



Complex Manifolds

Research Article

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The flux homomorphism on closed hyperbolic surfaces and Anti-de Sitter three-dimensional geometry

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Abstract: Given a smooth spacelike surface Σ of negative curvature in Anti-de Sitter space of dimension 3, invariant by a representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}$ where S is a closed oriented surface of genus ≥ 2 , a canonical construction associates to Σ a diffeomorphism ϕ_Σ of S . It turns out that ϕ_Σ is a symplectomorphism for the area forms of the two hyperbolic metrics h and h' on S induced by the action of ρ on $\mathbb{H}^2 \times \mathbb{H}^2$. Using an algebraic construction related to the flux homomorphism, we give a new proof of the fact that ϕ_Σ is the composition of a Hamiltonian symplectomorphism of (S, h) and the unique minimal Lagrangian diffeomorphism from (S, h) to (S, h') .

Keywords: Anti-de Sitter geometry, Hamiltonian symplectomorphisms, Hyperbolic surfaces

MSC: 57M50, 53D05, 30F45

1 Introduction

Anti-de Sitter space is a real Lorentzian three-manifold of constant negative sectional curvature, which can be defined as the Lie group $\mathrm{PSL}_2\mathbb{R}$ endowed with the Lorentzian metric induced by the Killing form. In recent times, since the groundbreaking paper [12] of Mess, its study has spread widely, mostly motivated by the relations between Anti-de Sitter space (which is denoted AdS^3) and Teichmüller theory of hyperbolic surfaces – and the present paper lies in this research direction as well.

Let us explain one of the instances of this relation. A remarkable construction permits to associate to a smooth spacelike surface Σ (topologically a disc) in AdS^3 a submanifold Λ_Σ of $\mathbb{H}^2 \times \mathbb{H}^2$, where \mathbb{H}^2 is the hyperbolic plane. This is essentially due to the fact that the space of timelike lines of AdS^3 is identified to $\mathbb{H}^2 \times \mathbb{H}^2$. Hence the submanifold Λ_Σ is the set of timelike lines orthogonal to Σ – roughly speaking, the analogue of the Gauss map in this context.

It turns out, as observed in [2] and [6], that Λ_Σ is always a *Lagrangian* submanifold, for the symplectic structure which makes $\mathbb{H}^2 \times \mathbb{H}^2$ a Kähler manifold, namely $\Omega = (\pi_l^* \Omega_{\mathbb{H}^2} - \pi_r^* \Omega_{\mathbb{H}^2})$, where $\Omega_{\mathbb{H}^2}$ is the area form of \mathbb{H}^2 , π_l, π_r are the projections to each factor of $\mathbb{H}^2 \times \mathbb{H}^2$, and $\tilde{\mathbb{H}}^2$ denotes \mathbb{H}^2 endowed with the opposite orientation. There are two classes of particular cases of this construction:

- If Σ has negative curvature, then Λ_Σ is the graph of a symplectomorphism ϕ_Σ of $(\mathbb{H}^2, \Omega_{\mathbb{H}^2})$ – this is the prototype of Lagrangian submanifolds of $\mathbb{H}^2 \times \mathbb{H}^2$. This case was first considered in [9].

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- If $\Sigma = \Sigma_0$ has vanishing mean curvature (also called *maximal*, which implies negative curvature), then ϕ_{Σ_0} is *minimal Lagrangian*, that is, it is a symplectomorphism and its graph is a minimal surface in the Riemannian product $\mathbb{H}^2 \times \mathbb{H}^2$. See [3, 4, 16–18] for results in this direction.

From the results of [2] and [5], it follows that being Lagrangian is essentially the only obstruction to inverting this construction, namely to realizing a submanifold of $\mathbb{H}^2 \times \mathbb{H}^2$ as the image of the Gauss map of a spacelike surface in AdS^3 .

However, the situation is extremely more interesting when we consider cocompact actions. More precisely, let S be a closed oriented surface of negative Euler characteristic. Suppose Σ is invariant by an action of the group $\pi_1(S)$, which preserves the orientation and the time-orientation of AdS^3 . This produces therefore a representation ρ of $\pi_1(S)$ in the isometry group $\text{Isom}(\text{AdS}^3)$, which (from the definition of AdS^3 as $\text{PSL}_2\mathbb{R}$ endowed with the bi-invariant metric) is naturally isomorphic to $\text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R}$. By the theory developed in [12], it turns out that $\rho = (\rho_l, \rho_r)$ where ρ_l and ρ_r are Fuchsian representations — that is, $\rho_l(\pi_1(S))$ and $\rho_r(\pi_1(S))$ act freely and properly discontinuously on \mathbb{H}^2 , with quotient a hyperbolic surface homeomorphic to S .

Let us identify the quotient hyperbolic surfaces $\mathbb{H}^2/\rho_l(\pi_1(S))$ and $\mathbb{H}^2/\rho_r(\pi_1(S))$ by (S, h_l) and (S, h_r) respectively, where h_l and h_r are hyperbolic metrics on S . Then the Lagrangian submanifold Λ_Σ of $\mathbb{H}^2 \times \mathbb{H}^2$ descends to a Lagrangian submanifold in the quotient $(S \times S, h_l \oplus h_r)$. In particular, if the cocompact surface Σ has negative curvature, then ϕ_Σ induces a symplectomorphism, which we denote again by $\phi_\Sigma : (S, \Omega_{h_l}) \rightarrow (S, \Omega_{h_r})$. On the other hand, again by the work of Mess, whenever one picks two Fuchsian representations ρ_l and ρ_r , there is abundance of embedded surfaces $\Sigma \subset \text{AdS}^3$ on which $\pi_1(S)$ acts freely and properly discontinuously by means of the representation $\rho = (\rho_l, \rho_r) : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R}$. For instance, there always exists a (unique) invariant maximal surface Σ_0 (i.e. of vanishing mean curvature, see [3]), which induces the minimal Lagrangian diffeomorphism isotopic to the identity:

$$\phi_{\Sigma_0} = \phi_{\text{ML}} : (S, h_l) \rightarrow (S, h_r).$$

It was already known from the results of [10] and [15] that such a minimal Lagrangian diffeomorphism ϕ_{ML} exists and is unique, for any two closed hyperbolic surfaces (S, h_l) and (S, h_r) .

Hence it is a natural question to characterize the symplectomorphisms

$$\phi_\Sigma =: (S, \Omega_{h_l}) \rightarrow (S, \Omega_{h_r}),$$

which arise as the symplectomorphism associated to a cocompact surface Σ in AdS^3 . The main result of this paper is a new proof of the following:

Theorem. *Let $\rho_l, \rho_r : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R}$ be Fuchsian representations and let $\Sigma \subset \text{AdS}^3$ be a smooth, embedded spacelike surface invariant for the representation $\rho = (\rho_l, \rho_r) : \pi_1(S) \rightarrow \text{Isom}_0(\text{AdS}^3)$, whose curvature is negative. Then*

$$\phi_\Sigma = \phi_{\text{ML}} \circ \psi,$$

where

- $\phi_\Sigma : (S, \Omega_{h_l}) \rightarrow (S, \Omega_{h_r})$ is the diffeomorphism associated to Σ ;
- $\phi_{\text{ML}} : (S, h_l) \rightarrow (S, h_r)$ is the unique minimal Lagrangian diffeomorphism isotopic to the identity;
- ψ is a Hamiltonian symplectomorphism, for the area form Ω_{h_l} .

Given a symplectic manifold (M, Ω) , a symplectomorphism $\psi : (M, \Omega) \rightarrow (M, \Omega)$ is Hamiltonian if there exists an isotopy ψ_t with $\psi_0 = \text{id}$ and $\psi_1 = \psi$ and a smooth family of functions $H_t : M \rightarrow \mathbb{R}$ such that the generating vector field X_t of the isotopy ψ_t is the symplectic gradient of H_t for every t . It turns out that Hamiltonian symplectomorphisms form a group, denoted $\text{Ham}(M, \Omega)$.

The above theorem is actually a consequence of the main result in [6], where it was proved that any submanifold Λ_Σ associated to an invariant surface Σ is in the same $\text{Ham}(S \times S, \pi_l^* \Omega_{h_l} - \pi_r^* \Omega_{h_r})$ -orbit of the submanifold $\text{graph}(\phi_{\text{ML}})$. This implies that, when Σ has negative curvature, and therefore Λ_Σ is the

graph of the symplectomorphism $\phi_\Sigma : (S, \Omega_{h_i}) \rightarrow (S, \Omega_{h_r})$, then ϕ_Σ and ϕ_M differ by a Hamiltonian symplectomorphism of (S, Ω_{h_i}) .

