Complex Manifolds

Research Article

Gianni Manno and Andreas Vollmer*

Benenti Tensors: A useful tool in Projective Differential Geometry

<https://doi.org/10.1515/coma-2018-0006>

Received October 12, 2017; accepted April 3, 2018.

Abstract: Two metrics are said to be *projectively equivalent* if they share the same geodesics (viewed as unparametrized curves). The degree of mobility of a metric g is the dimension of the space of the metrics projectively equivalent to g . For any pair of metrics $(g, \bar{g})$ on the same manifold one can construct a $(1, 1)$ -tensor $L(g, \bar{g})$ called the *Benenti tensor*. In this paper we discuss some geometrical properties of Benenti tensors when $(g, \bar{g})$ are projectively equivalent, particularly in the case of degree of mobility equal to 2.

Keywords: Projective connections, Benenti tensors, Projectively equivalent metrics, Levi-Civita metrics

MSC: 53A20, 53B10

1 Introduction

In the present paper, the word *metric* is used for both Riemannian and pseudo-Riemannian metrics, unless otherwise specified. The Einstein summation convention will be used.

Definition 1.1. We say that two symmetric affine connections on the same manifold M are projectively equivalent if they share the same geodesics (as unparametrized curves). The set of all connections projectively equivalent to a given connection Γ is called the projective class of such connection or the projective connection determined by Γ . Two metrics are projectively equivalent if their Levi-Civita connections are so.

We use the term *projective connection* mainly when we refer to a system of ordinary differential equations (ODE) that represents a certain projective class. Below we describe how to construct this representative system.

Definition 1.2. A projective transformation is a (local) diffeomorphism of M that sends geodesics into geodesics (where geodesics are to be understood as unparametrized curves). A vector field on M is projective if its (local) flow acts by projective transformations.

Locally, two symmetric affine connections Γ and $\bar{\Gamma}$ are projectively equivalent if and only if

$$\Gamma_{bc}^a = \bar{\Gamma}_{bc}^a - \delta_b^a \phi_c - \delta_c^a \phi_b \quad (1)$$

where ϕ_i are the components of a 1-form, cf. [1–3]. For a detailed explanation and proof of this statement see for instance [4]. From an ODE perspective, the projective class of a given symmetric affine connection Γ

Gianni Manno: Politecnico di Torino, DISMA, Corso Duca degli Abruzzi, 24, 10129 Torino TO, Italy,

E-mail: giovanni.manno@polito.it

***Corresponding Author: Andreas Vollmer:** Istituto Nazionale di Alta Matematica - Politecnico di Torino, DISMA, Corso Duca degli Abruzzi, 24, 10129 Torino TO, Italy, E-mail: andreasvollmer@gmail.com

can be understood as follows. Let $(u^1, u^2, \dots, u^N) = (x, u^2, \dots, u^N)$ be a system of coordinates on M . Then Γ gives rise to a system of second order ordinary differential equations

$$u_{xx}^k = -\Gamma_{11}^k - (2\Gamma_{1i}^k - \delta_i^k \Gamma_{11}^1)u_x^i - (\Gamma_{ij}^k - 2\delta_i^k \Gamma_{1j}^1)u_x^i u_x^j + \Gamma_{ij}^1 u_x^i u_x^j u_x^k, \quad k = 2, \dots, N, \quad i, j \geq 2, \quad (2)$$

obtained by eliminating the external parameter from the classical geodesic equations (see for instance [5]). We can interpret system (2) as the projective connection associated to Γ . In fact, as we said, two connections Γ and $\bar{\Gamma}$ are projectively equivalent if and only if they are related by (1) and connections linked by (1) give the same system (2); in other words, for any solution $(u^2(x), \dots, u^N(x))$ to (2), the curve $(x, u^2(x), \dots, u^N(x))$ is a geodesic of Γ up to reparametrization. From this perspective, local diffeomorphisms $(u^1, \dots, u^N) \rightarrow (u^1(v^1, \dots, v^N), \dots, u^N(v^1, \dots, v^N))$ preserving (2) (finite point symmetries) are projective transformations of Γ as they send geodesics into geodesics. Infinitesimal point symmetries of (2) are projective vector fields of Γ and generate a 1-parametric family of projective transformations.

Taking into account what we said so far, and the form of system (2), the following system of ODE defines a (N -dimensional) projective connection:

$$u_{xx}^k = f_{11}^k + f_{1i}^k u_x^i + f_{ij}^k u_x^i u_x^j + f_{ij}^1 u_x^i u_x^j u_x^k, \quad k = 2, \dots, N, \quad i, j \geq 2, \quad f_{ij}^k = f_{ji}^k. \quad (3)$$

Note that, for $N = 2$, system (2) reduces to a single ODE, namely the classical 2-dimensional projective connection associated to a 2-dimensional metric

$$u_{xx} = -\Gamma_{11}^2 + (\Gamma_{11}^1 - 2\Gamma_{12}^2)u_x - (\Gamma_{22}^2 - 2\Gamma_{12}^1)u_x^2 + \Gamma_{22}^1 u_x^3$$

where $(x, u, u_x, u_{xx}) := (u^1, u^2, u_x^2, u_{xx}^2)$. Furthermore, (3) reduces to a single ODE having on the right hand side a general polynomial of third degree in the first derivatives, i.e., a general 2-dimensional projective connection, extensively studied, for instance, in [6–8].

For $N > 2$, the right-hand side term of (3) is not a generic third order polynomial in the first derivatives u_x^i . Indeed, in any equation forming such a system, not all monomials of third order degree appear.

