Complex Manifolds

Research Article

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Kähler-Einstein metrics: Old and New

Abstract: We present classical and recent results on Kähler-Einstein metrics on compact complex manifolds, focusing on existence, obstructions and relations to algebraic geometric notions of stability (K-stability). These are the notes for the SMI course "Kähler-Einstein metrics" given by C.S. in Cortona (Italy), May 2017. The material is not intended to be original.

Introduction

Let us begin by presenting our main actors. The most "geometric" way to introduce them is as follow. Connected $2n$-dimensional Riemannian manifolds (M^{2n}, g) are called *Kähler-Einstein* (KE) if, as the name suggests, they are:

Kähler the holonomy $\mathrm{Hol}(M^{2n}, g)$ is contained in the unitary group $\mathrm{U}(n)$;

Einstein the Ricci curvature satisfies $\mathrm{Ric}(g) = \lambda g$ for $\lambda \in \mathbb{R}$.

In these lectures we are going to focus on *compact* KE manifolds. There are several possible motivations for the study such spaces:

1. For $n = 1$, KE metrics are precisely metrics with constant Gauss curvature on compact oriented surfaces. By the classical *Uniformization Theorem*, a jewel of XIX century mathematics due to works of Riemann, Poincaré and Koebe, three cases occur:
   - if $\lambda > 0$, then the manifold is the round sphere $(\mathbb{S}^2, g_{\mathrm{round}})$;
   - if $\lambda = 0$, then the manifold is a flat torus $(\mathbb{T}^2, g_{\mathrm{flat}})$;
   - if $\lambda < 0$, then the manifold is a compact hyperbolic surface $C_g \simeq \mathbb{H}^2/\Gamma$ where $\Gamma \subseteq \mathrm{PSL}(2; \mathbb{R})$ is a discrete subgroup that acts freely on $\mathbb{H}^2$.

   It is important to remark that, despite the *local* isometry class of such metrics depends only on the value of the constant Gauss curvature, there are, up to scalings, a real 2-dimensional continuous family of distinct (*i.e.*, non *globally* isometric) flat tori, and a $6g - 6$ family (the famous Riemann's *moduli* parameters) of distinct hyperbolic metrics on a surface of genus $g \geq 2$. KE manifolds are the most natural higher dimensional generalizations of such important geometric spaces.

2. In a curvature hierarchy, Einstein metrics lie between the better understood cases of full constant curvature and constant scalar curvature (such metrics exist in any conformal class, by the solution of the Yamabe problem). In real dimension bigger than three, Einstein metrics no longer have a unique local

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isometry model but they still form *finite* dimensional moduli spaces. They are also good candidates for being "canonical" metrics on a space and they are critical points of the natural *Einstein-Hilbert functional* $g \mapsto \int \mathrm{Sc}_g dV_g$, over $\int dV_g = 1$. Thus KE metrics provide an important example of such quite mysterious class of Einstein spaces to study.

3. The Kähler condition corresponds to the holonomy group $U(n)$ in the Berger's classification of reduced holonomy group of irreducible non-locally symmetric spaces. The other are the "generic" $SO(2n)$, the Calabi-Yau $SU(n)$, the hyper-Kähler $Sp(n)$, the quaternionic-Kähler $Sp(1)Sp(n)$ and the special groups G_2 and $Spin(7)$. Except than $SO(2n)$ and $U(n)$, the other cases are Einstein. Thus it is natural to look what happens when we combine the relatively soft $U(n)$ holonomy condition with the Einstein property: as we will see, such holonomy condition actually provides a great simplification in studying the Einstein equation.

The Kähler condition implies that there is a natural structure of complex manifold $X^n (\cong M^{2n})$ which is compatible with the metric. The Kähler-Einstein property implies even more:

– if $\lambda > 0$, then $c_1(X) > 0$; that is X^n is *Fano*;
– if $\lambda = 0$, then $c_1(X) = 0$; that is X^n is *Calabi-Yau*;
– if $\lambda < 0$, then $c_1(X) < 0$; that is X^n is *with ample canonical bundle*, in particular, it is *of general type*.

These mutually exclusive conditions are very strong and most of the complex manifolds are not of the above types (anyway, such special manifolds are, in a certain sense, supposed to be fundamental "building blocks" for complex (Kähler) manifolds).

In analogy with the uniformization of surfaces/complex curves mentioned above (note that in such case $c_1 = 2 - 2g$), it is very natural to ask if all compact complex manifolds whose first Chern class "has a sign" can be equipped with KE metrics. This and a related question was posed by Calabi in the ´50s (the famous *Calabi's conjectures*, [Cal54, Cal57]). However, it was immediately realized that some issues occur: Matsushima [Mat57] found an obstruction for the case $\lambda > 0$ in terms of reductivity of the Lie algebra of holomorphic vector fields. In particular, very simple Fano manifolds, such as the blow-up in a point of the projective plane, cannot admit KE metric.

However, for non-positive first Chern class, the situation turned out to much nicer. Actually, the best it could be:

Theorem 0.1 (Aubin [Aub76, Aub78], Yau [Yau77, Yau78]). *Let X^n be a compact complex manifold. If $c_1(X) < 0$, then there exists up to scaling a unique Kähler-Einstein metric on X^n.*

Theorem 0.2 (Yau [Yau77, Yau78]). *Let X^n be a compact complex Kähler manifold. If $c_1(X) = 0$, then there exists a unique Kähler-Einstein metric in any Kähler class on X^n.*

In the Fano case, the situation remained unclear. Starting from the ´80s there have been several discoveries of new obstructions: Futaki's invariant [Fut83], Tian's examples of a smooth Fano 3-fold with no non-trivial holomorphic vector fields with not KE metric [Tia97] based on obstructions coming from algebro-geometric notions of stability conditions (see also, *e.g.* [Don05]). Furthermore, some existence criteria and non-trivial examples were found in the last 30 years: α-invariant existence criterion [Tian87], study of highly symmetric cases (*e.g.* toric varieties [WZ04]), and a complete understanding in dimension two [Tia90a].

Motivated by the above important results and in analogy with Hitchin-Kobayashi correspondence for holomorphic vector bundles [Don87, UY86, LY87, LT95], it was conjectured, e.g. [Tia97, Don02b]), that the relations of the existence problem with some (to be understood) algebro-geometric notion of stability is indeed the crucial aspect to completely characterize, in purely algebro-geometric terms, the existence of KE metrics on Fano manifolds (and, even more generally, for constant scalar curvature Kähler metrics). Such conjecture is known as *Yau-Tian-Donaldson (YTD) conjecture*. By refining the Futaki's invariant, the notion

of *K-stability* has been introduced by Tian [Tia94, Tia97], and refined by Donaldson [Don02b]. This notion turned out to be the correct one:

Theorem 0.3. *(Yau-Tian-Donaldson conjecture for the Fano case; "if" Chen-Donaldson-Sun [CDS14, CDS15], "only if" Berman [Ber16]). A Fano manifold X^n admits a Kähler-Einstein metric if and only if X is K-polystable.*

The "only if" direction has been studied by Tian [Tia97], Donaldson [Don05], Mabuchi [Mab08], Stoppa [Sto09], and in full generality by Berman [Ber16]. The "if" direction is the celebrated breakthrough result obtained in 2012 by Chen-Donaldson-Sun [CDS14, CDS15].

In these six lectures we are going to describe some aspects of the above outlined "KE story". Due to the vastness of the subject, there is going to be plenty of omission of fundamental results. We apologize in advance. The first lectures are supposed to be more elementary and more detailed. In approaching the final lecture the "hand waving" is going to become more and more dominant. In any case, we hope that we at least succeed in describing some of the ideas behind the proofs of important results in the field, and that such "road map" can be of some use for the ones that would like to proceed further in the study by reading the original papers.

This is a more detailed description of the actual content of the lectures.

- Lecture 1. Review of basic Kähler geometry with emphasis on curvature formulas.
- Lecture 2. Calabi's conjectures: preliminaries, their proofs and some consequences.
- Lecture 3. Obstructions to the existence of KE metrics on Fanos: Matsushima's theorem and Futaki's invariant. Mabuchi's energy and geodesics: the "fake" proof of uniqueness.
- Lecture 4. First results on existence of KE metric on Fanos: energy functionals, properness and the α-invariant criterion.
- Lecture 5. Towards K-stability: review of moment maps and GIT, Donaldson-Fujiki's infinite dimensional picture, definition of K-stability.
- Lecture 6. YTD conjecture for Fanos: descriptions of Berman and Chen-Donaldson-Sun proofs. Compact Moduli spaces of KE Fanos.

1 Curvature formulas on Kähler manifolds

In this preliminary section, we recall some basic facts concerning differential complex geometry and curvature of Kähler metrics; references are *e.g.* [Szé14, Tia00, Mor07, Dem12, Voi07, Gau15, Huy05, MK71, GH78, Kod86, Wel08, Bal06, KN2].

1.1 Basic complex geometry

Let X^n be a *complex manifold*, *i.e.* a smooth manifold with an atlas whose transition functions are holomorphic. Examples include $\mathbb{CP}^n$, hypersurfaces in $\mathbb{CP}^n$ namely zero-set of a homogeneous polynomial, tori $\mathbb{C}^n/\Gamma$ where Γ is a discrete subgroup of maximal rank. We denote by M^{2n} the underlying smooth manifold, and we use local holomorphic coordinates $(z_j = x_j + \sqrt{-1}y_j)_j$, so $(x_j, y_j)_j$ are local differential coordinates and $(\partial_{x_j}, \partial_{y_j})_j$ denotes the corresponding local frame for the tangent bundle.

Multiplication by $\sqrt{-1}$ defines a tensor $J \in \mathrm{End}(TM^{2n})$ as $J\partial_{x_j} = \partial_{y_j}$ and $J\partial_{y_j} = -\partial_{x_j}$, which is globally defined and with the property that $J^2 = -\mathrm{id}$. Moreover, if we define the *Nijenhuis tensor* of such an endomorphism J as

$$N_J(Y, T) := \frac{1}{4} \left([JY, JT] - J[JY, T] - J[Y, JT] - [Y, T] \right),$$

for $Y, T \in TM^{2n}$, then clearly we have $N_J = 0$. Conversely, by the Newlander and Nirenberg theorem [NN57], if M^{2n} is a smooth manifold and $J \in \mathrm{End}(M^{2n})$ is such that $J^2 = -\mathrm{id}$ (say that J is an *almost-complex* structure), then J is *integrable* (*i.e.* it is induced by a holomorphic atlas) if and only if $N_J = 0$.

Consider the eigenspaces splitting

$$TM \otimes \mathbb{C} = TM^{(1,0)} \oplus TM^{(0,1)} = \langle \partial_{z_1}, \ldots, \partial_{z_n} \rangle \oplus \langle \partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n} \rangle,$$

where

$$\partial_{z_j} = \frac{1}{2} \left(\partial_{x_j} - \sqrt{-1} \partial_{y_j} \right), \qquad \partial_{\bar{z}_j} = \frac{1}{2} \left(\partial_{x_j} + \sqrt{-1} \partial_{y_j} \right).$$

Then we can identify

$$(TM, J) \overset{\simeq}{\to} (TM^{(1,0)}, \sqrt{-1})$$

by means of the maps

$$Y \mapsto Y^{(1,0)} := \frac{1}{2}(Y - \sqrt{-1}JY), \qquad 2\mathrm{Re}\, v = v + \bar{v} \hookleftarrow v.$$

A *holomorphic vector field* is a holomorphic section of $TM^{(1,0)}$; locally, it is given by $v \overset{\mathrm{loc}}{=} \sum v_j \partial_{z_j}$ where v_j are holomorphic function, *i.e.* smooth functions such that $\partial_{\bar{z}_k} v_j = 0$ for any k. We see easily that a smooth vector field $v \in TM^{(1,0)}$ is holomorphic if and only if $\mathcal{L}_{\mathrm{Re}\, v} J = 0$ (*i.e.* $\mathrm{Re}\, v$ is *real holomorphic*); see *e.g.* [Gau15, page 23].

Similarly, one can consider the splitting at the level of differential forms:

$$TM^{\vee} \otimes \mathbb{C} = \wedge^{1,0} M \oplus \wedge^{0,1} M = \langle dz_1, \ldots, dz_n \rangle \oplus \langle d\bar{z}_1, \ldots, d\bar{z}_n \rangle$$

where

$$dz_j = dx_j + \sqrt{-1} dy_j, \qquad d\bar{z}_j = dx_j - \sqrt{-1} dy_j.$$

Then

$$\wedge^j TM^{\vee} \otimes \mathbb{C} = \bigoplus_{p+q=j} \wedge^{p,q} M,$$

that is, locally, a (p, q)-*form* can be expressed as $\eta \overset{\mathrm{loc}}{=} \sum f_{I\bar{J}} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, where $f_{I\bar{J}} = f_{i_1,\ldots,i_p,\bar{j}_1,\ldots,\bar{j}_q}$ are smooth functions. We have $df = \partial f + \bar{\partial} f$ on functions, and this extends to forms. By $d^2 = 0$, we get $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$.

The *Dolbeault cohomology* is defined as

$$H^{p,q}_{\bar{\partial}}(X) = \frac{\ker(\bar{\partial} \colon \wedge^{p,q} X \to \wedge^{p,q+1} X)}{\mathrm{im}(\bar{\partial} \colon \wedge^{p,q-1} X \to \wedge^{p,q} X)}.$$

When X is compact, $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}(X) < +\infty$ are called the *Hodge numbers*. Note that $h^{p,0}$ is just the dimension of the space of holomorphic p-forms (*e.g.*, $h^{1,0} = g$ for compact complex curves of genus g). More generally, for people who know a bit of sheaf theory, if we denote by $\mathcal{A}^{p,q}_X$ the sheaf of germs of smooth (p, q)-forms, and define the sheaf of germs of *holomorphic p-forms* as $\Omega^p_X := \ker(\bar{\partial} \colon \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X)$, then it is a standard result that $H^{p,q}(X) \cong H^q(X, \Omega^p_X)$, where the last denotes sheaf cohomology [Dol53], see *e.g.* [Voi07].

1.2 Kähler manifolds

A *Hermitian metric* h on X^n is a smoothly varying family of Hermitian products on $TX = TM^{(1,0)}$. Namely, for any $p \in X$, it holds $h_p(v, v) > 0$ for any $v \neq 0$ and $h_p(v, w) = \overline{h_p(w, v)}$; locally, $h \overset{\mathrm{loc}}{=} \sqrt{-1} \sum h_{j\bar{k}} dz^j \otimes d\bar{z}^k$ where $(h_{j\bar{k}})$ is a positive-definite Hermitian matrix of functions.

A Hermitian manifold is called *Kähler* if the associated $(1, 1)$-form $\omega \overset{\mathrm{loc}}{:=} \sqrt{-1} \sum h_{j\bar{k}} dz_j \wedge d\bar{z}_k$ is closed:

$$d\omega = 0,$$

that is, ω is symplectic. Equivalently if the following PDE is satisfied:

$$\frac{\partial h_{j\bar{k}}}{\partial z_\ell} = \frac{\partial h_{\ell\bar{k}}}{\partial z_j}, \qquad \frac{\partial h_{j\bar{k}}}{\partial \bar{z}_\ell} = \frac{\partial h_{j\bar{\ell}}}{\partial \bar{z}_k}.$$

Examples include:

- $(\mathbb{C}^n, \omega_{\text{flat}})$ with $\omega_{\text{flat}} = \sum \frac{\sqrt{-1}}{2} dz_j \wedge d\bar{z}_j$ for (z_j) standard coordinates; and $\mathbb{C}^n/\Gamma$ flat tori.
- $(\mathbb{CP}^n, \omega_{FS})$ where the Fubini-Study metric is

$$\omega_{FS} \stackrel{\text{loc}}{=} \sqrt{-1}\partial\bar{\partial} \log \left(1 + \sum \frac{|Z_j|^2}{|Z_0|^2} \right)$$

  away from $\{Z_0 = 0\}$, where (Z_j) are homogeneous coordinates on $\mathbb{CP}^n$. For example, for $\mathbb{CP}^1$, we get the round metric $\omega = \sqrt{-1}\partial\bar{\partial}\log(1 + |\xi|^2) = \frac{\sqrt{-1}}{(1+|\xi|^2)^2} d\xi \wedge d\bar{\xi} > 0$. We remark here that, more generally, $(\mathbb{CP}^n, \omega_{FS})$ has constant *holomorphic sectional curvature*, that is, the sectional curvature is constant on *complex lines* in TX. See later for definitions.
- $(\mathbb{B}, \omega_{\mathbb{B}})$ where $\mathbb{B} \subset \mathbb{C}^n$ is the unit ball and $\omega_{\mathbb{B}} = \sqrt{-1}\partial\bar{\partial}\log(1 - \sum |z_j|^2)$ for (z_j) standard coordinates.
- Submanifolds of Kähler manifolds are naturally Kähler. In particular, algebraic manifolds (here meaning zeros of homogeneus polynomial in $\mathbb{CP}^n$) are Kähler. Note that submanifolds of $\mathbb{CP}^n$ with restricted ω_{FS} are almost never Kähler-Einstein for the induced metric.

We compare the real and the complex viewpoints. Starting from (X^n, h) Hermitian, we can define a Riemannian metric as

$$g(Y, Z) := 2\operatorname{Re} h(Y^{(1,0)}, Z^{(1,0)})$$

which is *Hermitian* in the sense that J is an isometry: $g(JY, JZ) = g(Y, Z)$. Conversely, we recover $h_{j\bar{k}} = \frac{1}{2}g^{\mathbb{C}}(\partial_{z_j}, \partial_{\bar{z}_k})$. Moreover,

$$\omega = g(J_-, \_).$$

In the following, we will confuse h, ω, g on a complex manifold X^n.

The Kähler condition $d\omega = 0$ gives a Kähler class $[\omega] \in H^2(X; \mathbb{R})$. Notice that, at p, we can always assume that $h_{j\bar{k}}(p) = \delta_{jk}$. From here, one can easily see that $dV_g = \frac{\omega^n}{n!}$. This implies that, if (X^n, ω) is compact Kähler, then the volume is cohomological: $\text{Vol}_\omega = \text{Vol}_{[\omega]}$, and the even Betti numbers are positive: $b_{2k} > 0$ for any $k \in \{0, \dots, n\}$.

We give now another interpretation of the Kähler condition. Take (M^{2n}, J, g) where g is a Riemannian metric and $J \in \text{End}(TM)$ a compatible almost complex structure, *i.e.* a pointwise isometry such that $J^2 = -\text{id}$.

Proposition 1.1. *Let g be a Riemannian metric and J be a compatible almost complex structure on M^{2n}. Then g is Kähler if and only if $\nabla^{LC}J = 0$, for the Levi Civita connection ∇^{LC}.*

Geometrically, this means that in the Kähler case, complex rotation by J commutes with parallel transport. That is, we immediately get:

Corollary 1.2. *Let g be a Riemannian metric and J be a compatible almost complex structure on M^{2n}. Then g is Kähler if and only if $\text{Hol}(M^{2n}, g) \subseteq U(n)$.*

The proof of Proposition 1.1 goes as follow. For the "$\Leftarrow$" direction, using that ∇^{LC} is torsion free, we can rewrite the Nijenhuis tensor as $4N_J(Y, T) = (\nabla^{LC}_{JY}J)(T) - J((\nabla^{LC}_{JY}J)(T)) - (\nabla^{LC}_{JT}J)(Y) + J((\nabla^{LC}_T J)(Y))$. Thus if J is parallel, N_J vanishes and J is integrable. Moreover, by expressing the differential of the associated two form $\omega = g(J_-, \_)$ using the covariant derivative $d\omega(Y, T, Z) = (\nabla^{LC}_Y \omega)(T, Z) - (\nabla^{LC}_T \omega)(Y, Z) + (\nabla^{LC}_Z \omega)(Y, T)$, we see that ω must be closed since ω is parallel, being so g and J. That is, g is Kähler. Conversely, it is easy (but slightly long) to check that $2g((\nabla^{LC}_X J)Y, Z) = d\omega(X, Y, Z) - d\omega(X, JY, JZ) - g(N_J(Y, JZ), X)$. Hence J has to be parallel if J is integrable and ω is closed.

Finally, a fundamental tool in the following section will be the:

Theorem 1.3 ($\sqrt{-1}\partial\bar{\partial}$-Lemma [DGMS75, pages 266–267]). *Let X^n be a compact Kähler manifold, and let η be a real $(1, 1)$-form such that $\eta = d\alpha$ for some 1-form α. Then there exists $f \in \mathbb{C}^\infty(X; \mathbb{R})$ such that $\eta = \sqrt{-1}\partial\bar{\partial}f$.*

Proof. We briefly review the proof. Write $\alpha = \beta + \bar{\beta}$ for $\beta \in \wedge^{1,0}X$ and note that $\partial\beta = 0$. Consider $\partial^\star$, the L^2-adjoint of ∂ with respect to the L^2-pairing $(\varphi, \psi)_{L^2} := \int_X g^{p\bar{j}} g^{k\bar{q}} \varphi_{p\bar{q}} \overline{\psi_{j\bar{k}}} dV_g$. Here Einstein's summation

convention to be understood and $g^{p\bar{j}}$ is the inverse of the Hermitian metric g. As for $\Delta_d = dd^\star + d^\star d$, we can analogously define the Laplacians Δ_∂ (and $\Delta_{\bar{\partial}}$). A fundamental fact is that for a Kähler metric, $\frac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\bar{\partial}}$ (this is a consequence of the so-called Kähler identities, see *e.g.* [Voi07, Theorem 6.7], [Huy05, Proposition 3.1.12]).

Since we have $\int_X \partial^\star \beta \frac{\omega^n}{n!} = 0$, we can solve the equation

$$\frac{1}{2}\Delta_d \varphi = -\partial^\star \beta$$

for φ smooth. Thus we get $\partial^\star(\beta + \partial\varphi) = 0$ and $\partial(\beta + \partial\varphi) = 0$, that is, $\beta + \partial\varphi$ is ∂-harmonic. Since $\Delta_\partial = \Delta_{\bar{\partial}}$, $\beta + \partial\varphi$ is also $\bar{\partial}$-harmonic; in particular, it is $\bar{\partial}$-closed: $\bar{\partial}\beta = \partial\bar{\partial}\varphi$. Finally, $\eta = d\alpha = \bar{\partial}\beta + \partial\bar{\beta} = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\mathrm{Im}\,\varphi$. □

This lemma allows to identify the space $\mathcal{K}_{[\omega]}$ of Kähler metrics in the Kähler class $[\omega] \in H^2(X;\mathbb{R})$ on X^n with the set

$$\left\{ \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \ : \ \varphi \in \mathcal{C}^\infty(X;\mathbb{R}) \right\},$$

once fixed some background metric ω. This is the first ingredient that explains why the Kähler-Einstein problem is easier than the general Einstein problem: in order to find a KE metric in a fixed cohomology class we need to look for a *function* φ and not for a *tensor*. The other reason is that there is a very nice formula for the Ricci curvature of a Kähler metric, as we are going to explain.

1.3 Curvatures of Kähler metrics

Consider (M^{2n}, g). In this section, we denote by $\nabla := \nabla_g^{LC}$ its Levi Civita connection, and by $\nabla^{\mathbb{C}}$ its complexification to $TM \otimes \mathbb{C}$. Note that $\nabla_{\bar{v}}^{\mathbb{C}}\bar{w} = \overline{\nabla_v^{\mathbb{C}}w}$. The complexified *Christoffel symbols* are defined as $\nabla_{\partial_{z_i}}^{\mathbb{C}}\partial_{z_j} = \sum_k \left(\Gamma_{ij}^k \partial_{z_k} + \Gamma_{ij}^{\bar{k}}\partial_{z_{\bar{k}}} \right)$ and similar. See *e.g.* [Szé14], [Tia00, Section 1.2] for more details on what follows.

Lemma 1.4. *Let g be a Kähler metric on M^{2n}. Then the only non-zero Christoffel symbols are Γ_{ij}^k and $\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = \overline{\Gamma_{ij}^k}$. Moreover, $\Gamma_{ij}^k = g^{k\bar{\ell}}\partial_{z_j}g_{i\bar{\ell}}$.*

Let us check $\Gamma_{ij}^{\bar{k}} = 0$, for the other is similar. Since $\nabla J = 0$, then $\nabla_{\partial_{z_i}}^{\mathbb{C}}(J^{\mathbb{C}}\partial_{z_j}) = J^{\mathbb{C}}(\nabla_{\partial_{z_i}}^{\mathbb{C}}(\partial_{z_j}))$, that is: $\sqrt{-1}\sum_k \left(\Gamma_{ij}^k \partial_{z_k} + \Gamma_{ij}^{\bar{k}}\partial_{z_{\bar{k}}} \right) = \sum_k \left(\sqrt{-1}\Gamma_{ij}^k \partial_{z_k} - \sqrt{-1}\Gamma_{ij}^{\bar{k}}\partial_{z_{\bar{k}}} \right)$. Thus, $\Gamma_{ij}^{\bar{k}} = 0$. Next, using $\nabla g = 0$ and the vanishing of mixed Christoffel, $\partial_{z_k}g_{i\bar{j}} = g(\nabla_{\partial_{z_k}}\partial_{z_i}, \partial_{\bar{z}_j}) = g(\Gamma_{ki}^l \partial_{z_l}, \partial_{\bar{z}_j}) = \Gamma_{ki}^l g_{l\bar{j}}$. By multiplying with the inverse the proof of the lemma is concluded.

