



Complex Manifolds

Review Article

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A survey on Inverse mean curvature flow in ROSSes

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Abstract: In this survey we discuss the evolution by inverse mean curvature flow of star-shaped mean convex hypersurfaces in non-compact rank one symmetric spaces. We show similarities and differences between the case considered, with particular attention to how the geometry of the ambient manifolds influences the behaviour of the evolution. Moreover we try, when possible, to give an unified approach to the results present in literature.

Keywords: Inverse mean curvature flow, Sub-Riemannian geometry, Webster curvature, Qc curvature

MSC: 53C17, 53C40, 53C44

1 Introduction

In the last decades geometric flows have been intensively studied in many contexts. Inverse mean curvature flow is a way to deform hypersurfaces in the normal direction with speed proportional to the inverse of the mean curvature. More precisely, fix a Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{g})$, called *ambient manifold* and a smooth hypersurface $F_0 : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$. The inverse mean curvature flow with initial datum F_0 is a smooth one-parameter family of smooth immersions $F : \mathcal{M} \times [0, T) \rightarrow \tilde{\mathcal{M}}$ such that

$$\begin{cases} \frac{\partial F}{\partial t} = \frac{1}{H}\nu, \\ F(\cdot, 0) = F_0, \end{cases} \quad (1)$$

where H is the mean curvature of $F_t = F(\cdot, t)$ and ν is the outward unit normal vector of F_t . The problem is well posed if the initial datum is mean convex, that is $H > 0$ everywhere on $\mathcal{M}_0 = F_0(\mathcal{M})$.

It can be checked that (1) is a *geometric* flow in the sense that it does not depend on the parametrization of the hypersurface. Then a solution of the inverse mean curvature flow will be often expressed as the family of immersed hypersurfaces $(\mathcal{M}_t)_{t \geq 0}$, where $\mathcal{M}_t = F(\mathcal{M}, t)$.

The inverse mean curvature flow is the leading example in the class of expanding flows, hence, if the flow can be defined for any positive time, then it explores the structure at infinity of the ambient space. It has many applications in General relativity, for example it was the main tool in proving the Penrose inequality in the celebrated paper [HI].

On the other hand, a very prolific and interesting research field is the study of real hypersurfaces in complex and quaternionic space forms. They are examples of rank one symmetric spaces, or ROSSes for short. In this survey we restrict our attention to the evolution of star-shaped hypersurfaces: an hypersurface of a Riemannian manifold is said star-shaped if there exists a point such that every geodesic passing from

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this point intersects the hypersurface in two distinct point in a trasversal way. In particular, any star-shaped hypersurface is an embedding of a sphere of suitable dimension.

The aim of this survey is to give a review on the state of art on the evolution by inverse mean curvature flow of star-shaped hypersurfaces in rank one symmetric spaces. We start reviewing classical results in (real) space forms [Ge1, Ge2, Ur] and the recent progresses in this subject: the fundamental remark of [HW] in case of the hyperbolic space and generalization to complex and quaternionic space forms of negative curvature [Pi2, Pi3].

The behaviour of the flow depends on the initial datum, of course, but the geometry of the ambient manifold has a crucial role too. One of the goal of this work is to show the interactions between the flow and the geometry of the ambient manifolds. We will discuss analogies and differences, in results and techniques, between the cases mentioned above. Moreover we try to introduce an unified approach as far as possible. The main similarities are that, in every case, star-shapedness and mean convexity are proprieties preserved by the flow, and that the flow can be defined for any positive time. In this way, the family of induced metrics on the evolving hypersurface produce a family of Riemannian metrics on the sphere of suitable dimension.

A first natural question is then to understand if such family converges to some nice limit when time goes to infinity. At this level we find the first important influence of the geometry of the ambient manifolds: in case of real space forms, we have a convergence to a Riemannian limit, but, considering the case of the other ROSSes, the family of metric degenerates in some special directions and the limit is only sub-Riemannian.

Once we have clarify the convergence, a second question is to classify the possible limits in terms of their curvature: in particular authors tried to identify conditions on the initial data in order to ensure a limit with constant scalar curvature. When the limit is Riemannian it is clear what we mean with curvature, however an interesting phenomenon appears: if the ambient manifold is the Euclidean space the limit is always the round metric on the sphere, when the ambient manifold is the hyperbolic space we have infinite examples of not round limit. It is the main result of [HW]. We describe the strategy for the construction of non round limits in Section 5.1. For hypersurfaces in the other ROSSes, as we said, the limit is only sub-Riemannian and the usual notion of curvature does not work, see Remark 3.5 below. The Webster and the quaternionic contact curvature were proposed in [Pi2, Pi3] as the right notion of curvature to use in this context. We introduced these notions and their main properties in Sections 4.2 and 4.3. Moreover, with a strategy inspired by [HW] is possible to see that in this setting too there are examples which produce limits with non constant scalar curvature.

The same kind of problem has been studied also in other ambient manifolds: for example rotationally symmetric in [Di] and asymptotically hyperbolic in [Ne] and warped products [Sc, Zh]. Evolution of star-shaped hypersurfaces in ROSSes was studied for different purpose in [KS] too.

2 Rank one symmetric spaces

The theory of symmetric spaces is very rich and well known. They can be seen in many ways, but in the present paper we privilege the Riemannian point of view. A classical introduction to these manifolds is Helgason's book [He]. Here we just recall some basic facts, focusing mostly on the small class of ROSSes.

Definition 2.1. A Riemannian manifold (\mathcal{M}, g) (or simply \mathcal{M} if it is clear what is the metric) is said symmetric if for every point $p \in \mathcal{M}$ there exists an isometry φ_p of (\mathcal{M}, g) such that

- (1) $\varphi_p^2 = id_{\mathcal{M}}$, where $id_{\mathcal{M}}$ is the identity map of \mathcal{M} ;
- (2) p is an isolated fixed point of φ_p .

Such φ_p is called symmetry at p . We say that \mathcal{M} is locally symmetric if conditions (1) holds just in a neighbourhood of p .

Obviously any symmetric manifold is also locally symmetric. It can be proved that the contrary is true if \mathcal{M} is simply connected. Symmetric spaces have many important properties. We list just few of them.

Proposition 2.2. *Let (\mathcal{M}, g) be a symmetric space, then the following hold:*

- (1) *for every $p \in \mathcal{M}$ and every geodesic γ such that $\gamma(0) = p$ we have $\varphi_p \circ \gamma(t) = \gamma(-t)$;*
- (2) *\mathcal{M} is complete;*
- (3) *\mathcal{M} is homogeneous, i.e. the isometry group acts transitively on \mathcal{M} ;*
- (4) *Let ∇ be the Levi-Civita connection of g and R its Riemann curvature tensor, then $\nabla R = 0$.*

Definition 2.3. *Let \mathcal{M} be a symmetric space, a totally geodesic submanifold of \mathcal{M} is said flat if it is isometric to the Euclidean space and it is said maximal if it is not contained properly in any other flat submanifold of \mathcal{M} . It can be checked that all such maximal flat submanifolds have the same dimension. We call this number the rank of \mathcal{M} .*

The simplest symmetric spaces is, of course, the Euclidean space \mathbb{R}^n . Trivially its rank is n . In general the rank is always greater or equal to 1 because geodesic curves can be considered as flat submanifolds of dimension 1. The rank one symmetric spaces is a well known class of manifolds. They are finitely many and they can be divided into compact and non-compact. They can be characterized in many ways. For example we have

Proposition 2.4.

- (1) *(\mathcal{M}, g) is a compact rank one symmetric space if and only if symmetric with strictly positive sectional curvature.*
- (2) *(\mathcal{M}, g) is a non-compact rank one symmetric space if and only if symmetric with strictly negative sectional curvature.*

The first class is often denoted with the acronym CROSSes and it is composed by the round sphere \mathbb{S}^n , and the projective spaces $\mathbb{K}\mathbb{P}^n$, where \mathbb{K} is one of the following algebras: the real \mathbb{R} , the complex \mathbb{C} , the quaternionic \mathbb{H} or the octonion numbers \mathbb{O} . In general n can be any integer, except to the last case where only $n = 2$ is allowed. $\mathbb{O}\mathbb{P}^2$ is called the *Cayley plane*. An excellent introduction to these spaces can be found in Chapter 3 of [Be].

In the present paper we focus mostly on non-compact rank one symmetric spaces. In the following, they will be called simply rank one symmetric spaces or with the acronym ROSSes. As suggested by Proposition 2.4, in the class of ROSSes we find the hyperbolic analogous of the CROSSes.

