**Complex Manifolds**

**Research Article**

Takumi Yamada*

# Complex structures on the complexification of a real Lie algebra

https://doi.org/10.1515/coma-2018-0010
Received April 28, 2018; accepted July 18, 2018.

**Abstract:** Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ be a Lie algebra with a direct sum decomposition such that $\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras. Then, we can construct an integrable complex structure $\tilde{J}$ on $\mathfrak{h} = {}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ from the decomposition, where ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ is a real Lie algebra obtained from $\mathfrak{g}^{\mathbb{C}}$ by the scalar restriction. Conversely, let $\tilde{J}$ be an integrable complex structure on $\mathfrak{h} = {}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Then, we have a direct sum decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ such that $\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras. We also investigate relations between the decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ and $\dim H^{s,t}_{\bar{\partial}_{\tilde{J}}}(\mathfrak{h}^{\mathbb{C}})$.

**Keywords:** Nilmanifold, Dolbeault cohomology group, Complex structure

**MSC:** 53C30, 57T15, 22E25

## 1 Introduction

S. Salamon [6] classified real 6-dimensional nilpotent Lie algebras for which the corresponding Lie group has a left-invariant complex structure, and estimated the dimensions of moduli spaces of such structures. The classification of complex structures on nilpotent Lie algebras provides the classification of invariant complex structures on nilmanifolds. Invariant complex structures on 6-dimensional nilmanifolds have been classified ([1, 2, 8]). Nakamura [4] has computed Hodge numbers of small deformations on compact complex parallelizable solvmanifolds to investigate rigidities of small deformations. However, for higher dimensional cases, the situation looks far from the completely understanding.

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$. In previous papers [10, 12], we considered the case where $\mathfrak{g}$ has a direct sum decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ such that $\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras of $\mathfrak{g}$. Then, we can construct an integrable complex structure $\tilde{J}$ on $\mathfrak{h} = {}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ from the decomposition, where ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ is a real Lie algebra obtained from $\mathfrak{g}^{\mathbb{C}}$ by the scalar restriction. We also studied relations between the decomposition and $\dim H^{s,t}_{\bar{\partial}_{\tilde{J}}}(\mathfrak{h}^{\mathbb{C}})$ for investigating the complex structure $\tilde{J}$ (see e.g. [10, Theorems 3.2, 3.3]), and constructed direct sum decompositions $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ on a nilpotent Lie algebra $\mathfrak{n}$ such that $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras by using root systems and T-root systems ([11, Section.4],[12]).

In this paper, we consider integrable complex structures on a real Lie algebra $\mathfrak{h} = {}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ and relations between the decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ and $\dim H^{s,t}_{\bar{\partial}_{\tilde{J}}}(\mathfrak{h}^{\mathbb{C}})$. We use the notation $H^{s,t}_{\bar{\partial}}(\mathfrak{g}_{\tilde{J}})$ instead of $H^{s,t}_{\bar{\partial}_{\tilde{J}}}(\mathfrak{h}^{\mathbb{C}})$ to emphasize $\mathfrak{g}$ and $\tilde{J}$. We mainly prove the following:

**Theorem 1.1.** *There exists one-to-one correspondence between the direct sum decompositions* $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ *such that* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are subalgebras and the integrable complex structures on* $\mathfrak{h} = {}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$.

***Corresponding Author: Takumi Yamada:** Department of Mathematics, Shimane University, Nishikawatsu-cho 1060, Matsue, 690-8504, Japan, E-mail: t_yamada@riko.shimane-u.ac.jp

**Theorem 1.2.**

$$H^{s,t}_{\bar\partial}(\mathfrak{g}_{\tilde J}) \cong H^{s,t}_{\bar\partial}(\mathfrak{g}_{-\tilde J})$$

*for each* $s, t$.

By this theorem, we have that if $\dim H^{s,t}_{\bar\partial}(\mathfrak{g}_{\tilde J})$ is expressed by data of $\mathfrak{a}$ and $\mathfrak{b}$, then we may change the roles of $\mathfrak{a}$ and $\mathfrak{b}$ (see Section 4 for the detail). Let $N$ be a simply connected real nilpotent Lie group and $\Gamma$ a lattice in $N$. If a left-invariant complex structure $J$ on $N$ is $\Gamma$-rational, then $H^{s,t}_{\bar\partial_J}(\Gamma\backslash N) \cong H^{s,t}_{\bar\partial_J}(\mathfrak{n}^{\mathbb{C}})$ for each $s, t$ ([3]). Thus, results on $H^{s,t}_{\bar\partial}(\mathfrak{n}^{\mathbb{C}})$ of the nilpotent Lie algebra with rational complex structures yield results on $H^{s,t}_{\bar\partial}(\Gamma\backslash N)$ of a compact nilmanifold with invariant rational complex structures.

## 2 Preliminaries

In this section, we recall integrability conditions of an almost left-invariant complex structure on a Lie group.

Let $H$ be a real Lie group and $\mathfrak{h}$ its Lie algebra. A left-invariant almost complex structure on $H$ can be identified with a linear mapping $J : \mathfrak{h} \longrightarrow \mathfrak{h}$ such that $J^2 = -\mathrm{id}$. We denote $\mathfrak{h}^{\pm}_J$ the vector spaces of the $\pm\sqrt{-1}$ eigenvectors of the almost complex structure $J$, respectively. We denote by $H^{*,*}_{\bar\partial_J}(\mathfrak{h}^{\mathbb{C}})$ the cohomology ring of the differential bigraded algebra $\wedge^{*,*}(\mathfrak{h}^{\mathbb{C}})^*$, associated to $\mathfrak{h}^{\mathbb{C}}$ with respect to the operator $\bar\partial_J$ in the canonical decomposition $d = \partial_J + \bar\partial_J$ on $\wedge^{*,*}(\mathfrak{h}^{\mathbb{C}})^*$ for an integrable complex structure $J$.

