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Strongly pseudo-convex CR space forms

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Abstract: For a contact manifold, we study a strongly pseudo-convex CR space form with constant holomorphic sectional curvature for the Tanaka-Webster connection. We prove that a strongly pseudo-convex CR space form M is weakly locally pseudo-Hermitian symmetric if and only if (i) dim $M = 3$, (ii) M is a Sasakian space form, or (iii) M is locally isometric to the unit tangent sphere bundle $T_1(\mathbb{H}^{n+1})$ of a hyperbolic space $\mathbb{H}^{n+1}$ of constant curvature -1.

Keywords: contact manifold, strongly pseudo-convex CR space form, pseudo-Hermitian symmetry

MSC: 53C15, 53C25, 53D10

Dedicated to Professor David E. Blair on the occasion of his 78th birthday

1 Introduction

Given a contact manifold $(M;\eta)$, two fundamental structures are playing crucial roles in its geometry. One is an associated Riemannian metric g and the other is the Levi form associated with an endomorphism J on the contact distribution $D(=\mathrm{Ker}\,\eta)$ such that $J^2 = -I$. Here, J defines an almost CR structure $\mathcal{H} = \{X - iJX : X \in D\}$, that is, each fiber $\mathcal{H}_p$, $p \in M$, is of complex dimension n and $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$. Then there is a one-to-one correspondence between the two associated structures by the relation

$$g = L + \eta \otimes \eta,$$

where we denote by the same letter L the natural extension ($i_\xi L = 0$) of the Levi form to a (0,2)-tensor field on M. For theoretical considerations, it is desirable to have integrability of the almost complex structure J (on D). If this is the case, we speak of an (*integrable*) *CR structure* and of a *CR manifold*. Looking at a contact manifold from the viewpoint of its pseudo-Hermitian CR structure, there exists a canonical affine connection, different from the Levi Civita connection ∇ of an associated metric. This is the *Tanaka-Webster connection* $\hat{\nabla}$ on a strongly pseudo-convex CR manifold. Using it, in earlier works [9], [13], [15], [16], [17] we started the intriguing study of interactions between the contact Riemannian structure and the strongly pseudo-convex pseudo-Hermitian structure.

A normal contact Riemannian manifold is called a Sasakian manifold. A Sasakian structure has another picture, namely, a contact strongly pseudo-convex CR structure whose characteristic vector field is a Killing vector field for its associated Riemannian metric. A Sasakian space form is a Sasakian manifold with constant holomorphic sectional curvature with respect to ∇. Then we find that a Sasakian space form has also constant holomorphic sectional curvature with respect to $\hat{\nabla}$. In [14] we defined a contact Riemannian space form which extends a Sasakian space form in the Riemannian view point. Corresponding to that, in [15] we introduced a notion, say, a *strongly pseudo-convex CR space form*, which is a contact strongly pseudo-convex CR manifold M of constant holomorphic sectional curvature c with respect to $\hat{\nabla}$, that is, M satisfies for any unit vector field X orthogonal to ξ

$$L(\hat{R}(X, JX)JX, X) = c.$$

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Exploring symmetry of contact manifolds, Takahashi [27] introduced the Sasakian local φ-symmetry as an odd-dimensional analogue of Hermitian locally symmetric spaces. Then we find that a Sasakian space form is locally φ-symmetric. On the other hand, a weak local pseudo-Hermitian symmetry on contact strongly pseudo-convex CR manifolds is defined by the condition

$$L((\hat{\nabla}_X \hat{R})(Y, Z)U, V) = 0$$

for all X, Y, Z, U, V orthogonal to ξ. In Section 5, we prove that a strongly pseudo-convex CR space form M is weakly locally pseudo-Hermitian symmetric if and only if (i) dim $M = 3$, (ii) M is a Sasakian space form, or (iii) M is locally isometric to the unit tangent sphere bundle $T_1(\mathbb{H}^{n+1})$ of a hyperbolic space $\mathbb{H}^{n+1}$ of constant curvature -1. In Section 6, we treat the three-dimensional case. Then, we find interesting examples of strongly pseudo-convex CR space forms other than contact Riemannian homogeneous spaces (see, Example 1 and Example 2).

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2 Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. We start by collecting some fundamental material about contact Riemannian manifolds and strongly pseudo-convex almost CR manifolds. We refer to [1] for further details.