2.1 Real hyperbolic space

The (real) hyperbolic space $\mathbb{R}\mathbb{H}^n$, or simply \mathbb{H}^n , is the best known example in the class of ROSSes. Probably, it does not need any presentations. However, we spend some words that will be useful later on. It is the space form of constant negative curvature. Up to a homothety we can consider that its sectional curvature is equal to -1 . A model for \mathbb{H}^n is (\mathbb{R}^n, \bar{g}) where, in polar coordinates, \bar{g} is given by

$$\bar{g} = d\rho^2 + \sinh^2(\rho)\sigma. \quad (2)$$

Here and in the following σ will denote always the usual round metric on \mathbb{S}^{n-1} with constant sectional curvature 1. The Riemann curvature tensor of \bar{g} has the following explicit expression:

$$\bar{R}(X, Y, Z; V) = -(\bar{g}(X, Z)\bar{g}(Y, V) - \bar{g}(X, V)\bar{g}(Y, Z)). \quad (3)$$

2.2 Complex and the quaternionic hyperbolic space

The complex hyperbolic space $\mathbb{C}\mathbb{H}^n$ and the quaternionic hyperbolic space $\mathbb{H}\mathbb{H}^n$ are respectively the complex and the quaternionic analogous of the real hyperbolic space. Like in the case of \mathbb{H}^n , there are many equivalent models. Let \mathbb{K} be one of the field \mathbb{C} or the associative algebra \mathbb{H} , and define

$$a = \dim_{\mathbb{R}}\mathbb{K} = \begin{cases} 2 & \text{if } \mathbb{K} = \mathbb{C}; \\ 4 & \text{if } \mathbb{K} = \mathbb{H}. \end{cases} \quad (4)$$

For any $z = (z_1, \dots, z_{n+1}), w = (w_1, \dots, w_{n+1}) \in \mathbb{K}^{n+1}$ let us define the following \mathbb{K} -analogous of the Minkowski metric in \mathbb{R}^{n+1} :

$$\langle z, w \rangle = \sum_{i=1}^n z_i \bar{w}_i - z_{n+1} \bar{w}_{n+1}.$$

Since for every z , $\langle z, z \rangle$ is real, we may define the set $V_- = \{z \in \mathbb{K}^{n+1} \mid \langle z, z \rangle < 0\}$. Clearly if $z \in V_-$, then all the subspace spanned by z belongs to V_- . If $z \neq 0$, we call this 1-dimensional subspace a *negative line*. $\mathbb{K}\mathbb{H}^n$ is $\mathbb{P}V_-$, i.e. the collection of all the negative lines. The product $\langle \cdot, \cdot \rangle$ induces a Riemannian metric on $\mathbb{K}\mathbb{H}^n$. When $\mathbb{K} = \mathbb{C}$ it is called *Bergman metric*. In this way $\mathbb{K}\mathbb{H}^n$ is a \mathbb{K} -manifold of dimension n . The structure of \mathbb{K} induces the complex structures of $\mathbb{K}\mathbb{H}^n$. We are interested mostly in its Riemannian structure, so we will consider it as a (real) manifold of dimension an .

In the following we will use another model, precisely like in (2), we wish to introduce polar coordinates. In order to do that, we need to define some special metrics on the sphere. Let us consider the sphere \mathbb{S}^{an-1} canonically embedded in \mathbb{R}^{an} and let ν be the outward unit normal vector of this immersion. \mathbb{R}^{an} can be identified with \mathbb{K}^n , and on \mathbb{K}^n we have the usual complex structures:

- (1) if $\mathbb{K} = \mathbb{C}$ we have $J = J_1$ the multiplication by the imaginary unit, in particular

$$J^2 = -id;$$

- (2) if $\mathbb{K} = \mathbb{H}$ we have J_1, J_2 and J_3 the multiplication by the three quaternionic units, in particular

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -id.$$

For every $1 \leq i \leq a - 1$ we can define $\xi_i = J_i \nu$. They are unit vector fields tangent to \mathbb{S}^{an-1} . Usually they are called *Hopf vector fields* because they are the vector fields tangent to the fibers of the Hopf fibration. The distribution \mathcal{V} generated by the Hopf vector fields is called the *vertical distribution*. The distribution $\mathcal{H} = \mathcal{V}^\perp$, where the \mathcal{V}^\perp denotes the orthogonal complement of \mathcal{V} with respect to the round metric σ , is called the *horizontal distribution*.

Definition 2.5. For any $\mu > 0$, the Berger metric on \mathbb{S}^{an-1} of parameter μ is the metric e_μ obtained with a deformation of σ on \mathcal{V} . For every tangent vector X and Y we impose:

$$\begin{aligned} e_\mu(X, Y) &= \sigma(X, Y) \quad \text{if } X, Y \in \mathcal{H}, \\ e_\mu(X, Y) &= 0 \quad \text{if } X \in \mathcal{H}, Y \in \mathcal{V}, \\ e_\mu(X, Y) &= \mu\sigma(X, Y) \quad \text{if } X, Y \in \mathcal{V}. \end{aligned}$$

When μ goes to infinity, the Berger metric degenerates on \mathcal{V} and it is well defined only on \mathcal{H} . However the horizontal distribution has the nice property of being bracket generating, i.e. $\mathcal{H} + [\mathcal{H}, \mathcal{H}]$ is the whole tangent space. This implies that we can join any two point of the sphere with *horizontal curves* (i.e. curves such that at any point the tangent vector belongs to \mathcal{H}). Hence a metric defined only on \mathcal{H} is enough to define a distance on the sphere: we use the usual definition length of a curve and of distance between points, but restricted to the class of horizontal curves. This distance is called the *Carnot-Carathéodory distance*.

Definition 2.6. A metric defined only on a bracket generating distribution is called sub-Riemannian. The limit $\lim_{\mu \rightarrow \infty} e_\mu$ is called the standard sub Riemannian metric on \mathbb{S}^{an-1} and it is denoted by σ_{sR} .

Remark 2.7. We introduced a common notation for both the possible value of \mathbb{K} . We point out that the two cases produce two different deformation and two different sub-Riemannian limit: for example \mathcal{V} has dimension $a - 1$, hence it depends on the choice of \mathbb{K} . If not specified explicitly, the right interpretation of e_μ will be clear from the context.

Now we have all the ingredients to describe properly the metric of $\mathbb{K}\mathbb{H}^n$ in polar coordinates: on the underling manifold \mathbb{R}^{an} we consider the usual (real) polar coordinates, then the metric is

$$\bar{g} = d\rho^2 + \sinh^2(\rho)e_{\cosh^2(\rho)}, \tag{5}$$

where the Berger metric is defined according to Remark 2.7. Note that, in this case, \bar{g} is not a warped product because the metric $e_{\cosh^2(\rho)}$ changes with the radial coordinate too. Its Riemann curvature tensor has the following explicit expression

$$\begin{aligned} \bar{R}(X, Y, Z, W) = & -\bar{g}(X, Z)\bar{g}(Y, W) + \bar{g}(X, W)\bar{g}(Y, Z) \\ & - \sum_{i=1}^{a-1} \bar{g}(X, J_i Z)\bar{g}(Y, J_i W) + \bar{g}(X, J_i W)\bar{g}(Y, J_i Z) \\ & - 2 \sum_{i=1}^{a-1} \bar{g}(X, J_i Y)\bar{g}(Z, J_i W), \end{aligned} \tag{6}$$

where the J_i 's are the complex structure of $\mathbb{K}\mathbb{H}^n$ that, in this model, coincide with the usual one in $\mathbb{R}^{an} = \mathbb{K}^n$. It follows that the sectional curvature of a plane spanned by two orthonormal vectors X and Y is given by

$$\bar{K}(X \wedge Y) = -1 - 3 \sum_{i=1}^{a-1} \bar{g}(X, J_i Y)^2 = -1 - 3|pr_{Y\mathbb{K}}X|^2, \tag{7}$$

where $Y\mathbb{K}$ is the space spanned by $\{J_i Y\}_{i \leq a-1}$ and $pr_{Y\mathbb{K}}$ is the projection on $Y\mathbb{K}$. Then $-4 \leq \bar{K} \leq -1$ and it is equal to -1 (respectively to -4) if and only if X is orthogonal (respectively parallel) to $Y\mathbb{K}$. The fact that the sectional curvature of the planes spanned by X and $J_i X$ is constant for every X and i is synthesized saying that $\mathbb{K}\mathbb{H}^n$ is a \mathbb{K} -space form. Moreover $\mathbb{K}\mathbb{H}^n$ is Einstein with

$$\bar{Ric} = -(an + 3a - 4)\bar{g} \tag{8}$$

The last element of the set ROSSes is the *octonion hyperbolic plane* $\mathbb{O}\mathbb{H}^2$. It can be defined in a similar way using the algebra of octonions instead of \mathbb{K} . Since we will not use this space in the sequel we skip a precise definition of this manifold, it can be found for example in [Pa2]. More details and informations about the geometry ROSSes can be found in [Pa1, Pa2].

