1}$ over $\tilde{X}$ extends the morphism $\mu^* F_i \xrightarrow{\mu^* g_i} \mu^* F_{i-1}$ over $\tilde{X} \smallsetminus H$.

Proof. Suppose X connected and let N_i be the rank of F_i. So for $x \in X \smallsetminus Z$ the graph of $g_i(x)$ is a vector subspace of dimension N_i of $(F_i \oplus F_{i-1})_x$. Consider the Grassmann bundles $G(N_i, F_i \oplus F_{i-1}) \to X$ and the fiber product

$$G = G(N_n, F_n \oplus F_{n-1}) \times_X \dots \times_X G(N_0, F_0 \oplus F_{-1}) \xrightarrow{\pi} X.$$

This gives a holomorphic map $\varepsilon : X \smallsetminus Z \to G$. Note ξ_i the tautological vector bundle over $G(N_i, F_i \oplus F_{i-1})$ and $\mathrm{pr}_i^* \xi_i$ its inverse image over G which is a holomorphic vector sub-bundle of rank N_i of $\pi^*(F_i \oplus F_{i-1})$.

Since for $x \in X \smallsetminus Z$, $(\mathrm{pr}_i^* \xi_i)_{\varepsilon(x)} \simeq (\pi^* F_i)_{\varepsilon(x)}$ and the complex of the F_i is exact in $X \smallsetminus Z$ so we have

$$\sum_{0 \le i \le n} (-1)^i (\mathrm{pr}_i^* \xi_i)_{|\varepsilon(X \smallsetminus Z)} = \mathbb{C}$$

in the K-theory of $\varepsilon(X \smallsetminus Z)$.

According to [3], page 122, with $\overline{\varepsilon(X \smallsetminus Z)}$ the closure of $\varepsilon(X \smallsetminus Z)$ in G, then

$$\mu : \overline{\varepsilon(X \smallsetminus Z)} \xrightarrow{\pi} X$$

is the blow up of X along Z and in the K-theory of $\tilde{X}$, we still have $\sum_{0 \le i \le n} (-1)^i (\mathrm{pr}_i^* \xi_i)_{|\tilde{X}} = \mathbb{C}$.

Note $G_i = (\mathrm{pr}_i^* \xi_i)_{|\tilde{X}}$ for $0 \le i \le n$ and $G_{-1} = \mathbb{C}$. Over $\tilde{X} \smallsetminus H$, the morphism $\mu^* F_i \xrightarrow{\mu^* g_i} \mu^* F_{i-1}$ with the isomorphisms $\mu^* F_i \simeq G_i$ and $\mu^* F_{i-1} \simeq G_{i-1}$ is nothing other than the restriction

$$(\mu^* F_i \oplus \mu^* F_{i-1}) \supset G_i \xrightarrow{\mathrm{pr}_2} G_{i-1} \subset (\mu^* F_{i-1} \oplus \mu^* F_{i-2})$$

of the projection $\mathrm{pr}_2 : \mu^* F_i \oplus \mu^* F_{i-1} \to \mu^* F_{i-1}$. Over $\tilde{X}$, we still have $\mathrm{pr}_2(G_i) \subset G_{i-1}$ and therefore $\gamma_i : G_i \xrightarrow{\mathrm{pr}_2} G_{i-1}$ extends $\mu^* F_i \xrightarrow{\mu^* g_i} \mu^* F_{i-1}$ over $\tilde{X} \smallsetminus H$. Finally the resulting complex G_i is exact over $\tilde{X}$. □

When Z is singular, $\tilde{X}$ can be singular and we reduce to the case when $\tilde{X}$ is smooth considering a modification of $\tilde{X}$ which is a desingularization of $\tilde{X}$. The exact complex of holomorphic vector bundles on $\tilde{X}$ remains exact when the inverse images over the desingularization are taken.