A $(2n+1)$-dimensional manifold M^{2n+1} is a *contact manifold* if it is equipped with a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, which is called the *characteristic vector field* or the *Reeb vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there also exist a Riemannian metric g and a $(1, 1)$-tensor field φ such that

$$\eta(X) = g(X, \xi), \ d\eta(X, Y) = g(X, \varphi Y), \ \varphi^2 X = -X + \eta(X)\xi, \tag{1}$$

where X and Y are vector fields on M. From (1), it follows that

$$\varphi\xi = 0, \ \eta \circ \varphi = 0, \ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (1) is said to be a *contact Riemannian manifold* or *contact metric manifold* and is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M, we define a $(1, 1)$-tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \varphi$, where $\mathcal{L}_\xi$ denotes Lie differentiation with respect to ξ. The operator h is self-adjoint and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h, \tag{3}$$

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{4}$$

for all vector fields X, Y, Z on M, where ∇ is the Levi-Civita connection and R the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$$

for all vector fields X, Y, Z. From (3) and (4) we see that ξ generates a geodesic flow. Furthermore, we know that $\nabla_\xi \varphi = 0$ in general. From the second equation of (3) it follows also that

$$(\nabla_\xi h)\varphi = -\varphi(\nabla_\xi h). \tag{5}$$

Along the characteristic flow ξ, the Jacobi operator $\ell = R(\cdot, \xi)\xi$ is a symmetric $(1, 1)$-tensor field. We call it the *characteristic Jacobi operator*. From the definition of R by using (4) we have

$$\ell = -\varphi^2 + \varphi \nabla_\xi h - h^2. \tag{6}$$

From (6) using the 2nd equation of (3) and (5) we have

$$\nabla_\xi h = 1/2(\ell\varphi - \varphi\ell). \tag{7}$$

A contact Riemannian manifold for which ξ is Killing is called a *K-contact manifold*. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$ or, equivalently, $\ell = I - \eta \otimes \xi$. For a contact Riemannian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be *normal* or *Sasakian*. We note that every Sasakian manifold is also K-contact, but the converse is only true in dimension 3. A Sasakian structure is characterized by the following equation:

$$(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{8}$$

or

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{9}$$

for all vector fields X and Y on M.

Next, we recall the natural relation of contact metric manifolds with CR manifolds. For a contact Riemannian manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as the direct sum $T_pM = D_p \oplus \{\xi\}_p$, where we denote $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a $2n$-dimensional distribution orthogonal to ξ, which is called the *contact distribution* or the *contact subbundle*. For a contact Riemannian structure (η, g), its associated almost CR structure is given by the holomorphic subbundle

$$\mathcal{H} = \{X - iJX : X \in D\}$$

of the complexification $\mathbb{C}TM$ of the tangent bundle TM, where $J = \varphi|D$, the restriction of φ to D. We see that each fiber $\mathcal{H}_p$, $p \in M$, is of complex dimension n, $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$ and $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. The *Levi form* L is defined by

$$L : D \times D \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, JY)$$

where $\mathcal{F}(M)$ denotes the algebra of differentiable functions on M. Since $d\eta(X, Y) = g(X, \varphi Y)$, the Levi form is Hermitian and positive definite. So, the pair (η, J) is a *pseudo-Hermitian strongly pseudo-convex almost CR structure* on M. The associated almost CR structure is *integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. This property does not hold for a general contact metric manifold. In terms of the structure tensors, integrability is equivalent to the condition $\Omega = 0$, where Ω is the $(1, 2)$-tensor field on M defined as

$$\Omega(X, Y) = (\nabla_X\varphi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) \tag{10}$$

for vector fields X, Y on M (see [30, Proposition 2.1]). In this case, the pair (η, J) is called a *pseudo-Hermitian strongly pseudo-convex CR structure* and $(M; \eta, J)$ is called a *strongly pseudo-convex integrable pseudo-Hermitian manifold* or a *strongly pseudo-convex CR manifold*. From (8) and (10), we see that the associated pseudo-Hermitian structure of a Sasakian manifold is integrable. The same is true for any three-dimensional contact metric space.

A *pseudo-homothetic* (or a *D-homothetic*) *transformation* of a contact metric manifold [29] is a change of structure tensors of the form

$$\bar{\eta} = a\eta, \ \bar{\xi} = 1/a\,\xi, \ \bar{\varphi} = \varphi, \ \bar{g} = ag + a(a-1)\eta \otimes \eta, \tag{11}$$

where a is a positive constant. From (11), we have $\bar{h} = (1/a)h$. By using the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) \ &= \ Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

we have

$$\bar{\nabla}_X Y = \nabla_X Y + C(X, Y), \tag{12}$$

where C is the (1,2)-tensor defined by

$$C(X, Y) = -(a-1)[\eta(Y)\varphi X + \eta(X)\varphi Y] - \frac{a-1}{a} g(\varphi hX, Y)\xi.$$

Remark 1. Integrability of the associated almost CR structure is preserved under pseudo-homothetic transformations. In fact, by direct computations, we have

$$(\bar{\nabla}_X \bar{\varphi})Y = (\nabla_X \varphi)Y + (a-1)\eta(Y)\varphi^2 X - (a-1)/a g(X, hY)\xi.$$

From this, we easily see that $\Omega = 0$ implies $\bar{\Omega} = 0$.