3 Inverse mean curvature flow

In (1) we defined the inverse mean curvature flow. The very first problem to deal is to understand if this flow has a solution. In local coordinates (1) produces a weakly parabolic PDE. Standard theory ensure the following existence result.

Theorem 3.1. *If the initial datum \mathcal{M}_0 is closed (i.e. connected, embedded, compact and without boundary) and mean convex (i.e. $H > 0$ everywhere), then (1) has an unique solution, at least for times t small enough.*

A proof of this result which holds for a wide class of flows can be found in Theorem 3.1 of [HP].

The behaviour of such solution depends, of course, on the initial datum. The geometry of the ambient manifold has a crucial role too. As we will see in the next sections, even in the small setting of geodesic spheres in ROSSes we have very different behaviour.

3.1 Explicit examples: geodesic spheres in ROSSes

In the Euclidean space and in the ROSSes, included the compact ones, the principal curvature of a geodesic sphere are constants that depend only on the radius (this characteristic does not hold, for examples, for other symmetric spaces). In particular the mean curvature is a constant that depends only on the radius too. Together with the uniqueness of the solution, this suggests that, in these ambient manifolds, the evolution of geodesic spheres is a family of geodesic spheres and everything is reduced to an ODE on the radius. Moreover, trivially, every geodesic sphere is a star-shaped hypersurface. In the following examples let \mathcal{M}_t be a geodesic sphere of radius $\rho = \rho(t)$.

Example 3.2. $\tilde{\mathcal{M}} = \mathbb{R}^{n+1}$

In this case $H(\rho) = \frac{\rho}{n}$, hence \mathcal{M}_t is an evolution by inverse mean curvature flow if and only if

$$\frac{d}{dt}\rho = \frac{\rho}{n}.$$

Integrating we can easily find that the explicit solution is

$$\rho(t) = \rho(0)e^{\frac{t}{n}}.$$

Example 3.3. $\bar{\mathcal{M}} = \mathbb{H}^{n+1}$

In this space it is well known that the geodesic spheres are totally umbilical hypersurfaces with second fundamental form

$$h_i^j = \frac{\cosh(\rho)}{\sinh(\rho)}\delta_i^j.$$

Then, in this case, the radial function satisfies:

$$\frac{d}{dt}\rho = \frac{\sinh(\rho)}{n \cosh(\rho)}.$$

The explicit solution is the following

$$\rho(t) = \sinh^{-1}\left(\sinh(\rho(0))e^{\frac{t}{n}}\right).$$

By (2), it easy to see that the induced metric on the geodesic sphere is

$$g = \sinh^2(\rho)\sigma,$$

Hence, we have that the rescaled induced metric

$$\tilde{g} = |\mathcal{M}|^{-\frac{2}{n}}g$$

converges to a Riemannian metric \tilde{g}_∞ which is a constant multiple of σ . Therefore, in this very special case, the curvature of the limit is constant.

Example 3.4. $\bar{\mathcal{M}} = \mathbb{K}\mathbb{H}^n$ with $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$

In these ROSSes the geodesic sphere are not totally umbilical, indeed it is well known that a totally umbilical hypersurface does not exist (Theorem 5.1 in [NR]). A geodesic sphere has 2 distinct principal curvatures:

$$\begin{cases} \frac{\sinh(\rho)}{\cosh(\rho)} + \frac{\cosh(\rho)}{\sinh(\rho)} & \text{with multiplicity } a - 1 \text{ and eigenvectors } \mathcal{V}; \\ \frac{\cosh(\rho)}{\sinh(\rho)} & \text{with multiplicity } a(n - 1) \text{ and eigenvectors } \mathcal{H}. \end{cases}$$

See also formulas (27) below for the second fundamental form of a generic star-shaped hypersurface. Therefore, like in the previous examples, we can find the explicit solution of the evolution of a geodesic sphere solving the following equation for the radius:

$$\frac{d}{dt}\rho = \left((a - 1)\frac{\sinh(\rho)}{\cosh(\rho)} + (an - 1)\frac{\cosh(\rho)}{\sinh(\rho)} \right)^{-1}.$$

The solution of this ODE cannot be written nicely in terms of elementary functions, but we can see that the flow is defined for every time and that

$$\rho \approx \frac{t}{a(n + 1) - 2}, \quad \text{as } t \rightarrow \infty.$$

From (5) we have that the induced metric is

$$\sinh^2(\rho)e_{\cosh^2(\rho)},$$

and the rescaled induce metric

$$\tilde{g} = |\mathcal{M}|^{-\frac{2}{a(n+1)-2}}g$$

converges to $c\sigma_{sR}$, for some constant $c > 0$. We see, in this first special example, the new phenomenon which characterize the inverse mean curvature flow in $\mathbb{K}\mathbb{H}^n$. For every finite time \tilde{g} is obviously Riemannian, but, as the time grows, it degenerate in some directions, hence the limit is only sub-Riemannian.

Remark 3.5. *The two main theorems present in this survey (Theorems 5.1 and 6.1 below) describe the long time behaviour of the evolution of a star-shaped hypersurface considering the roundness of the possible limits. In any case, rescaling the induced metric, we produce a family of Riemannian metrics on a sphere of suitable dimension. In Euclidean and hyperbolic space this family converges to a Riemannian limit and hence it is clear what we mean when we talk about curvature and roundness. A clarification is needed when the limit is sub-Riemannian, like in the last Example. At first one can try to compute the curvature of the rescaled induced metric at any finite time and then let the time go to infinity. In this case we will find something that diverges: for example, the sectional curvature of the Berger metric e_μ assume any value between $4 - 3\mu$ and μ . This fact shows that a new notion of curvature is required. It is the subject of the Sections 4.2 and 4.3.*

3.2 Evolution equations

Starting from the definition of the flow (1), we can compute the evolution of the main geometric quantities of a hypersurface. The proof of this Lemma is similar to the computation of the analogous equations for the mean curvature flow which can be found in [Hu]. We use the following notations: let g_{ij} be the induced metric, and g^{ij} its inverse; the second fundamental form is denoted with h_{ij} , while the mean curvature is $H = h_{ij}g^{ij}$. Finally $|\mathcal{M}_t|$ denotes the volume.

Lemma 3.6. *For any hypersurface in a general ambient manifold the following evolution equations hold:*

$$\begin{aligned}
 (1) \quad & \frac{\partial g_{ij}}{\partial t} = \frac{2}{H} h_{ij}, \\
 (2) \quad & \frac{\partial g^{ij}}{\partial t} = -\frac{2}{H} h^{ij}, \\
 (3) \quad & \frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - 2 \frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\bar{\text{Ric}}(\nu, \nu)}{H}, \\
 (4) \quad & \frac{\partial h_{ij}}{\partial t} = \frac{\Delta h_{ij}}{H^2} - \frac{2}{H^3} \nabla_i H \nabla_j H + \frac{|A|^2}{H^2} h_{ij} - \frac{2}{H} \bar{R}_{i0j0} + \bar{\text{Ric}}(\nu, \nu) \frac{h_{ij}}{H^2} \\
 & \quad + \frac{1}{H^2} g^{lr} g^{ms} (2\bar{R}_{risj} h_{lm} - \bar{R}_{rmis} h_{jl} - \bar{R}_{rmjs} h_{il}) \\
 & \quad - \frac{1}{H^2} (\bar{\nabla}_j \bar{R}_{oli}{}^l + \bar{\nabla}_i \bar{R}_{0lj}{}^l), \\
 (5) \quad & \frac{\partial h_i^j}{\partial t} = \frac{\Delta h_i^j}{H^2} - \frac{2}{H^3} \nabla_i H \nabla_k H g^{kj} + \frac{|A|^2}{H^2} h_i^j - \frac{2}{H} \bar{R}_{i0k0} g^{kj} - 2 \frac{h_i^k h_k^j}{H} \\
 & \quad + \bar{\text{Ric}}(\nu, \nu) \frac{h_i^j}{H^2} + \frac{1}{H^2} g^{lr} g^{ms} g^{kj} (2\bar{R}_{risk} h_{lm} - \bar{R}_{rmis} h_{kl} - \bar{R}_{rmks} h_{il}) \\
 & \quad - \frac{1}{H^2} (\bar{\nabla}^j \bar{R}_{oli}{}^l + \bar{\nabla}_i \bar{R}_{0l}{}^{jl}), \\
 (6) \quad & \frac{d|\mathcal{M}_t|}{dt} = |\mathcal{M}_t|, \\
 (7) \quad & \frac{\partial \nu}{\partial t} = \frac{\nabla H}{H^2}.
 \end{aligned}$$

Here and in the following we are using Einstein convention on repeated indices. The operation of raising/lowering the indices is done with respect to the metric: for example $h_i^j = h_{ik}g^{kj}$. Moreover the index 0 is reserved to the normal direction ν .