Let's break down the exact sequence $0 \longrightarrow G_n \xrightarrow{\gamma_n} G_{n-1} \xrightarrow{\gamma_{n-1}} \dots \xrightarrow{\gamma_2} G_1 \xrightarrow{\gamma_1} G_0 \xrightarrow{\gamma_0} \mathbb{C} \longrightarrow 0$ into short exact sequences of holomorphic vector bundles

$$0 \longrightarrow S_0 \longrightarrow G_0 \xrightarrow{\gamma_0} \mathbb{C} \longrightarrow 0,$$

$$0 \longrightarrow S_i \longrightarrow G_i \xrightarrow{\gamma_i} S_{i-1} \longrightarrow 0$$

for $1 \le i \le n-2$,

$$0 \longrightarrow G_n \xrightarrow{\gamma_n} G_{n-1} \xrightarrow{\gamma_{n-1}} S_{n-2} \longrightarrow 0$$

with S_i defined in $\tilde{X}$.

We equip $G_i \subset \mu^* F_i \oplus \mu^* F_{i-1}$ with the Hermitian metric induced by the initial Hermitian metric of $\mu^* F_i \oplus \mu^* F_{i-1}$.

So in $\tilde{X} \smallsetminus H$ we have short exact sequences

$$0 \longrightarrow \mu^* K_0 \longrightarrow \mu^* F_0 \overset{\mu^* g_0}{\longrightarrow} \mathbb{C} \longrightarrow 0,$$

$$0 \longrightarrow \mu^* K_i \longrightarrow \mu^* F_i \overset{\mu^* g_i}{\longrightarrow} \mu^* K_{i-1} \longrightarrow 0$$

for $1 \leq i \leq n-2$,

$$0 \longrightarrow \mu^* F_n \overset{\mu^* g_n}{\longrightarrow} \mu^* F_{n-1} \overset{\mu^* g_{n-1}}{\longrightarrow} \mu^* K_{n-2} \longrightarrow 0$$

and the Hermitian metric on $\mu^* F_i$ induced by h_i and that obtained by the isomorphism with G_i are the same.

So for $0 \leq k \leq n$ we have the equality between Chern forms

$$c_k(\Theta_{G_i})_{|\tilde{X} \smallsetminus H} = c_k(\Theta_{\mu^* F_i})_{|\tilde{X} \smallsetminus H}$$

then $\tilde{X} \smallsetminus H$ being dense in $\tilde{X}$, we even have $c_k(\Theta_{G_i}) = c_k(\Theta_{\mu^* F_i})$.

To express Ψ, write $\Phi_i = \mu_* \check{\Phi}_i$ with $\check{\Phi}_i$ of class $\mathcal{C}^\infty$ in $\tilde{X}$ satisfying

$$\mu^* c(\Theta_{F_i}) - c(\Theta_{S_i, \check{h}_i}) c(\Theta_{S_{i-1}, \check{q}_{i-1}}) = -\mathrm{dd^c} \check{\Phi}_i \tag{13}$$

and given by the formula (9) written for the exact sequence $0 \longrightarrow S_i \longrightarrow G_i \overset{\gamma_i}{\longrightarrow} S_{i-1} \longrightarrow 0$ with $\check{h}_i$ (respectively $\check{q}_{i-1}$) the metric restriction on S_i (respectively quotient on S_{i-1}). On the other hand, write $\Lambda_i = \mu_* \check{\Lambda}_i$ with $\check{\Lambda}_i$ of class $\mathcal{C}^\infty$ in $\tilde{X}$ satisfying

$$c(\Theta_{S_i, \check{h}_i}) - c(\Theta_{S_i, \check{q}_i}) = -\mathrm{dd^c} \check{\Lambda}_i$$

given as in formula (10). Thus we have $\Psi = \mu_* \check{\Psi}$ with $\check{\Psi}$ of class $\mathcal{C}^\infty$ in $\tilde{X}$ satisfying

$$c_q(\Theta_{S_0}) - \mathrm{dd^c} \check{\Psi} = \text{component of bidegree } (q,q) \text{ of } \mu^* \prod_{1 \leq i \leq n} c(\Theta_{F_i})^{(-1)^{i-1}}.$$

We finally obtain the following result which explicitly gives a Green form of the algebraic cycle $\sum_j m_j Z_j$.