We now consider the *Riemannian curvature* operator:

$$\mathrm{Rm}^{\mathbb{C}}(v, w)z := \nabla_v^{\mathbb{C}}\nabla_w^{\mathbb{C}}z - \nabla_w^{\mathbb{C}}\nabla_v^{\mathbb{C}}z - \nabla_{[v,w]}^{\mathbb{C}}z.$$

Clearly, $\mathrm{Rm}(\bar{v}, \bar{w})\bar{z} = \overline{\mathrm{Rm}(v, w)z}$. By using that $\nabla J = 0$, we have that $\mathrm{Rm}(v, w)Jz = J\mathrm{Rm}(v, w)z$. This implies that $\mathrm{Rm}(z, w)(TM^{1,0}) \subseteq TM^{1,0}$, and also $\mathrm{Rm}(TM^{1,0}, TM^{1,0}) = 0$ thanks to the symmetries of Rm. This means that the only interesting coefficients of Rm are

$$R_{i\bar{j}k\bar{\ell}} := g(\mathrm{Rm}(\partial_{z_i}, \partial_{\bar{z}_j})\partial_{z_k}, \partial_{\bar{z}_\ell})$$

and $R_{\bar{i}j\bar{k}\ell} = \overline{R_{i\bar{j}k\bar{\ell}}}$. We have a nice expression for these coefficients:

Proposition 1.5. *Let g be a Kähler metric on M^{2n}. Then $R_{i\bar{j}k\bar{\ell}} = -g_{i\bar{j},k\bar{\ell}} + g^{m\bar{n}}g_{i\bar{n},k}g_{m\bar{j},\bar{\ell}}$.*

We have denoted with $g_{i\bar{j},k\bar{\ell}} := \partial_{z_k\bar{z}_\ell}^2 g_{i\bar{j}}$ (and similar) for short. Here the proof:

$$R_{i\bar{j}k\bar{\ell}} = -g(\nabla_{\bar{j}}(\nabla_i \partial_k), \partial_{\bar{\ell}}) = -g(\nabla_{\bar{j}}(\Gamma_{ik}^m \partial_m), \partial_{\bar{\ell}}) \overset{\text{Leibniz}}{=} -g(\Gamma_{ik,\bar{j}}^m \partial_m, \partial_{\bar{\ell}}) = \Gamma_{ik,\bar{j}}^m g_{m\bar{\ell}}.$$

But $\Gamma_{ik,\bar{j}}^m = -\partial_{\bar{j}}(g^{m\bar{q}}g_{i\bar{q},k}) = g_{,\bar{j}}^{m\bar{q}}g_{i\bar{q},k} + g^{m\bar{q}}g_{i\bar{q},k\bar{l}}$. Since $g^{i\bar{j}}_{,\bar{k}} = -g^{i\bar{m}}g^{l\bar{j}}g_{l\bar{m},\bar{k}}$, we get

$$R_{i\bar{j}k\bar{l}} = -g_{m\bar{l}}(-g^{m\bar{r}}g^{s\bar{q}}g_{s\bar{r},\bar{j}}g_{i\bar{q},k} + g^{m\bar{q}}g_{i\bar{q},n\bar{j}}) = -g_{i\bar{j},k\bar{l}} + g^{m\bar{n}}g_{i\bar{n},k}g_{m\bar{j},\bar{l}},$$

where for the last step we have used the Kähler property and renamed the variables.

Define now the *Ricci tensor*

$$\mathrm{Ric}_{i\bar{j}} := g^{k\bar{l}}R_{i\bar{j}k\bar{l}},$$

and the *scalar curvature*

$$\mathrm{Sc} := g^{i\bar{j}}\mathrm{Ric}_{i\bar{j}}.$$

For Kähler metrics, we have the following nice expression:

Proposition 1.6. *Let g be a Kähler metric on M^{2n}. Then* $\mathrm{Ric}_{i\bar{j}} = -\partial_{i\bar{j}}^2 \log \det(g_{k\bar{l}})$.

The proof follows by recalling the formula $\partial(\log \det A) = \mathrm{tr}(A^{-1}\partial A)$, and plugging in the above formula for the curvature: $-\partial_{i\bar{j}}^2 \log \det(g_{k\bar{l}}) = -\partial_i(g^{k\bar{l}}g_{k\bar{l},j}) = \mathrm{Ric}_{i\bar{j}}$.

Here we remark that clearly in such Kähler case the above are essentially the only non-vanishing terms of the (complexified) Riemannian Ricci tensor.

The expression for Ric in Proposition 1.6 is local, but it actually yields a global real $(1,1)$-form: define the *Ricci form* as

$$\mathrm{Ric}(\omega) := \sqrt{-1}\mathrm{Ric}_{j\bar{k}}dz_j \wedge d\bar{z}_k \overset{\mathrm{loc}}{=} -\sqrt{-1}\partial\bar{\partial}\log \omega^n.$$

Here, $\log \omega^n$ is short version for $\log \det(g_{k\bar{l}})$. Thus, we can say that in the Kähler case the Ricci curvature is simply given by the complex Hessian of the logarithm of the volume form! This beautiful expression will have important consequences in the next section, especially in combination with the above $\sqrt{-1}\partial\bar{\partial}$-lemma.

Thus the *Kähler-Einstein* (KE) condition can be phrased in term of $(1,1)$-forms by saying that

$$\mathrm{Ric}(\omega) = \lambda\omega,$$

for some "cosmological constant" $\lambda \in \mathbb{R}$. Note that, when $n \geq 2$ (the case $n = 1$ being described by the Uniformization Theorem), even taking $\lambda \in \mathcal{C}^\infty(X;\mathbb{R})$, the cosmological factor is in fact constant, since $0 = d\mathrm{Ric}(\omega) = d\lambda \wedge \omega \Rightarrow \lambda = $ constant. This is a short Kähler version of the Riemannian Schur lemma, see *e.g.* [Bes87, Theorem 1.97]. More generally, if only the scalar curvature is constant we say that ω is *cscK*.

Finally, let us note that if ω' is another Kähler metric, then one has

$$\mathrm{Ric}(\omega') - \mathrm{Ric}(\omega) = \sqrt{-1}\partial\bar{\partial}\log \frac{\omega^n}{\omega'^n}$$

where now $\log \frac{\omega^n}{\omega'^n}$ is a global function on M^{2n}. Being the Ricci form clearly closed, this means that any Kähler metric defines a unique class

$$[\mathrm{Ric}(\omega)] =: 2\pi c_1(X) \in H^2(X;\mathbb{R}) \cap H^{1,1}(X),$$

and we call $c_1(X)$ the *first Chern class* of the complex manifold X^n (that is, of the holomorphic tangent bundle, which is the same as the first Chern class of the anti-canonical line bundle $K_X^{-1} := \bigwedge^n TX$, *i.e.* the dual of the canonical bundle; of course, the definition will agree with any other definition of first Chern class you may have seen before).

2 Calabi's conjectures

2.1 Statement of Calabi's conjectures and basic Kähler-Einstein

We have seen that, for any compact Kähler manifold (X^n, ω), we have $\mathrm{Ric}(\omega) \in 2\pi c_1(X)$; then it is natural to ask what about the converse statement:

Theorem 2.1 (first Calabi conjecture, proven by Yau [Yau77, Yau78])**.** *Let (X^n, ω) be a compact Kähler manifold, and let $\rho \in 2\pi c_1(X) \in H^2(X; \mathbb{R}) \cap H^{1,1}(X)$. Then there exists a unique metric $\omega' \in [\omega]$ such that $\mathrm{Ric}(\omega') = \rho$.*

In particular, when $c_1(X) = 0$ (that is, when X is *Calabi-Yau*), the above theorem gives Kähler metric ω' with $\mathrm{Ric}(\omega') = 0$. Historically, these were the first non-flat examples of compact Ricci flat manifolds to have been found. More precisely, Yau's result implies:

Theorem 2.2 (Yau [Yau77, Yau78])**.** *Let X^n be a compact Kähler Calabi-Yau manifold, i.e. $c_1(X) = 0$. Then in any Kähler class there is a unique Kähler-Einstein metric with cosmological constant $\lambda = 0$.*

As regards the other Kähler-Einstein metrics, we first observe the following important fact:

Proposition 2.3 (after [Kod54])**.** *Let (X^n, ω) be a compact Kähler-Einstein manifold with cosmological constant $\lambda \neq 0$. Then X is projective algebraic (actually of a very special type, see next), i.e. it has an embedding $X \hookrightarrow \mathbb{CP}^N$.*

Remark 2.4.
– *This is not true for $\lambda = 0$ (e.g. the generic flat complex torus in dimension bigger than one).*
– *The KE metric* are not *given by the restriction of Fubini-Study metric.*

The proof requires a little, but important, digression. First, we recall that for any holomorphic Hermitian line bundle (L, h) on a complex manifold X^n (obvious definitions), there exists a unique connection ∇_h^{Ch} on L being compatible with the Hermitian structure and such that $\nabla^{0,1} = \bar{\partial}$: it is called the *Chern connection*, see *e.g.* [Szé14, Section 1.6], [Gau15, Section 1.7], [Dem12, Section V.12]. More precisely, let s be a local non-vanishing holomorphic section of L, and write $h(s) = |s|^2$, then the connection 1-form is $\alpha = \frac{\partial h}{h}$. Thus its curvature is $\sqrt{-1}F_{\nabla^{Ch}} = -\sqrt{-1}d\alpha = -\sqrt{-1}\partial\bar{\partial}\log h$, and it represents, by definition, the class $2\pi c_1(L)$ where $c_1(L) \in H^2(X; \mathbb{R}) \cap H^{1,1}(X)$ is the *first Chern class of L*.

Second, we say that the Hermitian line bundle (L, h) over X^n is *positive* if $\sqrt{-1}F_{\nabla^{Ch}}$ is a Kähler metric on X^n. Note that this is equivalent to ask that there exists a Kähler metric $\omega \in 2\pi c_1(L)$ (*a priori* a weaker requirement). Indeed, take any Hermitian structure $\tilde{h}$ on L, then by the $\sqrt{-1}\partial\bar{\partial}$-Lemma there exists φ such that $\omega - \sqrt{-1}F_{\nabla_{\tilde{h}}^{Ch}} = \sqrt{-1}\partial\bar{\partial}\varphi$, and then $\omega = \sqrt{-1}F_{\nabla_h^{Ch}}$ for $h := \exp(-\varphi)\tilde{h}$. By the way, this argument precisely says that we can always solve the *Hermitian-Einstein equation* for line bundles!

We also introduce another notion: a holomorphic line bundle L over X is called *ample* if there exists a holomorphic embedding $f \colon X \hookrightarrow \mathbb{CP}^N$ such that $L^{\otimes k} \simeq f^\star \mathcal{O}_{\mathbb{CP}^N}(1)$ for some k. Here, $\mathcal{O}_{\mathbb{CP}^N}(1)$ is the dual of the tautological line bundle $\mathcal{O}_{\mathbb{CP}^N}(-1)$ of $\mathbb{CP}^N$ (the one that takes the line $\ell \in \mathbb{C}^{n+1}$ over the point $[\ell] \in \mathbb{CP}^n$).

Next we recall the following very deep theorem by Kodaira that relates the two notions, see *e.g.* [Voi07, Theorem 7.11], [Dem12, Corollary VII.13.3]:

Theorem 2.5 (Kodaira [Kod54])**.** *A holomorphic line bundle over a compact complex manifold admits a positive Hermitian structure if and only if it is ample.*

Of course one direction is trivial: just take the restriction of Fubini-Study metric. The other requires considerably more work. We will describe some of the idea of the proof in subsequent sections using the "pick sections technique".

Proof of Proposition 2.3. If we have a KE metric with positive cosmological section $\mathrm{Ric}(\omega) = \lambda\omega > 0$, hence, by the discussion at the end of the last lecture the anticanonical bundle is positive. Thus, by Kodaira's theorem, X can be embedded in some projective space (hence it is algebraic, *i.e.* it is cut by homogeneous polynomial by Chow's theorem, *e.g.* [GH78]). The case of negative cosmological constant goes along the same lines. □

We say that a compact complex manifold is *Fano* if $c_1(X) > 0$, *i.e.* if the first Chern class can be represented by a Kähler form. This is equivalent to say that the anticanonical line bundle is ample, and it is thus a "trivial" necessary condition for a complex manifold to admit KE metric of positive scalar curvature. Note that as an immediate consequence of the first Calabi conjecture, Fano manifolds are precisely the compact

complex manifolds admitting Kähler metrics of positive (non necessary constant!) Ricci curvature. If $c_1(X) < 0$ (equivalenty, the canonical bundle is ample) the manifold is, in particular, of *general type* (*i.e.* of top Kodaira dimension, see *e.g.* [Voi07]). Again this condition is necessary for KE metrics with negative cosmological constant.

Fano manifolds are definitely fewer that manifolds with negative first Chern class. In particular, there are only finitely many diffeomorphism classes of Fano manifolds for fixed dimension [KMM92], see *e.g.* [Kol96]. The next proposition provides some simple examples Fanos, Calabi-Yaus, and algebraic manifolds with ample canonical bundle.

Proposition 2.6. *Let $X_d = \{F_d = 0\} \subseteq \mathbb{CP}^n$ be a smooth hypersurface of degree d. Then:*
- *if $d < n + 1$, then X_d is Fano;*
- *if $d = n + 1$, then X_d is Calabi-Yau;*
- *if $d > n + 1$, then X_d is with ample canonical bundle.*

Proof. We sketch here some ideas for the proof. See *e.g.* [GH78] for more details on the definitions.

For $Y \subseteq X$ a complex submanifold of codimension 1, we have the short exact sequence of holomorphic vector bundles

$$0 \to TY \to TX\rfloor_Y \to N_Y \to 0,$$

and its dual, whence the adjunction formula

$$K_Y^{-1} = K_X^{-1}\rfloor_Y \otimes N_Y,$$

where $N_Y^\vee = \mathcal{O}(Y)\rfloor_Y$. Here, given locally $Y \cap U_\alpha = \{f_\alpha = 0\}$, then $\mathcal{O}(Y)$ is the line bundle with transition functions $g_{\alpha\beta} := \frac{f_\alpha}{f_\beta}$. Since $df_\alpha = g_{\alpha\beta} df_\beta$, we have a non-vanishing global section of $N_Y^\vee \otimes \mathcal{O}(Y)$, that is, $N_Y^\vee \otimes \mathcal{O}(Y) \simeq \mathcal{O}$ the trivial line bundle.

In our case, we recall that the space of isomorphisms classes of holomorphic line bundles on $\mathbb{CP}^n$ is $\mathrm{Pic}(\mathbb{CP}^n) = \langle \mathcal{O}(1) \rangle \simeq \mathbb{Z}$. We have $K_{\mathbb{CP}^n} \simeq \mathcal{O}(-1)^{\otimes(n+1)}$ (*e.g.* since the standard holomorphic volume form on $\mathbb{C}^n$ has a pole of order $n + 1$ at infinity). Since an hypersurface of degree d is given by a section of $\mathcal{O}(d)$,

$$K_{X_d}^{-1} \simeq \mathcal{O}(n + 1 - d)\rfloor_{X_d}.$$

The statement follows. □

It is worth noting that by choosing different homogeneous polynomials in the above proposition we have different (non-biholomorphic) complex manifolds. For fixed degree d they are all diffeomorphic. The rough "count" of the number of complex moduli parameters considering the PGL(n) reparametrization is equal to $\binom{n+d}{d} - (n + 1)^2$. Clearly, from the point of view of studying existence of KE metrics on them, such complex manifolds are all to be considered different.

In our search for KE metrics, the next theorem is crucial:

Theorem 2.7 (second Calabi conjecture, proven by Aubin [Aub76, Aub78], Yau [Yau77, Yau78])**.** *Let X^n be a compact complex manifold with ample canonical bundle. Then there exists a unique Kähler metric $\omega \in 2\pi c_1(X)$ such that $\mathrm{Ric}(\omega) = -\omega$.*

Thus, thanks to Theorems 2.1 and 2.7, Proposition 2.6 gives many examples of (Kähler)-Einstein metrics: *any* hypersurface of degree $d \geq n + 1$ in $\mathbb{CP}^n$ is Kähler-Einstein.

By the way, as regards hypersurfaces in the Fano case, the problem is still open! It is conjectured they all do.

Remark 2.8. *It is known that KE metrics exists on hypersurfaces of Fermat type $\sum_i z_i^d = 0$ (e.g. [Tia00]). The case of hypersurfaces for $n = 3$, $d = 3$ was proved in [Tia90a]. Donaldson (also Odaka independently) showed that the set of Kähler-Einstein manifolds in the space of Fano manifolds is Zariski open under the assumption that the group of holomorphic automorphisms of each manifold is discrete (e.g. [Don15]), hypothesis which holds*

for hypersurfaces. Fujita recently proved existence of KE metrics on all smooth hypersurfaces of degree d = n in $\mathbb{CP}^n$ *[Fuj16b]. For n = 4 the conjecture follows by (very recent!) works [SS17, LX17]. Thus for $3 \leq d \leq n - 1$ and $n \geq 5$ the existence problem of KE metrics on all smooth hypersurfaces is open.*

2.2 Reduction of Calabi's conjectures to certain PDE's

In this section, we reduce first and second Calabi's conjectures, Theorems 2.1 and 2.7, to a PDE, whose solutions are going to be studied in the following sections.

Consider the first Calabi's conjecture, Theorem 2.1, that is, the problem of representing a chosen form ρ in $2\pi c_1(X)$ as the Ricci form of a metric in $[\omega]$, where ω is the fixed background metric. Thanks to the $\sqrt{-1}\partial\bar{\partial}$-Lemma, we are reduced to look for $\varphi \in \mathcal{C}^\infty(X; \mathbb{R})$ such that $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ has $\mathrm{Ric}(\omega_\varphi) = \rho$. Again by the $\sqrt{-1}\partial\bar{\partial}$-Lemma, we have $f \in \mathcal{C}^\infty(X; \mathbb{R})$ such that $\mathrm{Ric}(\omega) - \rho = \sqrt{-1}\partial\bar{\partial}f$; we can normalize the Ricci potential f in such a way that $\int_X \exp f \omega^n = \int_X \omega^n$. By the expression for the Ricci form of a Kähler metric, Proposition 1.6, we are reduced to solve $\sqrt{-1}\partial\bar{\partial}\log\frac{\omega_\varphi^n}{\omega^n} = \sqrt{-1}\partial\bar{\partial}f$. When X is compact, this reduces to

$$\omega_\varphi^n = \exp f \omega^n. \tag{1}$$

In other words, we are prescribing the volume form. In coordinates, this can be written as

$$\det(g_{i\bar{j}} + \partial^2_{z_i\bar{z}_j}\varphi) = \exp f \det g_{i\bar{j}},$$

which is a *complex Monge-Ampère equation* a very much studied second order non-linear PDE.

Remark 2.9. *As regards deep study of* complex *Monge-Ampère equations, see e.g. [Aub98, LNM2038, GZ17]). As regards* real *Monge-Ampère equations, see e.g. [GT98, CC95, Fig17] and the references therein.*

Similarly, the problem of finding Kähler-Einstein metrics with cosmological constant $\lambda \in \mathbb{R}$ reduces to the equation

$$\omega_\varphi^n = \exp(f - \lambda\varphi)\omega^n. \tag{2}$$

In particular, the second Calabi's conjecture, Theorem 2.7, corresponds to $\lambda = -1$, up to homothety:

$$\omega_\varphi^n = \exp(f + \varphi)\omega^n. \tag{3}$$

2.3 Uniqueness of solutions for Calabi's conjectures

In this section, we prove uniqueness of solutions for equations (1) and (3).

Consider first the equation (3) for the second Calabi's conjecture. In this case the proof of uniqueness follows easily by the maximum principle. Assume that ω and ω_φ are two different Kähler-Einstein metrics. We can assume $f = 0$ in (3), so we have $\omega_\varphi^n = \exp\varphi\,\omega^n$. Let $p_{\max}$ be a maximum point for φ. Then $\partial^2_{z_i\bar{z}_j}\varphi(p_{\max}) \leq 0$. By using $\omega_\varphi^n = \exp\varphi\,\omega^n$, we get that $\exp\varphi(p_{\max}) \leq 1$, then $\varphi(x) \leq \varphi(p_{\max}) \leq 0$ for any x. The same argument for a minimum point $q_{\min}$ gives $\varphi(x) \geq 0$ for any x. Thus $\varphi = 0$.

Uniqueness, up to additive constants, of solutions for first Calabi's conjecture (1) has been proven by Calabi [Cal57]. The proof now is more subtle, and it needs some non-pointwise estimates. Assume that $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ and $\omega_\psi = \omega + \sqrt{-1}\partial\bar{\partial}\psi$ are two different metrics in the same Kähler class $[\omega]$ with the same volume form $\omega_\varphi^n = \omega^n = \omega_\psi^n$. Consider the *Aubin functional* [Aub84] as a generalization of the classical Dirichlet energy (see also *e.g.* [Gau15, Section 4.2]) and [Tia00, page 46]):

$$I_\omega(\varphi) := \frac{1}{n!\mathrm{Vol}_\omega}\int_X \varphi(\omega^n - \omega_\varphi^n).$$

We have:

$$0 = \int_X (\varphi - \psi)(\omega_\psi^n - \omega_\varphi^n) = -\sqrt{-1}\int_X (\varphi - \psi)\partial\bar{\partial}(\varphi - \psi) \wedge \sum_{k=0}^{n-1}\omega_\varphi^k \wedge \omega_\psi^{n-k-1}$$

$$= \sqrt{-1} \int_X \partial(\varphi - \psi) \wedge \overline{\partial}(\varphi - \psi) \wedge \omega_\varphi^{n-1}$$

$$+ \sqrt{-1} \sum_{k=0}^{n-2} \int_X \partial(\varphi - \psi) \wedge \overline{\partial}(\varphi - \psi) \wedge \omega_\varphi^k \wedge \omega_\psi^{n-k}$$

$$\geq \sqrt{-1} \int_X \partial(\varphi - \psi) \wedge \overline{\partial}(\varphi - \psi) \wedge \omega_\varphi^{n-1}$$

$$= \frac{1}{n} \int_X |\partial(\varphi - \psi)|_{\omega_\varphi}^2 \omega_\varphi^n \geq 0,$$

yielding $\partial(\varphi - \psi) = 0$. Since $\varphi - \psi$ is real, this gives that $\varphi - \psi$ is constant and so $\omega_\varphi = \omega_\psi$.

As a consequence, we get:

Proposition 2.10. *Let X^n be a compact complex manifold with ample canonical bundle. Then any $F \in \mathrm{Aut}(X)$ is an isometry for the Kähler-Einstein metric.*

That is, Kähler-Einstein metrics are *canonical*: namely, the symmetries for the metric are the holomorphic symmetries. Recall that, in such a case, the group of isometries is finite, by Bochner formula, see *e.g.* [Bes87, Theorem 1.84] and discussions in the next section.

2.4 Existence of solutions for second Calabi's conjecture

We prove existence of a smooth solution for equation (3) by applying the *continuity method*. We consider the family of equations

$$\omega_\varphi^n = \exp(tf + \varphi)\omega^n \tag{$\star_t$}$$

varying $t \in [0, 1]$. Set

$$E := \left\{ t \in [0, 1] \; : \; \text{there exists } \varphi_s \in \mathcal{C}^\infty(X; \mathbb{R}) \text{ solution of } (\star_t) \text{ for all } 0 \leq s \leq t \right\}.$$

We prove that:

- $E \neq \varnothing$ (this is straightforward because $\varphi = 0$ is a solution for $(\star_t)$ when $t = 0$);
- E is open (this will follow by an implicit function argument);
- E is closed (this will follow by *a priori* estimates on the solutions).

In this case, since $[0, 1]$ is connected, we get $E = [0, 1]$, and in particular $1 \in E$, assuring that we have a solution for $(\star_t)$ for $t = 1$, which is exactly (3).