Remark 3.7. *Note that, integrating equation (6), we have that the inverse mean curvature flow is an expanding flow, precisely $|\mathcal{M}_t| = |\mathcal{M}_0| e^t$.*

Since we are considering only symmetric ambient space, as seen in Proposition 2.4, we have that $\bar{\nabla} \bar{R} = 0$, hence the formulas above can be simplified. They can be even simpler because we know the explicit expression of the curvature tensors, see (3) and (6).

4 The Yamabe problem

A central and classical subject in Geometric Analysis is the Yamabe problem: find, if they exist, the metrics with constant scalar curvature in a fixed conformal class. The detailed explanation of its solution goes beyond the purposes of the present work. Indeed there exist surveys devoted to it, we suggest [LP] and [BM] and the references therein.

Here we want to discuss only the special case of the conformal class of standard metric on the sphere. Where “standard” must be considered in a broad sense: we will discuss also the generalization of the (Riemannian) classical Yamabe problem to the class of CR and quaternionic contact structures on the sphere. The solution of this problem has been the starting point for the construction of the counterexamples in [HW, Pi2, Pi3]. We will describe these counterexamples in Sections 5.1 and 6.1

4.1 The classical case

Let (\mathbb{S}^n, σ) be the round sphere. We want to understand if there exist functions f such that the conformal change $\tilde{\sigma} = e^{2f}\sigma$ produces a metric with constant scalar curvature. It is well known that the sign of the scalar curvature is invariant, hence in our case it should be positive. The answer is given by this result of Obata [Ob]

Theorem 4.1. *The metric $\tilde{\sigma}$ has constant scalar curvature, then, up to a constant factor, it is obtained from σ by a conformal diffeomorphism on \mathbb{S}^n .*

In this way we have a full characterization of the functions f that realize the desired conformal multiple. We report Lemma 4 of [HW].

Lemma 4.2. *The metric $e^{2f}\sigma$ has constant scalar curvature if and only if e^{-f} is a linear combination of constants and first eigenfunctions on the sphere.*

4.2 CR-Yamabe problem

As noted in Remark 3.5, even in the very special case of a geodesic sphere of $\mathbb{C}\mathbb{H}^n$, the usual Riemannian curvature is not enough to understand the nature of the limit. In [Pi2] we proposed the Webster curvature as the right notion for investigate the roundness of the limit. It is a notion of CR-geometry, in fact, since $\mathbb{C}\mathbb{H}^n$ is a complex manifold, every real hypersurface has, in a natural way, a CR-structure: the associated 1-form is just $\theta(\cdot) = g(J\nu, \cdot)$, where ν is the unit normal vector of the hypersurface. The associated horizontal distribution is $\mathcal{H} = \ker\theta$. On \mathcal{H} we have the complex structure J induced by the complex structure of the ambient manifold. Let π be the canonical projection from the tangent space to \mathcal{H} and let $\xi_\theta = J\nu$. We have $\theta(\xi_\theta) = 1$ and $d\theta(\xi_\theta, \cdot) = 0$. We can extend J to the whole tangent space of the hypersurface requiring that $J\xi_\theta = 0$. For every $X, Y \in \mathcal{H}$ we can define $G_\theta(X, Y) = d\theta(X, JY)$. The metric $g_\theta = \pi G_\theta + \theta^2$ is called *Webster metric* associated to θ .

Theorem / Definition 4.3. *Let θ, J, ξ_θ and g_θ be as before. There exists an unique linear connection ∇^{TW} such that $\nabla^{TW}J = \nabla^{TW}\theta = \nabla^{TW}\xi_\theta = \nabla^{TW}g_\theta = 0$ and with torsion T of pure type, i.e. for every $X, Y \in \mathcal{H}$ the torsion satisfies:*

$$\begin{cases} T(X, Y) = d\theta(X, Y)\xi_\theta; \\ T(X, \xi_\theta) \in \mathcal{H} \\ g_\theta(T(X, \xi_\theta), Y) = g_\theta(T(Y, \xi_\theta), X) = -g_\theta(T(JX, \xi_\theta), JY). \end{cases}$$

This connection ∇^{TW} is called the Tanaka-Webster connection associated to θ .

The *Webster curvature* of θ is the curvature defined in the usual way, but using the Tanaka-Webster connection instead of the Levi-Civita connection. A natural question is the Yamabe problem in this new setting, the so called *CR-Yamabe problem*: find the functions f such that the conformal multiple $e^{2f}\theta$ has constant Webster scalar curvature.

As before, we will focus only on the case of the sphere, what follows can be said with much more generality. We refer to the monograph [DT] for all the details about CR-geometry and the general solution of the CR-Yamabe problem. On \mathbb{S}^{2n-1} we have a standard contact form: $\hat{\theta}(\cdot) = \sigma(\xi, \cdot)$. With respect to this form we have: $\xi_{\hat{\theta}} = \xi$, the Hopf vector field. $G_{\hat{\theta}} = \sigma_{sR}$ and $g_{\hat{\theta}} = \sigma$. Obviously the Webster curvature of $\hat{\theta}$ is constant. A metric of the form $e^{2f}\sigma_{sR}$ can be thought as the restriction to $\mathcal{H} \times \mathcal{H}$ of the Webster metric associated to the 1-form $e^{2f}\hat{\theta}$, and vice versa. Then we will talk indifferently about the Webster curvature of a sub-Riemannian metric or of a contact form.

The answer of the CR-Yamabe problem on the sphere is given by the following remarkable formula by Jerison and Lee [JL].

Lemma 4.4. *The Webster scalar curvature of $e^{2f}\sigma_{sR}$ is constant if and only if there are two positive real numbers u, c and $\zeta \in \mathbb{S}^{2n-1}$ such*

$$e^{-2f}(z) = c |\cosh(u) + \sinh(u)z \cdot \bar{\zeta}|^2, \quad \forall z \in \mathbb{S}^{2n-1}. \tag{9}$$

Here we are considering the odd-dimensional sphere immersed in \mathbb{C}^n and the norm and the product are the usual ones in \mathbb{C}^n .

4.3 qc-Yamabe problem

In the same spirit of the previous subsection, we need to clarify what we mean when we talk about curvature of a sub-Riemannian metric defined on the horizontal distribution of codimension 3 of \mathbb{S}^{4n-1} . In this context, the analogous of the CR-geometry, is the *quaternionic contact-geometry* (qc-geometry for short).

The notion of *qc-structure* has been introduced by Biquard in [Bi]. We refer also to the book [IV1] of Ivanov and Vassilev for further details. A qc-structure on a real $(4n - 1)$ -dimensional manifold (\mathcal{M}, g) is a codimension 3 distribution \mathcal{H} locally given as the kernel of a 1-form $\eta = (\eta_1, \eta_2, \eta_3)$ with values in \mathbb{R}^3 such that for every $i = 1, 2, 3$ $d\eta_{i|\mathcal{H}}(\cdot, \cdot) = 2g(I_i \cdot, \cdot)$, where the I_i 's are three almost complex structure on \mathcal{H} satisfying the identities of the imaginary unit quaternions and which are compatible with the metric g . Such η is determined up to the action of $SO(3)$ on \mathbb{R}^3 and a conformal factor. Hence \mathcal{H} is equipped with a conformal class $[g]$ of quaternionic Hermitian metrics. To every metric in the conformal class of g one can associate a linear connection with torsion preserving the qc-structure called the *Biquard connection*. It has been defined by Biquard in [Bi] if $n > 2$ and by Duchemin in [Du] if $n = 2$. They proved that this connection is unique. Using the Biquard connection we can define the qc-Ricci tensor as in [Bi]: it is a symmetric tensor and its trace is called the *qc-scalar curvature*. Once again, with this new notion of curvature, we can define in the usual way the *qc-Yamabe problem*. This problem too has been solved in great generality, see [IV2] for a survey. In our case $\mathcal{M} = \mathbb{S}^{4n-1}$, $g = \sigma_{sR}$ and for every i $\eta_i(\cdot) = \sigma(J_i \nu, \cdot)$. In [IMV] Ivanov, Michev and Vassilev fully characterized the solution of the qc-Yamabe problem in the special case of the quaternionic Heisenberg group. As they noticed, with the Cayley transform, we can find the corresponding solutions on $(\mathbb{S}^{4n-1}, [\sigma_{sR}])$. The result is very similar to Lemma 4.4.