The almost complex structure $J$ is said to be *integrable* if

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0$$

for all $X, Y \in \mathfrak{h}$. We shall refer to a pair $(\mathfrak{h}, J)$ consisting of a Lie algebra and an integrable almost complex structure simply as a *Lie algebra with a complex structure*. The equation $N_J(X, Y) = 0$ holds for all $X, Y \in \mathfrak{h}$ if and only if $d(\wedge^{1,0}_J) \subset \wedge^{2,0}_J \oplus \wedge^{1,1}_J$ (cf. [6, pp.313–pp.314]). On the other hand, the following equivalent condition is also well known (see e.g. [7, Section.1], [5, Proposition.1]):

**Proposition 2.1.** *A real Lie group $H$ has an integrable left-invariant complex structure if and only if $\mathfrak{h}^{\mathbb{C}}$ admits a direct sum decomposition*

$$\mathfrak{h}^{\mathbb{C}} = \mathfrak{q} \oplus \tau\mathfrak{q},$$

*where $\mathfrak{q}$ is a complex subalgebra of $\mathfrak{h}^{\mathbb{C}}$, and $\tau$ is the complex conjugation.*

In fact, let $\mathfrak{q}$ be a subalgebra of $\mathfrak{h}^{\mathbb{C}}$ which satisfies the condition in the proposition. Then, $\tilde J = -\sqrt{-1}\,\mathrm{id}_{\mathfrak{q}} \oplus \sqrt{-1}\,\mathrm{id}_{\tau\mathfrak{q}}$ is an integrable complex structure. Conversely, for a given integrable left-invariant complex structure $\tilde J$, let $\mathfrak{q}$ be the subspace of $\mathfrak{h}^{\mathbb{C}}$ defined by

$$\mathfrak{q} = \{X + \sqrt{-1}\tilde J X \,|\, X \in \mathfrak{h}\}.$$

Then, $\mathfrak{q}$ is a subalgebra satisfying $\mathfrak{h}^{\mathbb{C}} = \mathfrak{q} \oplus \tau\mathfrak{q}$.

**Proposition 2.2.** *Let $J$ be an Ad-invariant complex structure on $\mathfrak{h}$. Assume that there exist subalgebras $\mathfrak{k}$ and $\mathfrak{m}$ of $\mathfrak{h}$ such that*

$$\mathfrak{h} = \mathfrak{k} + \mathfrak{m}, \mathfrak{k} \cap \mathfrak{m} = \{0\}, J(\mathfrak{k}) \subset \mathfrak{k}, J(\mathfrak{m}) \subset \mathfrak{m}.$$

*Put*

$$\mathfrak{q} = \mathrm{span}_{\mathbb{C}}\{X - \sqrt{-1}JX, Y + \sqrt{-1}JY \,|\, X \in \mathfrak{k}, Y \in \mathfrak{m}\}.$$

*Then, $\mathfrak{q}$ is a subalgebra of $\mathfrak{h}^{\mathbb{C}}$ satisfying $\mathfrak{h}^{\mathbb{C}} = \mathfrak{q} \oplus \tau\mathfrak{q}$, and $\tilde J = -\sqrt{-1}\,\mathrm{id}_{\mathfrak{q}} \oplus \sqrt{-1}\,\mathrm{id}_{\tau\mathfrak{q}}$ is an integrable complex structure.*

*Proof.* Since $\mathrm{ad}(X) \circ J = J \circ \mathrm{ad}(X)$, $\mathfrak{q}$ is a subalgebra of $\mathfrak{h}^{\mathbb{C}}$. □

In the proposition, note that

$$X-\sqrt{-1}JX=X+\sqrt{-1}\tilde{J}X,\ Y+\sqrt{-1}JY=Y+\sqrt{-1}\tilde{J}Y$$

for $X\in\mathfrak{k}$, $Y\in\mathfrak{m}$.

# 3 Decompositions and Complex structures

In this section, we consider the case of $\mathfrak{h}$ is a real Lie algebra obtained from the scalar restriction of the complexification of a real Lie algebra.

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$. We can consider the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ as a real Lie algebra ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ with the Ad-invariant complex structure $J$, where ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ is the scalar restriction of $\mathfrak{g}^{\mathbb{C}}$. Put $\mathfrak{h}={}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. We will denote the extension of $J$ to a $\mathbb{C}$-linear transformation of $\mathfrak{h}^{\mathbb{C}}$ by the same letter $J$. Then, we have a direct sum decomposition

$$\mathfrak{h}^{\mathbb{C}}=\mathfrak{h}_J^{+}+\mathfrak{h}_J^{-},\quad \tau\mathfrak{h}_J^{+}=\mathfrak{h}_J^{-}.$$