Proposition 8.2. *The differential forms Ψ and $\Gamma = (-1)^{q-1}\Psi + \psi' + (\log \|s\|)(\mathrm{dd^c} \log \|s\|)^{q-1}$ are with coefficients of class $\mathcal{C}^\infty$ in $X \smallsetminus Z$, $\mathrm{L}^1_{\mathrm{loc}}$ in X and we have in X in the sense of currents*

$$\sum_j m_j [Z_j] - \mathrm{dd^c}\Gamma = c_q(\Theta_E) + (-1)^{q-1} \times \text{component of bidegree } (q,q) \text{ of } \prod_{1 \leq i \leq n} c(\Theta_{F_i})^{(-1)^{i-1}},$$

E being equipped with the Hermitian metric h and each F_i with the Hermitian metric h_i.

Since

$$c(E^*) = c(\mathcal{O}_X/\mathcal{I})^{-1} \prod_{1 \leq i \leq n} c(F_i)^{(-1)^{i-1}}$$

with $c(\mathcal{O}_X/\mathcal{I}) = 1 + c_q(\mathcal{O}_X/\mathcal{I}) + \ldots$, Proposition 8.2 gives effectively the formula of Grothendieck $\sum_j m_j \{Z_j\} = (-1)^{q-1} c_q(\mathcal{O}_X/\mathcal{I})$ expressing the cohomology class of the algebraic cycle $\sum_j m_j Z_j$ using the q^{th} Chern class of the coherent sheaf $\mathcal{O}_X/\mathcal{I}$.

Note also that if one uses only the existence of an exact sequence

$$0 \longrightarrow G_n \overset{\gamma_n}{\longrightarrow} G_{n-1} \overset{\gamma_{n-1}}{\longrightarrow} \ldots \overset{\gamma_2}{\longrightarrow} G_1 \overset{\gamma_1}{\longrightarrow} G_0 \overset{\gamma_0}{\longrightarrow} \mathbb{C} \longrightarrow 0$$

of holomorphic vector bundles on $\tilde{X}$ satisfying $G_{i|\tilde{X} \smallsetminus H}$ isomorphic to $(\mu^* F_i)_{|\tilde{X} \smallsetminus H}$ with the property *(iii)* of Proposition 8.1, then one has for any Hermitian metric on G_i, instead of the previous equality $c_k(\Theta_{G_i}) = \mu^* c_k(\Theta_{F_i})$, the equality

$$c_k(\Theta_{G_i}) = \mu^* c_k(\Theta_{F_i}) + \mathrm{dd^c} A_{i,k} + j_* R_{i,k}$$

where $A_{i,k}$ is a differential $(k-1,k-1)$-form with coefficients L^1_{loc} in $\tilde{X}$, $j : H \to \tilde{X}$ is the canonical injection and $R_{i,k}$ is a $(k-1,k-1)$-current closed in H. The formula (13) becomes

$$\mu^* c(\Theta_{F_i}) - c(\Theta_{S_i,\check{h}_i}) c(\Theta_{S_{i-1},\check{q}_{i-1}}) = -\mathrm{dd^c}(\check{\Phi}_i + \sum_k A_{i,k}) - \sum_k j_* R_{i,k}$$

so that $\mu^* c(\Theta_{F_i})$ is replaced by $\mu^* c(\Theta_{F_i}) + \sum_k j_* R_{i,k}$ and Proposition 8.2 does not generalize a priori because in order to calculate $\check{\Psi}$ from the formula generalizing the formula (11), we must make products of currents.