Lemma 4.5. *The metric $e^{2f}\sigma_{sR}$ has constant qc-scalar curvature if and only if there are $\zeta \in \mathbb{S}^{4n-1}$ and positive constant sc and u such that*

$$e^{-2f}(z) = c |\cosh(u) + \sinh(u)z \cdot \bar{\zeta}|^2, \quad \forall z \in \mathbb{S}^{4n-1}, \tag{10}$$

Here we are considering the sphere of real codimension one immersed in $\mathbb{R}^{4n} \cong \mathbb{H}^n$ and the norm and the product are the usual ones in \mathbb{H}^n .

5 The case of the Euclidean and the Hyperbolic space

From the Definition 2.3 of rank, it is evident that the Euclidean space is not in the class of ROSSes. We decided to discuss this case too for two main reasons. First of all the Euclidean space is the simplest ambient manifold, then it is also the first case studied in literature: many ideas developed in ROSSes find their origin in the early works of Gerhardt [Ge1] and Urbas [Ur]. The two authors studied independently and in the same time the inverse mean curvature flow of star-shaped hypersurfaces of the Euclidean space. Moreover we choose to describe this case together with the case of the hyperbolic space because both these manifolds can be seen as a warped product of the real line and the round sphere. The inverse mean curvature flow in the hyperbolic space was studied first by Gerhardt in [Ge2], but recently an unified approach for warped products of this kind has been introduced by Scheuer [Sc]. In this paper we prefer to describe this strategy when possible. We refer also to [Zh] for inverse mean curvature flow in warped products. Studying the evolution of a star-shaped hypersurfaces, there are many common features between the two non-compact space forms, but a big difference appears when we consider the roundness of the limit as shown by Hung and Wang in [HW]. The ambient manifolds considered in [Sc] are warped products of the type

$$([0, \infty) \times \mathbb{S}^n, \tilde{g}),$$

where

$$\tilde{g} = d\rho^2 + \alpha^2(\rho)\sigma,$$

for some function α satisfying some reasonable properties (see Assumption 1.1, 1.2 and 1.3 in [Sc]). For $\alpha(\rho) = \rho$ we have the Euclidean space \mathbb{R}^{n+1} , while if $\alpha(\rho) = \sinh(\rho)$ then we find the hyperbolic metric (2). In this class of ambient manifolds there are also many other interesting examples, one of them is the de Sitter-Schwarzschild manifold.

From this point of view, the results of [Ge1, Ge2, HW, Ur] can be summarized in the following way.

Theorem 5.1. *Let $\tilde{\mathcal{M}}$ be either the Euclidean space \mathbb{R}^{n+1} or the hyperbolic space \mathbb{H}^{n+1} . Let \mathcal{M}_0 be a star-shaped, closed and mean convex hypersurface in $\tilde{\mathcal{M}}$ and let $(\mathcal{M}_t)_{t \geq 0}$ be its evolution by inverse mean curvature flow. Then:*

- (1) \mathcal{M}_t is star-shaped and mean convex for any time t the flow is defined;
- (2) the flow is defined for any positive time;
- (3) the second fundamental form converges exponentially to that one of an horosphere: precisely there exists a positive constant c such that

$$\left| h_i^j - \frac{\alpha'}{\alpha} \delta_j^i \right| \leq c e^{-\frac{t}{n}}.$$

- (4) there is a function $f : \mathbb{S}^n \rightarrow \mathbb{R}$ such that

$$\tilde{g}_t = |\mathcal{M}_t|^{-\frac{2t}{n}} g_t \text{ converges to } \tilde{g}_\infty = e^{2f} \sigma.$$

As seen in Lemma 4.2, the nature of the function f describes the shape of the limit. We have:

- (5) if $\tilde{\mathcal{M}} = \mathbb{R}^{n+1}$ then f is always constant, hence \tilde{g}_∞ is always round;
- (6) if $\tilde{\mathcal{M}} = \mathbb{H}^{n+1}$ there are infinitely many examples such that \tilde{g}_∞ has not constant scalar curvature.

The construction of the counterexample in (6) is the remarkable result present in [HW]. In the following of this section we sketch the main point of the proof of Theorem 5.1, focusing mostly on reasons why the difference between (5) and (6) appears and describing the strategy of Hung and Wang.

We start introducing some notations. Fix an auxiliary function $\varphi(\rho)$ such that $\frac{d\varphi}{d\rho} = \frac{1}{\alpha(\rho)}$. Fix a basis (Y_1, \dots, Y_n) tangent on \mathbb{S}^n , for any i, j we define

$$\begin{aligned} \varphi_i &= \nabla_{\sigma Y_i}(\varphi) = Y_i(\varphi), \\ \varphi_{ij} &= \nabla_{\sigma Y_i Y_j}^2(\varphi) = Y_j Y_i(\varphi) - \nabla_{\sigma Y_i} Y_j(\varphi), \end{aligned}$$

where ∇_σ is the Levi-Civita connection of σ . Analogous notation are used for the derivatives of every function. Then the set of vectors $\{V_i = Y_i + \rho_i \partial \rho\}_{1 \leq i \leq n}$ is a basis of the tangent space of \mathcal{M}_t at any time t . In these coordinates we have:

$$g_{ij} = \alpha^2 (\varphi_i \varphi_j + \sigma_{ij}), \tag{11}$$

$$g^{ij} = \frac{1}{\alpha^2} \left(\sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2} \right), \tag{12}$$

$$h_i^j = \frac{1}{v\alpha} \left(-\tilde{\sigma}^{jk} \varphi_{kj} + \alpha' \delta_i^j \right), \tag{13}$$

where $\varphi^i = \varphi_k \sigma^{ki}$, $\tilde{\sigma}^{jk} = \alpha^2 g^{jk}$, $\alpha' = \frac{d\alpha}{d\rho}$ and

$$v = \bar{g}(\nu, \partial \rho)^{-1} = \sqrt{1 + |\nabla_\sigma \varphi|_\sigma^2}.$$

Taking the trace of (13) we have:

$$H = \frac{1}{v\alpha} \left(-\tilde{\sigma}^{ij} \varphi_{ji} + n\alpha' \right). \tag{14}$$

Starting from the definition of the flow (1), it easy to check that the radial function satisfies the scalar flow:

$$\frac{\partial \rho}{\partial t} = \frac{v}{H}, \tag{15}$$

and that the evolution of hypersurfaces is defined at least as long as (15).

We prefer to work with the auxiliary function φ , hence its evolution is given by

$$\frac{\partial \varphi}{\partial t} = \frac{d\varphi}{d\rho} \frac{\partial \rho}{\partial t} = \frac{v}{H\alpha} =: \frac{1}{F}. \tag{16}$$

By (14) we have that

$$F = F(\varphi, \varphi_i, \varphi_{ij}) = \frac{1}{v^2} \left(-\tilde{\sigma}^{ij} \varphi_{ji} + n\alpha' \right)$$

The first step is to prove that the hypothesis on the initial datum are "good" properties, in the sense that they are preserved by the flow.

Proposition 5.2. *If \mathcal{M}_0 is star-shaped, then \mathcal{M}_t is star-shaped as far the flow exists.*

Proof. A hypersurface is star-shaped if the normal vector and the radial vector are never orthogonal in the ambient space. With the notations introduced above we have that this condition is equivalent to the existence of a constant c such that

$$v^2 \leq 1 + c \iff |\nabla_\sigma \varphi|_\sigma^2 \leq c.$$

Let us define $\omega = \frac{1}{2} |\nabla_\sigma \varphi|_\sigma^2 = \frac{1}{2} \varphi_i \varphi^i$. Using (16), we can compute the evolution equation of ω : let $a^{ij} = -\frac{\partial F}{\partial \varphi_{ij}} = \frac{\tilde{\sigma}^{ij}}{v^2}$, $b^i = \frac{\partial F}{\partial \varphi_i}$ and, for simplicity of notation, $\nabla = \nabla_\sigma$, then

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= \varphi^k \nabla_k \frac{\partial \varphi}{\partial t} \\ &= -\frac{1}{F^2} \left(-a^{ij} \varphi_{ijk} \varphi^k + b^i \varphi_{ik} \varphi^k + \frac{\partial F}{\partial \varphi} \varphi_k \varphi^k \right) \\ &= -\frac{1}{F^2} \left(-a^{ij} \varphi_{ijk} \varphi^k + b^i \omega_i + 2 \frac{\partial F}{\partial \varphi} \omega \right) \end{aligned}$$

We can apply the rule for interchanging derivatives:

$$\varphi_{ijk} = \varphi_{kji} + R^m{}_{ijk} \varphi_m,$$

where this time R is the Riemann curvature tensor of σ , i.e. $R_{sijk} = \sigma_{sj} \sigma_{ik} - \sigma_{sk} \sigma_{ij}$. Since a^{ij} is symmetric we get:

$$-a^{ij} \varphi_{ijk} \varphi^k = -a^{ij} \varphi_{kji} \varphi^k - a^{ij} (\delta_j^m \sigma_{ik} - \delta_k^m \sigma_{ij}) \varphi_m \varphi^k$$

$$= -a^{ij}\omega_{ij} + a^{ij}\varphi_{ik}\varphi_j^k - a^{ij}\varphi_i\varphi_j + 2a_i^i\omega.$$

The following inequality holds:

$$-a^{ij}\varphi_i\varphi_j + 2a_i^i\omega \geq 0.$$

From these estimates we get:

$$\frac{\partial\omega}{\partial t} \leq \frac{1}{F^2} \left(a^{ij}\omega_{ij} - a^{ij}\varphi_{ik}\varphi_j^k - b^i\omega_i - 2\frac{\partial F}{\partial\varphi}\omega \right)$$

A key property is that a^{ij} is positive definite, hence the evolution of ω is given by a parabolic equation and we can use the maximum principle to estimate ω itself.