In [9] Benenti introduced, in the context of Riemannian manifolds, a certain conformal Killing (2, 0)-tensor, that he called L -tensor in [10, 11]. After lowering one index by using the metric, one obtains a self-adjoint (1, 1)-tensor, whose eigenspaces play a central role in the theory of orthogonal separation coordinates. Most importantly, there exist separable coordinate systems associated to certain subspaces of Killing tensors (so-called Killing-Stäckel spaces, see Section 3 in [12]) for which the basis can be obtained from one single L -tensor, see e.g. Formula (2.21) and Remark 2.1 in [9] or, more recently, Theorem 8.1 in [12].

A number of subsequent papers is devoted to the historical context and geometric significance of this class of tensors, e.g. [13–15], from which we take the following definition.

Definition 1.3 ([13, 15]). *Let g be a Riemannian metric on a N -dimensional manifold M . A non-degenerate (1, 1)-tensor field L on M is called a Benenti tensor field with respect to g if L is self-adjoint and satisfies*

1. *The Nijenhuis tensor of L is identically zero;*
2. *For the functions $H := \frac{1}{2}g^{ij}p_i p_j$, $F := g^{ik}L_k^j p_i p_j$, we have*

$$\{H, F\} = 2H \left(\frac{\partial \text{trace}(L)}{\partial x^i} g^{ij} p_j \right)$$

where x^i, p_j are the standard coordinates on the cotangent bundle T^*M and $\{, \}$ is the standard Poisson bracket between functions on T^*M .

A link between Benenti tensors and projectively equivalent metrics is given by the following theorem.

Theorem 1.4 ([15] and Section 5 of [10]). *The Riemannian metrics $g, \bar{g}$ are projectively equivalent if and only if*

$$L(g, \bar{g}) := \left(\frac{\det(\bar{g})}{\det(g)} \right)^{\frac{1}{N+1}} \bar{g}^{-1} g \quad (4)$$

is a Benenti tensor for g .

An analogous result holds for pseudo-Riemannian metrics, see, e.g., [8]. In what follows, the pseudo-Riemannian case is going to be included. Following the terminology in [15], a tensor defined by (4) is referred to as a Benenti tensor. In contrast to Definition 1.3, however, we do not fix a metric.

Definition 1.5. We define the Benenti tensor for a (ordered) pair of (Riemannian or pseudo-Riemannian) projectively equivalent metrics $(g, \bar{g})$ by Formula (4).

Benenti tensors (4) that are constructed starting from a pair of projectively equivalent metrics become particularly useful in many areas of projective differential geometry. They have been successfully used, e.g., in the proof of the Lichnerowicz-Obata conjecture [4], the splitting-gluing construction of geodesically equivalent metrics [16], at the crossroads of geodesic equivalence and integrability [19], the solution of Lie's Second Problem [6], in the context of general relativity [4], and others.

In the present paper we show how Benenti tensors, as defined in Definition 1.5, are helpful for studying the structure of the space of metrics projectively equivalent to a given one. In Section 2 we basically characterize such a space as a complement of an algebraic variety. Then we focus our attention to the case when the projective class of the initial metric contains, roughly speaking, also non-proportional metrics. In Section 4, under this assumption, we study the case when there exists a projective vector field. In particular, we investigate the Lie derivative of Benenti tensors along such a vector field. In Section 5 we use the results of Sections 2 and 4 for studying 3-dimensional Riemannian metrics of Levi-Civita type admitting a projective vector field.

2 Degree of mobility, metrizable projective connections and Benenti tensors

A classical question is to see if a projective connection is *metrizable*, i.e., if there exists a Levi-Civita connection whose corresponding projective connection is the given one.

In local coordinates, the projective connection (3) is metrizable if there exists an N -dimensional metric g such that (2), where $\Gamma = \Gamma_{jk}^i$ is the Levi-Civita connection of g , is equal to (3). This is equivalent to the existence of a solution to the following system of $\frac{1}{2}N(N-1)(N+2)$ PDEs

$$-\Gamma_{11}^k = f_{11}^k, \quad -(2\Gamma_{1m}^k - \delta_m^k \Gamma_{11}^1) = f_{1m}^k, \quad -((2 - \delta_{jm})\Gamma_{jm}^k - 2\delta_m^k \Gamma_{1j}^1 - 2\delta_j^k \Gamma_{1m}^1) = f_{jm}^k, \quad \Gamma_{ij}^1 = f_{ij}^1, \\ k, i, j, m = 2, \dots, N \quad (5)$$

where the Christoffel symbols are given by

$$\Gamma_{jk}^i = \frac{1}{2}g^{ih}(g_{kh,j} + g_{hj,k} - g_{jk,h}).$$

System (5) is highly non-linear in the unknown functions g_{ij} , but, it turns out that if we perform the substitution

$$\sigma^{ij} := \det(g)^{\frac{1}{N+1}} g^{ij} \in S^2(M) \otimes (\Lambda^N(M))^{\frac{2}{N+1}}, \quad (6)$$

we obtain a *linear* system in the unknown variables σ^{ij} . Of course, (6) does not make sense if g is negative-definite. Therefore, without further mentioning, in (6) and in the remainder of the paper, the fractional exponent implies that we use the absolute value of the base expression unless stated otherwise. We have the following theorem.

Theorem 2.1 ([17]). *A metric g on an N -dimensional manifold lies in the projective class of a given connection Γ if and only if σ^{ij} defined by (6) is a solution of*

$$\nabla_a \sigma^{bc} - \frac{1}{N+1}(\delta_a^c \nabla_i \sigma^{ib} + \delta_a^b \nabla_i \sigma^{ic}) = 0, \quad (7)$$

where ∇ is the covariant derivative of Γ .