We prove *openness*. We can define the Hölder spaces $\mathcal{C}^{k,\alpha}(X)$ (on a compact manifold just take the usual definition in charts; one can also use a fixed background metric and compute the Hölder differences via parallel transport. All these possible constructions give the same spaces with equivalent norms). Consider

$$\mathcal{E} \colon \mathcal{C}^{k,\alpha}(X) \times [0, 1] \to \mathcal{C}^{k-2,\alpha}(X), \qquad (\varphi, t) \mapsto \log \frac{\omega_\varphi^n}{\omega^n} - \varphi - tf,$$

We can then rewrite: φ is a solution of $(\star_t)$ if and only if $\mathcal{E}(\varphi, t) = 0$. We compute

$$D_\varphi \mathcal{E}(\varphi, t)(\psi) := \frac{d}{ds}\big\lfloor_{s=0} \mathcal{E}(\varphi + s\psi, t)$$

$$= \mathrm{tr}_{\omega_\varphi} \sqrt{-1}\partial\overline{\partial}\psi - \psi = -\Delta_{\overline{\partial}}\psi - \psi.$$

Since $-\Delta_{\overline{\partial}}$ has non-positive eigenvalues, then the kernel of $D_\varphi \mathcal{E}(\varphi, t)$ is zero. Being self-adjoint,, the set of $\mathcal{C}_\omega^{k,\alpha}$-solutions is open by implicit function theorem. To prove smoothness of the solution, we can start from $k = 2$ using the Evans-Krylov theory [Eva82, Kry82], or from $k = 3$ using the Schauder theory (see *e.g.* [GT98, Chapter 6]). The idea of the Schauder theory is the following. Consider φ_t a $\mathcal{C}^{3,\alpha}$-regular solution of $\log \det(g_{j\bar{k}} + \partial_{z_j\bar{z}_k}^2 \varphi_t)_{j,k} - \log \det(g_{j\bar{k}})_{j,k} - \varphi_t - tf = 0$. By differentiating the equation in z_ρ, we get

$$g_{\varphi_t}^{j\bar{k}} \left(\partial_{z_\rho} g_{j\bar{k}} + \partial_{z_\rho z_j \bar{z}_k}^3 \varphi_t \right) - \partial_{z_\rho} \log \det(g_{j\bar{k}})_{j,k} - \partial_{z_\rho}\varphi_t - t\partial_{z_\rho}f = 0.$$

So locally $\partial_{z_\rho}\varphi_t$ is a $\mathcal{C}^2$-regular solution of

$$L\partial_{z_\rho}\varphi_t = h$$

where $L := \Delta_{\omega_{\varphi_t}} - 1$ (here and later $\Delta_\omega := -\frac{1}{2}\Delta_d$, as analyst would prefer) is a second-order linear elliptic operator with $\mathcal{C}^{1,\alpha}$-regular coefficients, and $h := -g^{j\bar{k}}_{\varphi_t}\partial_{z_\rho}g_{j\bar{k}} + \partial_{z_\rho}\log\det(g_{j\bar{k}})_{j,k} + t\partial_{z_\rho}f$ is $\mathcal{C}^{1,\alpha}$-regular. By Schauder estimates, see *e.g.* [GT98, Theorem 6.19], $d\varphi$ is $\mathcal{C}^{3,\alpha}$-regular, then φ is $\mathcal{C}^{4,\alpha}$-regular. By bootstrap, we get that φ is $\mathcal{C}^{k,\alpha}$-regular for any k and α, so φ is smooth.

We prove now *closedness*. (See also *e.g.* [Tia00, Chapter 5], [Szé14, Chapter 3].) More precisely, let $(\varphi_{t_j})_j$ be a sequence where φ_{t_j} is a solution of $(\star_t)$ for $t = t_j$. We want to prove that $\varphi_{t_j} \to \varphi_{\bar{t}}$ as $t_j \to \bar{t}$, and that $\varphi_{\bar{t}}$ is a solution of $(\star_t)$ for $t = \bar{t}$. This will follow by the following *a priori* estimates on the solutions of $(\star_t)$:

(i)   uniform $\mathcal{C}^0$-estimate;
(ii) uniform Laplacian estimate;
(iii) uniform higher-order estimate.

Indeed, if we have $\|\varphi_{t_j}\|_{\mathcal{C}^{3,\alpha}\omega(X)} < C$, then by the Ascoli-Arzelà theorem, see *e.g.* [Aub98, 3.15], we get that $\varphi_{t_j} \to \varphi_{\bar{t}} \in \mathcal{C}^{3,\alpha'}(X)$ for some $0 < \alpha' < \alpha$, and $\varphi_{\bar{t}}$ is still a solution of $(\star_t)$ for $t = \bar{t}$. Moreover, by the estimate in (ii), we will also get $g_\varphi > 0$.

Notice that (ii) and (iii), as well as the argument in the paragraph above, hold also for equation (1) and for equation (2). So the difficulty to solve the first Calabi's conjecture essentially arises in adapting the $\mathcal{C}^0$-estimate.

The $\mathcal{C}^0$-*estimate* is very easy. This follows directly by applying the maximum principle to $(\star_t)$ as we did in the proof of uniqueness. We get $\|\varphi_t\|_{L^\infty} \le C\|f\|_{L^\infty}$ for some constant $C = C(X^n, \omega)$.

We prove the *Laplacian estimate*. Roughly, we look for an estimate of the type

$$0 < n + \Delta_g\varphi \le C(\|\varphi\|_{L^\infty}),$$

where C is a function (exponential) so that together with the L^∞ bound give the control on the Laplacian. The crucial estimate is the following: denote $g_\varphi = g + \partial^2\varphi$, so $\mathrm{tr}_g g_\varphi = n + \Delta_g\varphi$; we prove that there exists a constant $B = B(X^n, g)$ such that

$$\Delta_{g_\varphi}\log\mathrm{tr}_g g_\varphi \ge -B\mathrm{tr}_{g_\varphi}g - \frac{\mathrm{tr}_g\mathrm{Ric}(g_\varphi)}{\mathrm{tr}_g g_\varphi}. \tag{4}$$

We prove (4). At a point p_0, we can assume that $g(p_0) = \mathrm{id}$ and $g_\varphi(p_0)$ is diagonal. We use also the Kähler condition so at the point p_0 we can further assume that we have complex normal coordinates, *i.e.* the first derivatives of the metric g at p_0 vanish. This can be done for any Kähler metric [GH78]. Then

$$\mathrm{tr}_g g_\varphi(p_0) = \sum_i (g_\varphi)_{i\bar{i}}(p_0), \qquad \mathrm{tr}_{g_\varphi}g(p_0) = \sum_j g^{j\bar{j}}_\varphi(p_0) = \sum_j (g_\varphi)_{j\bar{j}}(p_0)^{-1}.$$

At p_0, we compute:

$$\begin{aligned}
\Delta_{g_\varphi}\mathrm{tr}_g g_\varphi(p_0) &= g^{p\bar{q}}_\varphi\partial^2_{z_p\bar{z}_q}(g^{j\bar{k}}(g_\varphi)_{j\bar{k}}) \\
&= g^{p\bar{q}}_\varphi\partial^2_{z_p\bar{z}_q}g^{j\bar{k}}(g_\varphi)_{j\bar{k}} + g^{p\bar{q}}_\varphi g^{j\bar{k}}\partial^2_{z_p\bar{z}_q}(g_\varphi)_{j\bar{k}} \\
&= g^{p\bar{q}}_\varphi\partial^2_{z_p\bar{z}_q}g^{j\bar{k}}(g_\varphi)_{j\bar{k}} - g^{p\bar{q}}_\varphi g^{j\bar{k}}(\mathrm{R}_\varphi)_{j\bar{k}p\bar{q}} + g^{a\bar{b}}_\varphi g^{p\bar{q}}_\varphi g^{j\bar{k}}(\partial_{z_j}(g_\varphi)_{p\bar{b}})(\partial_{\bar{z}_k}(g_\varphi)_{a\bar{q}}) \\
&= g^{p\bar{q}}_\varphi\partial^2_{z_p\bar{z}_q}g^{j\bar{k}}(g_\varphi)_{j\bar{k}} - g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}} + g^{a\bar{b}}_\varphi g^{p\bar{q}}_\varphi g^{j\bar{k}}(\partial_{z_j}(g_\varphi)_{p\bar{b}})(\partial_{\bar{z}_k}(g_\varphi)_{a\bar{q}}) \\
&= g^{p\bar{p}}_\varphi\partial^2_{z_p\bar{z}_p}g^{j\bar{j}}(g_\varphi)_{j\bar{j}} - g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}} + g^{a\bar{b}}_\varphi g^{p\bar{q}}_\varphi g^{j\bar{k}}(\partial_{z_j}(g_\varphi)_{p\bar{b}})(\partial_{\bar{z}_k}(g_\varphi)_{a\bar{q}}) \\
&\ge -Bg^{p\bar{p}}_\varphi(g_\varphi)_{j\bar{j}} - g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}} + g^{a\bar{b}}_\varphi g^{p\bar{q}}_\varphi g^{j\bar{k}}(\partial_{z_j}(g_\varphi)_{p\bar{b}})(\partial_{\bar{z}_k}(g_\varphi)_{a\bar{q}}) \\
&= -B\mathrm{tr}_g g_\varphi\mathrm{tr}_{g_\varphi}g - g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}} + g^{a\bar{b}}_\varphi g^{p\bar{q}}_\varphi g^{j\bar{k}}(\partial_{z_j}(g_\varphi)_{p\bar{b}})(\partial_{\bar{z}_k}(g_\varphi)_{a\bar{q}}) \\
&= -B\mathrm{tr}_g g_\varphi\mathrm{tr}_{g_\varphi}g - g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}} + \sum_j g^{a\bar{a}}_\varphi g^{p\bar{p}}_\varphi|\partial_{z_j}(g_\varphi)_{p\bar{a}}|_g
\end{aligned}$$

where $-B = -B(X^n, g)$ is a lower bound for the holomorphic bisectional curvature of g at p_0 (which is finite since X is compact). We now compute, at p_0:

$$\Delta_{g_\varphi} \log \mathrm{tr}_g g_\varphi(p_0) = \frac{\Delta_{g_\varphi} \mathrm{tr}_g g_\varphi}{\mathrm{tr}_g g_\varphi} - \frac{g_\varphi^{p\bar{q}} \partial_p(\mathrm{tr}_g g_\varphi) \partial_{\bar{q}}(\mathrm{tr}_g g_\varphi)}{(\mathrm{tr}_g g_\varphi)^2}$$

$$\geq -B \mathrm{tr}_{g_\varphi} g - \frac{g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}}}{\mathrm{tr}_g g_\varphi} + \frac{1}{\mathrm{tr}_g g_\varphi} \sum_j g_\varphi^{a\bar{a}} g_\varphi^{p\bar{p}} |\partial_{z_j}(g_\varphi)_{p\bar{a}}|_g$$

$$- \frac{1}{(\mathrm{tr}_g g_\varphi)^2} g_\varphi^{p\bar{p}} \partial_p(g_\varphi)_{a\bar{a}} \partial_{\bar{p}}(g_\varphi)_{b\bar{b}}.$$

We estimate the last term by the Cauchy-Schwarz inequality twice:

$$g_\varphi^{p\bar{p}} \partial_p(g_\varphi)_{a\bar{a}} \partial_{\bar{p}}(g_\varphi)_{b\bar{b}} = (g_\varphi^{p\bar{p}})^{\frac{1}{2}} (\partial_p(g_\varphi)_{a\bar{a}})(g_\varphi^{p\bar{p}})^{\frac{1}{2}} (\partial_p(g_\varphi)_{b\bar{b}})$$

$$\leq \sum_{a,b} (\sum_p g_\varphi^{p\bar{p}} |\partial_p(g_\varphi)_{a\bar{a}}|^2)^{\frac{1}{2}} (\sum_p g_\varphi^{p\bar{p}} |\partial_q(g_\varphi)_{b\bar{b}}|^2)^{\frac{1}{2}}$$

$$= \sum_a \sum_p g_\varphi^{p\bar{p}} |\partial_p(g_\varphi)_{a\bar{a}}|^2$$

$$= (\sum_a ((g_\varphi)_{a\bar{a}})^{\frac{1}{2}} (\sum_p g_\varphi^{p\bar{p}} g_\varphi^{a\bar{a}} |\partial_p(g_\varphi)_{a\bar{a}}|^2)^{\frac{1}{2}})^2$$

$$\leq (\sum_a (g_\varphi)_{a\bar{a}})(\sum_b \sum_p g_\varphi^{p\bar{p}} g_\varphi^{b\bar{b}} |\partial_p(g_\varphi)_{b\bar{b}}|^2).$$

Thus we get

$$\Delta_{g_\varphi} \log \mathrm{tr}_g g_\varphi(p_0) \geq -B \mathrm{tr}_{g_\varphi} g - \frac{g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}}}{\mathrm{tr}_g g_\varphi} + \frac{1}{\mathrm{tr}_g g_\varphi} \sum_j g_\varphi^{a\bar{a}} g_\varphi^{p\bar{p}} |\partial_{z_j}(g_\varphi)_{p\bar{a}}|_g$$

$$- \frac{1}{(\mathrm{tr}_g g_\varphi)^2} g_\varphi^{p\bar{p}} \partial_p(g_\varphi)_{a\bar{a}} \partial_{\bar{p}}(g_\varphi)_{b\bar{b}}$$

$$\geq -B \mathrm{tr}_{g_\varphi} g - \frac{g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}}}{\mathrm{tr}_g g_\varphi} + \frac{1}{\mathrm{tr}_g g_\varphi} \sum_j g_\varphi^{a\bar{a}} g_\varphi^{p\bar{p}} |\partial_{z_j}(g_\varphi)_{p\bar{a}}|_g$$

$$- \frac{1}{\mathrm{tr}_g g_\varphi} g_\varphi^{p\bar{p}} g_\varphi^{a\bar{a}} |\partial_p(g_\varphi)_{a\bar{a}}|^2$$

$$\geq -B \mathrm{tr}_{g_\varphi} g - \frac{g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}}}{\mathrm{tr}_g g_\varphi} + \frac{1}{\mathrm{tr}_g g_\varphi} \sum_j g_\varphi^{a\bar{a}} g_\varphi^{p\bar{p}} |\partial_{z_j}(g_\varphi)_{p\bar{a}}|_g$$

$$- \frac{1}{\mathrm{tr}_g g_\varphi} \sum_j g_\varphi^{p\bar{p}} g_\varphi^{a\bar{a}} |\partial_p(g_\varphi)_{j\bar{a}}|^2$$

$$\geq -B \mathrm{tr}_{g_\varphi} g - \frac{g^{j\bar{k}}(\mathrm{Ric}(\omega_\varphi))_{j\bar{k}}}{\mathrm{tr}_g g_\varphi},$$

where we used the Kähler condition. This proves (4). We prove now the uniform equivalence between the metrics g and g_φ, that will yield to the Laplacian and to partial $\mathcal{C}^2$-estimates. More precisely, we prove that there exists a constant $C = C(X^n, \omega, \|f\|_{L^\infty}, \inf \Delta_\omega f, \|\varphi\|_{L^\infty}) > 0$ such that

$$C^{-1} g_{j\bar{k}} \leq (g_\varphi)_{j\bar{k}} \leq C g_{j\bar{k}},$$

by using the maximum principle. We plug-in $(\star_t)$ in the estimate (4) to get:

$$\Delta_{g_\varphi} \log \mathrm{tr}_g g_\varphi \geq -B \mathrm{tr}_{g_\varphi} g + \frac{\Delta f + \mathrm{tr}_g g_\varphi - n - \mathrm{Sc}_\omega}{\mathrm{tr}_g g_\varphi}.$$

By the Cauchy-Schwarz inequality, we have

$$\mathrm{tr}_g g_\varphi \cdot \mathrm{tr}_{g_\varphi} g \geq n^2.$$

Then, under the normalization for f, we have a constant C such that

$$\Delta_\varphi \log \mathrm{tr}_g g_\varphi \geq -C\mathrm{tr}_{g_\varphi} g.$$

Since $\Delta_{g_\varphi} \varphi = n - \mathrm{tr}_{g_\varphi} g$, then for $A \gg 1$:

$$\Delta_{g_\varphi}(\log \mathrm{tr}_g g_\varphi - A\varphi) \geq \mathrm{tr}_{g_\varphi} g - nA.$$

By the maximum principle, at a maximum point $p_{\max}$, we get

$$\mathrm{tr}_{g_\varphi} g(p_{\max}) \leq A.$$

On the other side, by the equation, we get, in suitable coordinates at $p_{\max}$, that $\prod g_{\varphi, i\bar{i}}(p_{\max}) = \exp(f(p_{\max}) + \varphi(p_{\max})) \leq C$ for some constant C, then

$$\mathrm{tr}_g g_\varphi(p_{\max}) \leq C.$$

Then, at any point,

$$\log \mathrm{tr}_g g_\varphi - A\varphi \leq C - A\varphi(p_{\max}),$$

whence

$$\sup \log \mathrm{tr}_g g_\varphi \leq C.$$

We now shortly discuss *higher-order estimates*. For the $\mathcal{C}^3$ estimates, after Phong-Sesum-Sturm [PSS07], it is convenient to look at estimates of the Christoffel symbols: this will suffice because g_φ and g are equivalent. Set

$$S(\varphi)^i_{jk} := \Gamma(g)^i_{jk} - \Gamma(g_\varphi)^i_{jk}.$$

A long computation as before yields,

$$\|S\|_{L^\infty} \leq C.$$

By the above $\mathcal{C}^0$-estimate, we have that g_φ is equivalent to g, then we get $|\partial^3_{z_j \bar{z}_k} \varphi| \leq C$. This gives $\partial_{z_j \bar{z}_k} \varphi \in \mathcal{C}^{0,\alpha}_\omega(X)$, for any j, k. By Schauder as before, we get $\partial_{z_j} \varphi \in \mathcal{C}^{2,\alpha}_\omega(X)$, whence finally $\partial^3 \varphi \in \mathcal{C}^{0,\alpha}_\omega(X)$.

2.5 Existence of solutions for first Calabi's conjecture

We consider now the first Calabi conjecture. Essentially, the only difference is the $\mathcal{C}^0$-estimate: the Laplacian and higher-order estimates, as well as the general argument, can be repeated with very small variations, see [Tia00].

Under the suitable normalization $\sup_X \varphi = -1$, we prove the Moser iteration:

$$\|\varphi\|_{L^\infty} \leq C(\|\varphi\|_{L^2_\omega} + 1).$$

By Poincaré inequality, we prove:

$$\|\varphi\|_{L^2_\omega} \leq C(\|\varphi\|_{L^1_\omega} + 1).$$

Finally, by Green function argument, we prove:

$$\|\varphi\|_{L^1_\omega} \leq C.$$

The essential steps in the Moser iteration are the following. Set $\varphi_- := -\varphi \geq 1$. Consider the equation

$$(\exp f - 1)\omega^n = \omega_\varphi^n - \omega^n$$
$$= -\sqrt{-1}\partial\bar{\partial}\varphi_- \wedge (\omega_\varphi^{n-1} + \omega_\varphi^{n-2} \wedge \omega + \cdots + \omega_\varphi \wedge \omega^{n-2} + \omega^{n-1});$$

now muplitply by φ_-^p and integrate as before:

$$\fint_X \varphi_-^p(\exp f - 1)\omega^n \geq p \cdot \fint_X \varphi_-^{p-1}\sqrt{-1}\partial\varphi_- \wedge \bar{\partial}\varphi_- \wedge \omega^{n-1}$$

$$= \frac{4p}{n(p+1)^2} \fint_X (|\nabla(\varphi_-^{\frac{p+1}{2}})|^2 \omega^n,$$

(where, hereafter, we use the notation $\fint_X f\omega^n = \frac{1}{\mathrm{Vol}_\omega} \int_X f\omega^n$,) and on the other side clearly

$$Cf \fint_X (\varphi_-)^{p+1} \omega^n \geq \fint_X \varphi_-^p (\exp f - 1)\omega^n.$$

We use the Sobolev embedding, see *e.g.* [Aub98, Theorem 2.21], [GT98, Theorem 7.10]:

$$W_\omega^{1,q} \hookrightarrow L_\omega^{\frac{q\cdot 2n}{2n-q}}.$$

Then, for $q = 2$:

$$\fint_X (\varphi_-)^{p+1} \omega^n \geq C_1 \frac{p}{(p+1)^2} \left(\fint_X |(\varphi_-)^{\frac{p+1}{2}}|^{\frac{2n}{n-1}} \right)^{\frac{n-1}{2n}} - C_2 \fint_X (|\varphi|^{\frac{p+1}{2}})^2,$$

which yields

$$\|\varphi_-\|_{L_\omega^{(p+1)\frac{n}{n-1}}} \leq (C(p+1))^{\frac{1}{p+1}} \|\varphi_-\|_{L_\omega^{p+1}}.$$

We iterate (observing that the constant in the above estimate stay under control) to get:

$$\|\varphi\|_{L^\infty} \leq \tilde{C} \|\varphi\|_{L_\omega^2}.$$

We now prove the reduction from L^2-control to L^1-control thanks to the Poincaré inequality, see *e.g.* [Aub98, Corollary 4.3], [GT98, page 164]:

$$C \fint_X |\varphi|\omega^n \geq \fint_X \varphi(1 - \exp f)\omega^n$$

$$\geq \frac{1}{n} \fint_X |\nabla\varphi|^2 \omega^n$$

$$\geq \frac{\lambda_j(\omega)}{n} \left(\fint_X |\varphi|^2\omega^n - (\fint_X \varphi\omega^n)^2 \right),$$

using the normalization: $\varphi \leq -1$.

We finally prove the L^2-control. By consider the Green function G_ω, see *e.g.* [Aub98, Theorem 4.13]: let $p_{\max}$ a maximum point, then

$$-1 = \varphi(p_{\max}) = -\fint_X |\varphi|\omega^n - \fint_X \Delta_\omega \varphi G_\omega(p, \_)\omega^n.$$

Then, using that $\Delta_\omega \varphi > -n$, we get:

$$-1 + \int_X |\varphi| \leq n \int_X G_\omega(p, \_)\omega^n \leq C.$$

This completes the $\mathcal{C}^0$ estimate for the first Calabi conjecture.

2.6 Consequences of Calabi's conjectures

We give here some applications of the Calabi conjecture.

Theorem 2.11 (Kobayashi [Kob74], Yau [Yau77]). *A Fano manifold is simply-connected.*

Proof. Consider $\pi : \tilde{X} \to X$ the universal cover. By assumption, X is Fano, that is, there exists a Kähler form $\omega \in 2\pi c_1(X) > 0$. By the first Calabi conjecture, Theorem 1, there exists a Kähler form ω' on X such that $\mathrm{Ric}(\omega') = \omega$. Consider the Kähler metric $\pi^*\omega'$ on $\tilde{X}$. It has $\mathrm{Ric}(\pi^*\omega') = \pi^*\omega > 0$ bounded below away from zero. Then, by the Myers theorem in Riemannian geometry, see *e.g.* [Bes87, Theorem 6.51], $\tilde{X}$ is compact, and so it is Fano too. By using that X is Fano, that is the anti-canonical bundle is ample whence positive, and thanks to the Kodaira

vanishing theorem, see *e.g.* [Voi07, Theorem 7.13], we have $h^{0,j}(X) = \dim H^j(X; \mathcal{O}_X) = \dim H^j(X; K_X \otimes K_X^{-1}) = 0$ for $j > 0$; it follows that the Euler characteristic for the structure sheaf $\mathcal{O}_X$ is $\chi(\mathcal{O}_X) := \sum_j (-1)^j h^{0,j}(X) = 1$. For the same reasons, also $\chi(\tilde{X}) = 1$. On the other side, the Euler characteristic is additive on a cover, for example, as a consequence of the Hirzebruch-Riemann-Roch theorem. Then $X = \tilde{X}$. □

As innocent as it seems, the above statement is kind of deep:

Remark 2.12.
- *The Calabi's conjecture is crucially used to prove that the* universal cover is compact. *All the other arguments would work the same for arbitrary finite covers, giving the weaker statement that X has trivial pro-finite completion of the fundamental group.*
- *There exists a fully algebraic prove that makes use of rationally connectivity of Fanos (hence also very deep results in algebraic geometry based on Mori's Bend-and-Break, which uses reduction in finite characteristic p (e.g. [Mor79, KM98, Deb01]).*

The next consequence is a very famous inequality between Chern numbers.

Theorem 2.13 (Bogomolov-Miyaoka-Yau inequality [VdV66, Bog78, Miy77, Yau77, Yau78]). *Let X^n be a compact Kähler manifold.*
- *If X^n is Fano Kähler-Einstein, then*

$$n c_1^n(X) \leq 2(n+1) c_1^{n-2}(X) c_2(X),$$

with equality if and only if $X = \mathbb{CP}^n$.
- *If X^n is with ample canonical bundle, and then Kähler-Einstein by Theorem 3, then*

$$n(-1)^{n-2} c_1^n(X) \leq 2(n+1)(-1)^{n-2} c_1^{n-2}(X) c_2(X),$$

with equality if and only if $X^n = \mathbb{B}^n / \Gamma$.