The reaction term can be estimated in the following way

$$\frac{\partial F}{\partial\varphi} = \frac{\partial F}{\partial\rho} \frac{\partial\rho}{\partial\varphi} = \frac{n\alpha\alpha''}{v^2} \geq 0.$$

Finally we have that

$$a^{ij}\varphi_{ik}\varphi_j^k = a^{ij}\sigma^{kl}\varphi_{ik}\varphi_{jl} \geq 0 \tag{17}$$

because, as showed in [Di], if A, B and C are symmetric matrices, with A and B positive definite, then $tr(ACBC) \geq 0$. The thesis follow by the maximum principle. \square

The following result shows that, in particular, also the mean convexity is preserved by the flow.

Proposition 5.3. *There are two constant depending only on n and the initial datum such that for any time the flow is defined*

$$0 \leq c_1 < H < c_2.$$

If the ambient manifold is the hyperbolic space, then c_1 can be chosen strictly positive.

Proof. Both the estimates are proved by maximum principle applied on suitable functions. The inequality $|A|^2 \geq \frac{H^2}{n}$ holds for any hypersurfaces. Since the ambient manifold has non-positive sectional curvature, by Lemma 3.6, we can deduce that

$$\frac{\partial H}{\partial t} \leq \frac{\Delta H}{H^2} - \frac{H}{n} + \frac{c}{H},$$

where $c \geq 0$. It follows by the maximum principle that H is bounded from above.

To prove that H is bounded from below we define $\psi = \frac{v}{\alpha H} e^{\frac{t}{n}} = \frac{1}{F} e^{\frac{t}{n}} = \frac{\partial\varphi}{\partial t} e^{\frac{t}{n}}$ and we prove that this function is bounded from above. Preceding like in the proof of Proposition 5.2:

$$\frac{\partial\psi}{\partial t} = \frac{1}{F^2} \left(a^{ij}\psi_{ij} - b^i\psi_i - \frac{\partial F}{\partial\varphi}\psi \right) + \frac{1}{n}\psi$$

From (17) we have that

$$\frac{\partial F}{\partial\varphi} \geq n \frac{\alpha''}{v^2} = \begin{cases} 0 & \text{if } \bar{\mathcal{M}} = \mathbb{R}^{n+1}; \\ n \frac{\sinh^2(\rho)}{v^2} & \text{if } \bar{\mathcal{M}} = \mathbb{H}^{n+1}. \end{cases}$$

Therefore

$$-\frac{1}{F^2} \frac{\partial F}{\partial\varphi}\psi + \frac{1}{n}\psi \leq \begin{cases} \frac{1}{n}\psi & \text{if } \bar{\mathcal{M}} = \mathbb{R}^{n+1}; \\ -c\psi^3 + \frac{1}{n}\psi & \text{if } \bar{\mathcal{M}} = \mathbb{H}^{n+1}. \end{cases} \tag{18}$$

for some constant c . By the maximum principle we deduce that ψ grows exponentially in the first case and it is bounded in the second case. By definition of ψ we have the thesis. \square

With an other application of the maximum principle is possible to prove that $|A|^2$ is uniformly bounded (see for a proof Lemma 3.8 in [Zh]). From the a priori estimates just described, we have that the flow (16) (or equivalently (15)) is uniformly parabolic and it satisfies the requirements to apply the regularity results of Krylov-Safonov (par. 5.5 of [Kr]). In this way we have that the flow is defined for any positive time and the solution is smooth as the initial datum.

The proof of part (3) of Theorem 5.1 is another application of maximum principle on the function $G = \frac{1}{2} \left| h_i^j - \frac{\alpha'}{\alpha} \delta_i^j \right|^2$. We prefer to skip it, the proof can be read in Theorem 4.8 of [Sc].

5.1 The construction of counterexamples in \mathbb{H}^{n+1}

The warping function has a fundamental role in the possible value of the conformal multiple that appears in $\tilde{g}_\infty = e^{2f}\sigma$. Since in the Euclidean case α' is bounded, only the constants can appear. On the other hand, when the ambient manifold is the hyperbolic space $\alpha' = \cosh(\rho)$ and, since it is unbounded, we have examples where f is not constant. In this case a natural problem is the classification of the possible limits according to their curvature. K.P. Hung and M.T. Wang proved in [HW] that the limit is not always round, indeed the constant scalar curvature of the limit is a very special case and there are infinitely many non trivial examples. The way to define such hypersurfaces is indirect and uses a notion that was born in general relativity.

They defined for any closed hypersurface \mathcal{M} the *modified Hawking mass*

$$m_H(\mathcal{M}) = |\mathcal{M}|^{-1+\frac{4}{n}} \int_{\mathcal{M}} |\mathring{A}|^2 d\mu, \tag{19}$$

where $\mathring{A} = A - \frac{H}{n}g$ is the trace-free part of the second fundamental form of \mathcal{M} . This quantity measures the umbilicity of the hypersurface: in fact it is easy to check that $|\mathring{A}|^2 = 0$ if and only if the hypersurface is totally umbilical. One of the main property of m_H is the following.

Proposition 5.4. [HW]. *Let f be a real function on \mathbb{S}^n , τ a positive number and let $\tilde{\mathcal{M}}^\tau$ be the star-shaped hypersurface defined by the radial function $\tilde{\rho} = \tau + f + o(1)$, then:*

$$\lim_{\tau \rightarrow \infty} m_H(\tilde{\mathcal{M}}^\tau) = \left(\int_{\mathbb{S}} e^{nf} d\mu_\sigma \right)^{-1+\frac{4}{n}} \int_{\mathbb{S}^n} e^{(n-2)f} |\mathring{\nabla}_\sigma e^{-f}|_\sigma^2 d\mu_\sigma,$$

where we are integrating with the measure induced by σ and the circle means again “trace-free part”. The limit of the induced rescaled metric is e^{2f} and, by the classification of Lemma 4.2, we have that

$$e^{2f} \text{ is round } \Leftrightarrow |\mathring{\nabla}_\sigma e^{-f}|_\sigma^2 = 0 \Leftrightarrow \lim_{\tau \rightarrow \infty} m_H(\tilde{\mathcal{M}}^\tau) = 0. \tag{20}$$

Now we need link the modified Hawking mass with the inverse mean curvature flow. By the formulas in Lemma 3.6, we can compute its evolution. In [HW] the following fundamental estimate is proved:

Lemma 5.5. *There exists a positive constant c such that along the inverse mean curvature flow $(\mathcal{M}_t)_{t \geq 0}$ we have*

$$\frac{d}{dt} m_H(\mathcal{M}_t) \geq -c e^{-\frac{2t}{n}}.$$

In other words, we don't know if m_H is monotone, but if it decreases, it decreases very slowly. Now we can collect all the results and describe the strategy of Hung and Wang to produce non trivial examples.

(1) Fix a positive constant c_0 big enough and choose a function $f : \mathbb{S}^n \rightarrow \mathbb{R}$ such that

$$m_H(f) := \left(\int_{\mathbb{S}} e^{nf} d\mu_\sigma \right)^{-1+\frac{4}{n}} \int_{\mathbb{S}^n} e^{(n-2)f} |\mathring{\nabla}_\sigma e^{-f}|_\sigma^2 d\mu_\sigma \geq 4c_0.$$

(2) Consider the family $\tilde{\mathcal{M}}^\tau$ defined in Proposition 5.4 and choose a τ big enough such that $\tilde{\mathcal{M}}^\tau$ is mean convex, and, by Proposition 5.4, $Q(\tilde{\mathcal{M}}^\tau) \geq 2c_0$.

(3) Let $(\mathcal{M}_t^\tau)_{t \geq 0}$ be the inverse mean curvature flow with initial datum $\tilde{\mathcal{M}}^\tau$.