Note that Theorem 2.1 implicitly contains the assumption $\det(\sigma) \neq 0$, as σ^{ij} otherwise does not correspond to a metric. Furthermore, note that, since σ is a weighted tensor field,

$$\nabla_a \sigma^{bc} = \sigma^{bc}_{,a} + \Gamma^b_{ad} \sigma^{dc} + \Gamma^c_{ad} \sigma^{db} - \frac{2}{N+1} \Gamma^d_{da} \sigma^{bc},$$

where “ $,a$ ” stand for the derivative w.r.t. the a -coordinate. Of course, the set of solutions to the linear system of PDEs (7) form a linear space. Note that in terms of solutions of system (7), taking into account (6), Formula (4) takes the compact form

$$L(\sigma, \bar{\sigma}) := \bar{\sigma} \sigma^{-1}.$$

Definition 2.2. Let g be a metric and Γ its Levi-Civita connection. The (linear) space of solutions to system (7) is denoted by $\mathfrak{A}(g)$ (or $\mathfrak{A}$ in short when there is no risk of confusion) and its dimension is called the degree of mobility of g .

Since the degree of mobility is the same for any choice of metric g within a given projective class, it is also reasonable to call the degree of mobility of g the degree of mobility of its projective class (or of its projective connection).

Let us recall that, in general, a solution σ to (7) can be such that $\det(\sigma) = 0$, implying that there is no metric corresponding to σ via (6). In the current section, we would like to understand the structure of the solution space of (7). We begin by showing that the ‘desired’ solutions lie dense among all solutions of the system (7). From now on we assume a metrizable projective connection.

Proposition 2.3. Assume a metrizable projective connection and let $\mathfrak{A}$ denote the (linear) space of solutions to (7). Then there is a basis of $\mathfrak{A}$ made up of full-rank solutions and the set

$$\{\sigma \text{ solution to (7) : } \det(\sigma) \neq 0\} \subset \mathfrak{A}$$

of full-rank solutions to (7) is dense in $\mathfrak{A}$.

Proof. By the hypothesis, there exists a metric g such that its Levi-Civita connection lies in the initial projective class. This means that, taking into account formula (6), $c \cdot \sigma$ is a full-rank solution to system (7) for each $c \in \mathbb{R} \setminus \{0\}$. This (punctured) line lies in an open subset of $\mathfrak{A}$ as the condition to be of maximal rank is an open one. This implies that we can choose k linearly independent solutions to system (7), where k is the degree of mobility of the metric g . Denote those solutions by $(\sigma_1, \dots, \sigma_k)$ and consider the equation

$$\det \left(\sum_i K_i \sigma_i \right) = 0, \quad K_i \in \mathbb{R}.$$

The above equation defines an algebraic variety in $\mathfrak{A}$ which is a zero-measure set in $\mathfrak{A} \sim \mathbb{R}^k$, so that its complement is dense in $\mathfrak{A}$. $\square$

Note that, in view of Proposition 2.3, the degree of mobility of a metric g is a “measure of the size” of the projective class of g .

Let us now consider the case when the projective class has degree of mobility 2, i.e. $\dim(\mathfrak{A}) = 2$. The assumption of degree of mobility 2 is justified for the following reasons: If the degree of mobility is less than 2, the situation is somehow trivial since all projectively equivalent metrics are proportional. Secondly, for compact connected manifolds of dimension $N \geq 3$, in [18] it is proved, under mild assumptions, that either the degree of mobility of the projective class is at most 2 or the projective class is made of affinely equivalent metrics. Lastly, for 2-dimensional metrics, there always exists a (at most) 2-dimensional subspace of $\mathfrak{A}$, invariant under the action of an essential (i.e., non-homothetic) projective vector field, and for 3-dimensional metrics of non-constant curvature, the degree of mobility is at most 2 (see again [18]).

We shall see that the properties of Benenti tensors are closely linked to lower-rank solutions of (7).

Proposition 2.4. *Consider a metrizable projective connection with degree of mobility 2. Let (g_1, g_2) be a pair of non-proportional, projectively equivalent metrics and L their Benenti tensor. Assume that L admits the constant eigenvalue t . Then any linear combination*

$$K_1 g_1^{-1} \det(g_1)^{\frac{1}{N+1}} + K_2 g_2^{-1} \det(g_2)^{\frac{1}{N+1}}, \quad K_i \in \mathbb{R}$$

with $K_1 = -K_2 t$ is a degenerate solution of (7), and vice versa.

Proof. Let σ_i be obtained from g_i as in (6). Eigenvalues of L are roots of the characteristic polynomial $\det(L - t \text{Id})$. Multiplying by $\det(\sigma_1) \neq 0$, we get

$$0 = \det(L - t \text{Id}) \det(\sigma_1) = \det(\sigma_2 - t \sigma_1).$$

The backwards direction of the proof follows by the same argument. $\square$

We have thus established that we can obtain lines of lower rank in $\mathfrak{A}$ by finding the roots of the characteristic polynomial of a Benenti tensor. Let us now shift our attention from the properties of a specific Benenti tensor to transformations between Benenti tensors. In Definition 1.5 the ordering of the pair (g_1, g_2) is important. For instance, if we exchange the ordering of the pair (g_1, g_2) in Definition 1.5, the Benenti tensor is replaced by its inverse. On the other hand, if we fix the direction of the first component of the pair (g_1, g_2) , we obtain the following.