Let as in the proof of Proposition 5.1 (U_α) be a finite covering of X with open sets of coordinate maps and let (λ_α) be a $\mathcal{C}^\infty$ partition of the unit subordinate to (U_α). With $\chi_\epsilon \geq 0$ of class $\mathcal{C}^\infty$ approximating the Dirac mass at 0 in $\mathbb{C}^n$, we set

$$\Gamma_\epsilon = \sum_\alpha \lambda_\alpha \big((\Gamma_{|U_\alpha}) * \chi_\epsilon \big)$$

which is a differential form of class $\mathcal{C}^\infty$ in X, weakly converging in X to Γ. Set

$$\tilde{W}_\epsilon = \mathrm{dd^c}\Gamma_\epsilon + c_q(\Theta_E) + (-1)^{q-1} \times \text{component of bidegree } (q,q) \text{ of } \prod_{1 \leq i \leq n} c(\Theta_{F_i})^{(-1)^{i-1}}.$$

In conclusion, we have the following result.

Proposition 8.3. *The differential forms $\tilde{W}_\epsilon$ are of class $\mathcal{C}^\infty$ closed in X, weakly converge in X to $\sum_j m_j [Z_j]$ and satisfy $\tilde{W}_\epsilon \geq -\tilde{U}_\epsilon$ where the $\tilde{U}_\epsilon$ are of class $\mathcal{C}^\infty$ in X and converge to 0 for the L^1_{loc} topology, when $\epsilon \to 0^+$.*

Proof. See the proof of Proposition 5.1. □

In particular since $\tilde{U}_\epsilon \to 0$ for the L^1_{loc} topology when $\epsilon \to 0^+$, there are a sequence $\epsilon_l \to 0^+$ and a function $g \geq 0$ and L^1_{loc} in X such that $\|\tilde{U}_{\epsilon_l}\| \leq g$ for all l.

9 Closed regularization with bounded negative part

Using the formula of King, we will show for the current $\sum_j m_j [Z_j]$ of integration the existence of a closed regularization with bounded negative part.

First $\log \|s\|$ is a quasi-plurisubharmonic function in X because $\mathrm{dd^c} \log \|s\| + \frac{\{\frac{i}{2\pi}\Theta_E(s), s\}}{\|s\|^2} \geq 0$ where $\Theta_E \in \mathcal{C}^\infty_{1,1}(X, E \otimes E^*)$ is the curvature of E, that satisfies $i\Theta_E$ Hermitian. We choose u a differential $(1,1)$-form $\mathcal{C}^\infty$ closed positive in X such that $\|s\|^2 u \geq \{\frac{i}{2\pi}\Theta_E(s), s\}$. For $\epsilon > 0$ we have by the Cauchy-Schwarz inequality

$$\frac{i}{2\pi}\partial\bar{\partial}\log(\|s\|^2 + \epsilon) + \frac{\{\frac{i}{2\pi}\Theta_E(s), s\}}{\|s\|^2 + \epsilon} \geq 0 \tag{14}$$

so $\frac{i}{2\pi}\partial\bar{\partial}\log(\|s\|^2 + \epsilon) + u \geq \frac{\epsilon u}{\|s\|^2 + \epsilon} \geq 0$.

King's formula is written as

$$\sum_j m_j [Z_j] = (\mathrm{dd^c} \log \|s\| + u)^q - (\mathrm{dd^c} \log \|s\| + u)^q_{|X \setminus Z}$$

since $(\mathrm{dd^c} \log \|s\|)^k = (\mathrm{dd^c} \log \|s\|)^k_{|X \setminus Z}$ for $0 \leq k < q$.

We have the regularization

$$\sum_j m_j [Z_j] = \lim_{\epsilon \to 0} T_\epsilon$$

with $T_\epsilon = (\frac{i}{2\pi}\partial\bar{\partial}\log(\|s\|^2 + \epsilon) + u)^q - R_\epsilon$ where R_ϵ is real $\mathcal{C}^\infty$ closed, weakly converging in X to $(\mathrm{dd^c} \log \|s\| + u)^q_{|X \setminus Z}$.

Proposition 9.1. *There is R_ϵ such that $\lim_{\epsilon \to 0} \int_X R_\epsilon \wedge \theta$ exists in $\mathbb{R}$ for every current $\theta \geq 0$ in X with* $\operatorname{supp} \theta \subset \cup_j Z_j$.