Note that $f:\mathfrak{g}^{\mathbb{C}}\longrightarrow\mathfrak{h}_J^{+};X\mapsto\frac{1}{2}(X-\sqrt{-1}JX)$ is an isomorphism between complex Lie algebras, and $\mathfrak{h}_J^{+}\cong\mathfrak{h}_J^{-}$. Indeed, let $\{X_1,\dots,X_n\}$ be a basis of $\mathfrak{g}$. Assume that $[X_i,X_j]=\sum_k c_{ij}^kX_k$ for each $i,j$, where $c_{ij}^k\in\mathbb{R}$. Then, we see

$$\begin{aligned}[X_i-\sqrt{-1}JX_i,X_j-\sqrt{-1}JX_j]&=2([X_i,X_j]-\sqrt{-1}J[X_i,X_j])\\&=4\sum_k c_{ij}^k\frac{1}{2}(X_k-\sqrt{-1}JX_k).\end{aligned}$$

Let $\mathfrak{g}=\mathfrak{a}+\mathfrak{b}$ be a direct sum decomposition such that $\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras. We define another complex structure $\tilde{J}$ on $\mathfrak{h}$ by

$$\tilde{J}=\begin{cases}-J & \text{on}\quad {}_{\mathbb{R}}(\mathfrak{a}^{\mathbb{C}})\\ J & \text{on}\quad {}_{\mathbb{R}}(\mathfrak{b}^{\mathbb{C}}).\end{cases}$$

Then, $\tilde{J}$ is integrable by Proposition 2.2 (cf. [11, 12]). Indeed, put

$$\mathfrak{q}=\operatorname{span}_{\mathbb{C}}\{X-\sqrt{-1}JX,\ Y+\sqrt{-1}JY\,\big|\,X\in{}_{\mathbb{R}}(\mathfrak{a}^{\mathbb{C}}),\ Y\in{}_{\mathbb{R}}(\mathfrak{b}^{\mathbb{C}})\}.$$

Then, $\mathfrak{q}$ is a complex subalgebera of $\mathfrak{h}^{\mathbb{C}}$, and $\tilde{J}$ satisfies

$$\tilde{J}=-\sqrt{-1}\,\mathrm{id}_{\mathfrak{q}}\oplus\sqrt{-1}\,\mathrm{id}_{\tau\mathfrak{q}}.$$

We shall call that $\tilde{J}$ is the complex structure corresponding to the decomposition $\mathfrak{g}=\mathfrak{a}+\mathfrak{b}$. Note that $f(\mathfrak{a})$ and $f(\mathfrak{b})$ are real subalgebras of $\mathfrak{h}_{\tilde{J}}^{\mp}$, respectively.

Conversely, let $\tilde{J}$ be another integrable complex structure on $\mathfrak{h}$. Then, we have another direct sum decomposition

$$\mathfrak{h}^{\mathbb{C}}=\mathfrak{h}_{\tilde{J}}^{+}+\mathfrak{h}_{\tilde{J}}^{-},\quad \tau\mathfrak{h}_{\tilde{J}}^{+}=\mathfrak{h}_{\tilde{J}}^{-}.$$

Since $\tilde{J}$ is integrable, $\mathfrak{h}_{\tilde{J}}^{+}$ and $\mathfrak{h}_{\tilde{J}}^{-}$ are subalgebras. Then, we have

$$\begin{aligned}\mathfrak{h}_J^{+}&=\mathfrak{h}_J^{+}\cap\mathfrak{h}^{\mathbb{C}}\\&=\mathfrak{h}_J^{+}\cap(\mathfrak{h}_{\tilde{J}}^{+}+\mathfrak{h}_{\tilde{J}}^{-})\\&=(\mathfrak{h}_J^{+}\cap\mathfrak{h}_{\tilde{J}}^{-})+(\mathfrak{h}_J^{+}\cap\mathfrak{h}_{\tilde{J}}^{+}).\end{aligned}$$

Notice that $\mathfrak{h}_J^{+}\cap\mathfrak{h}_{\tilde{J}}^{-}$ and $\mathfrak{h}_J^{+}\cap\mathfrak{h}_{\tilde{J}}^{+}$ are complex subalgebras. Put

$$\begin{aligned}\mathfrak{v}&=f(\mathfrak{g})\cap(\mathfrak{h}_J^{+}\cap\mathfrak{h}_{\tilde{J}}^{-})\subset\mathfrak{h}_{\tilde{J}}^{-}\cap f(\mathfrak{g}),\\ \mathfrak{w}&=f(\mathfrak{g})\cap(\mathfrak{h}_J^{+}\cap\mathfrak{h}_{\tilde{J}}^{+})\subset\mathfrak{h}_{\tilde{J}}^{+}\cap f(\mathfrak{g}).\end{aligned}$$

Then, we have that $\mathfrak{v}$ and $\mathfrak{w}$ are real subalgebras which satisfy that

$$\mathfrak{v} + \mathfrak{w} = f(\mathfrak{g}), \mathfrak{v} \cap \mathfrak{w} = \{0\},$$

and

$$\mathfrak{v}^{\mathbb{C}} = f(\mathfrak{g})^{\mathbb{C}} \cap (\mathfrak{h}_J^+ \cap \mathfrak{h}_{\tilde{J}}^-) = \mathfrak{h}_J^+ \cap \mathfrak{h}_{\tilde{J}}^- \subset \mathfrak{h}_{\tilde{J}}^-,$$
$$\mathfrak{w}^{\mathbb{C}} = f(\mathfrak{g})^{\mathbb{C}} \cap (\mathfrak{h}_J^+ \cap \mathfrak{h}_{\tilde{J}}^+) = \mathfrak{h}_J^+ \cap \mathfrak{h}_{\tilde{J}}^+ \subset \mathfrak{h}_{\tilde{J}}^+.$$

Put $\mathfrak{a} = f^{-1}(\mathfrak{v})$, $\mathfrak{b} = f^{-1}(\mathfrak{w})$. Then, $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{g}$ which satisfy that

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{g}, \mathfrak{a} \cap \mathfrak{b} = \{0\},$$

and the complex structure on $\mathfrak{h}$ corresponding to the decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ is $\tilde{J}$ because $\mathfrak{v} \subset \mathfrak{h}_{\tilde{J}}^-$, $\mathfrak{w} \subset \mathfrak{h}_{\tilde{J}}^+$ and $\dim_{\mathbb{C}} \mathfrak{h}_{\tilde{J}}^+ = \dim_{\mathbb{R}} \mathfrak{v} + \dim_{\mathbb{R}} \mathfrak{w}$.