Proposition 2.5. *Let L be a Benenti tensor field of the metric g . The transformation $L \rightarrow rL + t \text{Id}$, with $r \in \mathbb{R} \setminus \{0\}$ and $t \in \mathbb{R}$, maps a Benenti tensor of g to a Benenti tensor of g . Conversely, let g have degree of mobility 2. All Benenti tensors (4) of pairs $(g, \tilde{g})$ where $\tilde{g}$ is a metric projectively equivalent to g are then linked by a transformation of this kind.*

Proof. The first part is easily confirmed by checking the properties in Definition 1.3. For the second part, let $\tilde{g}$ be a metric projectively equivalent to g . Since the degree of mobility is 2, we have that any Benenti tensor of a pair with g as its first entry is given by an element of the family

$$L_{\alpha\beta} = (\alpha\sigma + \beta\bar{\sigma})\sigma^{-1} \quad (\alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\})$$

where we obtain, respectively, $\sigma, \bar{\sigma}$ via (6). Therefore $L_{\alpha\beta} = \alpha \text{Id} + \beta L$ with L being the Benenti tensor field of $(g, \tilde{g})$, which proves the assertion. $\square$

3 Comparison with Benenti's terminology

Let us briefly comment on Definitions 1.3 and 1.5. Definition 1.3 is deeply rooted in Riemannian geometry as, in fact, the construction of (orthogonal) separable coordinates is strongly related to the existence of real eigenvalues of the $(1, 1)$ -tensor L . In the present paper, however, pseudo-Riemannian metrics are usual rather than exceptional. Definition 1.5 takes account of this setting, and corresponds to the definition of a **J**-tensor in [10] by Benenti. The L -tensor of Definition 1.3 is a special **J**-tensor, compare also Section 11 of [10].

3.1 Tensors introduced by Benenti

In addition to **J** and **L**, there are other objects in the literature that are closely related to the objects studied here. We devote this subsection to a comparison of these objects with the ones used in the present paper. To keep the exposition clear and brief, without going too far astray from the actual purpose of this paper, we shall focus on reference [10] and compare the objects studied there with those introduced in the previous sections. Basically, [10] discusses four different kinds of special $(2, 0)$ -tensors, called A, B, J and L (the $(1, 1)$ -tensors

obtained by index-shifting are denoted by boldface letters $\mathbf{A}, \mathbf{B}, \mathbf{J}, \mathbf{L}$ in the reference). To facilitate an easy comparison between conventions, we use the notation of [10] within this subsection, but not outside it.

As already briefly touched upon, the Benenti tensors we introduced in Definitions 1.5 and 1.3 correspond, respectively, to $\mathbf{J}$ -tensors and $\mathbf{L}$ -tensors of [10].

As is remarked in item (i) on page 39 of [10], $\mathbf{J}$ -tensors form a linear space, while $\mathbf{A}$ and $\mathbf{B}$ do not. The reason is that $\mathbf{J}$ -tensors are very similar to the (weighted) tensors σ that appear after linearization of the metrizable problem, see (7) and also [6, 7, 17]. In this sense, the objects σ are more “natural” in this context.

Below we write a more specific correspondence between the objects we introduced, especially σ and (4), and the conformal Killing tensors introduced in [10]. To begin with, tensors J^{ij} of [10] satisfy similar equations as σ^{ij} (see Formula (6)), cf. Equation 8(c) and Remark 3.6 of [10] and Equation (2.3) of [17], and in addition Equation (7). In [10], as we said above, the $(1, 1)$ -tensor $\mathbf{J}$ is introduced by lowering one index of J^{ij} : $J_k^i = g_{kj} J^{ji}$. Now, let $\bar{g}$ be a projectively equivalent metric to g . In Sections 4 and 5 of [10], in this case, it is introduced the tensor $B = \mu \bar{g}$, where $\mu = \det(\mathbf{J})$. Also, $\mathbf{J} = \mathbf{B}^{-1}$. This allows us to compute the tensor $\mathbf{J}$ in terms of g and $\bar{g}$. More precisely

$$\mathbf{J} = J \cdot g = \mathbf{B}^{-1} g = \frac{1}{\mu} \bar{g}^{-1} g \quad (8)$$

so that

$$\mu = \det(\mathbf{J}) = \frac{1}{\mu^N} \frac{\det(g)}{\det(\bar{g})}$$

which, substituted into (8), gives (4).

In [10], another class of tensors is considered, defined by $\mathbf{A} = \text{adj}(\mathbf{J}) := \det(\mathbf{J}) \mathbf{J}^{-1}$. This $\mathbf{A}$ -tensor links $\mathbf{J}$ -tensors to integrals of motion, by grace of the following formula (see, e.g., [19]):

$$I(\xi) = \det(\mathbf{J}) g(\mathbf{J}^{-1}(\xi), \xi) = g(\mathbf{A}(\xi), \xi). \quad (9)$$

The function (9) is an integral of motion for the Hamiltonian $H = g(\xi, \xi)$, where $\xi^i = g^{ij} p_j$, with p_j denoting momenta, $p \in T^*M$.