Proof. If θ is a $(n-q,n-q)$-current in X, according to Poly, $\mu^*\theta$ exists as a current in $\tilde{X}$, in the sense that $\mu^*\theta$ is a current such that $\mu_*\mu^*\theta=\theta$ (see [19]). But $\theta\to\mu^*\theta$ is not weakly continuous. By Proposition 4.1 if $\theta\geq 0$ is closed, then $\mu^*\theta$ is ≥ 0 and closed.

If $\theta=\mathrm{d}\psi$ with ψ of class $\mathcal{C}^\infty$, then $\mu^*\theta=\mathrm{d}\mu^*\psi$. If $\theta=\mathrm{d}\psi$ with ψ a current, since $\mu:\tilde{X}\smallsetminus H\to X\smallsetminus Z$ is a submersion, $(\mu^*\mathrm{d}\psi-\mathrm{d}\mu^*\psi)_{|\tilde{X}\smallsetminus H}=0$. So $\mu^*\theta-\mathrm{d}\mu^*\psi$ has support in H then

$$\mu^*\theta=\mathrm{d}\mu^*\psi+j_*\nu$$

with $j:H\to\tilde{X}$ the canonical injection. So $\theta=\mu_*\mu^*\theta=\mathrm{d}\psi+\mu_*(j_*\nu)$ therefore $\mu_*(j_*\nu)=0$.

For θ not necessarily exact, $\mu^*\theta$ will be replaced by $\mu^*\theta-j_*\nu_\theta$ where ν_θ satisfies $\mu_*(j_*\nu_\theta)=0$ and $\nu_\theta=\nu$ if $\theta=\mathrm{d}\psi$. This will still be an inverse image of θ by μ, in the sense that its direct image by μ will be equal to θ.

We have the formula

$$(\mathrm{dd^c}\log\|s\|+u)^q_{|X\smallsetminus Z}\wedge\theta=\mu_*\big((-\xi+\mu^*u)^q\wedge\mu^*\theta\big)$$

claiming that $\mu_*((-\xi+\mu^*u)^q\wedge\mu^*\theta)$ is a current in X extending the current $(\mathrm{dd^c}\log\|s\|+u)^q_{|X\smallsetminus Z}\wedge\theta_{|X\smallsetminus Z}$ defined in $X\smallsetminus Z$.

If θ is $\mathcal{C}^\infty$ in X then $\int_{\tilde{X}}(-\xi+\mu^*u)^q\wedge\mu^*\theta=\int_X(\mathrm{dd^c}\log\|s\|+u)^q_{|X\smallsetminus Z}\wedge\theta$, so if θ is any current, the linear form

$$\theta\to\int_{\tilde{X}}(-\xi+\mu^*u)^q\wedge\mu^*\theta$$

extends $(\mathrm{dd^c}\log\|s\|+u)^q_{|X\smallsetminus Z}$. So we set

$$L(\theta)=\int_{\tilde{X}}(-\xi+\mu^*u)^q\wedge(\mu^*\theta-j_*\nu_\theta)$$

which still satisfies $L(\theta)=\int_X(\mathrm{dd^c}\log\|s\|+u)^q_{|X\smallsetminus Z}\wedge\theta$ if θ is $\mathcal{C}^\infty$, because then $\nu_\theta=0$. Moreover we have $L(\theta)=0$ if $\theta=\mathrm{d}\psi$ with ψ a current.

If there is R_ϵ closed of class $\mathcal{C}^\infty$ satisfying $\lim_{\epsilon\to 0}\int_X R_\epsilon\wedge\theta=L(\theta)$ for every current $\theta\geq 0$ with supp $\theta\subset\cup_j Z_j$ and for every θ smooth, then in particular $L(\theta_0-\theta_1)=0$ when $\theta_0\geq 0$ and $\theta_1\geq 0$ are such that supp $\theta_i\subset\cup_j Z_j$ and $\theta_0-\theta_1=\mathrm{d}\psi$ with ψ a current. This necessary condition is satisfied.