Thus, we have the following:

**Theorem 3.1.** *Let $\mathfrak{g}$ be a real Lie algbera. There exists one-to-one correspondence between the direct sum decompositions $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ such that $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras and the integrable complex structures on $\mathfrak{h} = {}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$.*

**Remarks 3.2.** *(i) In the paper [11], we consider the case of $\mathfrak{g} = (\mathfrak{t} \ltimes \mathfrak{k}) \ltimes \mathfrak{b}'$, where $\mathfrak{t}$ is abelian, $\mathfrak{k}$ is an ideal of $\mathfrak{t} \ltimes \mathfrak{k}$, and $\mathfrak{b}'$ is an ideal of $\mathfrak{g}$. This is a special case of a direct sum decomposition of the above form $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ because $\mathfrak{a} = \mathfrak{k}$ and $\mathfrak{b} = \mathfrak{t} \ltimes \mathfrak{b}'$ are subalgebras.*

*(ii) In the papers [11, 12], we construct direct sum decompositions $\mathfrak{n} = \mathfrak{a} + \mathfrak{b}$ on a nilpotent Lie algebras $\mathfrak{n}$ such that $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras by using root systems and T-root systems. For example, we can construct the following left-invariant complex structures on a real 6-dimensional nilpotent Lie group from the viewpoint of the root system of type $A_2$:*

$$\left\{\left.\begin{pmatrix}1 & z_1 & z_3\\ 0 & 1 & z_2\\ 0 & 0 & 1\end{pmatrix}\right| z_i \in \mathbb{C}\right\}, \quad \left\{\left.\begin{pmatrix}1 & \bar{z}_1 & z_3\\ 0 & 1 & z_2\\ 0 & 0 & 1\end{pmatrix}\right| z_i \in \mathbb{C}\right\},$$
$$\left\{\left.\begin{pmatrix}1 & z_1 & \bar{z}_3\\ 0 & 1 & \bar{z}_2\\ 0 & 0 & 1\end{pmatrix}\right| z_i \in \mathbb{C}\right\}, \quad \left\{\left.\begin{pmatrix}1 & \bar{z}_1 & \bar{z}_3\\ 0 & 1 & \bar{z}_2\\ 0 & 0 & 1\end{pmatrix}\right| z_i \in \mathbb{C}\right\}.$$

# 4 Hodge numbers

In this section, we consider relations between the decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ and $\dim H_{\bar{\partial}}^{s,t}(\mathfrak{g}_{\tilde{J}})$.

Let $\mathfrak{g}$ be a real Lie algebra with a direct decomposition

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{b},$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras of $\mathfrak{g}$. Take a basis of the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$:

$$\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{U_1^1, \dots, U_p^1\},$$
$$\mathfrak{b} = \mathrm{span}_{\mathbb{R}}\{V_1^1, \dots, V_q^1\}.$$

Consider the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Since $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$, the scalar restriction ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ has the following basis:

$$\{U_1^1, \dots, U_p^1, V_1^1, \dots, V_q^1, U_1^2, \dots, U_p^2, V_1^2, \dots, V_q^2\},$$

where $U_i^2 = \sqrt{-1}U_i^1$, $V_j^2 = \sqrt{-1}V_j^1$. Let

$$\{\alpha_1^1, \dots, \alpha_p^1, \beta_1^1, \dots, \beta_q^1, \alpha_1^2, \dots, \alpha_p^2, \beta_1^2, \dots, \beta_q^2\}$$

be the dual basis of

$$\{U^1_1, \dots, U^1_p, V^1_1, \dots, V^1_q, U^2_1, \dots, U^2_p, V^2_1, \dots, V^2_q\}.$$

Let $J$ be the complex structure on $\mathfrak{h} = {}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ defined by

$$JU^1_i = U^2_i \ (JU^2_i = -U^1_i),\ JV^1_j = V^2_j \ (JV^2_j = -V^1_j)$$

for each $i, j$. Put

$$\lambda_i = \alpha^1_i + \sqrt{-1}\alpha^2_i,\ \mu_j = \beta^1_j + \sqrt{-1}\beta^2_j$$

for each $i, j$. We can assume that

$$d\lambda_i = -\sum_{k,h} C^i_{kh}\lambda_k \wedge \lambda_h - \sum_{k,s} D^i_{ks}\lambda_k \wedge \mu_s,\ d\mu_j = -\sum_{s,t} E^j_{st}\mu_s \wedge \mu_t - \sum_{k,s} F^j_{ks}\lambda_k \wedge \mu_s$$

for each $i, j$, where $C^i_{kh}, D^i_{ks}, E^j_{st}, F^j_{ks} \in \mathbb{R}$. Note that $({}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}), J)$ is a complex Lie algebra.