However, we provide here a new proof, when Σ has negative curvature. This relies on the construction of a map $C_{h,h'}$, defined on $\text{Symp}_0(S, \Omega_h, \Omega_{h'})$ with values in the quotient of the de Rham cohomology group $H^1_{\text{dR}}(S, \mathbb{R}) \cong \text{Hom}(\pi_1(S), \mathbb{R})$ by the discrete subgroup of 1-forms with periods integer multiples of 2π , which is identified to $\text{Hom}(\pi_1(S), 2\pi\mathbb{Z})$. This map is given by a rather algebraic construction, which involves differential-geometric invariants of the two hyperbolic surfaces h and h' . Hence *a priori* it depends on the choice of the two hyperbolic metrics. However, we then show that, when $h = h'$, this map coincides with the so-called *flux homomorphism* (see [7], [1] and [13, Chapter 6]):

$$\text{Flux} : \text{Symp}_0(S, \Omega) \rightarrow H^1_{\text{dR}}(S, \mathbb{R}) ,$$

post-composed with the projection to the quotient of $H^1_{\text{dR}}(S, \mathbb{R})$ by $\text{Hom}(\pi_1(S), 2\pi\mathbb{Z})$. In particular, when $h = h'$, then the map $C_{h,h}$ turns out to be a homomorphism (this is proved directly, and is an important step of the proof), and only depends on the area form of h . The proof of the main theorem then follows from interpreting such map $C_{h,h'}$ in terms of Anti-de Sitter geometry, and showing essentially that it vanishes when the symplectomorphism ϕ_Σ is associated to an invariant spacelike surface Σ . One then obtains our main result as a consequence of the classical fact that the group of Hamiltonian symplectomorphisms $\text{Ham}(M, \Omega)$ is exactly the kernel of the flux homomorphism Flux, applied to the symplectic manifold $(M, \Omega) = (S, \Omega_{h_i})$.

The paper then terminates with some remarks about the converse statement of our main theorem. Namely, given ϕ of the form $\phi_M \circ \psi$, for $\psi \in \text{Ham}(S, \Omega_{h_i})$, one can construct an invariant surface Σ in AdS^3 , which is the candidate to be a surface such that $\phi_\Sigma = \phi$. This is indeed the case if Σ is embedded. However, such surface Σ will in general develop singularities. Although this generality is not considered here, in [6] it was showed that one can always construct a smooth lift of Σ to the unit tangent bundle of the universal cover of AdS^3 , thus partially reversing the implication proved here.

2 Algebraic construction for the flux homomorphism

In this paper, S will denote a closed oriented surface of Euler characteristic $\chi(S) < 0$. Moreover, we will usually fix two hyperbolic metrics (i.e. Riemannian metrics of constant curvature -1) h and h' on S .

In the first section, we will recall the classical definition of flux homomorphism, for a symplectic surface (S, Ω) , from the point of view of symplectic geometry. Then we will give the definition of the alternative map $C_{h,h'}$ which *a priori* depends on the choice of hyperbolic metrics h and h' on S . We will study some of its properties, for instance the fact that it is a homomorphism in the case $h = h'$, and we will show that the two homomorphisms essentially coincide in this case.

The flux homomorphism

In the definitions below, we will only suppose that S is a closed oriented surface endowed with a symplectic form Ω . Later, we will consider $\Omega = \Omega_h$ as the area form induced by a hyperbolic metric h . Let us denote by $\text{Symp}_0(S, \Omega)$ the group of symplectomorphisms $\psi : (S, \Omega) \rightarrow (S, \Omega)$ isotopic to the identity.

Let us recall the definition of Hamiltonian symplectomorphisms (see [13, Chapter 6] as a reference):

Definition 2.1. *Let (S, Ω) be a closed symplectic manifold. Then a symplectomorphism $\psi : (S, \Omega) \rightarrow (S, \Omega)$ is Hamiltonian if there exists a smooth isotopy*

$$\psi_\bullet : [0, 1] \rightarrow \text{Symp}_0(S, \Omega) ,$$

with $\psi_0 = \text{id}$ and $\psi_1 = \psi$, and a smooth map

$$H_\bullet : [0, 1] \times S \rightarrow \mathbb{R} ,$$

such that (if we denote $H_t(p) = H(t, p)$):

$$\Omega(X_t, \cdot) = dH_t,$$

where X_t is the generating vector field of the isotopy ψ_t , namely:

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t.$$

It turns out that the space of Hamiltonian symplectomorphisms is a group, which we denote by $\text{Ham}(S, \Omega)$. Let us now define the flux homomorphism

$$\text{Flux} : \text{Symp}_0(S, \Omega) \rightarrow H_{\text{dR}}^1(S, \mathbb{R}).$$

Definition 2.2. Given a symplectomorphism $\psi : (S, \Omega) \rightarrow (S, \Omega)$ isotopic to the identity, take an isotopy ψ_t with $\psi_0 = \text{id}$ and $\psi_1 = \psi$, and define

$$\text{Flux}(\psi) = \int_0^1 [\Omega(X_t, \cdot)] dt \in H_{\text{dR}}^1(S, \mathbb{R}),$$

where X_t is the generating vector field of the isotopy ψ_t , namely:

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t.$$

It turns out that, if there exists a homotopy in $\text{Symp}_0(S, \Omega)$ between the two paths

$$\psi_\bullet, \psi'_\bullet : [0, 1] \rightarrow \text{Symp}_0(S, \Omega)$$

with fixed endpoints, then the value of $\text{Flux}(\psi)$ does not change. Since the group $\text{Symp}_0(S, \Omega)$ is simply connected, see [14, Section 7.2], Flux is well-defined on $\text{Symp}_0(S, \Omega)$. In fact, by the Moser isotopy argument (see [11]), the inclusion of $\text{Symp}(S, \Omega)$ into $\text{Diff}(S)$ is a homotopy equivalence, and $\text{Diff}_0(S)$ is contractible by [8].

The flux homomorphism provides the following characterization of Hamiltonian symplectomorphisms:

Theorem 2.3. *The sequence*

$$1 \longrightarrow \text{Ham}(S, \Omega) \xrightarrow{i} \text{Symp}_0(S, \Omega) \xrightarrow{\text{Flux}} H_{\text{dR}}^1(S, \mathbb{R}) \longrightarrow 1$$

is a short exact sequence of groups. In particular, a symplectomorphism $\psi : (S, \Omega) \rightarrow (S, \Omega)$ is Hamiltonian if and only if $\text{Flux}(\psi) = 0$.

An alternative homomorphism

Let us now fix two hyperbolic metrics h and h' on S , let Ω_h and $\Omega_{h'}$ be the area forms induced by h and h' , and let $\psi : (S, \Omega_h) \rightarrow (S, \Omega_{h'})$ be a symplectomorphism.

We denote by $\text{Isom}(TS, \psi^*h', h)$ the subbundle of the bundle $\text{End}(TS)$, whose fibers over $x \in S$ are linear orientation-preserving automorphisms of $T_x S$ which are isometries between the metrics ψ^*h' and h .

Now, given a symplectomorphism ψ and a smooth section b of $\text{Isom}(TS, \psi^*h', h)$, we will define a smooth 1-form

$$\eta_{\psi, b} \in Z_{\text{dR}}^1(S, \mathbb{R}).$$

The 1-form $\eta_{\psi, b}$ will *a priori* depend on the hyperbolic metrics h, h' , even when h and h' coincide.

We start by defining the 1-form locally. Let $\{v_1, v_2\}$ be an oriented orthonormal frame for h , on an open subset U of S , and let ω be the associated connection form for the Levi-Civita connection ∇ of h , which is defined by:

$$\nabla_v v_1 = \omega(v)v_2.$$

Analogously, since $\psi^* h' = h(b \cdot, b \cdot)$, let ω' be the connection form associated to the oriented $\psi^* h'$ -orthonormal frame $\{v'_1, v'_2\} := \{b^{-1}v_1, b^{-1}v_2\}$ and to the Levi-Civita connection ∇' of $\psi^* h'$, that is:

$$\nabla'_v(b^{-1}v_1) = \omega'(v)(b^{-1}v_2).$$

Then define

$$\eta_{\psi,b} = \omega' - \omega.$$

The following lemma shows that this definition does not depend on the choice of the orthonormal frame, and therefore this local definition provides a well-defined global 1-form on S .