Thus one concludes that there is R_ϵ closed $\mathcal{C}^\infty$ satisfying $\lim_{\epsilon\to 0}\int_X R_\epsilon\wedge\theta=\int_{\tilde{X}}(-\xi+\mu^*u)^q\wedge\mu^*\theta\in\mathbb{R}$ for every current $\theta\geq 0$ with supp $\theta\subset\cup_j Z_j$ and weakly converging to $(\mathrm{dd^c}\log\|s\|+u)^q_{|X\smallsetminus Z}$.

This does not necessarily mean the convergence of the integral $\int_X(\mathrm{dd^c}\log\|s\|+u)^q_{|X\smallsetminus Z}\wedge\theta$ since the $R_\epsilon\wedge\theta$ do not necessarily converge in X to the product $(\mathrm{dd^c}\log\|s\|+u)^q_{|X\smallsetminus Z}\wedge\theta$ of currents. □

We have

$$\int_X(\mathrm{dd^c}\log\|s\|)^q\wedge\theta=\int_X\mathrm{dd^c}\Big\{-\frac{1}{2(q-1)}\Big(\frac{\mathrm{dd^c}\|s\|^2}{2\|s\|^2}\Big)^{q-1}\Big\}\wedge\theta=-\int_X\frac{1}{2(q-1)}\Big(\frac{\mathrm{dd^c}\|s\|^2}{2\|s\|^2}\Big)^{q-1}\wedge\mathrm{dd^c}\theta$$

and this last integral can be defined by means of the Hörmander-Lojasiewicz division theorem, when θ is a current. Thus by changing the first term in T_ϵ, we can assume that $\lim_{\epsilon\to 0}\int_X T_\epsilon\wedge\theta$ exists in $\mathbb{R}$ for every current $\theta\geq 0$ in X with supp $\theta\subset\cup_j Z_j$.

With ω a Kähler metric in X, set $C(\theta)=\sup_{\epsilon>0}(-\frac{\int_X T_\epsilon\wedge\theta}{\int_X\omega^q\wedge\theta})$. If there is a sequence of smooth differential forms $\theta_j\geq 0$ such that $\lim_j C(\theta_j)=+\infty$, replacing θ_j by $(\int_X\omega^q\wedge\theta_j)^{-1}\theta_j$ which is bounded by mass and using an argument of double limit, it can be assumed θ_j converging to a current $\theta\geq 0$ such that $\lim_{\epsilon\to 0}\int_X T_\epsilon\wedge\theta$ is $\leq$ thus $=-\infty$ and such that supp $\theta\subset\cup_j Z_j$ by an application of the Lebesgue-Nikodym theorem. This is a contradiction and there exists a constant $C\geq 0$ such that $-\int_X T_\epsilon\wedge\theta\leq C\int_X\omega^q\wedge\theta$ for every smooth differential form $\theta\geq 0$. Finally $T_\epsilon\geq -C\omega^q$.

Proposition 9.2. *There exists a sequence T_l of closed smooth differential (q,q)-forms in X that weakly converge in X towards $\sum_j m_j[Z_j]$ and satisfy $T_l\geq -C\omega^q$ for all l, where C is a certain constant ≥ 0.*

Remark. *Since the restriction of* $[Z]$ *in* $X \smallsetminus Z$ *is* 0*, we can write* $[Z] = \lim_{\epsilon \to 0} T_\epsilon$ *with* T_ϵ *smooth closed in* X *such that*
(i) $\int_X T_\epsilon \wedge \theta$ *converges for all positive current* θ *in* X *with* $\operatorname{supp} \theta \subset Z$*,*
(ii) $\int_X T_\epsilon \wedge \theta$ *converges to* 0 *for all current* θ *in* X *with compact support in* $X \smallsetminus Z$*.*

As a consequence, $T_{\epsilon|\Omega}$ *converges to* 0 *in the space of smooth differential* (q,q)*-forms in* Ω*, for all relatively compact open subset* $\Omega \subset X \smallsetminus Z$*.*