Proof. For more details see *e.g.* [Tos17]. First of all, we fix ω Kähler-Einstein, with cosmological constant λ, and we notice that we can rewrite the two inequalities that we want to prove as

$$\left(\frac{2(n+1)}{n} c_2 - c_1^2 \right) [\omega]^{n-2} \geq 0,$$

and we also want to prove that equality holds if and only if X has constant holomorphic sectional curvature. Indeed, recall that simply-connected Kähler manifolds with constant holomorphic sectional curvature are holomorphically isometric to either $\mathbb{C}^n$, or $\mathbb{B}^n$, or $\mathbb{CP}^n$, up to homothety, see *e.g.* [KN2, Theorems IX.7.8, IX.7.9] (the proof is very similar to the corresponding result in the real case). Recall that the *holomorphic sectional curvature* is $H(X) := K(X \wedge JX) := R(X, JX, JX, X)$ for $|X| = 1$. Then H is constant if and only if the traceless part R^0 of R vanishes, where:

$$R_{ijk\bar{\ell}}^0 = R_{ijk\bar{\ell}} - \frac{\lambda}{n+1}(g_{ij}g_{k\bar{\ell}} + g_{i\bar{\ell}}g_{kj}).$$

Using the KE condition, we immediately see that

$$|R^0|^2 = |R|^2 - \frac{2\lambda^2 n}{n+1} \tag{5}$$

Let $\Omega_i^j = \sqrt{-1} \sum R_i{}^j{}_{k\bar{l}} dz_k \wedge d\bar{z}_l$ be the curvature seen as a form valued endomorphism of the tangent bundle. By Chern-Weil theory we can represent the first Pontryagin class (if you want, this a possible definition of such class) as:

$$(2\pi^2)^{-1} \text{tr}(\Omega \wedge \Omega) := (2\pi^2)^{-1} \sum_{i,k} \Omega_i^k \wedge \Omega_k^i \in p_1(X) = c_1^2(X) - 2c_2(X).$$

Then, using the KE condition again, we compute:

$$n(n-1) \text{tr}(\Omega \wedge \Omega) \wedge \omega^{n-2} = (\lambda^2 n - |R|^2) \omega^n \tag{6}$$

Finally, integrating (5) and (6) on X,

$$\frac{1}{n(n-1)4\pi^2} \int_X |R^0|^2 \omega^n = \left(2c_2(X) - \frac{n}{n+1}c_1^2(X)\right)[\omega]^{n-2},$$

whence we get the statement.   □

Remark 2.14. *We would like to make some observations on the Bogomolov-Miyaoka-Yau (BMY) inequality.*
– *For compact complex surfaces, the BMY inequalities reduce to*

$$c_1^2 \le 3c_2$$

where $c_2 = e(X)$ is the topological Euler characteristic and $c_1^2 = 2e(X) + 3\tau(X)$ with $\tau(X)$ the signature of the intersection form on the second cohomology. Hence it is purely topological. Surfaces which realize the equality are hard to construct (e.g. Mumford Fake projective spaces [Mum79]).
– *For Fano KE threefolds, the BMY inequality reduces to*

$$c_1^3(X) \le 64 = c_1^3(\mathbb{CP}^3)$$

thanks to the fact that $c_1(X)c_2(X) = 24 \int_X \mathrm{td}(X) = \chi(\mathcal{O}_X) = 1$. In dimension $n \ge 4$, it is proven that for KE Fano manifolds [Fuj15]

$$c_1^n(X) \le (n+1)^n = c_1^n(\mathbb{CP}^n)$$

*continues to hold (*not *as a consequence of BMY, but using K- stability considerations). We should remark that there are Fano manifolds whose volume is bigger than $c_1^n(\mathbb{CP}^n)$: e.g. certain $\mathbb{CP}^1$-bundles over $\mathbb{CP}^3$.*

3 First obstructions to existence of Kähler-Einstein metrics

In this section, we discuss two obstructions to the existence of Kähler-Einstein metrics:

Matsushima the automorphisms group of Kähler-Einstein Fano manifolds is reductive (namely, it is the complexification of a maximal compact subgroup);

Futaki the functional F defined on the space of holomorphic vector fields as in (8) vanishes when there exists a Kähler-Einstein metric.

More in general, these obstructions hold for *constant scalar curvature Kähler metric* (cscK). Note that, if $\omega \in 2\pi c_1(X)$ is a cscK metric on X^n Fano, then ω is in fact KE. Indeed, by the $\sqrt{-1}\partial\bar{\partial}$-Lemma, we have that $Ric(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}f$. Taking trace with the metric, we find that $0 = Sc(\omega) - n = \Delta_\omega f$, *i.e.* f has to be constant.

We are going to prove the statements in the more general cscK case, since they require only a bit more effort than in the KE case.

3.1 Matsushima-(Lichnerowicz) theorem

On a (X^n, g) Hermitian manifold, we consider the Lie algebra $\mathfrak{h}$ of *holomorphic vector fields* (that is, real vector fields V such that $\mathcal{L}_V J = 0$, equivalently, $\bar{\partial} V^{(1,0)} = 0$).

We also consider the Lie algebra $\mathfrak{k}_g$ of the isometry group $\mathrm{Isom}(X, g)$ (that is, $\mathfrak{k}_g$ contains vector fields V such that $\mathcal{L}_V g = 0$, called *Killing vector fields*). If (X^n, g) is compact Kähler, then $\mathfrak{k}_g \subseteq \mathfrak{h}$, see *e.g.* [Gau15, Proposition 2.2.1]. Indeed, notice that the associated $(1,1)$-form ω to g is harmonic, *e.g.* since $d \star \omega = \frac{1}{(n-1)!}d\omega^{n-1} = 0$. Take any $\psi \in \mathrm{Isom}_0(X, g)$, the connected component of the identity. Then $\psi^\star[\omega] = [\omega]$. By uniqueness of the harmonic representative, $\psi^\star \omega = \omega$. Thus, we have proven that, for any $V \in \mathfrak{k}_g$, it holds $\mathcal{L}_V \omega = 0$. This, together with $\mathcal{L}_V g = 0$, gives $\mathcal{L}_V J = 0$.

We also consider the Lie algebra $\mathfrak{a}_g$ of *parallel vector fields* (that is, $\mathfrak{a}_g$ contains vector fields V such that $\nabla V = 0$, where ∇ is the Levi-Civita connection of g). Clearly, $\mathfrak{a}_g \subseteq \mathfrak{k}_g$ and $\mathfrak{a}_g$ is Abelian, since ∇ is torsion-free. If g is Kähler, then $\mathfrak{a}_g$ is complex, that is, if $V \in \mathfrak{a}_g$ then also $JV \in \mathfrak{a}_g$ (just using that J is parallel).

Finally, we consider the space $\mathfrak{h}_0$ of *holomorphic vector fields with zeroes*, consisting of $V \in \mathfrak{h}$ such that $V(p) = 0$ at some point p, and $\mathfrak{k}_0$ the space of Killing vector fields with zeros.

Before stating the next theorem, we recall that the *Albanese map* [Bla56], also called Jacobi map, associated to a compact Kähler manifold X^n is

$$A \colon X \to \mathrm{Alb}(X) := H^0(X; \Omega_X^1)^\vee / H_1(X; \mathbb{Z}) \simeq \mathbb{C}^{h^{1,0}} / \Gamma,$$

$$p \mapsto \int_{p_0}^p {\_},$$

where p_0 is fixed. It is well defined since any holomorphic form is closed on a compact Kähler manifold. Any morphism from X to a compact complex torus factors uniquely through the Albanese map. It yields an isomorphism

$$A^\star \colon H^0(\mathrm{Alb}(X); \Omega^1_{\mathrm{Alb}(X)}) \xrightarrow{\widetilde{}} H^0(X; \Omega^1_X). \tag{7}$$

We also consider the operator that acts on $f \in \mathcal{C}^\infty(X; \mathbb{C})$ as

$$\partial^\sharp f := (\mathrm{grad}_g f)^{1,0} \overset{\mathrm{loc}}{=} g^{i\bar{j}} f_{,\bar{j}} \partial_{z_i}.$$

Theorem 3.1 (Matsushima [Mat57], Carrell-Lieberman [CL73], LeBrun-Simanca [LBS94]). *Let (X^n, ω) be a compact Kähler manifold. Then the following properties are equivalent for a holomorphic vector field $Z \in \mathfrak{h}$:*

(i) $Z \in \mathfrak{h}_0$;

(ii) Z is tangent to the fibres of the Albanese map;

(iii) $Z = \partial^\sharp f$ for some $f \in \mathcal{C}^\infty(X; \mathbb{C})$.

Proof. We prove that *(i) implies (ii)*. We want to prove that Z is tangent to the fibres of the Albanese map, equivalently, $\mu(dA(Z)) = 0$ for any $\mu \in H^0(\mathrm{Alb}(X), \Omega^1_{\mathrm{Alb}(X)})$. By the isomorphism (7), this is equivalent to show that $\eta(Z) = 0$ for any $\eta \in H^0(X, \Omega^1_X)$, where $\eta = A^\star \mu$. We conclude by noticing that $\eta(Z)$ is a holomorphic function on X compact, then it is constant, in fact, $\eta(Z) = 0$ because Z has non-empty zero-set.

We prove that *(ii) implies (iii)*. Define the following $(0, 1)$-form:

$$\varphi := \sqrt{-1} \iota_Z \omega \overset{\mathrm{loc}}{=} \sqrt{-1} Z^k g_{k\bar{j}} d\bar{z}^j.$$

By the Cartan formula, we have $\mathcal{L}_{\mathrm{Re}\, Z} \omega = -\sqrt{-1} d\varphi$, where the right-hand side is a $(1,1)$-form since Z is holomorphic: then $\overline{\partial} \varphi = 0$. We now use Hodge theory to construct the potential f. Take any $\alpha \in \mathcal{H}^{0,1}_\omega$, the space of harmonic $(0,1)$-form; set $\beta := \bar{\alpha}$. Then $(\varphi, \alpha)_\omega = g^{k\bar{j}} \varphi_k \overline{\alpha_j} = \sqrt{-1} Z^j \beta_j = \sqrt{-1} \beta(Z) = 0$ by the assumption that Z is tangent to the fibres of the Albanese map. We have proven that $\varphi \perp \mathcal{H}^{0,1}_\omega$. By Hodge theory, we have $\varphi = \overline{\partial} f + \overline{\partial}^\star h$. But $\|\overline{\partial}^\star h\|^2_{L^2} = (\overline{\partial}^\star h, \overline{\partial}^\star h)_{L^2} = (h, \overline{\partial}(\varphi - \overline{\partial} f))_{L^2} = 0$. So $\varphi = \overline{\partial} f$, and $Z = \partial^\sharp f$.

We prove that *(iii) implies (i)*. Take $Z = \partial^\sharp f$ holomorphic. (Note that f is complex-valued, and not just real-valued, so we cannot apply the maximum principle directly.) We have that $Z(\bar{f}) = g^{i\bar{j}} f_{,\bar{j}} \overline{f_{,i}} = |Z|^2 \geq 0$. Set $c = \min |Z|^2 > 0$ and $C = \max |Z|^2$. We have to prove that $c = 0$. Fix $p \in X$. Construct $F \colon \mathbb{C} \to X^n$ with $F(0) = p$ and $dF(\frac{d}{dz}) = Z$. Define $h := f \circ F \colon \mathbb{C} \to \mathbb{C}$. We have $0 \leq |h| \leq C$ and $\frac{d\bar{h}}{dz_j} = Z(\bar{f}) \geq c$. Apply the Stokes theorem to a disk $\triangle_r$ of radius $r > 0$. We have $\frac{1}{2\pi} \int_0^{2\pi} h(r \exp(\sqrt{-1}\vartheta)) \exp(\sqrt{-1}\vartheta) d\vartheta = \frac{1}{2\pi r} \int_{\partial \triangle_r} h\, dz = \frac{2\sqrt{-1}}{8\pi r} \int \frac{d\bar{h}}{dz} dx dy \geq cr$. Letting $r \to +\infty$ we see that $c = 0$, since the first integral is clearly bounded. $\qquad\square$

Remark 3.2.

– When X^n is Fano, then $\mathfrak{h} = \mathfrak{h}_0$ (this follows either by $\mathrm{Alb}(X)$ being trivial by the vanishing result, or because X is simply-connected).

– $\mathfrak{h}_0$ is an ideal in $\mathfrak{h}$ when the manifold is Kähler, but there are counterexamples in the non-Kähler case where it is not a linear subspace (consider for example the Hopf surface $\mathbb{C}^2 \setminus 0 / \langle z \mapsto 2z \rangle$, with local coordinates $(z, w) \in \mathbb{C}^2$, then both $v = z\partial_z \in \mathfrak{h}_0$ and $t = w\partial_w \in \mathfrak{h}_0$, but $v + t \notin \mathfrak{h}_0$).

Clearly $Z = \partial^\sharp f$ is holomorphic if and only if $f \in \ker(\overline{\partial} \partial^\sharp)^\star \overline{\partial} \partial^\sharp$. Such fourth order operator is called *Lichnerowicz operator* and by a (slightly tedious) computation can be show to be equal to

$$\mathbb{L}_\omega(\varphi) := \Delta^2_\omega \varphi + (\mathrm{Ric}(\omega), \partial \overline{\partial} \varphi)_\omega + \partial^\sharp \varphi(\mathrm{Sc}_\omega),$$

see *e.g.* [Szé14, page 59], [Gau15, page 63, Proposition 2.6.1].

In general, $\mathbb{L}_\omega$ is not a real operator, that is, $\overline{\mathbb{L}_\omega(\varphi)} \neq \mathbb{L}_\omega(\bar{\varphi})$. We will define the complex conjugate

$$\tilde{\mathbb{L}}_\omega(\varphi) := \overline{\mathbb{L}_\omega(\bar{\varphi})}.$$

We have $\tilde{\mathbb{L}}_\omega(\varphi) - \mathbb{L}_\omega(\varphi) = \partial^\sharp \varphi(\mathrm{Sc}_\omega) - \bar{\partial}^\sharp \bar{\varphi}(\mathrm{Sc}_\omega)$.

Proposition 3.3. *Let (X^n, ω) be a compact Kähler manifold. Then the space of Killing vector fields with zeroes is given by*

$$\mathfrak{k}_0 = \left\{ J\mathrm{grad}_g f \; : \; f \in \mathbb{C}^\infty(X; \mathbb{R}) \text{ such that } \partial^\sharp f \in \mathfrak{h}_0 \right\}.$$

Proof. The inclusion "$\supseteq$" follows since $J\mathrm{grad}_g f$ is a real holomorphic vector field being also Hamiltonian, whence it is Killing. The inclusion "$\subseteq$" is as follows. Take $X \in \mathfrak{k}_0$. Then $JX \in \mathfrak{h}_0$. Consider $Z = JX + \sqrt{-1}X$ with $\bar{\partial}Z = 0$. We know that $Z = \partial^\sharp f$ where $f \in \mathbb{C}^\infty(X; \mathbb{C})$, say $f = u + \sqrt{-1}v$. Then $\mathrm{Re}\,\partial^\sharp f = \mathrm{grad}_g u + J\mathrm{grad}_g v$. Then $X = -\mathrm{grad}_g v + J\mathrm{grad}_g u$. We claim that $\mathrm{grad}_g v = 0$. By Cartan, $0 = \mathcal{L}_X\omega = d\iota_{-\mathrm{grad}_g v + J\mathrm{grad}_g u}(\omega) = dg(-J\mathrm{grad}_g v, \_) - d(g(\mathrm{grad}_g u, \_)) = dJdv = 2\sqrt{-1}\partial\bar{\partial}v$. Since X is compact, then v is constant. $\qquad\square$

Theorem 3.4 (Matsushima [Mat57], Lichnerowicz [Lic58])**.** *Let ω be a cscK metric on X^n compact complex manifold. Then*

$$\mathfrak{h}_0 = \mathfrak{k}_0 \oplus J\mathfrak{k}_0.$$

Proof. Clearly, $\mathfrak{k}_0 \oplus J\mathfrak{k}_0 \subseteq \mathfrak{h}_0$. Take $Z = \partial^\sharp f$ with $\mathbb{L}_\omega(f) = 0$. Since ω is cscK, then $\mathbb{L} = \tilde{\mathbb{L}}_\omega$ is a real operator. Then $\mathbb{L}_\omega(\mathrm{Re}\,f) = \mathbb{L}_\omega(\mathrm{Im}\,f) = 0$. Then we can conclude by the previous proposition. (See also *e.g.* [Gau15, Theorems 3.6.1-2], [Szé14, Proposition 4.18]). $\qquad\square$

Remark 3.5. *By Bochner [Boc46], if V is Killing, then $\frac{1}{2}\Delta_g|V|^2 = -\mathrm{Ric}(g)(V, V) + |\nabla V|^2$. Thus:*
– *if $\mathrm{Ric}(\omega) = 0$, then $\mathfrak{h} = \mathfrak{a}$;*
– *if $\mathrm{Ric}(\omega) = -\omega$, then $\mathfrak{h} = 0$.*

Remark 3.6. *Recall that $\mathrm{Isom}(M, g)$ is always a compact Lie group [MS39]. In particular, $\mathfrak{h}_0$ is reductive.*

Example 3.7. *Consider $\hat{X}^2 = \mathrm{Bl}_p\mathbb{CP}^2$, that is, locally on $U \ni p$, it is $\hat{X} \overset{loc}{=} \{((z, w), [u : v]) \in U \times \mathbb{CP}^1 \; : \; zv - wu = 0\}$. We have $K_{\hat{X}} = \pi^\star K_{\mathbb{CP}^2} \otimes \mathcal{O}(E)$, where $E \simeq \mathbb{CP}^1$ is the exceptional divisor. $K_{\hat{X}}^{-1}$ is still ample. Then*

$$\mathrm{Aut}(\hat{X}) = \left\{ \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix} \right\} \Big/ \mathbb{C}^\star$$

and

$$\mathfrak{h}(\hat{X}) = \left\{ \begin{pmatrix} 0 & \star & \star \\ 0 & \star & \star \\ 0 & \star & \star \end{pmatrix} \right\} \simeq \mathfrak{gl}(2; \mathbb{C}) \oplus \mathbb{C}^2$$

which cannot be given by a complexification of a compact Lie group. Then, by the Matsushima-Lichnerowicz theorem, $\hat{X}$ is an example of a Fano manifold that does not admit cscK metric, in particular, does not admit Kähler-Einstein metrics.

Even if the blow-up in a point of $\mathbb{CP}^2$ is not KE, such space is related to two important (positive) Einstein metrics.
– The *Page metric* [Pag79] (see *e.g.* [Bes87, 9.125]) constructed as $g_P = \mathrm{Sc}_{\mathrm{extr}}^{-2}g_{\mathrm{extr}}$ where g_{extr} is the explicit Calabi's extremal metric (that is, with holomorphic grandient of the scalar curvature) in $2\pi c_1(X)$.
– The $Y^{2,1}$ irregular Sasaki-Einstein metric constructed on the smooth bundle $S^1 \hookrightarrow K \to \mathrm{Bl}_p\mathbb{CP}^2$ by deforming the natural $\mathbb{C}^\star$-action. This metric was first constructed by physicists studying the AdS-CFT correspondence [MS06].

3.2 Futaki invariant

Let (X^n, ω) be a compact Kähler metric. Set $\hat{S}$ the value of the eventual constant scalar curvature of some metric in $[\omega]$. Note that $\hat{S}$ is a cohomological invariant depending just on $c_1(X)$ and $[\omega]$, determined by

$$[\omega]^n \hat{S} = \int_X \mathrm{Sc}_\omega \omega^n = n \int_X Ric(\omega) \wedge \omega^{n-1} = 2\pi n c_1(X).[\omega]^{n-1}.$$

For $Z = \partial^\sharp f$, define

$$F(Z, \omega) := \int_X f(\mathrm{Sc}_\omega - \hat{S})\omega^n. \tag{8}$$

Theorem 3.8 (Futaki [Fut83]). *If ω and ω' are cohomologous, then $F(Z, \omega) = F(Z, \omega')$. Therefore $F(Z, \omega) = F(Z, [\omega])$. In particular, if there exists $\omega' \in [\omega]$ cscK, then $F = 0$.*

Proof. Note that the space of Kähler metrics in $[\omega]$ is connected. Take $(\omega_t)_t$ a path of metrics in $[\omega]$, for example of the form $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}(t\varphi)$. We show that $\dot{F}(Z, \omega_t) := \frac{d}{dt}\lfloor_{t=0} F(Z, \omega_t) = 0$. We compute:

- $\dot{f}_t = \partial^\sharp f(\varphi)$:
  Indeed, $\bar{\partial}\dot{f}_t = \bar{\partial}(Z^j\varphi_{,j})$ since Z is holomorphic;
- $\dot{\mathrm{Sc}}_{\omega_t} = -\tilde{\mathbb{L}}_\omega(\varphi) + \partial^\sharp \mathrm{Sc}_\omega(\varphi)$:
  indeed, $Ric(\omega_t) = Ric(\omega) - \sqrt{-1}t\partial\bar{\partial}\Delta\varphi + o(t)$, then $\mathrm{Sc}_{\omega_t} = g_t^{i\bar{j}}(Ric(\omega_t))_{i\bar{j}} = (g^{i\bar{j}} - tg^{i\bar{\ell}}g^{k\bar{j}}\varphi_{,k\bar{\ell}})(Ric_{i\bar{j}} - t(\Delta\varphi)_{,i\bar{j}}) + o(t) = \mathrm{Sc}_\omega - t\Delta^2\varphi - tg^{i\bar{\ell}}g^{k\bar{j}}\varphi_{,k\bar{\ell}}Ric_{i\bar{j}} + o(t) = \mathrm{Sc}_\omega - t\tilde{\mathbb{L}}_\omega + t\partial^\sharp \mathrm{Sc}_\omega(\varphi) + o(t)$;
- $\dot{\omega}_t^n = \Delta\varphi\,\omega^n$.

Thus

$$\dot{F}_t = \int_X (\partial^\sharp f(\varphi)(\mathrm{Sc}_\omega - \hat{S}) - f\tilde{\mathbb{L}}_\omega(\varphi) + f\partial^\sharp \mathrm{Sc}_\omega(\varphi) + f(\mathrm{Sc}(\omega) - \hat{S})\Delta\varphi)\omega^n$$
$$= -\int_X f\tilde{\mathbb{L}}_\omega(\varphi)\omega^n = -(f, \mathbb{L}_\omega(\varphi))_{L^2}$$
$$= -(\mathbb{L}_\omega(f), \varphi)_{L^2} = 0,$$

by integrating by part and by using that $\mathbb{L}_\omega(f) = 0$, since f is holomorphic. $\qquad\square$

The Futaki invariant will be central in the definition of the notion of K-stability, see Section 5.1.

Remark 3.9 (Other equivalent formulations of the Futaki invariant). *Let ψ a solution of $\mathrm{Sc}_\omega - \hat{S} = \Delta_\omega\psi$, so, for $Z = \partial^\sharp f$, we have*

$$F(Z, [\omega]) = \int_X f(\mathrm{Sc}(\omega) - \hat{S})\omega^n$$
$$= (f, \bar{\partial}^*\bar{\partial}\psi) = (\bar{\partial}f, \bar{\partial}\psi)$$
$$= \int_X \partial^\sharp f(\psi) = \int_X Z(\psi)\omega^n$$

so that we can write

$$F(Z, [\omega]) = \int_X Z(\psi)\omega^n.$$

3.3 Mabuchi functional and evidences for uniqueness

On the space $\mathcal{K}_{[\omega]} = \{\varphi : \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$ of Kähler metrics in $[\omega]$, consider the following 1-form:

$$\alpha_\varphi(\psi) := \int_X \psi(\hat{S} - \mathrm{Sc}_{\omega_\varphi})\omega_\varphi^n.$$

We compute

$$\frac{d}{dt}(\alpha_{\varphi+t\psi_2}(\psi_1)) = \int (\psi_1(\mathbb{L}_{\omega_\varphi}\psi_2 - \partial^\sharp\psi_2(\mathrm{Sc}_{\omega_\varphi})) + \psi_1(\hat{S} - \mathrm{Sc}_{\omega_\varphi})\Delta_{\omega_\varphi}\psi_2)\omega_\varphi^n$$

$$= \int_X (\psi_1\mathbb{L}_{\omega_\varphi}\psi_2 - (\hat{S} - \mathrm{Sc}_{\omega_\varphi})\partial^\sharp\psi_1(\psi_2))$$

which is symmetric in ψ_1 and ψ_2, since they are both real. Then

$$d\alpha = 0.$$

Then there exists $\mathcal{M}\colon \mathcal{K}_{[\omega]} \to \mathbb{R}$ such that

$$\alpha = d\mathcal{M}.$$

Such $\mathcal{M}$ is called *Mabuchi K-energy*. (See also *e.g.* [Gau15, Section 4.10], [Szé14, Proposition 4.23].)

The *Mabuchi metric* [Mab87, Sem92, Don99] on $\mathcal{K}_{[\omega]}$ is defined as

$$\langle \psi_1, \psi_2 \rangle_\varphi := \int_X \psi_1\psi_2\omega_\varphi^n.$$

We look at the geodesics for such metric: let $(\varphi_t)_t$ a path in $\mathcal{K}_{[\omega]}$ connecting $\varphi_0 = 0$ and $\varphi_1 = \varphi$, and consider the energy functional

$$E(\varphi) = \int_0^1 \int_X \dot{\varphi}_t^2 \omega_{\varphi_t}^n dt.$$

We look at geodesics, that is, critical point of E. We compute

$$\frac{d}{ds}\bigg|_{s=0} E(\varphi + s\psi) = \int_0^1 \int_X -2\psi_t(\ddot{\varphi}_t - \partial^\sharp\dot{\varphi}_t(\dot{\varphi}_t))\omega_{\varphi_t}^n$$

so the geodesic equation is

$$\ddot{\varphi}_t - |\partial\dot{\varphi}_t|^2_{\omega_{\varphi_t}} = 0.$$

Remark 3.10. *Consider now $\hat{X} = X \times [0, 1] \times \mathbb{S}^1$. Donaldson [Don99] and Semmes [Sem92] observed that, by extending*

$$\Phi(p, t, s) := \varphi_t(p)$$

then the geodesic equation can be given as

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^{n+1} = 0$$

on $\hat{X}$, from which one can use pluripotential theory to study geodesics [Che00].