Lemma 2.4. *The definition of $\eta_{\psi,b}$ does not depend on the choice of the oriented orthonormal frame v_1, v_2 .*

Proof. Let $\{\widehat{v}_1, \widehat{v}_2\}$ be another oriented orthonormal frame for h , on the open set U , which we can assume simply connected. Then there exists a smooth function $\theta : U \rightarrow \mathbb{R}$ (unique up to multiples of 2π) such that

$$\widehat{v}_i = R_\theta(v_i),$$

where $R_\theta \in \Gamma^\infty(U, \text{Isom}(TS, h))$ is the section given, at every point $x \in U$, by counterclockwise rotation fixing x of angle θ for the metric h . By a direct computation,

$$\nabla_v \widehat{v}_1 = R_\theta \nabla_v v_1 + d\theta(v) J_h R_\theta v_1,$$

where $J_h = R_{\pi/2}$ is the almost-complex structure of h . Hence

$$\widehat{\omega}(v) = h(\nabla_v \widehat{v}_1, \widehat{v}_2) = h(R_\theta \nabla_v v_1, R_\theta v_2) + d\theta(v) h(R_\theta v_2, R_\theta v_2) = \omega(v) + d\theta(v).$$

Observing that, since $\psi^* h' = h(b \cdot, b \cdot)$, the rotation R'_θ for the metric $\psi^* h'$ coincides with $b^{-1} R_\theta b$, the orthonormal frame $\{\widehat{v}'_1, \widehat{v}'_2\}$ is $\{R'_\theta v'_1, R'_\theta v'_2\}$, and thus by the same computation:

$$\widehat{\omega}'(v) = \omega'(v) + d\theta(v).$$

Therefore $\widehat{\omega}' - \widehat{\omega} = \omega' - \omega$ on U , and this concludes the proof. \square

Hence we defined a 1-form $\eta_{\psi,b}$ associated to the pair (ψ, b) . Let us show that this form is closed:

Lemma 2.5. *The 1-form $\eta_{\psi,b}$ is closed.*

Proof. By the well-known formula for the curvature form, one has

$$d\omega = -\Omega_h,$$

and analogously

$$d\omega' = -\Omega_{\psi^* h'} = -\Omega_h,$$

since ψ is a symplectomorphism. Therefore $d\eta_{\psi,b} = d\omega' - d\omega = 0$. \square

Unfortunately, the cohomology class of $\eta_{\psi,b}$ defined in this way is not independent of the choice of b , once the symplectomorphism ψ isotopic to the identity is fixed. However, it can only differ in a controlled way, namely by cohomology classes with integer periods:

Lemma 2.6. *Let b and \widehat{b} be smooth sections of $\text{Isom}(TS, \psi^* h', h)$. Then*

$$[\eta_{\psi,b}] - [\eta_{\psi,\widehat{b}}] \in H^1_{\text{dR}}(S, 2\pi\mathbb{Z}).$$

Let us first explain the meaning of $H^1_{\text{dR}}(S, 2\pi\mathbb{Z})$. Recall that there is a standard isomorphism $e : H^1_{\text{dR}}(S, \mathbb{R}) \rightarrow \text{Hom}(\pi_1(S), \mathbb{R})$, which is defined by

$$e([\eta]) : [\tau] \mapsto \int_{\tau} \eta.$$

Then we denote $H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$ as the subgroup of $H_{\text{dR}}^1(S, \mathbb{R})$ such that

$$H_{\text{dR}}^1(S, 2\pi\mathbb{Z}) = e^{-1}(\text{Hom}(\pi_1(S), 2\pi\mathbb{Z})) .$$

That is, $H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$ consists of cohomology classes of 1-forms with periods which are integer multiples of 2π .

Proof of Lemma 2.6. Given b and \widehat{b} , there exists a smooth function $\vartheta : S \rightarrow \mathbb{S}^1$ such that $\widehat{b} = R_\vartheta \circ b$. Consider the composition of vector space homomorphisms

$$C^\infty(S, \mathbb{S}^1) \longrightarrow H_{\text{dR}}^1(S, \mathbb{R}) \xrightarrow{e} \text{Hom}(\pi_1(S), \mathbb{R})$$

where the first arrow is

$$\vartheta \mapsto [d\vartheta] ,$$

as $d\vartheta$ is locally well-defined as a 1-form.

Let us observe that the composition

$$F : C^\infty(S, \mathbb{S}^1) \rightarrow \text{Hom}(\pi_1(S), \mathbb{R})$$

can be expressed as

$$F(\vartheta) = 2\pi\vartheta_* : \pi_1(S) \rightarrow \mathbb{R} ,$$

where ϑ_* is the map induced by ϑ from $\pi_1(S)$ to $\pi_1(\mathbb{S}^1) \cong \mathbb{Z} < \mathbb{R}$. Hence F has image in $\text{Hom}(\pi_1(S), 2\pi\mathbb{Z})$.

Now, pick an oriented orthonormal frame $\{v_1, v_2\}$ for h . Then let $\widehat{\omega}'$ be the connection form of ψ^*h' with respect to the frame $\{\widehat{v}'_1, \widehat{v}'_2\} = \{\widehat{b}^{-1}v_1, \widehat{b}^{-1}v_2\} = \{b^{-1}R_{-\vartheta}v_1, b^{-1}R_{-\vartheta}v_2\}$, which coincides with $\{R'_{-\vartheta}b^{-1}v_1, R'_{-\vartheta}b^{-1}v_2\} = \{R'_{-\vartheta}v'_1, R'_{-\vartheta}v'_2\}$.

If ω' is the connection form of ψ^*h' with respect to the frame $\{v'_1, v'_2\}$, by the same computation as in Lemma 2.4, we have $\widehat{\omega}' = \omega' - d\vartheta$. Therefore $\eta_{\psi, b}$ and $\eta_{\psi, \widehat{b}}$ differ by the 1-form $d\vartheta$, which has integer coefficients as remarked above. \square

Remark 2.7. From the construction in the above proof, it follows that, given b and \widehat{b} , $[\eta_{\psi, b}] = [\eta_{\psi, \widehat{b}}]$ in $H_{\text{dR}}^1(S, \mathbb{R})$ if and only if

$$\widehat{b} = R_\theta \circ b$$

for $\theta : S \rightarrow \mathbb{R}$. That is, if and only if the function $\vartheta : S \rightarrow \mathbb{S}^1$ above can be lifted to a \mathbb{R} -valued map.

Lemma 2.6 enables us to give the following definition:

Definition 2.8. Given two hyperbolic metrics h and h' on S with area forms Ω_h and $\Omega_{h'}$, we define

$$C_{h, h'} : \text{Symp}_0(S, \Omega_h, \Omega_{h'}) \rightarrow H_{\text{dR}}^1(S, \mathbb{R})/H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$$

by $C_{h, h'}(\psi) := [\eta_{\psi, b}]$, where b is any smooth section of $\text{Isom}(TS, \psi^*h', h)$.

Some properties of the 1-form $\eta_{\psi, b}$.

We will now derive an equivalent expression for the previously defined 1-form $\eta_{\psi, b}$, which will be used later.

Lemma 2.9. Let h be a hyperbolic metric on S , let $b \in \Gamma^\infty(\text{End}(TS))$ non-singular. If ∇ is the Levi-Civita connection of h and ∇' is the Levi-Civita connection of $h' = h(b \cdot, b \cdot)$, then:

$$\nabla'_v w = b^{-1} \nabla_v b(w) + \alpha(v) J_{h'}(w) , \tag{1}$$

where $J_{h'}$ is the almost-complex structure induced by the metric h' and α is the 1-form defined by

$$\alpha(v) = h(b^{-1} J_h(\star d^\nabla b), J_h v) ,$$

if J_h is the almost-complex structure induced by h .

Recall that $d^\nabla b$ denotes the exterior derivative, which is the 2-form with values in TS defined by:

$$d^\nabla b(v, w) = \nabla_v(b(w)) - \nabla_w(b(v)) - b[v, w], \quad (2)$$

where v and w are any two vector fields. Furthermore, $\star d^\nabla b$ is the Hodge dual of the 2-form $d^\nabla b$, which can be defined as

$$\star d^\nabla b = d^\nabla b(v_1, v_2),$$

where $\{v_1, v_2\}$ is an oriented orthonormal frame (and this expression does not depend on the chosen oriented orthonormal frame).