We consider another complex structure $\tilde{J}$ on $\mathfrak{h} = {}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$, which appeared in the previous section, defined by

$$\tilde{J}U^1_i = -U^2_i \ (\tilde{J}U^2_i = U^1_i),\ \tilde{J}V^1_j = V^2_j \ (\tilde{J}V^2_j = -V^1_j)$$

for each $i, j$. Put

$$\xi_i = \alpha^1_i - \sqrt{-1}\alpha^2_i,\ \eta_j = \beta^1_j + \sqrt{-1}\beta^2_j$$

for each $i, j$. Then, $\xi_i, \eta_j \in \bigwedge^{1,0}_{\tilde{J}}$, and

$$\begin{aligned} \bar{\partial}\xi_i &= -\sum_{k,s} D^i_{ks}\xi_k \wedge \bar{\eta}_s,\ \bar{\partial}\eta_j = -\sum_{k,s} F^j_{ks}\bar{\xi}_k \wedge \eta_s, \\ \bar{\partial}\bar{\xi}_i &= -\sum_{k,h} C^i_{kh}\bar{\xi}_k \wedge \bar{\xi}_h,\ \bar{\partial}\bar{\eta}_j = -\sum_{s,t} E^j_{st}\bar{\eta}_s \wedge \bar{\eta}_t. \end{aligned} \tag{1}$$

We can consider a complex structure $-\tilde{J}$ instead of $\tilde{J}$, which corresponds to changing the roles of $\mathfrak{a}$ and $\mathfrak{b}$. Then, we have the following:

**Theorem 4.1.**

$$H^{s,t}_{\bar{\partial}}(\mathfrak{g}_{\tilde{J}}) \cong H^{s,t}_{\bar{\partial}}(\mathfrak{g}_{-\tilde{J}})$$

*for each $s, t$.*

*Proof.* Note that $\xi_i = \bar{\lambda}_i, \eta_j = \mu_j \in \bigwedge^{1,0}_{\tilde{J}}$, and $\xi'_i = \lambda_i, \eta'_j = \bar{\mu}_j \in \bigwedge^{1,0}_{-\tilde{J}}$. Since $C^i_{kh}, D^i_{ks}, E^j_{st}, F^j_{ks}$ are real numbers, we have

$$\begin{aligned} d\xi_i = d\bar{\lambda}_i &= -\sum_{k,h} C^i_{kh}\bar{\lambda}_k \wedge \bar{\lambda}_h - \sum_{k,s} D^i_{ks}\bar{\lambda}_k \wedge \bar{\mu}_s \\ &= -\sum_{k,h} C^i_{kh}\xi_k \wedge \xi_h - \sum_{k,s} D^i_{ks}\xi_k \wedge \bar{\eta}_s, \\ d\eta_j = d\mu_j &= -\sum_{s,t} E^j_{st}\eta_s \wedge \eta_t - \sum_{k,s} F^j_{ks}\bar{\xi}_k \wedge \eta_s, \\ d\xi'_i = d\lambda_i &= -\sum_{k,h} C^i_{kh}\xi'_k \wedge \xi'_h - \sum_{k,s} D^i_{ks}\xi'_k \wedge \bar{\eta}'_s, \\ d\eta'_j = d\bar{\mu}_j &= -\sum_{s,t} E^j_{st}\bar{\mu}_s \wedge \bar{\mu}_t - \sum_{k,s} F^j_{ks}\lambda_k \wedge \bar{\mu}_s \\ &= -\sum_{s,t} E^j_{st}\eta'_s \wedge \eta'_t - \sum_{k,s} F^j_{ks}\bar{\xi}'_k \wedge \eta'_s. \end{aligned}$$

Thus, $\{\xi_i, \eta_j\}$ and $\{\xi'_i, \eta'_j\}$ satisfy the same structure equations. Hence, $H^{s,t}_{\bar{\partial}}(\mathfrak{g}_{\tilde{J}}) \cong H^{s,t}_{\bar{\partial}}(\mathfrak{g}_{-\tilde{J}})$ for each $s, t$. □

By this theorem, we see that if $\dim H^{s,t}_{\bar{\partial}}(\mathfrak{g}_{\tilde{J}})$ is expressed by data of $\mathfrak{a}$ and $\mathfrak{b}$, then we may change the roles of $\mathfrak{a}$ and $\mathfrak{b}$ as the following theorems. We denote $\dim H^{s,t}_{\bar{\partial}}(\mathfrak{g}_{\tilde{J}})$ by $h^{s,t}(\mathfrak{g}_{\tilde{J}})$.

Let $\mathfrak{g}_\mathfrak{a}$, $\mathfrak{g}_\mathfrak{b}$ be real Lie algebras such that $\mathfrak{g}_\mathfrak{a}^* = \mathrm{span}\{\mu_1^0, \ldots, \mu_p^0, \nu_1^0, \ldots, \nu_q^0\}$ and $\mathfrak{g}_\mathfrak{b}^* = \mathrm{span}\{\mu_1^1, \ldots, \mu_p^1, \nu_1^1, \ldots, \nu_q^1\}$ have the structure equations

$$d\mu_i^0 = -\sum_{k,s} D_{ks}^i \mu_k^0 \wedge \nu_s^0, \quad d\nu_j^0 = -\sum_{s,t} E_{st}^j \nu_s^0 \wedge \nu_t^0, \tag{2}$$

$$d\mu_i^1 = -\sum_{k,h} C_{kh}^i \mu_k^1 \wedge \mu_h^1, \quad d\nu_j^1 = -\sum_{k,s} F_{ks}^j \mu_k^1 \wedge \nu_s^1, \tag{3}$$