(4) Since c_0 is big enough, Lemma 5.5 ensures that

$$\lim_{t \rightarrow \infty} m_H(\mathcal{M}_t^\tau) \geq c_0 > 0.$$

(5) The thesis follows from Proposition 5.4.

6 The case of the Complex and Quaternionic Hyperbolic space

The extension of Theorem 5.1 to the other manifolds in ROSSes has been faced by the author of the present survey in [Pi2] for $\mathbb{C}\mathbb{H}^n$ as ambient manifold and in [Pi3] for $\mathbb{H}\mathbb{H}^n$. Once again there are similarities with the previous cases, but also some new phenomena appear in this context. The most important is that, even after rescaling, the induced metric do not converges to a Riemannian limit but only to a sub-Riemannian metric defined of a distribution of codimension $a - 1$. The main reason for this different behaviour could be found in the fact that the metric (5) is not a warped product, but the metric on the sphere changes with ρ and degenerates at the boundary at infinity of the ambient manifold (i.e. when ρ diverges). Moreover since for both value of \mathbb{K} , the “warping” function in (5) has unbounded derivative, it is possible to produce examples which develop limits with non constant scalar curvature: for the motivation explained in Remark 3.5, the roundness of the limit is studied in term of Webster and qc scalar curvature. The precise statement is the following:

Theorem 6.1. *Let \mathcal{M}_0 be a closed, mean convex and \mathbb{S}^{a-1} -invariant star-shaped hypersurface in $\mathbb{K}\mathbb{H}^n$, where \mathbb{K} is either \mathbb{C} or \mathbb{H} , and let \mathcal{M}_t be its evolution by inverse mean curvature flow. Then:*

- (1) \mathcal{M}_t is \mathbb{S}^{a-1} -invariant, star-shaped and mean convex for any time t ;
- (2) the flow is defined for any positive time;
- (3) let

$$\lambda = \frac{2}{a(n+1) - 2}, \tag{21}$$

then there is a smooth \mathbb{S}^{a-1} -invariant function f such that the rescaled induced metric $\tilde{g}_t = |\mathcal{M}_t|^{-\lambda} g_t$ converges to a sub-Riemannian metric $\tilde{g}_\infty = e^{2f} \sigma_{SR}$ (i.e. a conformal multiple of the standard sub-Riemannian metric on the sphere \mathbb{S}^{an-1});

- (4) *there are examples of \mathcal{M}_0 such that the limit does not have constant Webster (if $\mathbb{K} = \mathbb{C}$) or qc (if $\mathbb{K} = \mathbb{H}$) scalar curvature.*

The goal of this section is to describe the main differences between Theorem 5.1 and Theorem 6.1, in terms of result and techniques used in the proofs. Moreover we give an unified approach for both the value of \mathbb{K} .

First of all we need to clarify the new hypothesis. \mathbb{S}^{a-1} can be identify with a group of isometries of $(\mathbb{S}^{an-1}, \sigma)$ acting in the following way:

$$S : (\gamma, z) \in \mathbb{S}^{a-1} \times \mathbb{S}^{an-1} \subset \mathbb{K} \times \mathbb{K}^n \mapsto \gamma z \in \mathbb{S}^{an-1}. \tag{22}$$

The projection

$$\pi : z \in \mathbb{S}^{an-1} \mapsto [z] = \{ \gamma z \mid \gamma \in \mathbb{S}^{a-1} \} \in \mathbb{K}\mathbb{P}^{n-1}$$

is the well known *Hopf fibration*. Since star-shaped hypersurfaces are identified with their radial function $\rho : \mathbb{S}^{an-1} \rightarrow \mathbb{R}^+$, we have the following natural definition.

Definition 6.2. *We say that a star-shaped hypersurface in $\mathbb{K}\mathbb{H}^n$ is \mathbb{S}^{a-1} -invariant if its radial function is invariant by the action (22).*

This new hypothesis is required for proving the analogous of Proposition 5.2 in the new setting. Moreover the symmetries beehive very well with geometric flows. The paper [Pi1] is about their interaction with the mean curvature flow. Lemma 3.1 of [Pi1] can be reproduced with minor modification for the inverse mean curvature flow showing:

Lemma 6.3. *Let \mathcal{M}_0 be an \mathbb{S}^{a-1} -invariant hypersurface in $\mathbb{K}\mathbb{H}^n$, then \mathcal{M}_t is \mathbb{S}^{a-1} -invariant for any time the flow is defined.*

We describe the main geometric quantity for an \mathbb{S}^{a-1} -invariant star-shaped hypersurface in $\mathbb{K}\mathbb{H}^n$. Since the ambient space is not isotropic, we have to fix a particular basis tangent to the sphere. Let (Y_1, \dots, Y_{an-1}) be

a σ -orthonormal basis on \mathbb{S}^{an-1} with $Y_i = J_i \nu$ for every $1 \leq i \leq a - 1$. The induced metric, the quantity v and the unit normal vector have the following expression in this new context:

$$g_{ij} = \sinh^2(\rho)(\varphi_i \varphi_j + e_{ij}), \tag{23}$$

$$g^{ij} = \frac{1}{\sinh^2(\rho)} \left(e^{ij} - \frac{\varphi^i \varphi^j}{v^2} \right), \tag{24}$$

$$v = \sqrt{1 + |\nabla_e \varphi|_e^2}, \tag{25}$$

$$\nu = \frac{1}{v} \left(\frac{\partial}{\partial \rho} - \frac{\nabla_e \rho}{\sinh^2(\rho)} \right), \tag{26}$$

where for short $e_{ij} = (e_{\cosh^2(\rho)})_{ij}$. Note that in case of \mathbb{S}^{a-1} -invariant hypersurfaces we have that $\nabla_e \varphi = \nabla_\sigma \varphi$, hence

$$v = \sqrt{1 + |\nabla_\sigma \varphi|_\sigma^2}.$$

Notation 6.4. We introduce the following notation in order to distinguish between derivatives of a function with respect to different metrics. For any given function $f : \mathbb{S}^{2n-1} \rightarrow \mathbb{R}$, let f_{ij} (\hat{f}_{ij} respectively) be the components of the Hessian of f with respect to σ (e_μ respectively). The value of μ will be clear from the context. The indices go up and down with the associated metric: for instance $\hat{f}_i^k = \hat{f}_{ij} e_\mu^{jk}$, while $f_i^k = f_{ij} \sigma^{jk}$. Analogous notations can be used for higher order derivatives.

In case of \mathbb{S}^{a-1} -invariance the second fundamental form has the following expression:

$$h_i^k = -\frac{\hat{\varphi}_{ij} \tilde{e}^{jk}}{v \sinh(\rho)} + \begin{cases} \frac{\sinh^2(\rho) + \cosh^2(\rho)}{v \sinh(\rho) \cosh(\rho)} \delta_i^k & \text{if } 1 \leq i \leq a - 1, \\ \frac{\cosh(\rho)}{v \sinh(\rho)} \delta_i^k & \text{if } i \geq a. \end{cases} \tag{27}$$

Taking the trace we have the mean curvature:

$$H = -\frac{\hat{\varphi}_{ij} \tilde{e}^{ji}}{v \sinh(\rho)} + \frac{\hat{H}}{v}, \tag{28}$$

where

$$\hat{H} = (an - 1) \frac{\cosh(\rho)}{\sinh(\rho)} + (a - 1) \frac{\sinh(\rho)}{\cosh(\rho)}. \tag{29}$$

Note that, when \mathcal{M} is a geodesic sphere, that is ρ is constant, the mean curvature is exactly \hat{H} .

The following lemma is useful to simplify the expression of the mean curvature.