Proof of Lemma 2.9. Observe that $b^{-1}\nabla b$ is a connection on S compatible with the metric h' — but not symmetric in general. Hence the difference between ∇' and $b^{-1}\nabla b$ is a 1-form with values in the bundle of h' -skew-symmetric endomorphisms of TS . Hence we can write

$$\nabla'_v w = b^{-1}\nabla_v b(w) + \alpha(v)J_{h'}(w),$$

where α is a 1-form and

$$J_{h'} = b^{-1}J_h b \quad (3)$$

is the almost-complex structure induced by the metric h' . Then from the definitions, we have

$$\begin{aligned} d^\nabla b(v, w) &= \nabla_v b(w) - \nabla_w b(v) - b[v, w] \\ &= (b\nabla'_v w - \alpha(v)bJ_{h'}(w)) - (b\nabla'_w v - \alpha(w)bJ_{h'}(v)) - b[v, w] \\ &= b(\nabla'_v w - \nabla'_w v - [v, w]) - \alpha(v)J_h b(w) + \alpha(w)J_h b(v) \\ &= -\alpha(v)J_h b(w) + \alpha(w)J_h b(v), \end{aligned}$$

since we are assuming ∇' is torsion-free.

Let us choose an oriented orthonormal frame $\{v_1, v_2\}$ for h . Recall that the Hodge dual of the 2-form $d^\nabla b$ equals $\star d^\nabla b = d^\nabla b(v_1, v_2)$. Then

$$\begin{aligned} \star d^\nabla b &= -\alpha(v_1)J_h bJ_h(v_1) - \alpha(v_2)J_h bJ_h(v_2) \\ &= -(J_h bJ_h)(\alpha(v_1)v_1 + \alpha(v_2)v_2). \end{aligned}$$

Therefore, for every vector v ,

$$\begin{aligned} \alpha(v) &= -h((J_h bJ_h)^{-1}(\star d^\nabla b), v) \\ &= -h(J_h b^{-1}J_h(\star d^\nabla b), v) \\ &= h(b^{-1}J_h(\star d^\nabla b), J_h v). \end{aligned}$$

This concludes the proof. \square

Using Equation (1), we can now express the 1-form $\eta_{\psi, b}$ in the following way: if $\{v_1, v_2\}$ is an oriented orthonormal frame, then:

$$\begin{aligned} \eta_{\psi, b}(w) &= \omega'(w) - \omega(w) \\ &= h'(\nabla'_w b^{-1}v_1, b^{-1}v_2) - h(\nabla_w v_1, v_2) \\ &= h(b\nabla'_w b^{-1}v_1, v_2) - h(\nabla_w v_1, v_2) \\ &= h(\alpha(w)J_h v_1, v_2) = \alpha(w), \end{aligned}$$

since $J_h v_1 = v_2$, and we have used Equation (3). Therefore, using Lemma 2.9, we have the following:

Corollary 2.10. *Given two hyperbolic metrics h and h' on S with area forms Ω_h and $\Omega_{h'}$, a symplectomorphism $\psi \in \text{Symp}_0(S, \Omega_h, \Omega_{h'})$, and a smooth section b of $\text{Isom}(TS, \psi^*h', h)$,*

$$\eta_{\psi, b} = h(b^{-1}J_h(\star d^\nabla b), J_{h'}), \quad (4)$$

Observe that, from Equation (4), $\eta_{\psi,b}$ vanishes if and only if $\star d^\nabla b$ vanishes identically, which is equivalent to $d^\nabla b = 0$. Hence from Corollary 2.10, we obtain:

Corollary 2.11. *Given two hyperbolic metrics h and h' on S with area forms Ω_h and $\Omega_{h'}$, a symplectomorphism $\psi \in \text{Symp}_0(S, \Omega_h, \Omega'_h)$, and a smooth section b of $\text{Isom}(TS, \psi^* h', h)$,*

$$\eta_{\psi,b} = 0 \quad \text{if and only if} \quad d^\nabla b = 0.$$

The group structure

Let us now study how the map $C_{h,h'}$ (defined in Definition 2.8) transforms under composition of symplectomorphisms. In particular, this will prove that $C_{h,h} : \text{Symp}_0(S, \Omega_h) \rightarrow H_{\text{dR}}^1(S, \mathbb{R})/H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$ is a group homomorphism.

Lemma 2.12. *Let h and h' be hyperbolic metrics on S . Then for $\psi \in \text{Symp}_0(S, \Omega_h)$ and $\widehat{\psi} \in \text{Symp}_0(S, \Omega_h, \Omega_{h'})$,*

$$C_{h,h'}(\widehat{\psi} \circ \psi) = C_{h,h}(\psi) + C_{h,h'}(\widehat{\psi}).$$

Proof. Let b and \widehat{b} be smooth sections of $\text{End}(TS)$ such that $\widehat{\psi}^* h' = h(\widehat{b}, \widehat{b})$ and $\psi^* h = h(b, b)$. Let us denote $\psi^* \widehat{b} = (d\psi)^{-1} \widehat{b}(d\psi)$. Then $\psi^* \widehat{b} \in \Gamma^\infty(\text{Isom}_\Sigma(\psi^* h, \psi^* \widehat{\psi}^* h'))$ and

$$(\widehat{\psi}\psi)^* h' = h((b \circ \psi^* \widehat{b}), (b \circ \psi^* \widehat{b})).$$

To compute $[\eta_{\widehat{\psi}\psi}]$, let us pick an orthonormal frame $\{v_1, v_2\}$ for h , let ω be the associated connection form and ω'' the connection form of $(\widehat{\psi}\psi)^* h'$ with respect to the frame $\{(b \circ \psi^* \widehat{b})^{-1} v_1, (b \circ \psi^* \widehat{b})^{-1} v_2\}$. Let ω' be the connection form of $\psi^* h$ for the frame $\{b^{-1} v_1, b^{-1} v_2\}$. Then

$$\eta_{\widehat{\psi}\psi, (b \circ \psi^* \widehat{b})} = \omega'' - \omega = (\omega'' - \omega') + (\omega' - \omega).$$

Now clearly $\omega' - \omega = \eta_{\psi,b}$. On the other hand, ω' and ω'' are the pull-backs, via ψ , of the connection forms of h and $\widehat{\psi}^* h'$, and thus $\omega'' - \omega' = \psi^* \eta_{\widehat{\psi}, \widehat{b}}$. This shows that:

$$\eta_{\widehat{\psi}\psi, (b \circ \psi^* \widehat{b})} = \eta_{\psi,b} + \psi^* \eta_{\widehat{\psi}, \widehat{b}}.$$

Taking cohomology classes, $[\psi^* \eta_{\widehat{\psi}, \widehat{b}}] = [\eta_{\widehat{\psi}, \widehat{b}}]$ since ψ is isotopic to the identity, and therefore

$$C_{h,h'}(\widehat{\psi} \circ \psi) = C_{h,h}(\psi) + C_{h,h'}(\widehat{\psi}),$$

which concludes the proof. □

In particular, if we choose $h = h'$, then Lemma 2.12 shows that, if $\psi, \widehat{\psi} \in \text{Symp}_0(S, \Omega_h)$,

$$C_{h,h}(\widehat{\psi} \circ \psi) = C_{h,h}(\psi) + C_{h,h}(\widehat{\psi}).$$

Hence $C_{h,h} : \text{Symp}_0(S, \Omega_h) \rightarrow H_{\text{dR}}^1(S, \mathbb{R})/H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$ is a group homomorphism. Moreover, if ψ varies smoothly in $\text{Symp}_0(S, \Omega_h)$, then the associated form $[\eta_{\psi,b}]$ varies smoothly. Hence $C_{h,h}$ is smooth, for the structure of infinite-dimensional Lie group of $\text{Symp}_0(S, \Omega_h)$. To summarize, we have:

Corollary 2.13. *Given a hyperbolic metric h on S , the map*

$$C_{h,h} : \text{Symp}_0(S, \Omega_h) \rightarrow H_{\text{dR}}^1(S, \mathbb{R})/H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$$

is a Lie group homomorphism.