Lemma 3.11. *The Mabuchi functional is convex along smooth geodesics.*

Proof. Let $(\varphi_t)_t$ be a geodesic. We compute

$$\frac{d^2}{dt^2}\mathcal{M}(\varphi_t) = \frac{d}{dt}\int_X \dot{\varphi}_t(\hat{S} - \mathrm{Sc}_{\omega_{\varphi_t}})\omega_{\varphi_t}^n$$

$$= \int_X (\ddot{\varphi}_t(\hat{S} - \mathrm{Sc}_{\omega_{\varphi_t}}) + \dot{\varphi}_t(\mathbb{L}_\omega(\varphi_t) - \partial^\sharp\varphi_t(\mathrm{Sc}_{\omega_{\varphi_t}})) + \dot{\varphi}_t(\hat{S} - \mathrm{Sc}_{\omega_{\varphi_t}})\Delta_{\omega_{\varphi_t}}\dot{\varphi}_t)\omega_{\omega_{\varphi_t}}^n$$

$$= \int_X (|\bar{\partial}\partial^\sharp\dot{\varphi}_t|^2_{\omega_{\varphi_t}} + (\hat{S} - \mathrm{Sc}_{\omega_{\varphi_t}})(\ddot{\varphi}_t - |\partial\dot{\varphi}_t|^2))\omega_{\omega_{\varphi_t}}^n$$

$$= \int |\bar{\partial}\partial^\sharp\dot{\varphi}_t|^2_{\omega_{\varphi_t}}\omega_{\omega_{\varphi_t}}^n \geq 0.$$

$\square$

Now, assume that any φ_1, φ_2 can be connected by smooth geodesics [Don99]. The previous result would then yield uniqueness of cscK metrics. Indeed, if there existed two cscK metrics, by convexity we get that $\bar{\partial}\partial^\sharp\varphi_t = 0$, and then there exists F automorphism such that $F^\star\omega_2 = \omega_1$ (*i.e.* the metric would be unique up to automorphisms). But the assumption on the regularity of geodesics is false: the optimal regularity is $C^{1,1}$ as proven by [DL12], which follows previous works [Don02a, LV13]. In any case, such approach turned out to be the correct one, at the price of working with weaker regularity [CTW17, BB17]. In the KE Fano case, the first proof of uniqueness (up to automorphisms) was given by Bando and Mabuchi with different techniques [BM87].

4 A criterion for the existence of Kähler-Einstein metrics on Fano manifolds

We try to construct a Kähler-Einstein metric in $2\pi c_1(X) > 0$, *i.e.*, $\mathrm{Ric}(\omega) = \omega$. This is equivalent to solve the Monge-Ampère (2) with $\lambda = 1$, namely, the equation

$$\omega_\varphi^n = \exp(f - \varphi)\omega^n \tag{9}$$

for $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$, where f is the Ricci potential normalized such that $\fint_X \exp f\,\omega^n = 1$.

We set up the continuity method along the *Aubin path*

$$\omega_{\varphi_t}^n = \exp(f - t\varphi)\omega^n \tag{10}$$

varying $t \in [0, 1]$. This equation is equivalent to asking

$$\mathrm{Ric}(\omega_{\varphi_t}) = (1 - t)\omega + t\omega_{\varphi_t} \geq t\omega_{\varphi_t} > 0,$$

Note that we (crucially!) get a lower bound on the Ricci curvature along the path.

Remark 4.1. *Define the* Székelyhidi invariant *[Tia92, Szé11] of (X^n, ω) as*

$$R(X) := \sup\{t \in [0, 1] \ : \ \mathrm{Ric}(\omega_{\varphi_t}) \geq t\omega_t\} \in (0, 1],$$

where $(\omega_t)_t$ is the Aubin path starting at the background metric ω. In fact, $R(X)$ does not depend on the background ω [Szé11, Theorem 1]. For example, $R(\mathrm{Bl}_p\mathbb{CP}^2) = \frac{6}{7}$. Explicit computations for toric manifolds can be found in [Li11]. It was conjectured [Szé11, Conjecture 13] that $R(X) = 1$ if and only if K-semi-stable. Such conjecture has been proven by C. Li [Li15b].

Solution for $t = 0$ the solvability of the Aubin's path follows by first Calabi's conjecture, Theorem 2.1. Openness is studies by means of the linearization

$$D\mathcal{E}(\_, t) = \Delta_{\omega_{\varphi_t}} + t,$$

with the same notation as at page 210. It is relative easy to show that the first eigenvalue of the Laplacians satisfies $-\lambda_1(t) \geq t$, with equality only for $t = 1$ see *e.g.* [Tia00, Remark 6.13]. Thus we have that the kernel is trivial for $t \in (0, 1)$. For $t = 1$, we have that the kernel is given by holomorphic vector fields with zeros.

4.1 Some *a priori* estimates

The Laplacian and higher order estimates can be repeated. We study now the $\mathcal{C}^0$-estimates, which we know they would fail in general, but we can hope to find some sufficient condition for them to hold.

On (X^n, ω) compact Kähler manifold, set $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi$, and consider the functional

$$I_\omega(\varphi) = \fint_X \varphi(\omega^n - \omega_\varphi^n).$$

We have $I_\omega(\varphi) \geq 0$. In fact, I_ω is equivalent to the *generalized energy* functional

$$J_\omega(\varphi) := \sqrt{-1} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \fint_X \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^k \wedge \omega_\varphi^{n-k} \geq 0.$$

More precisely, [Tian87, Lemma 2.2],

$$\frac{1}{n+1} I_\omega(\varphi) \leq J_\omega(\varphi) \leq \frac{n}{n+1} I_\omega(\varphi).$$

Theorem 4.2 (Tian [Tian87]). *Let (X^n, ω) be a compact Fano Kähler manifold. Along the Aubin path $\{\omega_{\varphi_t} := \omega + \sqrt{-1}\partial\overline{\partial}\varphi_t\}_{t \in [0,1]}$, we have the a priori estimates*

$$\|\varphi_t\|_{L^\infty} \leq C_t(1 + I_\omega(\varphi_t)),$$

with $C_t \leq C(\epsilon)$ for all $t > \epsilon$ for any chosen $\epsilon > 0$.

Proof. The equation for the Aubin path yields $\fint_X \exp(t\varphi)\omega_\varphi^n = \fint_X \exp f \omega^n = 1$, where recall that f is the Ricci potential of ω with the normalization above. It follows that $\inf \varphi \leq 0$ and $\sup \varphi \geq 0$. We claim that

$$\|\varphi\|_{L^\infty} \leq \operatorname{osc}\varphi := \sup \varphi - \inf \varphi \leq C_t(1 + I_\omega(\varphi_t)).$$

As for $\sup \varphi$, an estimate follows by using the Green function as for first Calabi's conjecture:

$$\sup \varphi \leq C \left(1 + \fint_X \varphi_t \omega^n \right).$$

Indeed, if G is the Green function such that $G \geq 0$, then write $\varphi(x) = \fint_X \varphi\omega^n + \fint -\Delta_\omega \varphi G(x, \_)\omega^n \leq C(1 + \fint_X \varphi\omega^n)$ since $-\Delta_\omega \varphi \leq n$.

We prove now an estimate for $\inf \varphi$. Take $1 \leq \tilde{\varphi}_t := \max\{1, -\varphi_t\}$. Note that $\tilde{\varphi}_t^n(n - \Delta_{\omega_{\varphi_t}} \varphi_t) \geq 0$, and then integrate on X with respect to the moving metric ω_{φ_t} to get

$$\fint_X \left| \nabla_{\omega_{\varphi_t}}(\tilde{\varphi}_t)^{\frac{p+1}{2}} \right| \omega_{\varphi_t}^n \leq C(p, n) \fint_X \tilde{\varphi}^{\frac{p+1}{2}} \omega_{\varphi_t}^n.$$

Use the Sobolev embedding with respect to $\omega_{\varphi_t}^n$. Since $\operatorname{Ric}(\omega_{\varphi_t}) \geq t$, then we can take a Sobolev constant that depends only on C and n. Then, by Moser's iteration, and using the recalled first eigenvalue estimate:

$$\sup \tilde{\varphi}_t \leq \|\tilde{\varphi}_t\|_{L^2} \leq \frac{C}{t}(\|\tilde{\varphi}_t\|_{L^1} + 1)$$

$$\leq \frac{C}{t}(1 - \fint_X \varphi_t \omega_{\varphi_t}^n),$$

where the second inequality follows by the Poincaré inequality. At the end, we get

$$\|\varphi_t\|_{L^\infty} \leq \sup \varphi_t - \inf \varphi_t \leq C(1 + \fint_X \varphi_t \omega^n) + \frac{C}{t}(1 - \fint_X \varphi_t \omega_{\varphi_t}^n),$$

proving the statement. See also *e.g.* [Tia00, Lemma 6.19]. $\qquad\square$

Definition 4.3 (Tian's properness [Tian87]). *Let (X^n, ω) be a compact Kähler manifold. Consider the space $\mathcal{K}_{[\omega]}$ of Kähler metrics in the Kähler class $[\omega]$. A functional F_ω on $\mathcal{K}_{[\omega]}$ is called* proper *if*

$$F_\omega(\varphi) \geq f(I_\omega(\varphi))$$

for some $f(t)$ increasing function as $t \to +\infty$.

We take the Mabuchi functional $\mathcal{M}_\omega$. Recall that it is the primitive of the closed 1-form $\alpha(\psi) = (\psi, \operatorname{Sc}_{\omega_\varphi} - \hat{S})_{L^2_{\omega_\varphi}}$, and it can be expressed as, see *e.g.* [Tia00, page 95],

$$\mathcal{M}_\omega(\varphi) = \fint_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n + \fint_X \left(f(\omega^n - \omega_\varphi^n) - (I_\omega - J_\omega)(\varphi) \right),$$

where f is the Ricci potential of ω. Here the first integral represents the "entropy" and the second integral is the "energy".

Theorem 4.4 (Tian). *Let (X^n, ω) be a compact Fano Kähler manifold. If the Mabuchi functional $\mathcal{M}_\omega$ is proper, then there exists a Kähler-Einstein metric.*

Proof. Let φ_t a solution of (10). We compute

$$
\begin{aligned}
\frac{d}{dt}\mathcal{M}_\omega(\varphi_t) &= -\frac{d}{dt}\fint_X t\varphi_t\omega_{\varphi_t}^n - \frac{d}{dt}(I_\omega - J_\omega)(\varphi_t) \\
&= -\fint_X \varphi_t\omega_{\varphi_t}^n - t\fint_X \dot\varphi_t\omega_{\varphi_t}^n - t\fint \varphi_t\Delta_{\omega_{\varphi_t}}\dot\varphi_t\omega_{\varphi_t}^n - \frac{d}{dt}(I_\omega - J_\omega)(\varphi_t) \\
&= \fint_X \Delta_{\omega_{\varphi_t}}\dot\varphi_t\omega_{\varphi_t}^n - t\fint_X \varphi_t\Delta_{\omega_{\varphi_t}}\dot\varphi_t\omega_{\varphi_t}^n - \frac{d}{dt}(I_\omega - J_\omega)(\varphi_t) \\
&= -t\fint_X \varphi_t\Delta_{\omega_{\varphi_t}}\dot\varphi_t\omega_{\varphi_t}^n - \frac{d}{dt}(I_\omega - J_\omega)(\varphi_t),
\end{aligned}
$$

where we are using that $\Delta_{\omega_{\varphi_t}}\dot\varphi_t = -\varphi_t - t\dot\varphi_t$ as follows by differentiating the equation. By the equation, we compute

$$
\begin{aligned}
\frac{d}{dt}(I_\omega - J_\omega)(\varphi_t) &= -\fint_X \varphi_t\Delta_{\omega_{\varphi_t}}\dot\varphi_t\omega_{\varphi_t}^n \\
&= \fint_X (\Delta_{\omega_{\varphi_t}}\dot\varphi_t + t\dot\varphi_t)\Delta_{\omega_{\varphi_t}}\dot\varphi_t \ge 0.
\end{aligned}
$$

At the end, we get

$$
\frac{d}{dt}\mathcal{M}_\omega(\varphi_t) = -(1 - t)\frac{d}{dt}(I_\omega - J_\omega)(\varphi_t) \le 0.
$$

If we assume that $\mathcal{M}_\omega$ is proper, then $I_\omega - J_\omega$ is bounded along the Aubin path. Then, by Theorem 4.2, we get $\|\varphi_t\|_{L^\infty} \le C$. (See also *e.g.* [Tia00, Theorem 7.13].) $\qquad\square$

Remark 4.5. *It is proven by Tian in [Tia97] that for manifold with discrete automorphism also the viceversa of the above Theorem holds. This was used for proving that there exists a Fano threefold with no holomorphic vector fields which has no KE metric (the famous deformation of the Mukai-Umemura manifold [Tia97, Section 7]). This result was crucial for the understanding of the relations between existence of KE metrics and stability conditions, and it sees in the work of Berman we will describe in the last lecture its ultimate incarnation.*

4.2 α-invariant criterion

For a compact Kähler manifold (X^n, ω), define the α-*invariant* [Tian87] as

$$
\begin{aligned}
\alpha(X) := \sup\{\alpha > 0 \ : \ &\text{there exists } C_\alpha \text{ such that,} \\
&\text{for any } \omega + \sqrt{-1}\partial\overline\partial\varphi > 0, \\
&\text{it holds } \fint_X \exp(-\alpha(\varphi - \sup\varphi))dV_g \le C_\alpha\}.
\end{aligned} \tag{11}
$$

where dV_g is any smooth volume form.

Theorem 4.6 (Tian [Tian87, Theorem 2.1]). *Let (X^n, ω) be a compact Fano Kähler manifold. If $\alpha(X) > \frac{n}{n+1}$, then there exists a Kähler-Einstein metric.*

Proof. Let $\alpha < \alpha(X)$ be such that the condition in (11) holds. By definition, there exists C_α such that, for any $\omega + \sqrt{-1}\partial\overline\partial\varphi > 0$,

$$
\begin{aligned}
\log C_\alpha &\ge \log\fint_X \exp(-\alpha(\varphi - \sup\varphi))\exp f\omega^n \\
&= \log\fint_X \exp\left(-\alpha(\varphi - \sup\varphi) - \log\frac{\omega_\varphi^n}{\omega^n} + f\right)\omega_\varphi^n
\end{aligned}
$$

$$\geq -\fint_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n + \alpha (\sup \varphi - \fint_X \varphi \omega_\varphi^n)$$

$$\geq -\fint_X \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n + \alpha I_\omega(\varphi).$$

So we have

$$\mathcal{M}_\omega(\varphi) \geq -(I_\omega - J_\omega)(\varphi) + \alpha I_\omega(\varphi) - C'_\alpha$$

$$\geq (\alpha - \frac{n}{n+1}) I_\omega(\varphi) - C''_\alpha,$$

which proves that the Mabuchi functional is proper. The conclusion follows by Theorem 4.4. □

Remark 4.7. *The same result holds by considering the action of a finite group G: one can define an invariant α_G, and if $\alpha_G > \frac{n}{n+1}$, then there exists a G-invariant Kähler-Einstein metric [Tian87, Theorem 4.1].*

Remark 4.8. *If $\alpha = \frac{n}{n+1}$, then Fujita [Fuj16b] proved K-stability. The existence of a KE metrics follows by the crucial result of [CDS15] we will describe in the last lecture.*

Remark 4.9. *Demailly (see appendix of [CS08]) identified $\alpha(X) = \mathrm{glct}(X)$ as the* global log canonical threshold. *It is defined as*

$$\mathrm{glct}(X) := \sup\{\lambda > 0 \ : \ (X, \lambda D) \text{ is } \log\text{-canonical}$$

$$\text{for all } D \text{ effective } \mathbb{Q}\text{-divisor with } D' \sim_\mathbb{Q} -K_X,$$

$$\text{i.e. } D \in |-mK_X| \text{ such that } D = \frac{1}{m} D'\}$$

The point is that, for $D = s_D^{-1}(0) \in |-mK_X|$, then $\eta = \frac{1}{m} \log |s_D|^2$ is a singular metric in $c_1(X)$ that can be used (by Demailly's approximation) to estimate the alpha invariant.

For example, Chetsov [Che08] computed the α-invariant for Del Pezzo surfaces, i.e. two dimensional Fanos, of degree $c_1^2(X) = 1$ (which have small anticanonical linear system): if there are no anti-canonical curves with cusps, then $\alpha(X) = 1$; if there are cusps, then $\alpha(X) = \frac{5}{6}$, which still satisfies the criterion for existence of KE metrics.

5 Towards K-stability

We are aimed at the Yau-Tian-Donaldson conjecture on existence of Kähler-Einstein metrics in the Fano case Theorem 0.3, which gives equivalence between the existence of KE metrics on Fano manifolds and the algebraic geometric notion of *K-polystability*.

Now some reasons why we should expect something as Theorem 0.3.

- *Historical:* The problem is the analogous in the case of varieties of the problem of equipping vector bundles with Hermitian-Einstein metrics. This is addressed by the Hitchin-Kobayashi correspondence, which indeed show that the existence of such special metric on the vector bundles is equivalent to stability in the sense of Mumford [Don87, UY86, LY87, LT95]. In a certain sense, the Yau-Tian-Donaldson conjecture is a "more non-linear" version of the Hitchin-Kobayashi correspondence.
- *Moduli:* From a more heuristic point of view, recall the following fact concerning *jump of complex structures* for Fano manifolds. Let $\mathcal{X} \xrightarrow{\pi} \Delta$ be a holomorphic submersion. There are cases in which $X_t = \pi^{-1}(t)$ are all smooth Fano, with $X_t \cong X_{t'}$ for any $t, t' \neq 0$, but $X_0 \not\cong X_t$. It follows that the moduli space of complex structures cannot be Hausdorff, since there would be non-closed points that cannot be separated: $[X_0] \in \overline{[X_t]}$. But the moduli space of Einstein *metrics* are Hausdorff. This suggests that by "removing" non KE Fanos we could hopefully get a nice (algebraic) moduli space of Fano manifolds, with topology compatible with distance (Gromov-Hausdorff, see next 6.3) induced by the KE metrics. Algebro-geometric notion of stabilities are good for constructing separated moduli in algebraic geometry.

Now we move to some more mathematical reasons which should motivate why the notion of K-stability is a natural one.

5.1 Variational point of view

The space $\mathcal{K}_\omega$ of Kähler potentials (say on a Fano, even if this description is more general for the cscK problem) looks like an infinite dimensional negative curvature space in the Mabuchi metric: $K_\varphi(\phi_1, \phi_2)\phi_3 = -\{\{\phi_1, \phi_2\}_\varphi, \phi_3\}_\varphi$, where $\{, \}_\varphi$ is the Poisson's bracket for ω_φ. As we have seen in section 3.3 such space admits geodesics for which the Mabuchi energy is convex. Note the following: take f such that grad f is holomorphic. Denote by ψ_t the flux of grad f: then $\psi_t^\star \omega = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ where $\dot{\varphi}_t = f$. Essentially by definition, the Futaki invariant is the derivative of the Mabuchi's energy:

$$\frac{d}{dt}\mathcal{M}_\omega(\varphi_t) = F((\text{grad} f)^{1,0}).$$

Thus we can make the following "speculation" (since we are on an infinite dimensional space, so care must be considered...): since the Mabuchi's energy is convex on $\mathcal{K}_\omega$ and its critical points are precisely the KE metrics, if at the "infinity" of $\mathcal{K}_\omega$, $\mathcal{M}_\omega$ has positive slope, we can suspect that a critical point actually do exists. In order to check the "slope at infinity" we could look for something like:

$$\lim_{t\to+\infty}\frac{d\mathcal{M}_\omega(\varphi_t)}{dt} =: \tilde{DF}((\varphi_t)_t)$$

where $\tilde{DF}$ should be a kind of "Futaki invariant".

Thus, we would have the following "geodesic stability picture": if we know that $\tilde{DF} > 0$ for all geodesic rays probing infinity, then we expect to have a critical point: in this case, we would say that X is *stable*, and hopefully there exists some minimum, *i.e.*, a Kähler-Einstein metric. If $\tilde{DF} \geq 0$, we would say that X is *semistable*. The case in which we can find rays such that $\tilde{DF} < 0$ is *unstable* (and there is no minimum, *i.e.*, a KE metric should not exist).

The precise mathematical definition of $\tilde{DF}$ will be given for special (algebraic) rays: *test configurations*. In such case, the limits at infinity are singular algebraic varieties with a $\mathbb{C}^*$ action, and $\tilde{DF}$ will be precisely equal to the Futaki invariant on the singular limits at infinity if such limits are not too singular (actually this was used in the first definition of K-stability proposed by Tian [Tia97]). Moreover, as we will see, such Futaki invariant admits a purely algebro geometric definition which extends to very singular limits (after Donaldson [Don02b]). It is worth noting that, thanks to deep algebro-geometric results, to test stability is sufficient to consider only mildly singular degenerate limits for which the original Tian's definitition works [LX14].

Thus we should expect that if we have a KE metric, it should be relatively "easier" to show that on such algebraic geodesic rays we have positivity of the DF invariant (this is the content of Berman's result we will discuss in Section 6). The other direction seems to be harder: we are imposing a condition only on certain algebraic directions in $\mathcal{K}_\omega$, which are much less than all possible infinities. The fact that this will be sufficient is deep. The proof of CDS, as we will see, is essentially arguing by contradiction by constructing destabilizing (algebraic) test configurations, but the actual way to do this is quite involved. Anyway, notice that a more pure variational viewpoint has been recently proved to be also successful to show existence for K-stable Fano manifolds, at least in the case of no holomorphic vector fields, see Berman-Boucksom-Jonsson [BBJ15].

5.2 Moment map picture for cscK

Let (M^{2n}, ω) be a compact symplectic manifold. When $\pi_1(X) = \{1\}$, then $\mathcal{L}_X\omega = 0$ yields that there exists h such that $dh = -\iota_X\omega$. Let G be a compact group in Sympl(M, ω). A moment map for the action of G is given by

$$\mu: M \to \mathfrak{g}^\vee$$

a G-equivariant map such that, for any $v \in \mathfrak{g}$,

$$d\langle \mu, v \rangle = -\omega(\rho(v), \_),$$

where ρ denotes the derivative of the action. The *symplectic quotient* is then defined as

$$M /^{\mathrm{sym}} G := \mu^{-1}(0)/G.$$

For example, $\mathbb{C}^n /^{\mathrm{sym}} U(1) = \mu^{-1}(0)/U(1) \cong S^{2n-1}/S^1 \cong \mathbb{CP}^{n-1}$ with $\mu(z) = |z|^2 - 1$.

Consider now (X^n, ω) a compact Kähler manifold. We focus on ω the symplectic structure on the underlying smooth manifold M^{2n}. Consider the infinite-dimensional Kähler manifold

$$\mathfrak{I}_\omega = \{\omega\text{-compatible complex structures on } M\}.$$

Its tangent space is

$$T_J \mathfrak{I}_\omega = \{A \in \mathrm{End}(TM) \ : \ AJ + JA = 0, \quad \omega(Ax, y) + \omega(x, Ay) = 0\}.$$

There is a tautological complex structure

$$\mathfrak{J}_J A = JA.$$

There is a natural L^2-metric:

$$\langle A, B \rangle_J = \int_M g_J(A, B) \frac{\omega^n}{n!},$$

where $g_J = \omega(\_, J\_)$. Then

$$(\mathfrak{I}_\omega, \mathfrak{J}, \langle \_ | \_ \rangle)$$

is Kähler. (See also *e.g.* [Tia00, Chapter 4], [Szé14, Section 4.1].)

Consider the action of $\mathfrak{G}_\omega$ the group of (Hamiltonian) symplectomorphisms. Its tangent space can be identified as $T\mathfrak{G}_\omega \simeq \mathfrak{C}_0^\infty(M; \mathbb{R})$ the space of zero-average smooth functions. Then:

Theorem 5.1 (Donaldson [Don97], Fujiki [Fuj90]). *Let (M^{2n}, ω) be a Kähler manifold, and consider the action $\mathfrak{G}_\omega$ of (Hamiltonian) symplectomorphism on the space $\mathfrak{I}_\omega$ of ω-compatible complex structures. Then the action $\mathfrak{G}_\omega \circlearrowleft \mathfrak{I}_\omega$ is symplectic, with moment map*

$$\mu : \mathfrak{I}_\omega \to \mathfrak{C}_0^\infty(M; \mathbb{R}), \qquad J \mapsto \mathrm{Sc}_{g_J} - \hat{S}$$

where $g_J := \omega(\_, J\_)$ and $\hat{S}$ is a cohomological invariant.

This means we have two operators $R : \mathfrak{C}_0^\infty(M; \mathbb{R}) \to T_J \mathfrak{I}_\omega$ defined as $h \mapsto \mathcal{L}_{X_h} J$ (the infinitesimal action) and $S : T_J \mathfrak{I}_\omega \to \mathfrak{C}_0^\infty(M; \mathbb{R})$ defined as $D(A) = DSc_J(A)$ (the derivative of the moment map), such that $(S(A), h)_{L^2} = -\Omega(R(h), A) :=< JA, R(A) >$. That is, $S^* = -JR$.

Then the symplectic quotient $\mu^{-1}(0)/\mathfrak{G}_\omega$ would be the "moduli space of cscK metrics in $[\omega]$". Let's us elaborate a bit more about this. If F is a symplectomorphism, then $F^\star \omega = \omega$, and $g_{F^\star J} = F^\star g$, *i.e.* nothing interesting is really happing about the metric (the tensor changes, but the underlying metric spaces are still isometric). To do something metrically non trivial we should look at "the complexification of the symplectomorphisms". While such group does not exist, at least infinitesimally is clear how it should act:

$$R(ih) := JR(h) = J\mathcal{L}_{X_h} J = \mathcal{L}_{JX_h} J.$$

Of course this does not preserve the symplectic form, but instead, as we computed previously: $\mathcal{L}_{JX_h}\omega = 2\sqrt{-1}\partial\bar{\partial}h$, Thus it acts precisely by deforming within the Kähler class! So looking at zero of the moment map within a complexified orbit corresponds to the problem of searching for a KE metric (or more generally, for a csck metric) in the space of Kähler potentials as we did in previous sections. From this point of view, the Mabuchi functional should be seen as a convex norm in such orbit (see discussion of Kempf-Ness theorem in the next section), whose critical points are the zeros of the moment map. Of course, if we find a J within

the complexified orbit which is inside the zero of the moment map (*i.e.* a cscK metric), we can apply Moser's theorem [Mos65, CdS01] to find a diffeomorphism which pull back the new symplectic Kähler form to the fixed symplectic background one (while keeping the abstract biholomorphism type of the complex manifold unchanged). Thus the complexified orbits for which we can find such points should be "special" (read "stable"), see previous discussion on "geodesic stability".