respectively. Since $\bar\partial^2 = 0$ on $\bigwedge_{\bar J}^{*,*}(\mathfrak{h}^\mathbb{C})^*$, we have that $d^2 = 0$ on $\bigwedge^1 \mathfrak{g}_\mathfrak{a}^*$ and $\bigwedge^1 \mathfrak{g}_\mathfrak{b}^*$, which imply that $\mathfrak{g}_\mathfrak{a}$, $\mathfrak{g}_\mathfrak{b}$ are Lie algebras. Put

$$Z_d^k(\mathfrak{g}_\mathfrak{a})|_\mathfrak{a} = Z_d^k(\mathfrak{g}_\mathfrak{a}) \cap \bigwedge{}^*\langle \mu_1^0, \ldots, \mu_p^0 \rangle,$$
$$Z_d^k(\mathfrak{g}_\mathfrak{b})|_\mathfrak{b} = Z_d^k(\mathfrak{g}_\mathfrak{b}) \cap \bigwedge{}^*\langle \nu_1^1, \ldots, \nu_q^1 \rangle,$$

where $Z_d^k(\mathfrak{g}_\mathfrak{a})$ and $Z_d^k(\mathfrak{g}_\mathfrak{b})$ are the set of d-closed $k$-forms on $\mathfrak{g}_\mathfrak{a}$ and $\mathfrak{g}_\mathfrak{b}$, respectively.

Let $F$ be the homomorphism

$$F : \bigoplus_r \Big( \bigoplus_{s+t=r} \bigwedge^{s,t} (\mathfrak{h}^\mathbb{C})^* \Big) \longrightarrow \bigoplus_r \bigwedge^r ((\mathfrak{g}_\mathfrak{a} \times \mathfrak{g}_\mathfrak{b})^\mathbb{C})^*)$$

induced by the linear isomorphism $(\mathfrak{h}^\mathbb{C})^* \longrightarrow ((\mathfrak{g}_\mathfrak{a} \times \mathfrak{g}_\mathfrak{b})^\mathbb{C})^*$ defined by

$$\bar\xi_i \mapsto \mu_i^1, \ \eta_j \mapsto \nu_j^1, \ \xi_i \mapsto \mu_i^0, \ \bar\eta_j \mapsto \nu_j^0 \ (i = 1, \ldots, p, j = 1, \ldots, q).$$

From the equations of (1), (2), (3), we have that $F$ is an isomorphism of differential graded algebras. Thus, we have the following theorems and proposition:

**Theorem 4.2** ([9]). *For each $t$,*

$$h^{0,t}(\mathfrak{g}_{\bar J}) = \dim H^t(\mathfrak{a} \times \mathfrak{b}).$$

Indeed, by the isomorphism $F$ of differential graded algebras, we see

$$F\Big(\bigwedge^t \langle \bar\xi_1, \ldots, \bar\xi_p, \bar\eta_1, \ldots, \bar\eta_q \rangle\Big) = \bigwedge^t \langle \mu_1^1, \ldots, \mu_p^1, \nu_1^0, \ldots, \nu_q^0 \rangle \cong \bigwedge^t ((\mathfrak{a} \times \mathfrak{b})^\mathbb{C})^*.$$

**Theorem 4.3** ([12]). *For each $r$,*

$$\sum_{s+t=r} h^{s,t}(\mathfrak{g}_{\bar J}) = \dim H^r(\mathfrak{g}_\mathfrak{a} \times \mathfrak{g}_\mathfrak{b}).$$

*In particular, if $\mathfrak{b}$ is an abelian ideal, then*

$$\sum_{s+t=r} h^{s,t}(\mathfrak{g}_{\bar J}) = \sum_{s+t=r} h^{s,t}(\mathfrak{g}_J).$$

*Proof.* If $\mathfrak{b}$ is an abelian ideal, then $D_{ks}^i = E_{st}^j = 0$ for each $i, j, k, s, t$. Then, we have $\mathfrak{g}_\mathfrak{a} \cong \mathbb{R}^{\dim \mathfrak{g}}$ and $\mathfrak{g}_\mathfrak{b}^\mathbb{C} \cong \mathfrak{g}^\mathbb{C} \cong \mathfrak{h}_J^+ \cong \mathfrak{h}_J^-$. Since

$$h^{s,t}(\mathfrak{g}_J) = \dim_\mathbb{C} \bigwedge^s (\mathfrak{h}_J^+)^* \cdot \dim_\mathbb{C} H^t(\mathfrak{h}_J^-) = \dim_\mathbb{R} H^s(\mathbb{R}^{\dim \mathfrak{g}}) \cdot \dim_\mathbb{R} H^t(\mathfrak{g}_\mathfrak{b}),$$

we have

$$\sum_{s+t=r} h^{s,t}(\mathfrak{g}_{\bar J}) = \sum_{s+t=r} h^{s,t}(\mathfrak{g}_J)$$

for each $r$. □

For corollaries of the theorems, see [9–11].

**Proposition 4.4.** *For each $s$,*

$$h^{s,0}(\mathfrak{g}_{\bar J}) = \sum_{s_1+s_2=s} \dim Z_d^{s_1}(\mathfrak{g}_\mathfrak{a})|_\mathfrak{a} \cdot \dim Z_d^{s_2}(\mathfrak{g}_\mathfrak{b})|_\mathfrak{b}.$$

*In particular, if* $\mathfrak{b}$ *is an ideal, then*

$$h^{s,0}(\mathfrak{g}_{\tilde{J}}) = \sum_{k=0}^{s} \binom{p}{k} \cdot \dim Z_d^{s-k}(\mathfrak{g}_{\mathfrak{b}})|_{\mathfrak{b}},$$

*where* $p = \dim \mathfrak{a}$.