3.2 Pencils of Benenti tensors

Propositions 2.4 and 2.5 use a certain freedom in the definition of the Benenti tensor. If we fix a point $\sigma_1 \in \mathfrak{A}$ then any non-proportional $\sigma_2 \in \mathfrak{A}$ defines a (non-trivial) Benenti tensor $L = L(\sigma_1, \sigma_2)$ by (4). Moreover, we can rescale σ_1 and σ_2 by non-zero constants. This defines a pencil

$$L_{s,t} = s(L + t \text{Id}) \quad \text{for } t \in \mathbb{R}, s \in \mathbb{R} \setminus \{0\}, \quad (10)$$

which gives rise to a family of integrals of motion,

$$I_{s,t}(\xi) = \det(L_{s,t}) g(L_{s,t}^{-1}(\xi), \xi).$$

This formula works, too, if $-t \in \mathbb{R}$ coincides with an eigenvalue of L , since the (classical) adjoint of a matrix is well-defined even for matrices with vanishing determinant. However, often certain values must be excluded to ensure that $L_{s,t}$ is non-degenerate (i.e., to make sure it is linked to a metric), see also item (ii) after Remark 3.2 in [10]. Note that, following Definition 1.5, a Benenti tensor is non-degenerate. Nonetheless, the pencil (10) yields a degenerate tensor if $-t \in \mathbb{R}$ is an eigenvalue of L . A more thorough discussion of the pencils (10) can be found, for instance, in [10, 11] (see also references therein). We mention that there is a related notion of L -pencils, which is discussed in detail in [11].

Finally, let us remark that the freedom in (10) is not the only freedom that we allow. Indeed, we do not need to fix $\sigma_1 \in \mathfrak{A}$ (up to scaling), and according to Definition 1.5, an arbitrary change of the bases is possible (as long as we do not use degenerate solutions). Most importantly we are thus, in contrast to (10), free to swap the basis entries. This latter freedom is made use of in many of our computations.

4 Derivative of Benenti tensors along projective vector fields

Recall that we work in degree of mobility 2. Let $(g, \bar{g})$ be a pair of non-proportional, projectively equivalent metrics, with corresponding solutions $\sigma, \bar{\sigma}$ in $\mathfrak{A}$. Let $L = L(g, \bar{g})$ be their Benenti tensor. In addition to this pair we are now going to assume the existence of a projective vector field w . In general, if we have pairs (g_1, g_2) and (g_3, g_4) with all $g_i, i \in \{1, \dots, 4\}$, in the same projective class, the respective Benenti tensors can be very different. Let (σ_1, σ_2) correspond to (g_1, g_2) via (6), then the Benenti tensor with respect to another pair $(\alpha_1\sigma_1 + \beta_1\sigma_2, \alpha_2\sigma_1 + \beta_2\sigma_2)$, $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ and $\det(\alpha_i\sigma_1 + \beta_i\sigma_2) \neq 0$, with α_i and $\beta_i \in \mathbb{R}$, for $i = 1, 2$, is

$$L_{\text{new}} = (\alpha_2\sigma_1 + \beta_2\sigma_2)(\alpha_1\sigma_1 + \beta_1\sigma_2)^{-1}.$$

So the Benenti tensor (4) heavily depends on the choice of the pair $(g, \bar{g})$. In the current section, we pose the question whether there is a ‘best’ choice of the pair $(g, \bar{g})$ within their projective class.

We consider this question under the assumption of additional structure on the projective connection. Particularly, we assume the existence of a projective vector field.

Remark 4.1. Let $\mathfrak{A}$ be the solution space of (7) for a projective connection. Let w be a projective vector field for it. Then $\mathfrak{A}$ is invariant under the action of the Lie derivative w.r.t. w , i.e. for any $\sigma \in \mathfrak{A}$ the Lie derivative $\mathcal{L}_w \sigma \in \mathfrak{A}$. Therefore, after choosing a basis of $\mathfrak{A}$, the Lie derivative can be represented by a $k \times k$ -matrix where k denotes the dimension of $\mathfrak{A}$.

For the Lie derivative of a Benenti tensor w.r.t. a projective vector field, the following is well-known:

Lemma 4.2 (e.g. [16]). Consider a metrizable projective connection with degree of mobility 2 and Lie derivative $\mathcal{L}_w$. Let w be a non-zero projective vector field. Moreover, let L be a Benenti tensor obtained from a pair of metrics (g_1, g_2) . Then $\mathcal{L}_w L$ is a quadratic polynomial in L .

Proof. We start with the invariance of the Equations (7) under $\mathcal{L}_w$ (see Remark 4.1), which implies

$$\mathcal{L}_w \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

for some $A, B, C, D \in \mathbb{R}$ where σ_i correspond to g_i via (6). The Benenti tensor corresponding, via (6), to the pair (σ_1, σ_2) is $L = \sigma_2 \sigma_1^{-1}$. Using $\mathcal{L}_w \sigma^{-1} = -\sigma^{-1} (\mathcal{L}_w \sigma) \sigma^{-1}$, we thus have

$$\begin{aligned} \mathcal{L}_w L &= \mathcal{L}_w (\sigma_2 \sigma_1^{-1}) = (\mathcal{L}_w \sigma_2) \sigma_1^{-1} + \sigma_2 \mathcal{L}_w \sigma_1^{-1} = (\mathcal{L}_w \sigma_2) \sigma_1^{-1} - \sigma_2 \sigma_1^{-1} (\mathcal{L}_w \sigma_1) \sigma_1^{-1} \\ &= (C\sigma_1 + D\sigma_2) \sigma_1^{-1} - L(A\sigma_1 + B\sigma_2) \sigma_1^{-1} = C \text{Id} + (D - A)L - BL^2 \end{aligned} \quad (11)$$

□

There is an obvious generalisation of Lemma 4.2 for cases with degree of mobility $k \geq 2$, yielding $\frac{k(k-1)}{2}$ multivariate quadratic polynomials in $\frac{k(k-1)}{2}$ Benenti tensors.

Can we modify the basis (σ_1, σ_2) in such a way that the polynomial on the right-hand side of $\mathcal{L}_w L = aL^2 + bL + c$ becomes particularly simple?