Remark 5.2. *Note that smooth hypersurfaces in $\mathbb{CP}^n$ are all symplectomorphic (for the natural symplectic structure induced by the Fubini-Study form), and the existence problem of KE metric on them fits the above picture.*

The next section is going to describe the classical Geometric Invariant Theory (GIT) notion of stability. This is crucial to motivate algebraically the notion of K-stability (that is, the rigorous notion of stability needed to make sense of the vague "stability" notion we described so far).

5.3 Introduction to GIT

For a gentle introduction see [Tho06]. More advanced one are [New78] and [Mum77]. The standard complete reference is [MFK94]. The following example contains the main ideas of GIT:

Example 5.3. *Consider the action $\mathbb{C}^* \circlearrowleft \mathbb{C}^2$ given by*

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Note that the topological quotient $\mathbb{C}^2/^{top}\mathbb{C}^*$ is not Hausdorff. From the algebro-geometric point of view, we have

$$\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[x, y].$$

We know what the functions on "$\mathbb{C}^2/\mathbb{C}^*$" should be: simply look at the $\mathbb{C}^*$-invariant polynomials, that is, $\mathbb{C}[x, y]^{\mathbb{C}^*} = \mathbb{C}[xy] = \mathbb{C}[t]$. Then

$$\mathbb{C}^2 /\!\!/ \mathbb{C}^* = \operatorname{Spec} \mathbb{C}[x, y]^{\mathbb{C}^*} = \operatorname{Spec} \mathbb{C}[t] = \mathbb{C}.$$

We have the map

$$\mathbb{C}^2 \to \mathbb{C}, \qquad (x, y) \mapsto xy.$$

Substantially, you use invariants to define quotients.

More generally: take G reductive acting as $G \circlearrowleft \operatorname{Spec} R = X$ on the algebraic variety X, where R is a finitely generated graded $\mathbb{C}$-algebra. Then define

$$X /\!\!/ G := \operatorname{Spec} R^G$$

where R^G is finitely generated thanks to reductivity.

Consider now compact quotients.

Definition 5.4. *Let $G \subseteq \operatorname{SL}(N + 1)$ act as $G \circlearrowleft X \subseteq \mathbb{CP}^N$, and assume that G is reductive. Take $x \in X$, and let $\hat{x}$ be a lift to $\mathbb{C}^{N+1}$. We say that $x \in X$ is:*
- GIT-semistable *if $0 \notin \overline{G \cdot \hat{x}}$;*
- GIT-stable *if $G \cdot \hat{x}$ is closed and its stabilizer is finite;*
- GIT-polystable *if $G \cdot \hat{x}$ is closed.*

This is the fundamental result of GIT:

Theorem 5.5 ([Mum77]). *Let $G \circlearrowright X$ be reductive. Define the compact space by taking invariant sections:*

$$Y := \text{Proj} \bigoplus_k H^0(X; \mathcal{O}_X(k))^G.$$

Then there exists

$$\Phi \colon X \supset X^{ss} \to Y =: X /\!\!/ G$$

surjective such that, for $x, y \in X^{ss}$ the set of GIT-semistable points:

$$\Phi(x) = \Phi(y) \quad \text{if and only if} \quad \overline{G \cdot x} \cap \overline{G \cdot y} \cap X^{ss} \neq \varnothing;$$

and, for $x, y \in X^s$ the set of GIT-stable points:

$$\Phi(x) = \Phi(y) \quad \text{if and only if} \quad x = g \cdot y \text{ for some } g \in G.$$

There is only one closed orbit in semi-stable-equivalence classes, and it is the polystable orbit. Thus, $X^s/G \subseteq X^{ss}/\!\!/G$ is a compactification of the moduli space of stable points, for which we can think at $X^{ss}/\!\!/G$, set-theoretically, as the moduli space of polystable objects. In Example 5.3, the origin has to be considered as a polystable point. The points in the axes as semistable ones. The points in the hyperbolas $xy = s \neq 0$ as stable.

We introduce now the *Hilbert-Mumford criterion*. Let $G \circlearrowright X \subseteq \mathbb{CP}^N$ reductive, and consider a 1-parameter subgroup $\lambda \colon \mathbb{C}^* \to G$ of G. Take a point $x \in X$ and consider $\lim_{t \to 0} \lambda(t) \cdot x = x_0 \in X$. Look at the complex line $\mathbb{C}_{x_0}$ over x_0. There is a $\mathbb{C}^*$-action on it: $\lambda(t) \circlearrowright \mathbb{C}_{x_0}$ as some power t^{-w}, where w is called the *weight*. Define

$$\mu(x, \lambda) := w.$$

Theorem 5.6 (Hilbert-Mumford criterion [Mum77]). *Let $G \circlearrowright X$ be reductive, and $x \in X$. Then:*
- *x is GIT-stable if and only if $\mu(x, \lambda) > 0$ for any λ one-parameter subgroup.*
- *x is GIT-semistable if and only if $\mu(x, \lambda) \geq 0$ for any λ one-parameter subgroup.*

The implication "x stable implies $\mu(x, \lambda) > 0$ for any λ" follows directly by "pictures". The implication "$\mu(x, \lambda) \geq 0$ for any λ implies x semistable" is hard; see [MFK94, Theorem 2.1].

Example 5.7. *For example, for $\text{SL}(2; \mathbb{C})$ acting on binary forms identified with $\text{Sym}^2(\mathbb{C}^2)$, GIT-stability is related to polynomials with multiple roots. This is a nice exercise to do in order to practice with GIT stability (use Hilbert-Mumford criterion).*

We are now going to quickly review the *Kempf-Ness theorem*, which relate finite dimensional symplectic quotient to GIT. Moreover, we discuss the relation with the infinite dimensional picture of the scalar curvature as a moment map.

Look at $\mathbb{C}^* \circlearrowright (\mathbb{C}^2, \omega_{\text{std}})$, with $U(1) \subset \mathbb{C}^*$ Hamitonian action ($h = |x|^2 - |y|^2$). We clearly get

$$\mathbb{C}^2/^{\text{sym}}U(1) \overset{\text{top}}{\cong} \mathbb{C}^2/\!\!/\mathbb{C}^*,$$

i.e. the GIT quotient and the symplectic quotient are identified!

More generally, the Kempf-Ness theorem states that if we have some compact $K \subseteq \text{SU}(N + 1)$ with $K^{\mathbb{C}} \subseteq \text{SL}(N + 1)$, and K acting symplectically on (X, ω_{FS}) with moment map $\mu(Z) = \sqrt{-1} \frac{Z_j \bar{Z}_j}{|Z|^2}$. Then

$$X /\!\!/ K^{\mathbb{C}} \cong X/^{\text{sym}}K.$$

The idea of the proof consists in studying the *log norm* function $m = \log |g \cdot \hat{x}|^2$ on $K^{\mathbb{C}}/K$. The crucial point is that such function is *convex* on the complexified orbits, and its critical points coincide with the zero of the moment map. Thus, if the slope at infinity of m is positive (which holds if the orbit is stable, since the weights at infinity are positive), we can find a critical point (that is, a zero of the moment map).

Let us go back to the infinite dimensional case. In such situation the log norm corresponds to the Mabuchi energy on the "complexified orbits" of the symplectomorphisms. Thus the complexified orbits which intersect the zero of the moment map (cscK) should be "stable". Thus, in this infinite dimensional setting, we should still have:

$$\text{“}\mu^{-1}(0)/\mathcal{G}_\omega \cong \mathcal{I}_\omega^{\text{stab}}/\mathcal{G}_\omega^{\mathbb{C}} =: \mathcal{I}_\omega /\!\!/ \mathcal{G}_\omega^{\mathbb{C}} \text{”}.$$

In the KE case, the rigorous notion of stability condition is going to be, of course, K-stability. "$\mathcal{I}_\omega /\!\!/ \mathcal{G}_\omega^{\mathbb{C}}$" should be a purely *algebraic* moduli spaces of stable varieties (K-moduli), which could also be rigorously constructed in the Fano case (*e.g.* see the survey [Spo17]).

Remark 5.8. *This moduli picture can be thought as an higher dimensional analogous of the identification between moduli spaces of metric with constant Gauss curvature on surface of genus g and the complex Deligne-Mumford moduli space of curves.*

We are now ready to finally define *K*-stabilty.

5.4 Definition of K-stability

For (X, K_X^{-1}) Fano manifold, we want to define a "GIT-like" notion of stability, following the dictionary:
–   1-parameter subgroups λ correspond to test configurations $\mathcal{X} \to \mathbb{C}$;
–   weight $\mu(x, \lambda)$ corresponds to Donaldson-Futaki invariant $DF(X, \mathcal{X})$.

In analogy with the Hilbert-Mumford criterion, we define:
– X is *K-stable* if $DF(X, \mathcal{X}) > 0$ for any test-configuration $\mathcal{X}$;
– X is *K-semistable* if $DF(X, \mathcal{X}) \geq 0$ for any test-configuration $\mathcal{X}$;
– X is *K-polystable* if X is K-semistable and $DF(X, \mathcal{X}) = 0$ if and only if $\mathcal{X} = X \times \mathbb{C}$.

So let us define what is a test configuration and what is the Donaldson-Futaki invariant.

A *test configuration* of exponent $\ell \in \mathbb{N}$ is a flat family

$$
\begin{array}{c}
\mathcal{L} \\
\mathbb{C}^* \circlearrowleft \Big\downarrow \\
\mathcal{X} \xrightarrow{\ \ \pi\ \ } \mathbb{C}
\end{array}
$$

such that, for $X_1 = \pi^{-1}(1)$:

1. $\mathcal{L}$ is relative ample and $\mathcal{L}\lfloor_{X_1} \simeq K_{X_1}^{-\ell}$;
2. π is $\mathbb{C}^*$-equivariant.

Although a priori quite abstract, test configurations are concrete:

Theorem 5.9 (Ross-Thomas [RT07])**.** *All test configurations are realized by 1-parameters subgroups in $PGL(N + 1)$ for embedding of $X \hookrightarrow \mathbb{CP}^N$.*

Remark 5.10. *Note that N in the above result is not fixed! In principle, we should consider test configurations arising as $N \to +\infty$. This is why K-stability is very hard to check! K-stability should be considered as a (new, non-GIT) stability condition defined on the "stack" of Fano varieties, where the DF invariant can be seen as the weigh of a stacky line bundle, the so-called CM line [PT09]. This fancier approach is relevant for precise moduli considerations.*

The next is an example of a test configuration for $\mathbb{CP}^1$.

Example 5.11. *For example, consider $X = \{xy = z^2\}$ obtained as the image of the Veronese embedding $\mathbb{CP}^1 \ni$ $[x : y] \mapsto [x^2 : y^2 : xy] \in \mathbb{CP}^2$. Conider the $\mathbb{C}^*$-action*

$$\lambda(t) = \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix},$$

and define

$$X_t := \lambda(t) \cdot X = \{xy = tz^2\}.$$

As $t \to 0$, we get $X_0 = \{xy = 0\}$.

Now we introduce the *Donaldson-Futaki invariant*. Consider a test configuration. We get an action $\mathbb{C}^* \circlearrowleft X_0$ inducing an action $\mathbb{C}^* \circlearrowleft H^0(X_0; \mathcal{L}\lfloor_{X_0}^{\otimes k})$ for any k. By the Riemann–Roch theorem, we write the Hilbert polynomial

$$h_k := \dim H^0(X_1; \mathcal{L}\lfloor_{X_1}^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + \mathcal{O}(k^{n-2}).$$

Similarly, we write the weight for the action $\mathbb{C}^* \circlearrowleft \wedge^{\text{top}} H^0(X_0; \mathcal{L}_0^{\otimes k})$ as:

$$w_k = b_0 k^{n+1} + b_1 k^n + \mathcal{O}(k^{n-1}).$$

Set the Donaldson-Fujiki invariant as

$$DF(X, \mathcal{X}) := \frac{a_1 b_0 - a_0 b_1}{a_0}.$$

Let us compute the *DF* invariant for the previous Example 5.11 of test configuration.

Example 5.12. *Recall that $X_0 = \text{Proj}\,\mathbb{C}[x, y, z]/\langle xy \rangle$. We have*

$$H^0(\mathcal{O}_{X_0}(1)) = \langle x, y, z \rangle,$$
$$H^0(\mathcal{O}_{X_0}(2)) = \langle x^2, xz, y^2, yz, z^2 \rangle,$$
$$H^0(\mathcal{O}_{X_0}(3)) = \langle x^3, x^2 z, xz^2, y^3, y^2 z, yz^2, z^3 \rangle,$$

and so on. In general,

$$h_k := \dim\{\text{polynomials of degree } k\} - \dim\{\text{polynomials of degree } k - 2\}$$
$$= 2k + 1.$$

Similar computations for the weight give:
$$w_k = \frac{k(k-1)}{2}.$$

At the end, we get:
$$DF(\mathbb{CP}^1, \mathcal{X}) = \left(\frac{1}{2} + 2 \cdot \frac{1}{2} \right) \cdot \frac{1}{2} = \frac{3}{4} > 0,$$

as it should be, since X is K-stable because it is KE.

Finally, let us see that at least for smooth central fiber, the *DF* invariant recovers the classical Futaki invariant.

Proposition 5.13. *If the central fiber X_0 is smooth, then the algebro-geometric Futaki invariant F and the DF invariant are essentially the same: if Z generates the $\mathbb{C}^*$-action*

$$DF(X_0, \mathcal{X}) = \frac{1}{4\pi} F(Z).$$

Proof. For more details on this approach via *Kähler quantization*, see *e.g.* [Szé14, Proposition 7.15].

Take $L \to (X_0, \omega)$ be a Kähler manifold with L ample. Consider $H^0(X_0; L^{\otimes k})$ and take $\underline{s}^k = (s_0^k, \ldots, s_{N_k}^k)$ an orthonormal basis of holomorphic sections with respect to the L^2-metric induced by ω. Denote by h the Hermitian structure on L with positive curvature ω. Introduce the *Bergman kernel*

$$b_h(p) = \sum_{j=0}^{N_k} |s_j^k(p)|_h^2 > 0,$$

where $N_k + 1 = \dim H^0(X_0; L^{\otimes k})$. It is easy to see that does not depend on the choice of the orthonormal basis.

Since L is ample,

$$\varphi_{\underline{s}^k} : X_0 \hookrightarrow \mathbb{CP}^{N_k}, \qquad p \mapsto [s_0^k(p) : \cdots : s_{N_k}^k(p)]$$

is well-defined for $k \gg 1$, and we compute

$$\varphi_{\underline{s}^k}^* \omega_{FS} = k\omega + \sqrt{-1}\partial\bar{\partial} \log b_{h^n}.$$

The important (and not trivial!) fact proven by Tian [Tia90b], Ruan [Rua98], Zelditch [Zel98], Lu [Lu00], Catlin [Cat99] is that

$$b_{h^k} = 1 + \frac{\mathrm{Sc}_\omega}{4\pi} \cdot \frac{1}{k} + \mathcal{O}\left(\frac{1}{k^2}\right)$$

As a consequence,

$$\omega = \frac{1}{2\pi k} \varphi_k^* \omega_{FS} + \mathcal{O}\left(\frac{1}{k^2}\right),$$

and we recover the Hilbert polynomial expression:

$$h_k = k^n \int_{X_0} \frac{\omega^n}{n!} + \frac{k^{n-1}}{4\pi} \int_{X_0} \mathrm{Sc}_\omega \frac{\omega^n}{n!} + \mathcal{O}(k^{n-2}).$$

This gives

$$a_0 = \int_{X_0} \frac{\omega^n}{n!}, \qquad a_1 = \frac{1}{4\pi} \int_{X_0} \mathrm{Sc}_\omega \frac{\omega^n}{n!}$$

Let us do similar computation for the weights. We assume that the $\mathbb{C}^*$ action comes from complexification of a U(1)-action for the embedding:

$$
\begin{array}{ccccc}
\mathbb{C}^* & \hookrightarrow & \mathrm{SL}(n+1) & \longrightarrow & \mathrm{Aut}(X_0 \subseteq \mathbb{CP}^{n+1}) \\
\uparrow & & \uparrow & & \\
\cup & & \cup & & \\
\mathrm{U}(1) & \hookrightarrow & \mathrm{U}(n+1). & &
\end{array}
$$

Recall that the moment map for ω_{FS} and the $U(n+1)$-action is $\mu(Z) = \frac{\sqrt{-1}Z_i \bar{Z}_j}{|Z|^2}$. Let f Hamiltonian for ω, and $f_k = -2\pi (A_k)_{i,j} \frac{(s_i^k, s_j^k)_{h^k}}{b_h^k}$ is Hamiltonian for $\omega_{FS|k}$. Then

$$f - \frac{1}{2\pi k} f_k = \mathcal{O}(k^{-2}).$$

We compute

$$\int_{X_0} f b_{h^k} \frac{\omega^n}{n!} = \frac{1}{2\pi k} \int_{X_0} f_k b_{h^k} \frac{(k\omega)^n}{k^n n!} + \mathcal{O}(k^{-2})$$

$$= \frac{1}{2\pi k} \int_{X_0} f \left(1 + \frac{\mathrm{Sc}_\omega}{4\pi} \frac{1}{k} + \mathcal{O}(k^{-2})\right) \frac{\omega^n}{n!}.$$

Then

$$w_k = -k^{n+1} \left(\int_{X_0} f \frac{\omega^n}{n!} + \frac{1}{4\pi k} \int_{X_0} \mathrm{Sc}_\omega \omega^n + \mathcal{O}(k^{-2}) \right),$$

whence we get

$$b_0 = -\int_{X_0} f \frac{\omega^n}{n!}, \qquad b_1 = -\frac{1}{4\pi k} \int_{X_0} \mathrm{Sc}_\omega \omega^n.$$

By plugging-in these expressions in *DF*, we get the statement. □

Remark 5.14.

– *In the Fano case, the same holds for mildly singular (klt) central fiber X_0.*
– *By analyzing in deeper detail the Kähler quantization picture, Donaldson proved [Don01] that cscK manifolds with no holomorphic vector fields are asymptotically balanced, condition which coincides with certain asymptotic GIT stability (Chow stability). However, there are examples which show that asymptotic Chow stability is not enough to capture the cscK or KE existence problem in general.*
– *With similar technique, Donaldson showed [Don05] a lower bound on the L^2 norm of the scalar curvature: $\|Sc_\omega - \overline{Sc}\|_{L^2} \geq -DF/\|\mathcal{X}\|$. This is clearly enough to conclude K-semistability for the general cscK case (improved to K-polystability by Stoppa in the case with no holomorphic vector fields [Sto09])*

With the above results we have closed the circle between the rough "geodesic stability" and the properly defined algebraic notion of K-stability. Moreover, the infinite moment map picture makes quite clear that it is natural to expect that such stability condition can precisely capture the existence problem. As we will discuss in the next section, this is precisely the case in our Fano situation.

6 Equivalence between existence of Kähler-Einstein and K-stability

The goal of this last section is to describe some of the ideas in the proof of equivalence between existence of KE metrics on Fano manifolds and K-stability, as well as to shortly describe compact moduli spaces of such manifolds and some explicit examples.

6.1 Berman's result

In [Ber16] it is proved that K-polystability is a necessary condition for the existence of a KE metrics on smooth (or even mildly singular) Fanos. Previous results in this direction are given by works of Tian, Donaldson, and Stoppa.

We do not discuss in detail Berman's proof, but we simply emphasize the main points. The argument is based on a *crucial formula* for the Donaldson-Futaki invariant:

$$DF(X, \mathcal{X}) = \lim_{t \to \infty} \frac{d}{dt} \mathcal{D}_\omega(\varphi_t) + \epsilon. \tag{12}$$

Let us explain its meaning. Here $\mathcal{X} \to \Delta_\tau \subseteq \mathbb{C}^*$ is a test configuration for a Fano variety $X \cong X_1$. Starting from a Kähler metric $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ on X_1 one can construct a weak geodesic ray in the Mabuchi metric (*i.e.* a weak solution of the homogeneus Monge-Ampère equation, compare Remark 3.10) with boundary datum ϕ. Let us denote with $\phi_t := \rho_\tau^* \psi_\tau$ such geodesic ray emanating from ϕ, where $t = -\log|\tau|^2$ and ρ_τ is the $\mathbb{C}^*$-action which identifies the general fibres.

$\mathcal{D}_\omega$ denotes the *Ding functional* [Din88]. The Ding functional has the advantage that it can be defined for metric of less regularity with respect to the Mabuchi energy. It has the property that its critical points are again the KE metrics as for the Mabuchi functional. It takes the explicit form

$$\mathcal{D}_\omega(\varphi) = -\frac{1}{(n+1)!} \sum_{j=0}^{n} \fint_X \varphi \omega_\varphi^j \wedge \omega^{n-j} - \log\left(\fint_X \exp(f - \varphi)\omega^n\right),$$

where f is the Ricci potential of ω.

The Berman's formula thus states that, similarly to what we discussed for the Mabuchi's energy, the Donaldson-Futaki invariant is the slope at infinity in the space of Kähler potentials of the Ding functional plus an error term denoted by ϵ which has the crucial property of being *non-negative*. Actually the term ϵ can be explicitly given in terms of purely algebro-geometric data of the test-configuration and its resolutions.

Thanks to the this formula (12), the idea of proof is now very clear. If we start the geodesic ray ϕ_t from the KE potential $\phi_0 = \phi^{KE}$, using that the KE equation is the Euler-Lagrange equation of the Ding functional,

we have $\frac{d}{dt}\mathcal{D}_\omega(\phi_t)_{|t=0} = 0$ (more precisely, Berman proved that is non-negative, the issue being related to regularity properties of the ray). Moreover, since also the Ding functional is convex along the ray, we have that its slope at infinity $\lim_{t\to\infty}\frac{d}{dt}\mathcal{D}_\omega(\phi_t)$ is still non negative. Hence, by the formula and the positivity of the error term, we find that the Donaldson-Futaki invariant of an arbitrary test configuration is non-negative (*i.e.* X is K-semistable). It is then possible to analyze in further details what happens in the the case of DF invariant equal to zero, to conclude that the error term vanishes and X_1 is biholomorphic to X_0 (the point is that one can show that the Ding function has to be affine, and the test configuration generated by an holomorphic vector field).

Remark 6.1.

– *The Ding functional can be defined also for mildly singular Fano varieties (more precisely klt Fanos, see later). Thus Berman's formula gives K-polystability also for such very natural class of singular varieties. As we will see, this is the class of singularities which need to be considered for moduli spaces compactification of smooth KE Fano manifolds.*

– *The formula for ϵ was used to define the notion of* Ding stability*, a notion that is a priori stronger but then equivalent to K-stability. This notion has been relevant for some of the newest developments in the area, for example in the works of K. Fujita (e.g. [Fuj15, Fuj16a]).*

6.2 K-polystability implies existence of KE metrics: Donaldson's cone angle continuity path

The goal of the next sections is to give a "map" of the main steps in the proof of the existence of KE metrics on K-polystable Fano manifolds. We hope that such description could be of some help for readers of the seminal papers [CDS15].

The first idea in the proof is to consider a "specialization" of the Aubin's continuity path $\text{Ric}(\omega_t) = t\omega_t + (1-t)\omega$, see (10), with $\omega \in 2\pi c_1(X)$, by replacing the smooth background Kähler form ω with a current of integration δ_D along a smooth divisor D given as zero of a smooth section of the line bundle K_X^{-1}. That is, we consider the following path, called *Donaldson's cone angle continuity path*:

$$\text{Ric}(\omega_\beta) = \beta\omega + 2\pi(1-\beta)\delta_D, \tag{13}$$

for $\beta \in (0, 1]$.

Note that the metric ω_β is exactly KE with Einstein constant equal to β on $X \setminus D$, but the price to pay is that is not longer smooth near D: it has so-called conical singularities. Roughly, this means that near the divisor D, if we choose local coordinates $(z_j)_j$ so that D is given by $z_n = 0$, the metric looks like (is uniformly equivalent to) the model flat cone (transverse to D) metric:

$$\omega_\beta \sim \frac{\sqrt{-1}}{|z|^{2(1-\beta)}} dz_n \wedge d\bar{z}_n + \sqrt{-1}\sum_{j=1}^{n-1} dz_j \wedge d\bar{z}_j.$$

Since it is not always possible to find a *smooth* anti-canonical sections, one need to consider plurianti-canonical sections, that is sections of powers $K_X^{-\lambda}$ of the anti-canonical bundle. Then the generic sections are smooth by Bertini's theorem . The value of the Einstein constant in the corresponding conical continuity path is given by $r(\beta) = 1 - \lambda(1 - \beta)$, and it is positive as soon as $\beta > 1 - \frac{1}{\lambda}$.