The coincidence in the quotient of Flux and $\mathcal{C}_{h,h}$

In this subsection we restrict to the case in which the two metrics h and h' coincide. In this case, the map $\mathcal{C}_{h,h}$ is a group homomorphism, as already observed. The main result of this part is the fact that the map $\mathcal{C}_{h,h}$ coincides with the flux homomorphism, composed with the projection from $H^1_{\text{dR}}(S, \mathbb{R})$ to $H^1_{\text{dR}}(S, \mathbb{R})/H^1_{\text{dR}}(S, 2\pi\mathbb{Z})$.

Proposition 2.14. *Given a hyperbolic metric h on a closed oriented surface S , let Ω_h be the induced area form. Then the following diagram is commutative:*

$$\begin{array}{ccc}
 \text{Symp}_0(S, \Omega_h) & \xrightarrow{\text{Flux}} & H^1_{\text{dR}}(S, \mathbb{R}) \\
 & \searrow \mathcal{C}_{h,h} & \downarrow \pi \\
 & & H^1_{\text{dR}}(S, \mathbb{R})/H^1_{\text{dR}}(S, 2\pi\mathbb{Z})
 \end{array} \tag{5}$$

Surprisingly, Proposition 2.14 implies that the homomorphism $\mathcal{C}_{h,h}$ (when the two hyperbolic metrics coincide) does not depend on the hyperbolic metric h , but only on its area form Ω_h .

To prove Proposition 2.14, we will first prove that the two maps coincide at the infinitesimal level. Recall that the Lie algebra $\mathfrak{symp}(S, \Omega_h)$ consists of smooth vector fields satisfying $\text{div}X = \text{tr} \nabla X = 0$. The exponential map $\exp : \mathfrak{symp}(S, \Omega_h) \rightarrow \text{Symp}_0(S, \Omega_h)$ coincides with the flow of the vector field X , namely $\exp(tX) = \psi_t \in \text{Symp}_0(S, \Omega_h)$ where

$$\frac{d}{dt} \psi_t = X \circ \psi_t .$$

Moreover, since $H^1_{\text{dR}}(S, 2\pi\mathbb{Z})$ is discrete, the tangent space to $H^1_{\text{dR}}(S, \mathbb{R})/H^1_{\text{dR}}(S, 2\pi\mathbb{Z})$ at the identity element is naturally identified with $H^1_{\text{dR}}(S, \mathbb{R})$ itself. With this convention:

Lemma 2.15. *Let $X \in \mathfrak{symp}(S, \Omega_h)$, and let $\psi_t \in \text{Symp}_0(S, \Omega_h)$ be the flow of X . Then*

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{C}_{h,h}(\psi_t) = [\Omega_h(X, \cdot)] .$$

Proof. To compute $\mathcal{C}_{h,h}(\psi_t)$, recall that $\mathcal{C}_{h,h}$ is defined as the class of the 1-form η_{ψ_t, b_t} , where we can choose a smoothly varying family $b_t \in \text{Isom}(TS, \psi_t^* h, h)$.

First of all, observe that, by differentiating the condition

$$h(d\psi_t, d\psi_t) = h(b_t, b_t)$$

we obtain that

$$h(\nabla_v X, w) + h(v, \nabla_w X) = h(\dot{b}(v), w) + h(v, \dot{b}(w)) ,$$

where

$$\dot{b} = \left. \frac{d}{dt} \right|_{t=0} b_t$$

and as usual ∇ is the Levi-Civita connection of h . In other words, the symmetric part of the operators ∇_X and \dot{b} coincide. Since ψ_t preserves the area, the generating field X is divergence-free, that is $\text{tr} \nabla X = 0$. On the other hand, since $\det b_t = 1$, we have $\text{tr} \dot{b} = 0$. Therefore there exists a function $f : S \rightarrow \mathbb{R}$ such that

$$\nabla X = \dot{b} + fJ_h ,$$

where J_h is the almost-complex structure determined by the metric h . Now, let us take exterior derivatives, applied to an orthonormal frame $\{v_1, v_2\}$. Using that the curvature of h is identically -1 , one obtains

$$d^\nabla(\nabla X)(v_1, v_2) = J_h X ,$$

while on the other hand, using that J_h is parallel (namely $\nabla J_h = 0$), we have

$$d^\nabla(fJ_h)(v_1, v_2) = -df(v_1)v_1 - df(v_2)v_2 = -\text{grad} f .$$

Hence we have the following formula for the Hodge duals, which will be used later in the proof:

$$*d^\nabla \dot{b} = J_h X - \text{grad} f . \quad (6)$$

Now from Corollary 2.10, we have:

$$\left. \frac{d}{dt} \right|_{t=0} \eta_{\psi_t, b_t} = \left. \frac{d}{dt} \right|_{t=0} h(b_t^{-1} J_h(*d^\nabla b_t), J_h \cdot)$$

Recall that b_0 is the identity operator, hence $d^\nabla b_0 = 0$. Hence

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \eta_{\psi_t, b_t} &= h(J_h(*d^\nabla \dot{b}), J_h \cdot) \\ &= h(*d^\nabla \dot{b}, \cdot) \\ &= h(J_h X - \text{grad} f, \cdot) \\ &= \Omega_h(X, \cdot) + df . \end{aligned}$$

This shows that

$$\left[\left. \frac{d}{dt} \right|_{t=0} \eta_{\psi_t, b_t} \right] = [\Omega_h(X, \cdot)] ,$$

and thus concludes the proof. \square

The proof of Proposition 2.14 will be a consequence of the following lemma about (infinite-dimensional) Lie groups.

Lemma 2.16. *Let G and H be Lie groups, possibly of infinite dimension. Let $F_1, F_2 : G \rightarrow H$ be Lie group homomorphism and let $(F_1)_*, (F_2)_* : \mathfrak{g} \rightarrow \mathfrak{h}$ be the induced Lie algebra homomorphisms. If $(F_1)_* = (F_2)_*$, then $F_1 = F_2$.*

Let us now conclude the proof of Proposition 2.14.

Proof of Proposition 2.14. Both $\pi \circ \text{Flux}$ and $\mathcal{C}_{h,h}$ (the latter by Corollary 2.13) are Lie groups homomorphisms from $\text{Symp}_0(S, \Omega)$ to $H_{\text{dR}}^1(S, \mathbb{R})/H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$. In Lemma 2.15, we showed that the differential of $\mathcal{C}_{h,h}$ is

$$(\mathcal{C}_{h,h})_*(X) = [\Omega_h(X, \cdot)] \in H_{\text{dR}}^1(S, \mathbb{R}) ,$$

for any $X \in \mathfrak{symp}(S, \Omega_h)$. On the other hand, observe that $\psi_t = \exp(tX)$, where X is the generating vector field of the 1-parameter subgroup ψ_t , and \exp is the Lie group exponential map of $\text{Symp}_0(S, \Omega_h)$. Hence, by definition

$$\text{Flux}(\psi_t) = \int_0^1 [\Omega_h(tX, \cdot)] ds = t[\Omega_h(X, \cdot)] .$$

Therefore

$$(\text{Flux})_*(X) = \left. \frac{d}{dt} \right|_{t=0} \text{Flux}(\psi_t) = [\Omega_h(X, \cdot)] .$$

By applying Lemma 2.16, the proof follows. \square

The following is a direct consequence of Proposition 2.14 and the fact that $H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$ is discrete in $H_{\text{dR}}^1(S, \mathbb{R})$:

Corollary 2.17. *Given a hyperbolic metric h on the closed oriented surface S , the connected component of the identity in $\text{Ker}(\mathcal{C}_{h,h})$ coincides with $\text{Ham}(S, \Omega_h)$, where Ω_h is the area form induced by h .*

3 Application to Anti-de Sitter geometry

In this section, we will start by providing the necessary preliminary notions on Anti-de Sitter geometry, from the basic definitions concerning Anti-de Sitter space. Then we will show how a symplectomorphism of hyperbolic surfaces is associated to equivariant surfaces in Anti-de Sitter space, with the special case of minimal Lagrangian diffeomorphisms. We conclude with the proof of our main result and some remarks.