Remark 4.3. The simplest change of the basis (σ_1, σ_2) is to just exchange the roles of the basis members. We see that this (roughly speaking) exchanges the quadratic and the zero-degree part of the polynomial. More precisely: $L^2 \leftrightarrow -1, L \rightarrow -L$.

Lemma 4.4. If there is a metric g such that $\mathcal{L}_w g = \lambda g$ for some constant λ , then there is a full-rank basis (σ_1, σ_2) of $\mathfrak{A}$ such that $\mathcal{L}_w L$ is a linear polynomial in $L = L(\sigma_1, \sigma_2)$.

Proof. Let $\sigma^{ij} = \det(g)^{\frac{1}{N+1}} g^{ij}$ as in Formula (6). Then

$$\mathcal{L}_w \sigma^{ij} = \det(g)^{\frac{1}{N+1}} \left(\frac{1}{N+1} \text{tr}(g^{-1} \mathcal{L}_w g) g^{ij} + \mathcal{L}_w g^{ij} \right) = \det(g)^{\frac{1}{N+1}} \left(\frac{N\lambda}{N+1} g^{ij} - \lambda g^{ij} \right) = -\frac{\lambda}{N+1} \sigma^{ij}.$$

Hence we can find a basis (σ_1, σ_2) of $\mathfrak{A}$ such that

$$\mathcal{L}_w \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

with $A, C, D \in \mathbb{R}$. This proves the claim in view of Formula (11). $\square$

Theorem 4.5. *In any metrizable projective class with degree of mobility 2 that admits a projective vector field w , there exists a pair of metrics $(g, \bar{g})$ such that the Lie derivative $\mathcal{L}_w L$ of the Benenti tensor $L = L(g, \bar{g})$ takes one of the following forms.*

1. $\mathcal{L}_w L = L$
2. $\mathcal{L}_w L = L + \text{Id}$
3. $\mathcal{L}_w L = L^2 + c \text{ Id}$, $c \in \mathbb{R}$
4. $\mathcal{L}_w L = 0$.

Furthermore, if no eigenvalue of L is constant, the normal forms can be reduced to one of the following cases:

- If $\mathcal{L}_w$ has no real eigenvalue: $\mathcal{L}_w L = L^2 + \text{Id}$
- If $\mathcal{L}_w$ has exactly one real eigenvalue of geometric multiplicity 1: $\mathcal{L}_w L = \text{Id}$ (or, by reordering the pair and rescaling w , $\mathcal{L}_w L = L^2$)
- If $\mathcal{L}_w$ has two different real eigenvalues: $\mathcal{L}_w L = L$
- If $\mathcal{L}_w$ has one real eigenvalue of geometric multiplicity 2: $\mathcal{L}_w L = 0$.

Proof. In view of Lemma 4.2, $\mathcal{L}_w L = aL^2 + bL + c \text{ Id}$ for some $a, b, c \in \mathbb{R}$. Since a projective vector field is defined up to multiplication by a non-zero constant, and in view of Proposition 2.5, let us consider the transformations

$$w \rightarrow qw, \quad L \rightarrow r(L + s \text{ Id}),$$

where $q, r \in \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}$ is different from any eigenvalue of L . For the new L and w we obtain

$$\mathcal{L}_w L = \frac{ar}{q} (L + s \text{ Id})^2 + \frac{b}{q} (L + s \text{ Id}) + \frac{c}{qr} \text{ Id} = \frac{ar}{q} L^2 + \left(\frac{2ars}{q} + \frac{b}{q} \right) L + \left(\frac{ars^2}{q} + \frac{bs}{q} + \frac{c}{qr} \right) \text{ Id}.$$

Let us first assume $a \neq 0$.

In this case we can choose $q = ar$, such that we obtain

$$\mathcal{L}_w L = L^2 + \left(2s + \frac{b}{ar} \right) L + \left(s^2 + \frac{bs}{ar} + \frac{c}{ar^2} \right) \text{ Id}.$$

- If $b \neq 0$, let us further choose $r = -\frac{b}{2as}$ (this choice narrows down our remaining freedom by $s \neq 0$). We get

$$\mathcal{L}_w L = L^2 + s^2 \left(\frac{4ac}{b^2} - 1 \right) \text{ Id},$$

so we arrive to normal form 3.

- If $b = 0$, we choose $s = 0$. If also $c = 0$, we have $\mathcal{L}_w L = L^2$. If $c \neq 0$, let $r = \sqrt{\left| \frac{c}{a} \right|}$, such that we obtain again an instance of normal form 3,

$$\mathcal{L}_w L = L^2 \pm \text{Id}.$$

Now let us assume $a = 0$.

In this case we set $s = 0$, which gives

$$\mathcal{L}_w L = \frac{b}{q} L + \frac{c}{qr} \text{ Id}.$$

- If $b \neq 0$, set $q = b$. Then, if $c = 0$, we have normal form 1. Otherwise, if $c \neq 0$, we choose $r = \frac{c}{b}$, to obtain normal form 2.
- If $b = 0$, set $q = 1$. Then, if $c = 0$, $\mathcal{L}_w L = 0$. Otherwise, if $c \neq 0$, $\mathcal{L}_w L = \text{Id}$ by an obvious choice of $r \neq 0$. In this latter case, due to Remark 4.3, we arrive at $\mathcal{L}_w L = L^2$.

Note that if $a = 0$, $b \neq 0$ and no eigenvalue of L is constant, we could also have chosen $q = b$ and $s = -\frac{c}{br}$ (with any choice of $r \neq 0$) to obtain $\mathcal{L}_w L = L$.