The idea is now simply to prove that the set of cone angles βs, for which we can solve equation 13 is *non-empty*, *open* and *closed* under the assumption of *K*-polystability of X. Then, letting $\beta \to 1$, the cone angles can "open-up" to a genuinely smooth KE metric on X (of course there is plenty of things to be proven to make such statement precise!). As a historical comment, let us point out that a similar strategy to deform cone angle Einstein metrics (with cone singularities along smooth *real* curves) to smooth Einstein metric was proposed by Thurston as a way to attack the 3D Poincarè conjecture [Thu82].

The fact that cone angle metrics (with no assumption on the curvature) always exist is easy: take $\omega = \omega_0 + \varepsilon\sqrt{-1}\partial\bar{\partial}|s_D|^{2\beta}_{h_0}$ with ω_0 smooth background metric (the curvature of the Chern connection of some Hermitian metric h_0 on the plurianti-canonical bundle) and $\varepsilon \ll 1$. Of course they do not satisfy our equation (13). Thus non-emptiness of the set of solutions for small cone angles follows by taking $\lambda > 0$ and apply the orbifold version of the negative KE problem for $\beta = \frac{1}{m} < 1 - \frac{1}{\lambda}$, see [DK01, Section 6.2], [CDS15, Part III, Section 5], using the above conical ω as background. If $\lambda = 1$ one could instead use a result of Berman [Ber13] which says that for sufficiently small βs, a log version of the alpha invariant criterion ensures existence of conical KE metrics along D of positive Einstein constant.

Even if not needed in the proof, we would like to make some remarks about the *behaviour* of the conical KE metrics for *small* cone angles.

Remark 6.2. *For $\lambda > 1$ the conical (negative) KE metrics converge as $\beta \to 0$ to complete cuspidal KE metrics (with hyperbolic cuspidal behaviour transverse to D) as proven in [Gue15].*

For $\lambda = 1$, an interesting well-known conjecture states that the KE cone metrics converge to the complete Calabi-Yau metrics on $X \setminus D$ constructed by Tian and Yau in [TY90, TY91].

If the anti-canonical bundle admits roots, we can consider also $\lambda < 1$. However, in such case small angle metrics do not exist: it is a (log)-K-unstable regime, as proved in [LS14]. Heuristically it is clear: we would expect convergence to complete metrics with strictly positive Ricci tensor, and this contradicts Myers' theorem.

Openness is definitely more sophisticated. The idea is to use some *implicit function theorem*, combined with the fact that there are no holomorphic vector fields tangential to D [Don12, Section 4.4] (which would be related to infinitesimal isometries as we have discussed in section 3.1, and hence would give obstructions to the linearization of the problem). However, using naive (weighted) Hölder spaces will not work. The actual function spaces to be consider are given by a variation of usual Hölder spaces where one considers norms of only certain second order derivatives [Don12, Section 4.3] (essentially the one which appear in the complex Hessian). In any case, in [Don12, Theorem 2] it is proved that if we assume existence of conical KE metrics, we can move the cone angle a bit and still find conical KE metrics.

Remark 6.3. *Detailed study of conical KE metrics (generalized energy functionals, alpha invariant, asymptotic analysis, etc…) can be found in the works of Jeffres, Mazzeo, and Rubinstein [JMR16].*

From now on, we are taking for granted that we have found conical KE metric for small cone angles, and that the KE condition is open. Thus we need to understand what happens to conical KE spaces (X, ω_{β_i}) as $\beta_i \nearrow \underline{\beta}$. The crucial point is now to establish a *weak geometric compactness result*, which will ensure that, after eventually passing to a subsequence, we can find a limit space X_∞ with sufficiently good property (via partial regularity results). In particular, if X_∞ can be proved to be an *algebraic* space we can try to use the *K*-stability hypothesis to show that bad behaviours cannot happens and that X_∞ is still biholomorphic to X and the limit metric structure is a cone angle metric on X along D with cone angle $\underline{\beta}$. By the continuity argument we could then show that β can go to one, and in this case the metric converge to a smooth KE metric.

The right geometric notion of weak limits we need to consider is given by the so-called *Gromov-Hausdorff topology*. This is a very weak topology on the class of isomorphism class of compact metric spaces, which has the advantage to make precise what one means for convergence of a smooth Riemannian manifold to a (potentially) singular space. Thus, in the next section we will recall some of the basic definitions and properties of such notion.

The next sections can be understood as a geometric way to do *a priori* estimates and study regularity for our KE equations.

6.3 Gromov-Hausdorff basics ("weak limits")

Let (X, d_X) and (Y, d_Y) be two compact metric spaces. Define the following *Gromov-Hausdorff* (GH) (pseudo-)distance:

$$d_{GH}(X, Y) = \inf_{Z \hookleftarrow_{\text{isom}} X, Y} \inf\{\varepsilon \geq 0 \mid \mathcal{N}_\varepsilon^Z(X) \supseteq Y, \, \mathcal{N}_\varepsilon^Z(Y) \supseteq X\},$$

where $\mathcal{N}_\varepsilon^Z(-)$ denotes the ε-neighborhood of a set in Z. It can be easily shown that $d_{GH}(X, Y) = 0$ implies that X and Y are isometric. Thus, the space of compact metric spaces modulo isometries is itself a metric space. Such "universal" space can be used to study degeneration of Riemannian manifolds to singular spaces. A usual way to bound the GH distance to study convergence in such "weak" GH topology is by finding functions $f : X \to Y$ which are ε-dense and ε-isometries.

A crucial question is to find criteria of convergence in such GH topology (*pre-compactness*), *i.e.* given a sequence (X_i) of compact metric spaces, find conditions which guarantee that such sequence subconverges to a compact metric space X_∞.

The fundamental result is the Gromov's pre-compactness theorem. A set $\mathcal{X}$ of compact metric spaces is *uniformly totally bounded* (UTB) if:

– $\exists D > 0$ such that $\text{diam}(X) < D$ for all $X \in \mathcal{X}$;
– $\forall \varepsilon > 0$, $\exists N = N(\varepsilon)$ such that for all $X \in \mathcal{X}$, $\exists S_X$ ε-dense set of cardinality at most N.

Theorem 6.4 (Gromov's pre-compactness). *If $\mathcal{X}$ is UTB, then $\mathcal{X}$ is pre-compact.*

The proof is simple (for more details and for more references on GH topology see [BBI01]). The idea is to uniformly discretize the spaces in the family, apply a diagonalization argument, and take a metric completion to get a compact limit space. More precisely, let $\mathcal{X} \in X_n \supseteq S_n := S_{n,1} \cup S_{n,\frac{1}{2}} \cup \cdots = \{x_{i,n}\}_{i=1}^\infty$ where $S_{n,\frac{1}{j}}$ is a finite $\frac{1}{j}$-dense set with at most $N(\frac{1}{j})$ points independent of n. Since $d_{ij}^n := d_{X_n}(x_{i,n}, x_{j,n}) < D$, by diagonalization we can find a subsequence n_k such that $d_{ij}^{n_k} \to d_{ij}$. Take S a countable set and equip it with the pseudo-distance d_{ij}. Then its metric completion after identification between points of zero distance $X_\infty = \overline{S/d}$ is a compact metric space which is the GH limit of the sequence X_{n_k}.

Thus, given families of Riemannian manifolds, it becomes interesting to find conditions which will implies that the family is UTB in the GH distance. The idea is that if we have a uniform control of the volume of small balls, we may hope to control uniformly the number of them we need to cover a manifolds. It is in such consideration that the Ricci curvature enters the game! Recall that in normal coordinates centered at p, the Ricci tensor exactly measure the second order deviation of the volume form from the flat Euclidean measure:

$$dV_g = \left(1 - \frac{1}{6}\sum_{i,j} \text{Ric}_{ij}(p)x_i x_j + O(x^3)\right) d\underline{x}.$$

A global result on the behaviour of volume under Ricci lower bounds is given by the following crucial result. Let H_c be the n-dimensional simply-connected space of constant sectional curvature equal to c. Then:

Theorem 6.5 (Bishop-Gromov monotonicity). *If $\text{Ric}(g) \geq (n-1)cg$, for $c \in \mathbb{R}$, then the function $\frac{\text{Vol}(B_g(p,r))}{\text{Vol}(B_{H_c}(r))}$ is non-increasing.*

See [BBI01] for its proof. Its immediate consequence is:

Theorem 6.6 (Riemannian Gromov's precompactness). *The set of n-dimensional Riemannian manifolds with $\text{diam}(M, g) \leq D$ and $\text{Ric}(g) \geq c(n-1)g$ is GH pre-compact.*

Proof. It is sufficient to check the UTB property. Take $\{x_i\}$ a maximally ε-separated set, *i.e.* $\{x_i\}$ is ε-dense and $B(x_i, \frac{\varepsilon}{2})$ are disjoint. Then, by monotonicity:

$$|\{x_i\}| \leq \frac{\text{Vol}(M, g)}{\text{Vol}(B_g(x_i, \frac{\varepsilon}{2}))} \leq \frac{\text{Vol}(B_{H_c}(D))}{\text{Vol}(B_{H_c}(\frac{\varepsilon}{2}))}.$$

Since the last term in the above expression depends only on c, D, and ε, we are done. □

If in the above theorem $c > 0$, then the diameter is automatically bounded thanks to Myers' theorem. In particular, let us note that the set of n-dimensional KE Fano manifolds is GH pre-compact, *i.e.* given any sequence of them we can extract a subsequence converging to a compact metric space.

Remark 6.7. *If the diameter bound does not hold, one still have pre-compactness in the* pointed GH sense*: by choosing a sequence of points* $p_i \in M_i$*, for all* $r > 0$*, closed* r*-balls centered at* p_i *sub-GH-converge to closed* r*-balls and center* p_∞ *in a metric space* X_∞ *(in general non-compact). In particular, this notion of convergence is important in the analysis of* metric bubbles *and* tangent cones *—both arising from "unbounded" rescalings (zoomings)— to study singularities of limits of* compact *Einstein spaces.*

In our problem, we need to consider limits of *conical* KE metrics. It is still possible to show that one can take limit in the GH topology also in this case (for example, but it is not trivial, it is possible to show that the cone angle can be "smoothed out" to genuine Riemannian metrics with lower bounds on the Ricci curvature and with control on their diameter [CDS15, part I]). Thus we can assume that our sequence in the Donaldson's continuity path (X, ω_{β_i}) subconverge in GH to a compact metric space X_∞. This is the "weak limit". Now we need to do some "regularity"!

6.4 Cheeger-Colding Theory ("smooth regularity")

In general, Cheeger-Colding(-Tian, for the Kähler case) theory, studies regularity of the GH limits X_∞ of sequences of Riemannian manifolds with Ricci bounded below. See survey [Che01]. In our situation we can (essentially) restrict to the Einstein case with volume *non-collapsing* hypothesis $\mathrm{Vol}(B(p, 1)) > C > 0$ (observe that if Ricci is positive, this is a simple consequence of Bishop-Gromov monotonicity). We focus on the absolute case, *i.e.* no conical manifolds. This case is interesting in itself (*e.g.* moduli compactifications in the KE Fano case), and still central in the more technical case of conical KE.

From a historical perspective, such theory generalizes results of Andersen, Bando, Kazue, Nakajima, and Tian of the ends of the eighties for the real four dimensional case, based on Ulhenbeck's ε-regularity techniques. In particular, under the above assumptions, they were able to show *orbifolds compactness*: *i.e.* X_∞ is a smooth Einstein space away from finitely many points which can be locally modeled on $\mathbb{R}^4/\Gamma_p$ (where $\Gamma_p \subset SO(4)$ is a finite subgroup acting freely on the 3-sphere, so that the metric pull-back to a smooth tensor (*orbifold smooth metric*).

In Cheeger-Colding theory "almost rigidity" theorems are used to define *stratification* of the limit space X_∞. A prototypical example of rigidity theorem in smooth Riemannian geometry is given by the Cheeger-Gromoll splitting theorem: a complete Riemannian manifold (M, g) with non-negative Ricci containing a *line* (*i.e.* an infinity length geodesic $\gamma : (-\infty, \infty) \to M$, whose subsegments are always minimizing) must be isometric to the split product $M \simeq N \times \mathbb{R}$. The idea of the proof consists in considering *Busemann functions* $b_{\gamma^\pm}(p) := \lim_{t \to \pm\infty} d(p, \gamma^\pm(t)) - t$. Using Laplacian comparison for the distance function under the Ricci curvature bound hypothesis, one can show that $b_{\gamma^\pm}$ are sub-harmonic. Since $b_{\gamma^+} + b_{\gamma^-} \geq 0$ taking value zero exactly on the image of γ, by the maximum principle the function $b_{\gamma^+} = -b_{\gamma^-}$ is harmonic. Finally, by Bochner's formula the unit length vector field ∇b_{γ^+} must be parallel, and thus its flow gives the desired splitting.

By performing an analysis of "approximate Busemann functions" via integral estimates, Cheeger-Colding theory provides the following "quantitative" version of the above theorem, known as *almost splitting theorem*. Let

– $\mathrm{Ric}(g) \geq -(n-1)\delta g$ (think $\delta \ll 1$);
– $p, q^+, q^- \in M$, with $d(p, q^\pm) \geq L \gg 1$ and $d(p, q^+) + d(p, q^-) - d(q^+, q^-) \leq \varepsilon \ll 1$.

Then $\exists$ a ball $B_R \subseteq \mathbb{R} \times X$, with X length space, such that $d_{GH}(B(p, R), B_R) \leq \phi(\delta, L^{-1}, \varepsilon | R)$, *i.e.* for fixed R, $\phi \to 0$, as $\delta, L^{-1}, \varepsilon, \to 0$.

As a corollary one sees that, by rescaling, pointed GH limits of sequences (M_i, g_i, p_i) with Ricci bounded below must split if they contain a line, since $\text{Ric}(\lambda_i g_i) \to 0$ for $\lambda_i \to \infty$.

More generally, this result, combined with "volume cones are metric cones", is the basis of the following structural picture for GH limits $(M_i, g_i) \to X_\infty$ of manifolds with Ricci bounded below, bounded diameter and volume non-collapsing.

Take $p \in X_\infty$ and define a (non-necessarily unique!) *metric tangent cone* at p to be $C_p(X_\infty) := \lim_{pGH}(X_\infty, p, \lambda_i d_\infty)$ as $\lambda_i \to \infty$. Then $C_p(X_\infty) = \mathbb{R}^k \times C(Y)$ with $C(Y)$ a *metric cone* (i.e. $d((r_1, y_1), (r_2, y_2)) = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(d_Y(y_1, y_2))}$ of diameter less than π and $n - k$ Hausdorff dimension. Note that the *link Y* may be singular. Define the *regular set*

$$\mathcal{R} := \{p \in X_\infty \mid \exists C_p(X_\infty) \cong_{\text{isom}} \mathbb{R}^n\},$$

which do not need to be open, and the *singular strata* for $k \in \mathbb{N}$,

$$\mathcal{S} := X_\infty \setminus \mathcal{R} \supseteq \mathcal{S}_k := \{p \in X_\infty \mid \text{no tangent cones at } p \text{ splits } \mathbb{R}^{k+1}\}.$$

Summarizing some of the main results of Cheeger-Colding theory:

Theorem 6.8. *Let $\mathcal{S}_0 \subseteq \cdots \subseteq \mathcal{S}_{n-2} = \mathcal{S} \subseteq X_\infty$ with Hausdorff dimension $\dim_{\mathcal{H}} \mathcal{S}_k \leq k$. If Ricci has two side bounds then $\mathcal{R}$ is open (actually a $C^{1,\alpha}$ manifold, C^∞ if the metrics are Einstein). If g_i are KE then $S = S_{n-4}$ (Cheeger-Colding-Tian).*

Let us go back to discuss in more details the Kähler situation.

6.5 Donaldson-Sun Theory ("algebraic regularity")

Let X_∞ be a limit of smooth complex n-dimensional KE Fano manifolds (M_i, g_i). The conical KE case is more technical, but part of the ideas can be already seen in this "absolute" situation. By Cheeger-Colding regularity we know that $\dim_{\mathcal{H}} X_\infty = 2n$, its singular set agrees with $\mathcal{S}_{2n-4}$ stratum, the regular set is open and on it we can find a (incomplete) smooth KE metric g_∞. Moreover, for all compact subset $K \subset\subset \mathcal{R}$ there exists diffeomorphisms onto their images $\psi_i : K \to M_i$ such that $\psi_i^\star g_i \to g_\infty$ in $C^{1,\alpha}$, and similarly for the complex structures J_i.

Not too surprisingly, the metric cones are now complex cones: $C_p(X_\infty) \cong_{isom} \mathbb{C}^k \times C(Y)$, with $C(Y)$ a Calabi-Yau cone, *i.e.* $\omega_{C(Y)} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} r_0^2$ where r_0 is the distance from the apex of the cone, and $Ric(\omega_{CY}) = 0$. The link Y is a potentially singular (Sasaki)-Einstein space with positive scalar curvature.

The main results of Donaldson and Sun [DS14] says the following:

Theorem 6.9. *The GH limit X_∞ is naturally* homeomorphic *to $W \subseteq \mathbb{CP}^n$ a singular Fano variety with $\mathcal{S}(X_\infty) = \text{Sing}(W)$.*

The rough (and very imprecise!) idea consists in proving a uniform Kodaira embedding using the pick sections method "near the singularities" ("on metric tangent cones"). The main technical result used in their proof is known as *partial C^0-estimates*. Let $b_k^{KE}(p) = \sum_i |s_i(p)|_{h_{KE}^k}^2$ be the Bergman Kernel using holomorphic sections L^2-orthonormal with respect to KE metrics. Then

Proposition 6.10. *$\exists k_0 = k_0(n, V)$, $b = b(n, V)$ so that $b_{k_0}^{KE} \geq b^2 > 0$ for all KE X^n of volume V.*

Then the identification of the GH limit with a complex variety follows by comparing the GH limit X_∞ with the "flat limit" obtaining as limit of the image $T_i^k(X_i)$ via Tian's L^2 orthonormal embedding T_i^k, for a fixed bounded k.

To bound the Bergman's kernel, we need to construct a pick section at any p (*i.e.* $s \in H^0(K_{X_i}^{-k_0})$ non vanishing at p), also, and crucially, near the singularity formations, with some uniform control. The idea

would be to trivialize the canonical on U_i near p with a local section $\sigma_0 = 1$. By cutting-off $\sigma := \chi\sigma_0$ we have a global smooth section picked at p and holomorphic near p and on the complement of U_i (where is identically zero). Next we want to solve the $\bar{\partial}$-equation $\bar{\partial}\tau = \bar{\partial}\sigma$ with estimates (of Hörmander's L^2-type), so that $s := \sigma - \tau$ will be fully holomorphic and picked. We need to find a nice U_i. The crucial idea is that, instead of working on X_i, we can work near the tip of a *metric tangent cone*, and then transplant the region on X_i via the diffeomorphism ϕ_i provided by Cheeger-Colding theory. Working on such a region it is possible to construct a σ_0 on the tangent cone (as outlined above), whose $\bar{\partial}$ is small in L^2 (via a good cut of function). Define $\tau := \bar{\partial}^\star \Delta_{\bar{\partial}^\star}^{-1} \bar{\partial}\phi_i^\star \sigma_0$. Here $\Delta_{\bar{\partial}^\star}$ is the Laplacian on $(1,0)$ forms valued the k-plurianticanonical bundle. It is uniformly invertible for k big enough, since by Bochner's formula $\Delta_{\bar{\partial}^\star} = \nabla^\star\nabla + \frac{1}{k}Ric(\omega) + 1$, we have $(\Delta_{\bar{\partial}^\star}(s), s)_{L^2_{k\omega}} \geq \frac{1}{2}|s|^2_{L^2_{k\omega}}$. Let $\sigma := \phi_i^\star\sigma_0 - \tau$. Then it is holomorphic. Moreover $|\tau|^2_2 \leq 2|\bar{\partial}\phi_i^\star\sigma_0|^2_2$ which is small. By Moser's iteration its $\mathcal{C}^1$-norm is also small. Thus $|\sigma(\phi_i^{-1}(p))| \geq C(k_0) > 0$, for a k_0 bounded by the geometry independent of i. Since also $\nabla\sigma$ is bounded uniformly, T_i are Lipschitz and converge to some continuous map T_∞ which identifies the GH limit X_∞ with the flat limit.

Remark 6.11. *In reality, to get Theorem 6.9 from Proposition 6.10 is necessary to raise further (in a bounded way) the k_0 constant (in order to really separate points in the limit). Of course there are many other difficulties we haven't discussed in the oversimplified sketch given above.*

Remark 6.12.

- *For the conical case, the limit is a Fano variety W equipped with a (singular) limit divisors D_∞ where, generically, the singular limit KE metric is going to have some cone angle singularities in the transverse directions to smooth point of D_∞.*
- *It can be proved that the singularities of the limit are Kawamata log terminal (klt), that is $K_{\hat{X}} = \phi^\star K_X + \sum a_i E_i$, with $a_i > -1$ for any divisorial resolutions of the singularities (since this is essentially equivalent to say that near the singularity there is a (root) of a holomorphic volume form Ω such that $\int_U \Omega \wedge \bar{\Omega} < +\infty$). In dimension two these singularities coincide with quotient singularities, thus recovering the orbifold theory. However, in higher dimension this is not the case, e.g. the ODP singularity $\sum_{i=0}^{n} x^2 = 0$ is not of quotient type as long as $n \geq 3$. Moreover, the understanding of the metric near such singularities is much more subtle, but there are very important recent results in such direction [DS17]. .*
- *It can be shown that the singular limit KE metrics are* weak KE *in the pluripotential theory sense [EGZ09].*

6.6 Summing-up: the idea of the proof

Let us now go back to the Donaldson continuity path and say that X_∞ is the GH limit of a sequence of increasing conical KE (M, ω_{β_i}), with $\beta_i \nearrow \underline{\beta}$ solving (13). Then, by the above regularity theories, we can assume that $X_\infty \cong (W, D_\infty, \omega_{\underline{\beta}})$ for a weak (log) KE Fano pair. We are now again in the domain of algebraic geometry! The main trick is now to use such $(W, D_\infty, \omega_{\underline{\beta}})$ to construct a test configuration so that we can use the K-stability hypothesis.

The first step consists in extending the Donaldson-Futaki invariant to the pair setting: $(\mathcal{X}, \mathcal{D}) \to \mathbb{C}$, with $DF((X, (1-\beta)D), (\mathcal{X}, (1-\beta)D)) := DF(X, \mathcal{X}) + (1-\beta)\tilde{F}(D)$ in a natural way: if (X, D, ω_β) is a singular space with a weak conical metric, then for a test configuration induced by holomorphic vector fields tangential to D, we have $DF(\mathcal{X}, (1-\beta)D) = F_D(v)$, for the natural extension of the Futaki type invariant [LS14], as in Proposition 5.13. In particular, if ω_β is conical KE, $DF(\mathcal{X}, (1-\beta)D) = 0$ for such test configurations. Note the linearity in β of the conical (log) version of the Futaki invariant.

If $(W, D_\infty) \cong (M, D)$ we would have done by uniqueness of the KE metric. If not, then by Donaldson-Sun we know that (W, D_∞) is realized as a flat limit of (M, D) with some "universal" projective space $\mathbb{CP}^{n_k}$, for fixed uniform k. In particular $(W, D_\infty) \in \overline{PGL(n_k).(M, D)}$. Since $(W, D_\infty, \omega_{\underline{\beta}})$ is KE, by an extension of Matsushima's reductivity, we have that $Aut(W, D_\infty)$ is reductive. Hence, by the standard Luna's slice theorem for reductive groups, we can assume that there exists a $\mathbb{C}^\star \hookrightarrow Aut(W, D_\infty) \subseteq PGL(n_k)$ so that $(W, D) = \lim_{t \to 0} t.(M, D)$, i.e. (W, D) is the central fibre for a test configuration! In particular, for such test configuration the log Futaki

invariant is zero for parameter β since the central fibre has a KE metric. By linearity of the Futaki invariant, this implies that the "absolute" Futaki invariant for X has to be negative. But this contradicts our K-stability hypothesis. Thus only case one is indeed possible, and we can repeat the argument until $\beta = 1$, and finally construct a smooth KE metric on X! Be aware that the fact that cone angle metrics open up to usual *smooth* KE metrics is far from being a triviality, even if it is very intuitive.

This concludes the sketch of the arguments for proving Theorem 0.3.

Remark 6.13.
– *There are now different proofs of the equivalence between KE metric and K-stability: via the original Aubin's path [DS16], via Ricci flow [CSW15], via Calculus of Variations [BBJ15, Dem17].*
– *Checking K-stability to construct new KE metric is still very hard (if not impossible at the present state-of-the art!) due to the too many test configurations which a priori needed to be checked. Understanding better this issue is definitely a crucial aspect of future investigations in the field.*

6.7 KE moduli spaces and explicit examples

Let
$$\overline{\mathcal{EM}}^{GH} := \overline{\{(X^n, \omega) \,|\, \omega \text{ is KE Fano}\}/\text{biholo-isom}}^{GH}$$

be the *compactification of the moduli space of KE Fano manifolds* obtained by adding the singular weak KE spaces coming from GH-degenerations. In some sense, these spaces can be thought as higher dimensional generalizations of the Deligne-Mumford moduli compactification in the curve case, but now for the *positive* KE case. Thus, it should be not too surprisingly that it can be shown that such GH compactifications admit a natural structure of complex variety. Sometime they are also known as K-moduli space $\mathcal{KM}$, for obvious reasons. Can we find explicit examples of such $\overline{\mathcal{EM}}^{GH}$? This is important since testing K-stability, as we said, is still very hard: if we can explicitly described such moduli spaces in concrete situations, then we can understand exactly which Fano varieties in *given families* are indeed KE/K-polystable.