Anti-de Sitter space

Let κ be the Killing form on the Lie group $\mathrm{PSL}_2\mathbb{R}$, which is a bi-invariant bilinear form on the Lie algebra $\mathfrak{sl}_2\mathbb{R}$ of signature $(2, 1)$. The bilinear form κ induces a Lorentzian metric on $\mathrm{PSL}_2\mathbb{R}$, which we will denote by g_κ . Then Anti-de Sitter space of dimension 3 is defined as:

$$\mathrm{AdS}^3 := \left(\mathrm{PSL}_2\mathbb{R}, \frac{1}{8}g_\kappa \right).$$

Therefore AdS^3 is a Lorentzian manifold, homeomorphic to a solid torus, of constant sectional curvature. It turns out that, due to the normalization factor $1/8$, its sectional curvature is -1 . From the construction, it follows that the identity component of the isometry group of AdS^3 is:

$$\mathrm{Isom}_0(\mathrm{AdS}^3) = \mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R},$$

where $(\alpha, \beta) \in \mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}$ acts on AdS^3 by

$$(\alpha, \beta) \cdot \zeta = \alpha \circ \zeta \circ \beta^{-1}.$$

The group $\mathrm{PSL}_2\mathbb{R}$ is isomorphic to the group of orientation-preserving isometries of \mathbb{H}^2 , the hyperbolic plane which we consider in the upper-half plane model:

$$\mathbb{H}^2 := \left(\{z \in \mathbb{C} \mid \Im(z) > 0\}, \frac{|dz|}{\Im(z)} \right).$$

Then, using the structure of the isometry group, one can show the following fact, which will be of fundamental importance in this paper. We recall that a differentiable curve $y : I \rightarrow \mathrm{AdS}^3$ is *timelike* if $g_\kappa(\dot{y}, \dot{y}) < 0$ at every point $y(t)$, *spacelike* if $g_\kappa(\dot{y}, \dot{y}) > 0$, and *lightlike* if $g_\kappa(\dot{y}, \dot{y}) = 0$.

Fact 3.1. *There is a 1-1 correspondence*

$$\{\text{timelike geodesics in } \mathrm{AdS}^3\} \leftrightarrow \mathbb{H}^2 \times \mathbb{H}^2,$$

which is defined by associating to $(x, y) \in \mathbb{H}^2 \times \mathbb{H}^2$ the timelike geodesic

$$L_{x,y} := \{\zeta \in \mathrm{PSL}_2\mathbb{R} \mid \zeta(y) = x\}. \quad (7)$$

This correspondence is equivariant with respect to the action of $\mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}$ on $\mathbb{H}^2 \times \mathbb{H}^2$ by isometries of \mathbb{H}^2 on each factor, and on the set of timelike geodesics induced by isometries of AdS^3 .

Indeed, L_{x_0, x_0} is a closed timelike geodesic for every $x \in \mathbb{H}^2$, since it is a maximal compact subgroup of the Lie group $\mathrm{PSL}_2\mathbb{R}$, hence the restriction of the bilinear form is negative definite. It is a geodesic since it is a 1-parameter group, and the Riemannian exponential map coincides with the Lie group exponential map for bi-invariant metrics as g_κ is. It also turns out that the length of L_{x_0, x_0} is π , and the arclength parameter is $1/2$ the angle of rotation of elliptic elements of $\mathrm{PSL}_2\mathbb{R}$ fixing x_0 .

The equivariance of such 1-1 correspondence can be easily checked, since:

$$(\alpha, \beta) \cdot L_{x,y} = L_{\alpha(x), \beta(y)}.$$

This also shows that every timelike geodesic of AdS^3 is of the form $L_{x,y}$ for some x, y , since it is the image of L_{x_0,x_0} under some isometry (α, β) of AdS^3 .

There is a natural notion of boundary at infinity of Anti-de Sitter space, topologically a torus, which is defined in the following way:

$$\partial_\infty \text{AdS}^3 := \mathbb{RP}^1 \times \mathbb{RP}^1,$$

where we declare that a sequence $\zeta_n \in \text{PSL}_2\mathbb{R}$ converges to a pair $(p, q) \in \mathbb{RP}^1 \times \mathbb{RP}^1$ if and only if there exists a point $x \in \mathbb{H}^2$ such that:

$$\begin{cases} \zeta_n(x) \rightarrow p \\ \zeta_n^{-1}(x) \rightarrow q \end{cases} \quad (8)$$

It is not difficult to check that, if Equation (8) holds for some point $x \in \mathbb{H}^2$, then it holds for every other point $x' \in \mathbb{H}^2$. It also turns out that every isometry of AdS^3 , of the form $(\alpha, \beta) \in \text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R}$, extends to the boundary, with the obvious action of (α, β) on $\mathbb{RP}^1 \times \mathbb{RP}^1$.

From invariant spacelike surfaces to symplectomorphisms of hyperbolic surfaces

Recall that in this paper S denotes a closed oriented hyperbolic surface of Euler characteristic $\chi(S) < 0$. A smoothly embedded surface Σ in AdS^3 is *spacelike* if its induced metric is a Riemannian metric. One of the main objects of this paper will be spacelike surfaces Σ which are preserved by a representation

$$\rho : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R},$$

which acts freely and properly discontinuously on Σ . Following Mess, there is a precise description of these objects:

Fact 3.2 ([12]). *Suppose Σ is a smooth spacelike surface in AdS^3 and*

$$\rho = (\rho_l, \rho_r) : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R},$$

is a representation which acts freely and properly discontinuously on Σ . Then:

- *The representations $\rho_l : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R}$ and $\rho_r : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R}$ are Fuchsian, that is, each of them acts freely and properly discontinuously on \mathbb{H}^2 .*
- *The frontier $\partial_\infty \Sigma$ of Σ in $\partial_\infty \text{AdS}^3 \cong \mathbb{RP}^1 \times \mathbb{RP}^1$ is the graph of the unique ρ_l - ρ_r -equivariant homeomorphism of \mathbb{RP}^1 .*

In general, given two representations ρ_1 and ρ_2 , we say that a map f is ρ_1 - ρ_2 -equivariant if, $f \circ \rho_1(\tau) = \rho_2(\tau) \circ f$ for every τ . If ρ_l and ρ_r are Fuchsian, it is well-known that there exists a unique homeomorphism $f : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ which is ρ_l - ρ_r -equivariant, which is obtained as the value induced on $\mathbb{RP}^1 = \partial_\infty \mathbb{H}^2$ by the lift to \mathbb{H}^2 of any diffeomorphism between the quotient closed surfaces $\mathbb{H}^2/\rho_l(\pi_1(S))$ and $\mathbb{H}^2/\rho_r(\pi_1(S))$ isotopic to the identity.

Hence, let ρ_l and ρ_r be Fuchsian representations, so that the oriented topological surface S is homeomorphic to the hyperbolic surfaces $\mathbb{H}^2/\rho_l(\pi_1(S))$ and $\mathbb{H}^2/\rho_r(\pi_1(S))$. We will denote by h and h' the hyperbolic metrics induced on S in this way. We will now define the symplectomorphism

$$\phi_\Sigma : (S, \Omega_{h_l}) \rightarrow (S, \Omega_{h_r}),$$

where Ω_{h_l} and Ω_{h_r} are the area forms induced by the metrics h_l and h_r and by the orientation of S . For this purpose, consider the map

$$\iota_\Sigma : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{H}^2, \quad (9)$$

which maps $\zeta \in \Sigma$ to the pair (x, y) which represents the unique timelike geodesic orthogonal to Σ at ζ , using Fact 3.1. Let $\pi_l, \pi_r : \mathbb{H}^2 \times \mathbb{H}^2 \rightarrow \mathbb{H}^2$ denote the projections on the first and second factor. Then we have:

Fact 3.3 ([4, 5, 9]). *Given two Fuchsian representations ρ_l and ρ_r and any smooth spacelike surface Σ in AdS^3 invariant by the representation*

$$\rho = (\rho_l, \rho_r) : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R},$$

- For $\zeta \in \Sigma$, if the curvature of the induced metric on Σ is different from zero at ζ , then $\pi_l \circ \iota_\Sigma$ and $\pi_r \circ \iota_\Sigma$ are local diffeomorphisms in a neighborhood of ζ .
- If the curvature of the induced metric on Σ is everywhere different from zero, then $\pi_l \circ \iota_\Sigma$ and $\pi_r \circ \iota_\Sigma$ are global diffeomorphisms which extend to homeomorphisms from $\partial_\infty \Sigma$ to \mathbb{RP}^1 .
- In this case, the composition

$$\tilde{\phi}_\Sigma = (\pi_r \circ \iota_\Sigma) \circ (\pi_l \circ \iota_\Sigma)^{-1}$$

induces a symplectomorphism

$$\phi_\Sigma : (S, \Omega_{h_l}) \rightarrow (S, \Omega_{h_r})$$

isotopic to the identity, where Ω_{h_l} and Ω_{h_r} are the area forms of h_l and h_r .