For the second part of the claim, we can (also) proceed as follows: $\mathcal{L}_w$ is, possibly after a change of basis, represented by one of the matrices, with $\lambda, \mu \in \mathbb{R}$,

$$\begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda - \mu & \\ \mu & \lambda \end{pmatrix},$$

for which by Formula (11) the corresponding polynomials are, respectively,

$$\mathcal{L}_w L = (\mu - \lambda) L, \quad \mathcal{L}_w L = -L^2, \quad \mathcal{L}_w L = \mu (L^2 + \text{Id}).$$

The normal forms are obtained after a rescaling of w . □

Corollary 4.6. *Consider a metrizable projective connection with degree of mobility 2. If the projective algebra is at least 2-dimensional, there is a projective vector field w such that $\mathcal{L}_w$ has a full-rank eigenvector and the normal form $\mathcal{L}_w L = L^2 + \text{Id}$ cannot be realized.*

Proof. Let (σ_1, σ_2) be a basis of $\mathfrak{A}$ such that $\det(\sigma_i) \neq 0$. Since the projective algebra is at least 2-dimensional, we find two non-proportional projective vector fields u, v that satisfy equations

$$\mathcal{L}_u \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \quad \mathcal{L}_v \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

We can assume $a_2 \neq 0$ and $b_2 \neq 0$, otherwise w would be either u or v and σ_1 the full-rank eigenvector we are looking for. Then, define $w = b_2 u - a_2 v$. We have that

$$\mathcal{L}_w \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} b_2 a_1 - a_2 b_1 & 0 \\ b_2 a_3 - a_2 b_3 & b_2 a_4 - a_2 b_4 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

Observe that this equation does, by Theorem 4.5, not allow for the normal form $\mathcal{L}_w L = L^2 + \text{Id}$. □

5 Example: Levi-Civita metrics in 3D

We illustrate the discussion of the previous sections by the example of Levi-Civita metrics, particularly for a 3-dimensional manifold. We can restrict to degree of mobility 2 by Corollary 3 of [18], as for degree of mobility 1 all metrics are proportional and the projective vector field is homothetic. Metrics of the type discussed in this section have first been constructed by Levi-Civita [20], see [21] for an English translation. Reference [22] discusses these metrics from the viewpoint of projective differential geometry. In [4] a description is given how to obtain Levi-Civita metrics via the method of splitting-gluing [16]. Locally, a Levi-Civita metric can be written in the form

$$g = \sum_{i=1}^n \varepsilon_i \prod_{j \neq i} (X_i - X_j) dx_i^2$$

where $\varepsilon_i \in \{\pm 1\}$ and where $X_i(x^i)$ are functions of one variable for each i . Particularly, in dimension 3, one obtains, for $0 < X_1 < X_2 < X_3$, the Riemannian metric

$$g = \begin{pmatrix} (X_2 - X_1)(X_3 - X_1) & 0 & 0 \\ 0 & (X_3 - X_2)(X_2 - X_1) & 0 \\ 0 & 0 & (X_3 - X_1)(X_3 - X_2) \end{pmatrix}$$

A metric projectively equivalent, but non-proportional to g can be written, maybe after a change of coordinates,

$$\tilde{g} = \sum_{i=1}^n \frac{\varepsilon_i \prod_{j \neq i} (X_i - X_j)}{X_i \prod_k X_k} dx_i^2$$

The Benenti tensor of $(g, \bar{g})$ has the particularly nice form

$$L(g, \bar{g}) = \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_N \end{pmatrix}. \quad (12)$$

Let us consider the corresponding weighted tensors defined by (6), say σ and $\bar{\sigma}$ for $g, \bar{g}$, respectively. Restricting to dimension 3, we find by a straightforward computation for a linear combination,

$$\det(k_1\sigma + k_2\bar{\sigma}) = \frac{(k_1 + k_2X_1)(k_1 + k_2X_2)(k_1 + k_2X_3)}{\sqrt{(X_3 - X_2)(X_3 - X_1)(X_1 - X_2)}}, \quad k_i \in \mathbb{R}.$$

A solution of lower rank is characterized by vanishing of this expression, and obviously this occurs, generically speaking, if and only if the equation $k_1 + k_2X_i = 0$ holds for some $i \in \{1, 2, 3\}$. Thus, we have to require at least one of the functions $X_1(x_1), X_2(x_2), X_3(x_3)$ to be constant. Obviously, this is in line with the results of Proposition 2.3 and corresponds, in view of (12), to Proposition 2.4. For brevity of exposition assume $X_1 = c \in \mathbb{R}$. Then g does not depend on the coordinate x_1 , implying that $\frac{\partial}{\partial x_1}$ is a Killing vector field. We see that in such case c is a constant eigenvalue of L , as required by Proposition 2.4.

Let us now turn our attention to Lemma 4.2. Assume g to be a Levi-Civita metric that admits a projective vector field w . For simplicity let us also assume that the Benenti tensor (12) does not admit a constant eigenvalue, i.e. $X'_i(x^i) \neq 0$ for any $i \in \{1, \dots, n\}$. Then we have by Lemma 4.2

$$\mathcal{L}_w L = aL^2 + bL + c \text{Id},$$

which by (12) amounts to

$$\mathcal{L}_w \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{pmatrix} = a \begin{pmatrix} X_1^2 & & \\ & \ddots & \\ & & X_n^2 \end{pmatrix} + b \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{pmatrix} + c \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