Lemma 6.5. Let $\varphi : \mathbb{S}^{2n-1} \rightarrow \mathbb{R}$ be an \mathbb{S}^1 -invariant smooth function. With respect to the basis introduced in the previous Lemma, the Hessian of φ with respect to e_μ is:

$$\hat{\varphi}_{ij} = \begin{pmatrix} 0 & \mu J_i Y_j(\varphi) \\ \mu J_j Y_i(\varphi) & \varphi_{ij} \end{pmatrix},$$

where we are using Notations 6.4. Taking the trace and the norm of the Hessian, in particular we have:

$$\begin{aligned} \Delta_e \varphi &= \Delta_\sigma \varphi; \\ |\nabla_e^2 \varphi|_e^2 &= |\nabla_\sigma^2 \varphi|_\sigma^2 + 2(a - 1)(\mu - 1) |\nabla_\sigma \varphi|_\sigma^2. \end{aligned}$$

The proof can be found in Section 2 of [Pi2] for $\mathbb{K} = \mathbb{C}$ and Section 2 of [Pi3] for $\mathbb{K} = \mathbb{H}$. Let $\tilde{\sigma}^{ji} = \sigma^{ji} - \frac{\varphi^i \varphi^j}{v^2}$, then by Lemma 6.5 we have that $\hat{\varphi}_{ij} \tilde{e}^{ji} = \varphi_{ij} \tilde{\sigma}^{ji}$. It follows that:

$$H = -\frac{\varphi_{ij} \tilde{\sigma}^{ji}}{v \sinh(\rho)} + \frac{\hat{H}}{v}. \tag{30}$$

The proof of part (1) and (2) of Theorem 6.1 are similar to the analogous results described in the previous section. Roughly speaking, equation (30) shows that the symmetries given by the action of \mathbb{S}^{a-1} helps to rule out the contribution of the special directions. Proposition 5.2 holds in the new case too with the modifications described below. In the new case we have that

$$\frac{\partial}{\partial t} \varphi = \frac{v}{H \sinh} =: \frac{1}{G}. \tag{31}$$

By (30) we have

$$G(\varphi, \varphi_i, \varphi_{ij}) = \frac{1}{v^2} \left(-\varphi_{ij} \tilde{\sigma}^{ij} + \sinh(\rho) \hat{H} \right).$$

The advantage to use the new hypothesis about the symmetries of the evolving hypersurface is that in this way $\tilde{\sigma}$ and v do not depend on ρ , hence, trying to repeat the proof of Proposition 5.2, we find that

$$\frac{\partial}{\partial \varphi} G = \frac{\sinh^2(\rho)}{v^2 \cosh^2(\rho)} \left[(a(n+1) - 1) \cosh^2(\rho) + a - 1 \right] \geq 0.$$

With minor modification of the type described above, we can prove that the mean convexity is preserved too and that the second fundamental form is uniformly bounded, hence the flow is defined for any positive time.

We would like to finish this section showing that the fact that the induced rescaled metric converges to a sub-Riemannian limit appears in a quite natural way. In Example 3.4 we saw that it is the case of a geodesic sphere, but this phenomenon happens for a general initial datum. Since the volume grows like $|\mathcal{M}_t| = |\mathcal{M}_0| e^t$, and the induced metric is given by (23) we have:

$$\tilde{g}_{ij} = c_0 \sinh^2(\rho) e^{-\frac{t}{n}} (\varphi_i \varphi_j + e_{ij}),$$

for some positive constant c_0 . The metric e_{ij} converges to $(\sigma_{sR})_{ij}$ by definition. Moreover one can prove that for any integer m there exists a positive constant c such that

$$|\nabla_{\tilde{\sigma}}^m \rho|_{\tilde{\sigma}}^2 \leq c.$$

See Sections 6 and 7 of [Pi2] for a proof. This fact, in particular implies that $\varphi_i \varphi_j$ goes to zero and that the factor $\sinh^2(\rho) e^{-\frac{t}{n}}$ converges to a positive smooth function.

6.1 The construction of counterexamples in $\mathbb{K}\mathbb{H}^n$

The strategy proposed in [Pi2] (resp. [Pi3]) for the construction of initial data which produce limits with non constant Webster (resp. qc) scalar curvature is a weaker version of the analogous problem described in Section 5.1. A well known result [NR] says that in our ambient manifold there are no totally umbilical hypersurfaces. Hence it is clear that the trace-less second fundamental form cannot have the strong meaning described in Proposition 5.4. The main tool used for investigating the roundness of the limit is the following Brown-York-like quantity: for any \mathbb{S}^{a-1} -invariant hypersurface \mathcal{M} we define

$$Q(\mathcal{M}) = |\mathcal{M}|^{-1+\lambda} \int_{\mathcal{M}} (H - \hat{H}) d\mu, \tag{32}$$

where λ was defined in (21) and \hat{H} in (29). Q gives a measure of how the hypersurface is far from being a geodesic sphere, but it is not a true measure, because we do not know its sign: it is trivially zero when \mathcal{M} is a geodesic sphere, but in general it is not true the opposite. One of the main properties of Q is the following. A proof can be found in [Pi2] if $\mathbb{K} = \mathbb{C}$ and in [Pi3] if $\mathbb{K} = \mathbb{H}$.

Proposition 6.6. *Let $\tilde{\mathcal{M}}^\tau$ be a family of hypersurfaces in $\mathbb{K}\mathbb{H}^n$ that are radial graph of the functions $\tilde{\rho}(z, \tau) = \tau + f(z) + o(1)$, for some fixed \mathbb{S}^{a-1} -invariant function $f : \mathbb{S}^{a-1} \rightarrow \mathbb{R}$. Then*

(1)

$$\lim_{\tau \rightarrow \infty} Q(\widetilde{\mathcal{M}}^\tau) = \left(\int_{\mathbb{S}^{an-1}} e^{\frac{2}{\lambda}f} d\sigma \right)^{-1+\lambda} \int_{\mathbb{S}^{an-1}} e^{\frac{2}{\lambda}f} \left(e^{-f} \Delta_\sigma e^{-f} - \frac{1}{\lambda} \left| \nabla_\sigma e^{-f} \right|_\sigma^2 \right) d\sigma.$$

(2) Moreover if $\lim_{\tau \rightarrow \infty} Q(\widetilde{\mathcal{M}}^\tau) \neq 0$, then $e^{2f} \sigma_{\mathbb{S}^R}$ - the limit of the rescaled metric on $\widetilde{\mathcal{M}}^\tau$ - does not have constant (Webster or qc) scalar curvature.

Let us define for brevity

$$Q(f) = \left(\int_{\mathbb{S}^{an-1}} e^{\frac{2}{\lambda}f} d\sigma \right)^{-1+\lambda} \int_{\mathbb{S}^{an-1}} e^{\frac{2}{\lambda}f} \left(e^{-f} \Delta_\sigma e^{-f} - \frac{1}{\lambda} \left| \nabla_\sigma e^{-f} \right|_\sigma^2 \right) d\sigma.$$

In order to prove property (2) we note that if $Q(f) \neq 0$, then for sure f cannot be constant, but, recalling that f is \mathbb{S}^{a-1} -invariant, an easy computation shows that f satisfies equations (9) (or (10)) if and only if it is constant.

A careful analysis about the evolution of Q says that if it decreases, then it decreases very slowly. In fact we have the following estimate.

Proposition 6.7. *Let \mathcal{M}_t be an \mathbb{S}^{a-1} -invariant star-shaped hypersurface of $\mathbb{K}\mathbb{H}^n$ evolving by inverse mean curvature flow. There is a positive constants c such that*

$$\frac{\partial Q}{\partial t} \geq -ce^{-t\lambda}.$$

Now we have all the ingredient to repeat the strategy of [HW] described in Section 6.1 and we can show the existence of counterexamples in $\mathbb{K}\mathbb{H}^n$ too.

(1) Fix a positive constant c_0 big enough and choose an \mathbb{S}^{a-1} -invariant function $f : \mathbb{S}^{an-1} \rightarrow \mathbb{R}$ such that

$$Q(f) \geq 4c_0.$$

(2) Consider the family of \mathbb{S}^{a-1} -invariant star-shaped hyperfurfaces $\widetilde{\mathcal{M}}^\tau$ defined by the radial function $\tilde{\rho}^\tau(z) = \tau + f(z)$. Fix a τ big enough such that $\widetilde{\mathcal{M}}^\tau$ is mean convex, and, by Proposition 6.6, $Q(\widetilde{\mathcal{M}}^\tau) \geq 2c_0$.

(3) Let the flow start from intial datum $\widetilde{\mathcal{M}}^\tau$.

(4) Since c_0 is big enough, Proposition 6.7 ensures that

$$\lim_{t \rightarrow \infty} Q(\mathcal{M}_t^\tau) \geq c_0 > 0,$$

where $(\mathcal{M}_t^\tau)_{t \geq 0}$ is the evolution of $\widetilde{\mathcal{M}}^\tau$.

(5) The thesis follows from Proposition 6.6.

Remark 6.8. *The analogous of Theorem 6.1 can be proved in the octonion hyperbolic plane $\mathbb{O}\mathbb{H}^2$ too. In this case we have to use \mathbb{S}^7 -invariance. The induced rescaled metric converges to a sub-Riemannian metric on \mathbb{S}^{15} defined on a distribution of codimension 7. Defining $\hat{H}(\rho) = 7 \frac{\sinh(\rho)}{\cos} + 15 \frac{\cosh(\rho)}{\sinh(\rho)}$, it is possible to define the analogous of the quantity Q (32) and produce, with the same strategy described above, a non-trivial example. The problem is that, for the best of our knowledge, there is no analogous of the Tanaka-Webster and Biquard connections in this context, hence it is not clear how to interpret the nature of the limit metric in terms of its curvature.*

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