In the second point, we remark that, since Σ is invariant by $\rho(\pi_1(S))$ with quotient homeomorphic to the closed surface S , with $\chi(S) < 0$, by the Gauss-Bonnet formula the curvature of the induced metric is always different from zero if and only if it is negative everywhere.

For the last point, it follows from the definitions that $\pi_l \circ \iota_\Sigma$ is ρ - ρ_l -equivariant, and $\pi_r \circ \iota_\Sigma$ is ρ - ρ_r -equivariant, hence the composition $(\pi_r \circ \iota_\Sigma) \circ (\pi_l \circ \iota_\Sigma)^{-1}$ induces a map in the quotient from $(S, h_l) = \mathbb{H}^2/\rho_l(\pi_1(S))$ to $(S, h_r) = \mathbb{H}^2/\rho_r(\pi_1(S))$.

In [4] and [5], some more precise properties of the map ϕ_Σ were given. That is:

Fact 3.4. *Given two Fuchsian representations ρ_l and ρ_r and any smooth spacelike surface Σ in AdS^3 invariant by the representation*

$$\rho = (\rho_l, \rho_r) : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R},$$

let $\phi_\Sigma : (S, \Omega_{h_l}) \rightarrow (S, \Omega_{h_r})$ be the associated symplectomorphism. Then there exists a smooth section $b \in \Gamma^\infty(\text{End}(TS))$ such that:

1. $\phi_\Sigma^* h_r = h_l(b \cdot, b \cdot)$;
2. $\det b = 1$;
3. $d^{\nabla_l} b = 0$, where ∇_l is the Levi-Civita connection of the metric h_l ;
4. $\text{tr } b \neq -2$.

We recall again that, for a connection ∇ , $d^{\nabla} b$ is a 2-form with values in TS , defined in Equation (2).

Remark 3.5. *Let us make some comments on these conditions. Condition (1) is equivalent to saying that b is a smooth section of the subbundle $\text{Isom}(TS, \phi_\Sigma^* h_r, h_l)$, whose fiber over the point $x \in S$ consists of the linear isometries $b_x : (T_x S, \phi_\Sigma^* h_r) \rightarrow (T_x S, h_l)$.*

First, such a section b is not uniquely determined. In fact, if b satisfies the conditions (1) and (2), then $R \circ b$ still satisfies (1) and (2), where $R \in \Gamma^\infty(\text{Isom}(TS, h_l, h_l))$ — that is, the tensor b can be post-composed with an isometry of the metric h_l at any point. Conditions (1) and (2) imply that ϕ_Σ preserves the area forms of h_l and h_r , hence $\phi_\Sigma : (S, \Omega_{h_l}) \rightarrow (S, \Omega_{h_r})$ is a symplectomorphism.

Further, if b satisfies (1), (2) and (3), then for any number $\theta_0 \in (0, 2\pi)$ also the section $R_{\theta_0} \circ b$ satisfies (1), (2) and (3), where

$$R_{\theta_0} = (\cos \theta_0) \text{id} + (\sin \theta_0) J_{h_l} \in \Gamma^\infty(\text{Isom}(TS, h_l, h_l))$$

is the section which at every point $x \in (S, h_l)$ rotates $T_x S$ counterclockwise of an angle θ_0 , for the metric h_l . (Here J_{h_l} is the almost-complex structure induced by the metric h_l .) Finally, as condition (4) is an open condition, if b satisfies (1), (2), (3) and (4), then $R_{\theta_0} \circ b$ still satisfies (1), (2), (3) and (4) for small θ_0 .

Finally, we sketch an argument explaining why Fact 3.4 is true. On the metric universal cover \mathbb{H}^2 of (S, h_l) , a particular section $\tilde{b} \in \Gamma^\infty(\text{End}(T\mathbb{H}^2))$ satisfying the above properties can be constructed explicitly as

$$\tilde{b} = (\text{id} + J_\Sigma B_\Sigma)^{-1} \circ (\text{id} - J_\Sigma B_\Sigma), \tag{10}$$

where B_Σ is the shape operator, and J_Σ is the almost-complex structure of the first fundamental form, both computed with respect to the embedding

$$\sigma := (\pi_l \circ \iota_\Sigma)^{-1} : \mathbb{H}^2 \rightarrow \text{AdS}^3.$$

It can be checked that \tilde{b} is ρ_l -invariant, and thus it defines a section $b \in \Gamma^\infty(\text{End}(TS))$, which always satisfies (1), (2), (3) and (4). See [5, Proposition 4.9] for more details.

Minimal Lagrangian diffeomorphisms

A particular example of the above correspondence occurs when Σ is a *maximal* surface, that is, its shape operator B_Σ satisfies $\text{tr } B_\Sigma \equiv 0$. In this case, the associated map $\phi_\Sigma : (S, h_l) \rightarrow (S, h_r)$ will be *minimal Lagrangian*, namely, it is a symplectomorphism for the induced area forms Ω_{h_l} and Ω_{h_r} , and the graph of ϕ_Σ is a minimal surface in $(S \times S, h_l \oplus h_r)$. Hence this is a notion which depends on the hyperbolic metrics h_l and h_r , not only on the symplectic form. The term *Lagrangian* comes from the fact that, if ϕ_Σ is a symplectomorphism, then the graph of ϕ_Σ is a Lagrangian surface in the symplectic manifold $(S \times S, \pi_l^* \Omega_{h_l} - \pi_r^* \Omega_{h_r})$.

Recall that an operator $b \in \Gamma^\infty(\text{End}(TS))$ is h -self-adjoint, for h a Riemannian metric on S , if

$$h(b \cdot, \cdot) = h(\cdot, b \cdot).$$

The correspondence between maximal surfaces and minimal Lagrangian diffeomorphisms is consequence of the following characterization (see [10]):

Lemma 3.6. *Let h_l and h_r be hyperbolic metrics on S . Then $\phi_{\mathbb{M}} : (S, h_l) \rightarrow (S, h_r)$ is minimal Lagrangian if and only if there exists a smooth h_l -self-adjoint section $b_L \in \Gamma^\infty(\text{End}(TS))$ such that:*

1. $\phi_\Sigma^* h_r = h_l(b_L \cdot, b_L \cdot)$;
2. $\det b_L = 1$;
3. $d^{\nabla_l} b_L = 0$, where ∇_l is the Levi-Civita connection of the metric h_l .

In this case, the tensor b_L (which is unique) is called Labourie operator.

Hence we have the following:

Fact 3.7. *Given two Fuchsian representations ρ_l and ρ_r , if Σ_0 is a maximal surface in AdS^3 invariant by the representation*

$$\rho = (\rho_l, \rho_r) : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R},$$

then the associated diffeomorphism $\phi_{\Sigma_0} : (S, h_l) \rightarrow (S, h_r)$ is minimal Lagrangian.

The proof of Fact 3.7 follows from checking that the tensor b defined in Equation (10) is h_l -self-adjoint if Σ_0 is a maximal surface, that is, if $\text{tr } B_{\Sigma_0} \equiv 0$. See [4].

The following is a theorem of existence and uniqueness of minimal Lagrangian maps isotopic to the identity between any two compact oriented hyperbolic surfaces. It was proved in [10] and [15], and it can also be inferred from the existence and uniqueness of the invariant maximal surface Σ_0 , see [3] and [4].

Theorem 3.8. *Given any two hyperbolic metrics h, h' on the compact oriented surface S , there exists a unique minimal Lagrangian diffeomorphism $\phi_{\mathbb{M}} : (S, h) \rightarrow (S, h')$ isotopic to the identity.*

Proof of the main theorem

Let us now prove the main result of this paper, which was already stated in the introduction. As usual, given two Fuchsian representations $\rho_l, \rho_r : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R}$, we denote $(S, h_l) = \mathbb{H}^2/\rho_l(\pi_1(S))$ and similarly $(S, h_r) = \mathbb{H}^2/\rho_r(\pi_1(S))$, and Ω_{h_l} and Ω_{h_r} are the induced area forms.

Theorem 3.9. *Let $\rho_l, \rho_r : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R}$ be Fuchsian representations and let $\Sigma \subset \text{AdS}^3$ be a smooth, embedded spacelike surface invariant for the representation $\rho = (\rho_l, \rho_r) : \pi_1(S) \rightarrow \text{Isom}_0(\text{AdS}^3)$, whose curvature is negative. Then*

$$\phi_\Sigma = \phi_{\text{ML}} \circ \psi ,$$

where

- $\phi_\Sigma : (S, \Omega_{h_l}) \rightarrow (S, \Omega_{h_r})$ is the diffeomorphism associated to Σ ;
- $\phi_{\text{ML}} : (S, h_l) \rightarrow (S, h_r)$ is the unique minimal Lagrangian diffeomorphism isotopic to the identity;
- ψ is a Hamiltonian symplectomorphism, for the area form Ω_{h_l} .