*Proof.* If $\mathfrak{b}$ is an ideal, then $D^i_{ks} = 0$ for each $i, k, s$, which implies that $d\mu_i^0 = 0$ for each $i$. Thus, we see that $Z_d^k(\mathfrak{g}_{\mathfrak{a}})|_{\mathfrak{a}} \cong Z_d^k(\mathbb{R}^p)$. □

## 5 Examples

In this section, we see examples of Theorems 4.2, 4.3 and Proposition 4.4.

**Example 5.1.** *Let* $H_{\mathbb{R}}(n)$ *be a* $(2n+1)$*-dimensional real Heisenberg group and* $\mathfrak{h}_{\mathbb{R}}(n)$ *its Lie algebra. Then,* $\mathfrak{h}_{\mathbb{R}}(n)$ *has a basis* $X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z$ *with the structure equations* $[X_i, Y_i] = Z$ $(i = 1, \ldots, n)$*. Consider the following Lie subalgebras of* $\mathfrak{h}_{\mathbb{R}}(n)$*:*

$$\mathfrak{a}_k = span\{X_1, \ldots, X_k\}$$
$$\mathfrak{b}_k = span\{X_{k+1}, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$$

*for each* $0 \le k \le n$*. Then,* $\mathfrak{a}_k$ *and* $\mathfrak{b}_k$ *are subalgebras which satisfy* $\mathfrak{h}_{\mathbb{R}}(n) = \mathfrak{a}_k + \mathfrak{b}_k$ *and* $\mathfrak{a}_k \cap \mathfrak{b}_k = \{0\}$*. Note that* $\mathfrak{b}_k$ *is an ideal of* $\mathfrak{h}_{\mathbb{R}}(n)$*. We have a complex structure* $\tilde{J}_k$ *corresponding to the decomposition* $\mathfrak{h}_{\mathbb{R}}(n) = \mathfrak{a}_k + \mathfrak{b}_k$*. Then,* $\mathfrak{g}_{\mathfrak{a}_k}$ *and* $\mathfrak{g}_{\mathfrak{b}_k}$ *have the following structure equations:*

$$d\mu_i^0 = 0,\ d\nu_j^0 = 0\ (i = 1, \ldots, k; j = 1, \ldots, 2n-k),$$
$$d\nu^0_{2n-k+1} = -\sum_{j=1}^{n-k} \nu_j^0 \wedge \nu^0_{n+j},$$
$$d\mu_i^1 = 0, d\nu_j^1 = 0\ (i = 1, \ldots, k; j = 1, \ldots, 2n-k),$$
$$d\nu^1_{2n-k+1} = -\sum_{j=1}^{k} \mu_j^1 \wedge \nu^1_{n-k+j}.$$

*Thus, we have*

$$\mathfrak{g}_{\mathfrak{a}_k} \cong \mathfrak{h}_{\mathbb{R}}(n-k) \times \mathbb{R}^{2k},\ \ \mathfrak{g}_{\mathfrak{b}_k} \cong \mathfrak{h}_{\mathbb{R}}(k) \times \mathbb{R}^{2(n-k)}.$$

*We also see*

$$\dim Z_d^h(\mathfrak{g}_{\mathfrak{b}_k})|_{\mathfrak{b}_k} = \binom{2n-k}{h}$$

*for k ≥ 2 (See [9, p.200]). Thus, for k ≥ 2,*

$$h^{s,0}(\mathfrak{h}_{\mathbb{R}}(n)_{\tilde{J}_k}) = \sum_{h=0}^{s} \binom{k}{h}\binom{2n-k}{s-h}.$$

*Since*

$$\mathfrak{a}_k \times \mathfrak{b}_k = \mathfrak{h}_{\mathbb{R}}(n-k) \times \mathbb{R}^{2k},\ \ \mathfrak{g}_{\mathfrak{a}_k} \times \mathfrak{g}_{\mathfrak{b}_k} = \mathfrak{h}_{\mathbb{R}}(k) \times \mathfrak{h}_{\mathbb{R}}(n-k) \times \mathbb{R}^{2n},$$

*we have*

$$h^{0,t}(\mathfrak{h}_{\mathbb{R}}(n)_{\tilde{J}_k}) = \dim H^t(\mathfrak{h}_{\mathbb{R}}(n-k) \times \mathbb{R}^{2k}),$$
$$\sum_{s+t=r} h^{s,t}(\mathfrak{h}_{\mathbb{R}}(n)_{\tilde{J}_k}) = \sum_{s+t=r} h^{s,t}(\mathfrak{h}_{\mathbb{R}}(n)_{\tilde{J}_{n-k}}).$$

**Example 5.2.** *Let $X_{ij} = E_{ij} \in M(4,\mathbb{R})$ $(1 \le i < j \le 4)$, where $E_{ij}$ is a matrix unit. Let $\mathfrak{n}(3) = \mathrm{span}_{\mathbb{R}}\{X_{ij}\}_{1\le i<j\le 4}$. Consider the following Lie subalgebras of $\mathfrak{n}(3)$:*

$$\mathfrak{a} = \mathrm{span}\{X_{13}, X_{23}\},\ \mathfrak{b} = \mathrm{span}\{X_{12}, X_{14}, X_{24}, X_{34}\}.$$