The idea to study such problem is via a "moduli continuity method": that is, given a family $\mathcal{X} \to H$ of Fano varieties, we would like to study the set of parameters for which the fibre variety X_t admits a KE metrics. If we can prove that there exists at least one fibre which is KE, that the KE condition is essentially open (in ideal situations) and that abstract GH limits are actually naturally embedded in our starting family, it is possible to conclude (via "stability comparison") that $\overline{\mathcal{EM}}_d^{GH} = \mathcal{H}_d /\!\!/ G$, where d is some numerical parameter (as the volume/degree) and G is a group acting on the parameter space giving a classical GIT quotient. In particular, K-stability would be equivalent to GIT for such family, and GIT stability can be in principle checked by hand. This strategy has been used for fully understanding the two dimensional case in [OSS16], extending results of [Tia90a, MM93].

For example one can study *cubic surfaces* in $\mathbb{CP}^3$. Their defining coefficient form a parameter space $\mathcal{H}_3 = \mathbb{CP}^{19}$ on which the group $SL(4, \mathbb{C})$ acts naturally by reparameterization. The Fermat cubic $x^3 + y^3 + z^3 + t^3 = 0$ is know to have alpha invariant bigger than $2/3$ and hence is KE. By implicit function theorem one can see that, among smooth cubic, the KE condition is open since they have only discrete automorphisms (actually, by [Tia90a] we also know that smooth ones are all KE). In any case let $X_\infty \cong W$ an abstract GH limit, which has only isolated orbifold singularities by the general regularity theory of degenerate limit recalled above. Then, by Bishop-Gromov inequality, we can compare the metric density at singular points with the total volume. This gives that $3 = c_1^2(W) < 12\sharp(\Gamma_p)^{-1}$. In particular, $\Gamma_p \cong \mathbb{Z}_2, \mathbb{Z}_3$. It can be shown that in such case only the $SU(2)$ representations survives (related to the notion of $\mathbb{Q}$-Gorenstein smoothability, which is a consequence of Donaldson-Sun theory). This means that W has only A_k singularities of the form $x^2 + y^2 = z^{k+1}$ with $k \leq 2$. It is a known result that two dimensional Fano varieties of degree three with such singularities have still very ample anti-canonical bundle, which thus provides embeddings $W \subseteq \mathbb{CP}^3$. By Berman results we know that W is K-polystable (since singular KE). It is then easy to see in such case that K-polystability implies GIT-polystabilty for the essentially unique linearization of the $SL(4, \mathbb{C})$ action. Thus, with some further work, $\overline{\mathcal{EM}}_3^{GH} \cong \mathcal{H}_3 /\!\!/ SL(4, \mathbb{C})$. The study of such GIT problem goes back to Hilbert. In particular, it is known

that cubics with at worse A_1-singularities are stable, and only (the orbit of) the toric $xyz = t^3$ (with $3A_2$ singularities), is strictly polystable. Thus this classifies also all the cubics which are KE.

Remark 6.14. *Very recently, new techniques (e.g. [Li15a, Liu16]) based on better understanding of the metric tangent cones in higher dimension and their relations with algebraic geometry, made possible to study in concrete situations such moduli spaces in higher dimensions too (e.g. [SS17, LX17]).*

References

[Aub76] Th. Aubin, Équations du type Monge-Ampère sur les variétés kähleriennes compactes, *C. R. Acad. Sci. Paris Sér. A-B* **283** (1976), no. 3, Aiii, A119–A121.

[Aub78] Th. Aubin, Équations du type Monge-Ampère sur les variétés kählériennes compactes, *Bull. Sci. Math. (2)* **102** (1978), no. 1, 63–95.

[Aub84] Th. Aubin, Réduction du cas positif de l'équation de Monge-Ampère sur les variétés kählériennes compactes à la démonstration d'une inégalité, *J. Funct. Anal.* **57** (1984), no. 2, 143–153.

[Aub98] Th. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.

[Bal06] W. Ballmann, *Lectures on Kähler manifolds*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2006.

[BM87] S. Bando, T. Mabuchi, Uniqueness of Einstein Kähler metrics modulo connected group actions, in *Algebraic geometry, Sendai, 1985*, 11–40, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.

[Ber13] R. J. Berman, A thermodynamical formalism for Monge-Ampère equations, Moser-Trudinger inequalities and Kähler-Einstein metrics, *Adv. Math.* **248** (2013), 1254–1297.

[Ber16] R. J. Berman, K-polystability of ℚ-Fano varieties admitting Kähler-Einstein metrics, *Invent. Math.* **203** (2016), no. 3, 973–1025.

[BB17] R. J. Berman, B. Berndtsson, Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics, *J. Amer. Math. Soc.* **30** (2017), no. 4, 1165–1196.

[BBJ15] R. Berman, S. Boucksom, M. Jonsson, A variational approach to the Yau-Tian-Donaldson conjecture, `arXiv:1509.04561`.

[Bes87] A. L. Besse, *Einstein manifolds*, Reprint of the 1987 edition, Classics in Mathematics. Springer-Verlag, Berlin, 2008.

[Bla56] A. Blanchard, Sur les variétés analytiques complexes, *Ann. Sci. Ecole Norm. Sup. (3)* **73** (1956), 157–202.

[Boc46] S. Bochner, Vector fields and Ricci curvature, *Bull. Amer. Math. Soc.* **52** (1946), 776–797.

[Bog78] F. A. Bogomolov, Holomorphic tensors and vector bundles on projective manifolds, *Izv. Akad. Nauk SSSR Ser. Mat.* **42** (1978), no. 6, 1227–1287, 1439. English translation in *Math. USSR-Izv.* **13** (1979), no. 3, 499–555.

[BBI01] D. Burago, Y. Burago, S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, **33**, American Mathematical Society, Providence, RI, 2001.

[CC95] L. A. Caffarelli, X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, **43**, American Mathematical Society, Providence, RI, 1995.

[Cal54] E. Calabi, The space of Kähler metrics, *Proc. Internat. Congress Math. Amsterdam* **2** (1954), pp. 206–207.

[Cal57]   E. Calabi, On Kähler manifolds with vanishing canonical class, in *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, pp. 78–89, Princeton University Press, Princeton, N. J., 1957.

[CdS01]   A. Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, **1764**, Springer-Verlag, Berlin, 2001.

[CL73]   J. B. Carrell, D. I. Lieberman, Holomorphic Vector fields and Kaehler Manifolds, *Invent. Math.* **21** (1973), 303–309.

[Cat99]   D. Catlin, The Bergman kernel and a theorem of Tian, in *Analysis and geometry in several complex variables (Katata, 1997)*, 1–23, Trends Math., Birkhäuser Boston, Boston, MA, 1999.

[Che01]   J. Cheeger, *Degeneration of Riemannian metrics under Ricci curvature bounds*, Lezioni Fermiane, Scuola Normale Superiore, Pisa, 2001.

[Che08]   I. A. Chel'tsov, Log canonical thresholds of del Pezzo surfaces, *Geom. Funct. Anal.* **18** (2008), no. 4, 1118–1144.

[CS08]   I. A. Chel'tsov, K. A. Shramov, Log-canonical thresholds for nonsingular Fano threefolds. With an appendix by J.-P. Demailly, *Uspekhi Mat. Nauk* **63** (2008), no. 5(383), 73–180; translation in *Russian Math. Surveys* **63** (2008), no. 5, 859–958.

[Che00]   X.x. Chen, The space of Kähler metrics, *J. Differential Geom.* **56** (2000), no. 2, 189–234.

[CDS14]   X.x. Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics and stability, *Int. Math. Res. Not. IMRN* **2014** (2014), no. 8, 2119–2125.

[CDS15]   X.x. Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities, II: Limits with cone angle less than 2π, III: Limits as cone angle approaches 2π and completion of the main proof, *J. Amer. Math. Soc.* **28** (2015), no. 1, 183–197, 199–234, 235–278.

[CSW15]   X.x. Chen, S. Sun, B. Wang, Kähler-Ricci flow, Kähler-Einstein metric, and K-stability, `arXiv:1508.04397`.

[CTW17]   J. Chu, V. Tosatti, B. Weinkove, On the $C^{1,1}$ Regularity of Geodesics in the Space of Kähler Metrics, *Ann. PDE* **3** (2017), no. 2, 3–15.

[DL12]   T. Darvas, L. Lempert, Weak geodesics in the space of Kähler metrics, *Math. Res. Lett.* **19** (2012), no. 5, 1127–1135.

[DS16]   V. Datar, G. Székelyhidi, Kähler-Einstein metrics along the smooth continuity method, *Geom. Funct. Anal.* **26** (2016), no. 4, 975–1010.

[Deb01]   O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001.

[DGMS75]   P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan, Real homotopy theory of Kähler manifolds, *Invent. Math.* **29** (1975), no. 3, 245–274.

[Dem12]   J.-P. Demailly, *Complex Analytic and Differential Geometry*, http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf, 2012.

[Dem17]   J.-P. Demailly, Variational approach for complex Monge-Ampère equations and geometric applications, Séminaire Bourbaki, Exposé 1112, *Astérisque* **390** (2017), 245–275.

[DK01]   J.-P. Demailly, J. Kollár, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, *Ann. Sci. École Norm. Sup. (4)* **34** (2001), no. 4, 525–556.

[Din88]   W. Y. Ding, Remarks on the existence problem of positive Kähler-Einstein metrics, *Math. Ann.* **282** (1988), no. 3, 463–471.

[Dol53]   P. Dolbeault, Sur la cohomologie des variétés analytiques complexes., *C. R. Acad. Sci. Paris* **236** (1953). 175–177.

[Don87]   S. K. Donaldson, Infinite determinants, stable bundles and curvature, *Duke Math. J.* **54** (1987), no. 1, 231–247.

[Don97]   S. K. Donaldson, Remarks on gauge theory, complex geometry and 4-manifold topology, in *Fields Medallists' lectures*, 384–403, World Sci. Ser. 20th Century Math., **5**, World Sci. Publ., River Edge, NJ, 1997.

[Don99]   S. K. Donaldson, Symmetric spaces, Kähler geometry and Hamiltonian dynamics, in *Northern California Symplectic Geometry Seminar*, 13–33, Amer. Math. Soc. Transl. Ser. 2, **196**, Adv. Math. Sci., **45**, Amer. Math. Soc., Providence, RI, 1999.

[Don01]   S. K. Donaldson, Scalar curvature and projective embeddings. I, *J. Differential Geom.* **59** (2001), no. 3, 479–522.

[Don02a]   S. K.Donaldson, Holomorphic discs and the complex Monge-Ampère equation, *J. Symplectic Geom.* **1** (2002), no. 2, 171–196.

[Don02b]   S. K. Donaldson, Scalar curvature and stability of toric varieties, *J. Differential Geom.* **62** (2002), no. 2, 289–349.

[Don05]   S. K. Donaldson, Lower bounds on the Calabi functional, *J. Differential Geom.* **70** (2005), no. 3, 453–472.

[Don12]   S. K. Donaldson, Kähler metrics with cone singularities along a divisor, in *Essays in mathematics and its applications*, 49–79, Springer, Heidelberg, 2012.

[Don15]   S. K. Donaldson, Algebraic families of constant scalar curvature Kähler metrics, in *Surveys in differential geometry 2014. Regularity and evolution of nonlinear equations*, 111–137, Surv. Differ. Geom., **19**, Int. Press, Somerville, MA, 2015.

[DS14]   S. K. Donaldson, S. Sun, Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, *Acta Math.* **213** (2014), no. 1, 63–106.

[DS17]   S. K. Donaldson, S. Sun, Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, II, *J. Differential Geom.* **107** (2017), no. 2, 327–371.

[Eva82]   L. C. Evans, Classical solutions of fully nonlinear, convex, second-order elliptic equations, *Comm. Pure Appl. Math.* **35** (1982), 333–363.

[EGZ09]   Ph. Eyssidieux, V. Guedj, A. Zeriahi, Singular Kähler-Einstein metrics, *J. Amer. Math. Soc.* **22** (2009), no. 3, 607–639.

[Fig17] A. Figalli, *The Monge-Ampère equation and its applications*, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2017.

[Fuj90] A. Fujiki, Moduli space of polarized algebraic manifolds and Kähler metrics [translation of Sûgaku **42** (1990), no. 3, 231–243], Sugaku Expositions **5** (1992), no. 2, 173–191.

[Fuj15] K. Fujita, Optimal bounds for the volumes of Kähler-Einstein Fano manifolds, `arXiv:1508.04578`.

[Fuj16a] K. Fujita, A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties, `arXiv:1602.00901`.

[Fuj16b] K. Fujita, K-stability of Fano manifolds with not small alpha invariants, `arXiv:1606.08261`.

[Fut83] A. Futaki, An obstruction to the existence of Einstein Kähler metrics, *Invent. Math.* **73** (1983), no. 3, 437–443.

[Gau15] P. Gauduchon, *Calabi's extremal Kähler metrics: An elementary introduction*, http://germanio.math.unifi.it/wp-content/uploads/2015/03/dercalabi.pdf.

[GT98] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[GH78] Ph. Griffiths, J. Harris, *Principles of algebraic geometry*, Reprint of the 1978 original, Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[LNM2038] *Complex Monge-Ampère equations and geodesics in the space of Kähler metrics*, Edited by Vincent Guedj, Lecture Notes in Mathematics, **2038**, Springer, Heidelberg, 2012.

[GZ17] V. Guedj, A. Zeriahi, *Degenerate complex Monge-Ampère equations*, EMS Tracts in Mathematics, **26**, European Mathematical Society (EMS), Zürich, 2017.

[Gue15] H. Guenancia, Kähler-Einstein metrics: from cones to cusps, `arXiv:1504.01947`.

[Huy05] D. Huybrechts, *Complex geometry. An introduction*, Universitext, Springer-Verlag, Berlin, 2005.

[JMR16] T. Jeffres, R. Mazzeo, Y. A. Rubinstein, Kähler-Einstein metrics with edge singularities, *Ann. of Math. (2)* **183** (2016), no. 1, 95–176.

[Kob74] S. Kobayashi, On compact Kähler manifolds with positive definite Ricci tensor, *Ann. of Math. (2)* **74** (1961), 570–574.

[KN2] S. Kobayashi, K. Nomizu, *Foundations of differential geometry. Vol. II*, Reprint of the 1969 original, Wiley Classics Library, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.

[Kod54] K. Kodaira, On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties), *Ann. Math. (2)* **60** (1954), 28–48.

[Kod86] K. Kodaira, *Complex manifolds and deformation of complex structures*, Translated from the 1981 Japanese original by Kazuo Akao, Reprint of the 1986 English edition, Classics in Mathematics, Springer-Verlag, Berlin, 2005.

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **3**, Folge, A Series of Modern Surveys in Mathematics, **32**, Springer-Verlag, Berlin, 1996.

[KMM92] J. Kollár, Y. Miyaoka, S. Mori, Rational Connectedness and Boundedness of Fano Manifolds, *J. Diff. Geom.* **36** (1992), 765–769.

[KM98] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original, Cambridge Tracts in Mathematics, **134**, Cambridge University Press, Cambridge, 1998.

[Kry82] N. V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations, *Izv. Akad. Nauk SSSR Ser. Mat.* **46** (1982), 487–523.

[LBS94] C. R. LeBrun, S. Simanca, Extremal Kähler metrics and complex deformation theory, *Geom. Funct. Analysis* **4** (1994), no. 3, 298–336.

[LV13] L. Lempert, L. Vivas, Geodesics in the space of Kähler metrics, *Duke Math. J.* **162** (2013), no. 7, 1369–1381.

[Li11] C. Li, Greatest lower bounds on Ricci curvature for toric Fano manifolds, *Adv. Math.* **226** (2011), no. 6, 4921–4932.

[Li15a] C. Li, Minimizing normalized volumes of valuations, `arXiv:1511.08164`.

[Li15b] C. Li, Remarks on logarithmic K-stability, *Commun. Contemp. Math.* **17** (2015), no. 2, 1450020, 17 pp..

[LS14] C. Li, S. Sun, Conical Kähler-Einstein metrics revisited, *Comm. Math. Phys.* **331** (2014), no. 3, 927–973.

[LX14] C. Li, C. Xu, Special test configuration and K-stability of Fano varieties, *Ann. of Math. (2)* **180** (2014), no. 1, 197–232.

[LY87] J. Li, S.-T. Yau, Hermitian-Yang-Mills connection on non-Kähler manifolds, in *Mathematical aspects of string theory (San Diego, Calif., 1986)*, 560–573, Adv. Ser. Math. Phys., **1**, World Sci. Publishing, Singapore, 1987.

[Lic58] A. Lichnerowicz, *Géométrie des groupes de transformation*, Travaux et Recherches Mathématiques **3**, Dunod (1958).

[Liu16] Y. Liu, The volume of singular Kähler-Einstein Fano varieties, `arXiv:1605.01034`.

[LX17] Y. Liu, C. Xu, K-stability of cubic threefolds, `arXiv:1706.01933`.

[Lu00] Z. Lu, On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch, *Amer. J. Math.* **122** (2000), no. 2, 235–273.

[LT95] M. Lübke, A. Teleman, *The Kobayashi-Hitchin correspondence*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[Mab87] T. Mabuchi, A Functional Integrating Futaki's Invariant, *Proc. Japan Acad. Ser. A* **61** (1985), no. 4, 119–120.

[Mab87] T. Mabuchi, Some symplectic geometry on compact Kähler manifolds. I, *Osaka J. Math.* **24** (1987), no. 2, 227–252.

[Mab08] T. Mabuchi, K-stability of constant scalar curvature polarization, `arXiv:0812.4093`.

[MM93] T. Mabuchi, S. Mukai, Stability and Einstein-Kähler metric of a quartic del Pezzo surface, in *Einstein metrics and Yang-Mills connections (Sanda, 1990)*, 133–160, Lecture Notes in Pure and Appl. Math., **145**, Dekker, New York, 1993.

[MS06]   D. Martelli, J. Sparks, Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals, *Comm. Math. Phys.* **262** (2006), no. 1, 51–89.

[Mat57]   Y. Matsushima, Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne, *Nagoya Math. J.* **11** (1957), 145–150.

[Miy77]   Y. Miyaoka, On the Chern numbers of surfaces of general type, *Invent. Math.* **42** (1977), 225–237.

[Mor79]   S. Mori, Projective manifolds with ample tangent bundles, *Ann. of Math.* **110** (1979), 593–606.

[Mor07]   A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts, **69**, Cambridge University Press, Cambridge, 2007.

[MK71]   J. Morrow, K. Kodaira, *Complex manifolds*, Reprint of the 1971 edition with errata, AMS Chelsea Publishing, Providence, RI, 2006.

[Mos65]   J. Moser, On the volume elements on a manifold, *Trans. Amer. Math. Soc.* **120** (1965), 286–294.

[Mum77]   D. Mumford, Stability of projective varieties, *Enseignement Math. (2)* **23** (1977), no. 1-2, 39–110.

[Mum79]   D. Mumford, An algebraic surface with K ample, $(K^2) = 9$, $p_g = q = 0$, *Amer. J. Math.* **101** (1979), no. 1, 233–244.

[MFK94]   D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory*, Third edition, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), **34**, Springer-Verlag, Berlin, 1994.

[MS39]   S. B. Myers, N. E. Steenrod, The group of isometries of a Riemannian manifold, *Ann. of Math. (2)* **40** (1939), no. 2, 400–416.

[NN57]   A. Newlander, L. Nirenberg, Complex analytic coordinates in almost complex manifolds, *Ann. of Math. (2)* **65** (1957), 391–404.

[New78]   P. E. Newstead, *Introduction to moduli problems and orbit spaces*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, **51**, Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978.

[OSS16]   Y. Odaka, C. Spotti, S. Sun, Compact moduli spaces of del Pezzo surfaces and Kähler-Einstein metrics, *J. Differential Geom.* **102** (2016), no. 1, 127–172.

[Pag79]   D. Page, A Compact Rotating Gravitational Instanton, *Phys. Lett. B* **79B** (1979), 235–238.

[PT09]   S. T. Paul, G. Tian, CM stability and the generalized Futaki invariant II, *Astérisque* **328 (2009)** (2010), 339–354.

[PSS07]   D. H. Phong, N. Sesum, J. Sturm, Multiplier ideal sheaves and the Kähler-Ricci flow, *Comm. Anal. Geom.* **15** (2007), no. 3, 613–632.

[RT07]   J. Ross, R. Thomas, A study of the Hilbert-Mumford criterion for the stability of projective varieties, *J. Algebraic Geom.* **16** (2007), no. 2, 201– 255.

[Rua98]   W.-D. Ruan, Canonical coordinates and Bergmann metrics, *Comm. Anal. Geom.* **6** (1998), no. 3, 589–631.

[Sem92]   S. Semmes, Complex Monge-Ampère and symplectic manifolds, *Amer. J. Math.* **114** (1992), no. 3, 495–550.

[Spo17]   C. Spotti, Kähler-Einstein Metrics on $\mathbb{Q}$-Smoothable Fano Varieties, Their Moduli and Some Applications, in *Complex and Symplectic Geometry*, 211–229, Springer INdAM Series 21, Springer, 2017.

[SS17]   C. Spotti, S. Sun, Explicit Gromov-Hausdorff compactifications of moduli spaces of Kähler-Einstein Fano manifolds, `arXiv:1705.00377`.

[Sto09]   J. Stoppa, K-stability of constant scalar curvature Kähler manifolds, *Adv. Math.* **221** (2009), no. 4, 1397–1408.

[Szé11]   G. Székelyhidi, Greatest lower bounds on the Ricci curvature of Fano manifolds, *Compos. Math.* **147** (2011), no. 1, 319–331.

[Szé14]   G. Székelyhidi, *An introduction to extremal Kähler metrics*, Graduate Studies in Mathematics, **152**, American Mathematical Society, Providence, RI, 2014.

[Tho06]   R. P. Thomas, Notes on GIT and symplectic reduction for bundles and varieties, *Surveys in differential geometry. Vol. X*, 221–273, *Surv. Differ. Geom.*, **10**, Int. Press, Somerville, MA, 2006.

[Thu82]   W. P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), no. 3, 357–381.

[Tian87]   G. Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, *Invent. Math.* **89** (1987), no. 2, 225–246.

[Tia90a]   G. Tian, On Calabi's conjecture for complex surfaces with positive first Chern class, *Invent. Math.* **101** (1990), no. 1, 101–172.

[Tia90b]   G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, *J. Differential Geom.* **32** (1990), no. 1, 99–130.

[Tia92]   G. Tian, On stability of the tangent bundles of Fano varieties, *Internat. J. Math.* **3** (1992), no. 3, 401–413.

[Tia94]   G. Tian, The K-energy on hypersurfaces and stability, *Comm. Anal. Geom.* **2** (1994), no. 2, 239–265.

[Tia97]   G. Tian, Kähler-Einstein metrics with positive scalar curvature, *Invent. Math.* **130** (1997), no. 1, 1–37.

[Tia00]   G. Tian, *Canonical metrics in Kähler geometry*, Notes taken by Meike Akveld, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2000.

[TY90]   G. Tian, S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature. I, *J. Amer. Math. Soc.* **3** (1990), no. 3, 579–609.

[TY91]   G. Tian, S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature. II, *Invent. Math.* **106** (1991), no. 1, 27–60.

[Tos17]   V. Tosatti, Uniqueness of $\mathbb{CP}^n$, *Expo. Math.* **35** (2017), no. 1, 1–12.

[UY86]   K. Uhlenbeck, S.-T. Yau, On the existence of Hermitian-Yang-Mills connections in stable vector bundles, *Frontiers of the mathematical sciences: 1985 (New York, 1985)*, Comm. Pure Appl. Math. **39** (1986), no. S, suppl., S257-–S293.

[VdV66] A. Van de Ven, On the Chern numbers of certain complex and almost complex manifolds, *Proc. Nat. Acad. Sci. U.S.A.* **55** (1966), 1624–1627.

[Voi07] C. Voisin, *Hodge theory and complex algebraic geometry. I*, Translated from the French by Leila Schneps, Reprint of the 2002 English edition, Cambridge Studies in Advanced Mathematics, **76**, Cambridge University Press, Cambridge, 2007.

[WZ04] X.-J. Wang, X. Zhu, Kähler-Ricci solitons on toric manifolds with positive first Chern class, *Adv. Math.* **188** (2004), no. 1, 87–103.

[Wel08] R. O. Wells, *Differential analysis on complex manifolds*, Third edition, With a new appendix by Oscar Garcia-Prada, Graduate Texts in Mathematics, **65**, Springer, New York, 2008.

[Yau77] S.-T. Yau, Calabi's conjecture and some new results in algebraic geometry, *Proc. Nat. Acad. Sci. U.S.A.* **74** (1977), no. 5, 1798–1799.

[Yau78] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), no. 3, 339–411.

[Zel98] S. Zelditch, Szegő kernels and a theorem of Tian, *Internat. Math. Res. Notices* **1998** (1998), no. 6, 317–331.