The non-trivial equations are

$$w_i X'_i = \mathcal{L}_w X_i = aX_i^2 + bX_i + c.$$

These equations have been found first by Solodovnikov in [22] as a (necessary) condition for the existence of a projective vector field. In fact, in presence of a homothetic vector field, Solodovnikov finds similar equations linear in X_i , and this is in line with our observations above, and with Lemma 4.4. By assumption, each of the functions X_i is non-constant. Thus, using the normal forms of the second part of Theorem 4.5, we find the three ODEs (omitting $\mathcal{L}_w L = 0$)

$$w_i X'_i = X_i^2 + 1, \quad w_i X'_i = X_i, \quad \text{and} \quad w_i X'_i = 1,$$

which can be solved explicitly, giving, respectively,

$$X_i(x_i) = \pm \frac{1}{\sqrt{2C - \int_1^{x_i} \frac{2dt}{w_i(t)}}}, \quad X_i(x_i) = \frac{1}{C - \int_1^{x_i} \frac{dt}{w_i(t)}}, \quad \text{and} \quad X_i(x_i) = C \exp \int_1^{x_i} \frac{dt}{w_i(t)}$$

for each i , with arbitrary integration constants C .

Acknowledgement: The authors wish to thank Vladimir Matveev for sharing his knowledge, and for hospitality. Gianni Manno is a member of GNSAGA of Istituto Nazionale di Alta Matematica (INdAM); Andreas Vollmer acknowledges his previous membership. The authors are grateful for support by the project FIR-2013 Geometria delle equazioni differenziali. Gianni Manno was also partially supported by "Starting grant per giovani ricercatori" of Politecnico di Torino, code 53_RSG16MANGIO. This work was written when Andreas Vollmer was a research fellow of INdAM. At the time of acceptance of this paper he is a research fellow of Deutsche Forschungsgemeinschaft and visiting fellow at the School of Mathematics and Statistics, University of New South Wales, Sydney, Australia.

References

- [1] Weyl H., Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung, Göttingen Nachr., 1921, 99–122.
- [2] Thomas T.Y., On the projective and equi-projective geometry of path, Proc. Nat. Acad. Sci., 1925, 11, 199–203
- [3] Veblen O., Projective and affine geometry of paths, Nat. Acad. Proc., 1922 8, 347–350
- [4] Matveev V.S., Geodesically equivalent metrics in general relativity, Journal of Geometry and Physics, 2012, vol. 62, 3, 675–691.
- [5] Manno G., On the geometry of Grassmannian equivalent connections, Adv. Geom., 2008, vol. 8, 3, 329–342.
- [6] Bryant, R.L., Manno G, Matveev V.S., A solution of a problem of Sophus Lie: normal forms of two-dimensional metrics admitting two projective vector fields, Mathematische Annalen, 2008, vol. 340, 2, 437–463.
- [7] Bryant R., Dunajski M., Eastwood M., Metrisability of two-dimensional projective structures, J. Differential Geom., 2009, vol. 83, 465–500.
- [8] Matveev V.S., Two-dimensional metrics admitting precisely one projective vector field, Mathematische Annalen, 2012, vol. 352, 4, 865–909.
- [9] Benenti S., Inertia tensors and Stäckel systems in Euclidean spaces 1992, vol. 50, 4, 315–341.
- [10] Benenti S., Special symmetric two-tensors, equivalent dynamical systems, cofactor and bi-cofactor systems, Acta Appl. Math., 2005, vol. 87, 1-3, 33–91.
- [11] Benenti S., Algebraic Construction of the Quadratic First Integrals for a Special Class of Orthogonal Separable Systems, in: Symmetries and Overdetermined Systems of Partial Differential Equations, 2008, Springer New York, New York, NY, 2008, 277–304.
- [12] Benenti S., Separability in Riemannian manifolds, SIGMA Symmetry Integrability Geom. Methods Appl., 2016, vol. 12, Paper No. 013, 21.
- [13] Ibrat A., Magri F., Marmo G., Bihamiltonian structures and Stäckel separability, Journal of Geometry and Physics, 2000, vol. 33, 3, 210–228.
- [14] Crampin M., Sarlet W., Thompson G., Bi-differential calculi and bi-Hamiltonian systems, Journal of Physics A: Mathematical and General, 2000, vol. 33, 20, pp. L177.
- [15] Bolsinov A.V., Matveev V.S., Geometrical interpretation of Benenti systems, Journal of Geometry and Physics, 2003, vol. 44, 4, 489–506.
- [16] Bolsinov A.V., Matveev V.S., Splitting and gluing constructions for geodesically equivalent pseudo-Riemannian metrics", Trans. Amer. Math. Soc., 2011, vol. 363, 4081–4107.
- [17] Eastwood M., Matveev V., Metric Connections in Projective Differential Geometry, in: Symmetries and Overdetermined Systems of Partial Differential Equations, Springer New York, New York, NY, 2008, 339–350.
- [18] Kiosak V., Matveev V.S., Proof of the Projective Lichnerowicz Conjecture for Pseudo-Riemannian Metrics with Degree of Mobility Greater than Two, Communications in Mathematical Physics, 2010, vol. 297, 2, 401–426.
- [19] Topalov P., Matveev V.S., Geodesic Equivalence via Integrability, Geometriae Dedicata, 2003, vol. 96, 1, 91–115.
- [20] Levi-Civita T., Sulle trasformazioni delle equazioni dinamiche, Annali di Matematica, 1896, vol. 24, 255–300.
- [21] Levi-Civita T., On the transformations of the dynamical equations, Regular and Chaotic Dynamics, 2009, vol. 14, 4, 580–614.
- [22] Solodovnikov A.S., Projective transformations of Riemannian spaces, Uspekhi Mat. Nauk, 1956, vol. 11, 4(70), 45–116.