*Then, $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras such that $\mathfrak{n}(3) = \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b} = \{0\}$. Thus, we have the following nilpotent Lie group with a complex structure:*

$$({}_{\mathbb{R}}(N(3)^{\mathbb{C}}), \tilde{J}) = \left\{ \begin{pmatrix} 1 & z_{12} & \bar{z}_{13} & z_{14} \\ 0 & 1 & \bar{z}_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle|\ z_{ij} \in \mathbb{C} \right\}.$$

*In general, we can construct complex structures on ${}_{\mathbb{R}}(\mathfrak{n}(n)^{\mathbb{C}})$, where $\mathfrak{n}(n) = \mathrm{span}_{\mathbb{R}}\{E_{ij}\}_{1\le i<j\le n+1}$, by using root systems (see [12] for details).*

*Then, we have*

$$\mathfrak{a} \times \mathfrak{b} \cong \mathfrak{h}_{\mathbb{R}}(1) \times \mathbb{R}^3,\ \mathfrak{g}_{\mathfrak{a}} \cong \mathfrak{g}_{\mathfrak{b}} \cong \mathfrak{h}(1,2) \times \mathbb{R}^1,$$

*where $\mathfrak{h}(1,2)$ is a 5-dimensional generalized Heisenberg Lie algebra, that is, $\mathfrak{h}(1,2) = \mathrm{span}_{\mathbb{R}}\{X_1, X_2, Y, Z_1, Z_2\}$ with the structure equations $[X_i, Y] = Z_i$ $(i = 1,2)$. Thus, we have*

$$h^{0,t}(\mathfrak{n}(3)_{\tilde{J}}) = \dim H^t(\mathfrak{h}_{\mathbb{R}}(1) \times \mathbb{R}^3),\ \sum_{s+t=r} h^{s,t}(\mathfrak{n}(3)_{\tilde{J}}) = \dim H^r(\mathfrak{h}(1,2) \times \mathfrak{h}(1,2) \times \mathbb{R}^2).$$

*By a straightforward computation, we see*

$$\dim Z_d^1(\mathfrak{g}_{\mathfrak{a}})|_{\mathfrak{a}} = \dim Z_d^2(\mathfrak{g}_{\mathfrak{a}})|_{\mathfrak{a}} = 1,$$
$$\dim Z_d^1(\mathfrak{g}_{\mathfrak{b}})|_{\mathfrak{b}} = 2,\ \dim Z_d^2(\mathfrak{g}_{\mathfrak{b}})|_{\mathfrak{b}} = \dim Z_d^3(\mathfrak{g}_{\mathfrak{b}})|_{\mathfrak{b}} = 3,\ \dim Z_d^4(\mathfrak{g}_{\mathfrak{b}})|_{\mathfrak{b}} = 1.$$

*Thus, we have*

$$h^{1,0}(\mathfrak{n}(3)_{\tilde{J}}) = 3,\quad h^{2,0}(\mathfrak{n}(3)_{\tilde{J}}) = 6,\quad h^{3,0}(\mathfrak{n}(3)_{\tilde{J}}) = 8,$$
$$h^{4,0}(\mathfrak{n}(3)_{\tilde{J}}) = 7,\quad h^{5,0}(\mathfrak{n}(3)_{\tilde{J}}) = 4,\quad h^{6,0}(\mathfrak{n}(3)_{\tilde{J}}) = 1.$$

**Acknowledgement:** The author would like to express his deep appreciation to Professor Yusuke Sakane for valuable advice and encouragement during his preparation of the paper. This work was supported by JSPS KAKENHI Grant number JP16K05131.

# References


[1] Andrada, A., Barberis, M.L., Dotti, I.: *Classification of abelian complex structures on 6-dimensional Lie algebras*, J. Lond. Math. Soc. **83** (2011), 232–255

[2] Ceballos, M., Otal, A., Ugarte, L., Villacampa, R.: *Invariant complex structures on 6-nilmanifolds: classification, Frölicher spectral sequence and special Hermitian metrics*, J. Geom. Anal. **26** (2016), 252–286

[3] Console, S. and Fino, A.: *Dolbeault cohomology of compact nilmanifolds*, Transform. Groups **6** (2001), 111–124

[4] Nakamura, I.: *Complex parallelisable manifolds and their small deformations*, J. Differential Geom. **10** (1975), 85–112

[5] Ovando, G.: *Invariant complex structures on solvable Lie groups*, Manuscripta Math. **103** (2000), 19–30

[6] Salamon, S.M.: *Complex structures on nilpotent Lie algebras*, J. Pure. Appl. Algebra **157** (2001), 311–333

[7] Snow, D.: *Invariant complex structures on reductive Lie groups*, J. Reine Angew. Math. **371** (1986), 191–215

[8] Ugarte, L.: *Hermitian structures on six-dimensional nilmanifolds*, Transform. Groups **12** (2007), 175–202

[9] Yamada, T.: *Hodge numbers and invariant complex structures of compact nilmanifolds*, Complex manifolds **3** (2016), 193–206

[10] Yamada, T.: *Remarks on Hodge numbers and invariant complex structures of compact nilmanifolds*, Complex manifolds **3** (2016), 271–281

[11] Yamada, T.: *Some relations between Hodge numbers and invariant complex structures on compact nilmanifolds*, Complex manifolds **4** (2017), 73–83

[12] Yamada, T.: *Invariant complex structures on compact nilmanifolds*, preprint