Proof. Let $\psi = \phi_{\text{ML}}^{-1} \circ \phi_\Sigma$. Let b the section of $\text{Isom}(TS, \phi_\Sigma^* h_r, h_l)$ given by Fact 3.4, and b_L the analogous section for ϕ_{ML} , which was also introduced in Lemma 3.6. Using Lemma 2.12, we have

$$C_{h, h'}(\phi_\Sigma) = C_{h, h'}(\phi_{\text{ML}}) + C_{h, h}(\psi) .$$

Now, since $d^{\nabla_l} b = d^{\nabla_l} b_L = 0$ by Fact 3.4 (and Lemma 3.6), using Corollary 2.11 we have $\eta_{\phi_\Sigma, b} = \eta_{\phi_{\text{ML}}, b_L} = 0$, hence $C_{h, h'}(\phi_\Sigma) = C_{h, h'}(\phi_{\text{ML}}) = 0$ and therefore $C_{h, h}(\psi) = 0$. From Definition 2.8 and Proposition 5, this implies that $\text{Flux}(\psi) \in H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$.

On the other hand, consider the maximal surface Σ_0 invariant by the representation ρ (as in Fact 3.7). By the results in [12], the surfaces Σ and Σ_0 induce closed embedded surfaces in a (so-called maximal globally hyperbolic) three-manifold M homeomorphic to $S \times \mathbb{R}$. Let us call $\bar{\Sigma}$ and $\bar{\Sigma}_0$ be the surfaces induced in the quotient. Both $\bar{\Sigma}$ and $\bar{\Sigma}_0$ are homotopic to $S \times \{\star\}$. Hence one can find a smooth isotopy $f_t : S \rightarrow M$ which is a homotopy equivalence at every time t , such that $f_0(S) = \bar{\Sigma}_0$, and $f_1(S) = \bar{\Sigma}$, and $f_t(S)$ is a smooth spacelike surface for every t . Lifting to AdS^3 , the surfaces Σ_t which project to $f_t(S)$ induce maps ϕ_{Σ_t} which vary smoothly. If we put $\phi_{\Sigma_t} = \phi_{\text{ML}} \circ \psi_t$, the corresponding classes $\text{Flux}(\psi_t)$ vary smoothly, and are all in $H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$ by the same argument as the first part of this proof. Moreover, ψ_0 is the identity, and therefore $\text{Flux}(\psi_0) = 0$. Since $H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$ is discrete, it then follows that $\text{Flux}(\psi_1) = \text{Flux}(\psi) = 0$. Hence by Theorem 2.3, ψ is Hamiltonian. This concludes the proof. □

Let us remark that the second part is exactly in the spirit of Corollary 2.17, by using the fact that there is a smooth interpolation between ϕ_Σ and ϕ_{ML} . By specializing to the case in which the two hyperbolic metrics coincide, we get the following:

Corollary 3.10. *Given a Fuchsian representation $\rho_0 : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{R}$, let*

$$\phi_\Sigma : \mathbb{H}^2/\rho_0(\pi_1(S)) \rightarrow \mathbb{H}^2/\rho_0(\pi_1(S))$$

be a symplectomorphism induced by a smooth embedded spacelike surface $\Sigma \subset \text{AdS}^3$ invariant by $(\rho_0, \rho_0)(\pi_1(S)) < \text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R}$. Then ϕ_Σ is Hamiltonian.

Reconstructing smooth spacelike surfaces

We conclude by a discussion about the question whether the condition that $\phi = \phi_{\text{ML}} \circ \psi$ for a symplectomorphism $\phi : (S, h_l) \rightarrow (S, h_r)$, where ϕ_{ML} is minimal Lagrangian and ψ is Hamiltonian, is also sufficient to obtain ϕ as the symplectomorphism associated to an invariant surface in AdS^3 . In fact, by Theorem 3.9 this is a necessary condition.

Using the tools from [5], given a symplectomorphism $\phi : (S, \Omega_l) \rightarrow (S, \Omega_r)$ and any section $b \in \Gamma(\text{Isom}(TS, \phi^* h_r, h_l))$, a map $\sigma_{\phi, b} : \mathbb{H}^2 \rightarrow \text{AdS}^3$ can be reconstructed explicitly, such that $\sigma_{\phi, b}$ is equivariant for the action of $\rho_l : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{R})$ on \mathbb{H}^2 and of the pair $\rho = (\rho_l, \rho_r)$ on AdS^3 .

The map $\sigma_{\phi, b} : \mathbb{H}^2 \rightarrow \text{AdS}^3$ is defined by the condition that $\sigma_{\phi, b}(x)$ is the unique isometry $\sigma \in \text{PSL}(2, \mathbb{R})$ (recalling that AdS^3 is the Lie group $\text{PSL}(2, \mathbb{R})$) such that

$$\begin{cases} \sigma(\phi(x)) = x \\ d\sigma_{\phi(x)} \circ d\phi_x = -b_x \end{cases} . \quad (11)$$

Moreover, such map is orthogonal to the foliation in timelike lines of the form $L_{x, \phi(x)}$ provided b satisfies $d^{\nabla^l} b = 0$ ([5, Corollary 5.7]), but in general it is not an immersion.

Therefore, given a section $b \in \Gamma(\text{Isom}(TS, \phi^* h_r, h_l))$, suppose $[\eta_{\phi, b}] = 0 \in H_{\text{dR}}^1(S, \mathbb{R})$. (This assumption is satisfied if $\phi = \phi_{\text{ML}} \circ \psi$, for $\phi_{\text{ML}} : (S, h_l) \rightarrow (S, h_r)$ the minimal Lagrangian diffeomorphism and $\psi : (S, \Omega_l) \rightarrow (S, \Omega_r)$ Hamiltonian, by an argument similar to above.) Then by the same argument of the proof of Lemma 2.6, one can find a function $\theta : S \rightarrow \mathbb{R}$ such that $\eta_{\phi, R_\theta b} = 0$, which is equivalent to $d^{\nabla^l}(R_\theta b) = 0$ by Corollary 2.11. If $\sigma_{\phi, R_\theta b}$ is an embedding, then the image $\Sigma = \sigma_{\phi, R_\theta b}(\mathbb{H}^2)$ is a smoothly embedded spacelike surface, invariant by the representation $\rho = (\rho_l, \rho_r)$, whose associated map is ϕ .

However, the map $\sigma_{\phi, R_\theta b}$ can be singular. Observe also that the function θ is not uniquely determined, but there are possible choices which differ by adding a constant. In general, $\sigma_{\phi, R_\theta b}$ might be non-singular only for some of these choices of θ (i.e. up to adding a constant function).

In [6], the authors showed that such a map $\sigma_{\phi, R_\theta b}$ always lifts to an embedding into the future timelike unit tangent bundle of the universal cover $\widetilde{\text{AdS}}^3$. Finally, let us remark that if one picks a section $b \in \Gamma(\text{Isom}(TS, \phi^* h_r, h_l))$ such that $[\eta_{\phi, \widehat{b}}] \in H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$, then one can still define a map $\sigma_{\phi, \widehat{b}}$ as in Equation (11). However, $\sigma_{\phi, b}$ and $\sigma_{\phi, \widehat{b}}$ will lift to maps from $\mathbb{H}^2 \rightarrow \widetilde{\text{AdS}}^3$ which are equivariant for *different* lifts to $\text{Isom}(\widetilde{\text{AdS}}^3)$ of the representation $\rho : \pi_1(S) \rightarrow \text{Isom}(\text{AdS}^3)$.

Lemma 2.1 in [6] shows that all the ρ_l - ρ_r -equivariant embeddings into AdS^3 lift to embeddings into $\widetilde{\text{AdS}}^3$ which are equivariant for the *same* lift of ρ . This implies that the map $\sigma_{\phi, \widehat{b}}$ will never be an embedding if $[\eta_{\phi, \widehat{b}}] \in H_{\text{dR}}^1(S, 2\pi\mathbb{Z})$ but $[\eta_{\phi, \widehat{b}}] \neq 0$.

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