Now for T a closed positive current of bidegree (q,q) in X, using the formula

$$T = \operatorname{pr}_{1*}([D_X] \wedge \operatorname{pr}_2^* T)$$

with D_X the diagonal in $X \times X$, we can write $T = \lim_{\epsilon \to 0} T_\epsilon$ with T_ϵ smooth closed in X satisfying $T_\epsilon \geq -C\omega^q$ for all $\epsilon > 0$, where C is a certain constant ≥ 0. Moreover if T is smooth on an open subset $U \subset X$, then $T_{\epsilon|\Omega}$ converges to $T_{|\Omega}$ in the space of smooth differential (q,q)-forms in Ω, for all relatively compact open subset $\Omega \subset U$.

For the proof, we write $[D_X] = \lim_{\epsilon \to 0} \Delta_\epsilon$ with Δ_ϵ smooth closed in $X \times X$ such that $\Delta_\epsilon \geq -C_0(\operatorname{pr}_1^*\omega + \operatorname{pr}_2^*\omega)^n$ for all $\epsilon > 0$, with $C_0 \geq 0$. We set

$$T_\epsilon = \operatorname{pr}_{1*}(\Delta_\epsilon \wedge \operatorname{pr}_2^* T)$$

which is smooth closed in X and weakly converges in X towards T. Moreover

$$T_\epsilon \geq -C_0 \operatorname{pr}_{1*}\big((\operatorname{pr}_1^*\omega + \operatorname{pr}_2^*\omega)^n \wedge \operatorname{pr}_2^* T\big)$$

and using the binomial theorem we have

$$\operatorname{pr}_{1*}\big((\operatorname{pr}_1^*\omega + \operatorname{pr}_2^*\omega)^n \wedge \operatorname{pr}_2^* T\big) = \sum_{k=0}^{n} C_n^k \operatorname{pr}_{1*}(\operatorname{pr}_1^*\omega^k \wedge \operatorname{pr}_2^*\omega^{n-k} \wedge \operatorname{pr}_2^* T) = \sum_{k=0}^{n} C_n^k \omega^k \wedge \operatorname{pr}_{1*}\operatorname{pr}_2^*(\omega^{n-k} \wedge T).$$

Since $\omega^{n-k} \wedge T$ is a current on X, we can take $n - k + q \leq n$ by considering the bidegree. Since the fibers of pr_1 are of dimension n, we can take $n \leq n - k + q$. In such a way, we are reduced to take $k = q$, therefore we conclude that

$$T_\epsilon \geq -C_0 C_n^q \omega^q \operatorname{pr}_{1*}\operatorname{pr}_2^*(\omega^{n-q} \wedge T)$$

where the constant $\operatorname{pr}_{1*}\operatorname{pr}_2^*(\omega^{n-q} \wedge T)$ is the volume of T with respect to ω.

On the other hand, let χ be a smooth function on X with compact support in U, equal to 1 on an open neighbourhood of $\overline{\Omega}$. We write

$$T_\epsilon = \operatorname{pr}_{1*}\big(\Delta_\epsilon \wedge \operatorname{pr}_2^*(\chi T)\big) + \operatorname{pr}_{1*}\big(\Delta_\epsilon \wedge \operatorname{pr}_2^*\big((1-\chi)T\big)\big)$$

and we use the hypocontinuity of the convolution of a distribution by a smooth function with compact support. Since $\operatorname{supp}(\chi T) \subset U$, χT is smooth in X thus $\operatorname{pr}_{1*}(\Delta_\epsilon \wedge \operatorname{pr}_2^*(\chi T))$ converges to χT in the space of smooth differential (q,q)-forms in X and $(\chi T)_{|\Omega} = T_{|\Omega}$. For the second term, we use that if $x \in \overline{\Omega}$ and $y \in \operatorname{supp}((1-\chi)T)$, then $y \notin \{\chi = 1\}$, thus $(x,y) \notin D_X$. Since Δ_ϵ converges to 0 in the space of smooth differential forms on every relatively compact open subset $\subset (X \times X) \smallsetminus D_X$, this second term converges to 0 in the space of smooth differential forms on Ω.

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