3 The generalized Tanaka-Webster connection

Now, we review the *generalized Tanaka-Webster connection* $\hat{\nabla}$ ([30]) on a contact Riemannian manifold $M = (M; \eta, g)$. It is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X, Y on M. Together with (4), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + A(X, Y), \tag{13}$$

where we put

$$A(X, Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Y)\xi. \tag{14}$$

We see that the generalized Tanaka-Webster connection $\hat{\nabla}$ has the torsion

$$\hat{T}(X, Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY. \tag{15}$$

In particular, for a K-contact manifold we get

$$A(X, Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi. \tag{16}$$

The generalized Tanaka-Webster connection can also be characterized differently.

Proposition 1 ([30]). *The generalized Tanaka-Webster connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M; \eta, g)$ is the unique linear connection satisfying the following conditions:*
(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$;
(iii-1) $\hat{T}(X, Y) = 2g(X, \varphi Y)\xi$, $X, Y \in D$;
(iii-2) $\hat{T}(\xi, \varphi Y) = -\varphi \hat{T}(\xi, Y)$, $Y \in D$;
(iv) $(\hat{\nabla}_X \varphi)Y = \Omega(X, Y)$, $X, Y \in TM$.

We note that the Tanaka-Webster connection ([28], [32]) was originally defined for a non-degenerate integrable pseudo-Hermitian manifold, in which case condition (iv) reduces to $\hat{\nabla}\varphi = 0$. The above definition is a natural generalization to the non-integrable case.

Proposition 2. ([9]) *The generalized Tanaka-Webster connection is invariant under a pseudo-homothetic transformation.*

Corollary 3. *The generalized Tanaka-Webster curvature tensor $\hat{R}$, its torsion tensor $\hat{T}$ and their covariant derivatives $\hat{\nabla}\hat{R}$ and $\hat{\nabla}\hat{T}$ are pseudo-homothetically invariant.*

We take a look at $\hat{R}(X,Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X,Y]}Z$ in some more detail for the general case. First, we have

Proposition 4.

$$\hat{R}(X,Y)Z = -\hat{R}(Y,X)Z,$$
$$L(\hat{R}(X,Y)Z, W) = -L(\hat{R}(X,Y)W, Z).$$

The first identity follows trivially from the definition of $\hat{R}$. Since the connection is metrical with respect to its associated metric g ($\hat{\nabla} g = 0$) the second identity is proved in a similar way as for the case of Riemannian curvature tensor. Since the Tanaka-Webster connection is not torsion-free, the Jacobi- or Bianchi-identities do not hold, in general. Before we study the curvature tensor $\hat{R}$, from (3), (13) and (14) we have

$$\begin{aligned}(\hat{\nabla}_X h)Y =&(\nabla_X h)Y + A(X,hY) - hA(X,Y)\\ =&(\nabla_X h)Y + 2\eta(X)\varphi hY + g((\varphi h + \varphi h^2)X, Y)\xi\\ &+ \eta(Y)(\varphi hX + \varphi h^2 X).\end{aligned} \tag{17}$$

From the definition of $\hat{R}$, together with (13), taking account of $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$, $\hat{\nabla} g = 0$ and (17), straightforward computations yield

$$\begin{aligned}\hat{R}(X,Y)Z =&R(X,Y)Z + \eta(Z)\Big(\Omega(X,Y) - \Omega(Y,X) + \Omega(X,hY) - \Omega(Y,hX)\\ &+ \varphi P(X,Y) + \varphi(A(X,Y) - A(Y,X)) + \varphi(A(X,hY) - A(Y,hX))\Big)\\ &- g\Big(\Omega(X,Y) - \Omega(Y,X) + \Omega(X,hY) - \Omega(Y,hX) + \varphi P(X,Y) + \varphi(A(X,Y) - A(Y,X))\\ &+ \varphi(A(X,hY) - A(Y,hX)), Z\Big)\xi - 2g(\varphi X, Y)\varphi Z\\ &- \eta(X)\big(\Omega(Y,Z) + \varphi A(Y,Z)\big) + \eta(Y)\big(\Omega(X,Z) + \varphi A(X,Z)\big)\\ &+ \eta(A(X,Z))(\varphi Y + \varphi hY) - \eta(A(Y,Z))(\varphi X + \varphi hX)\\ &+ g(\varphi X + \varphi hX, A(Y,Z))\xi - g(\varphi Y + \varphi hY, A(X,Z))\xi,\end{aligned}$$

where we put $P(X,Y) = (\nabla_X h)Y - (\nabla_Y h)X$. By using (1), (2), (3) and (14), we have

$$\hat{R}(X,Y)Z = R(X,Y)Z + B(X,Y)Z, \tag{18}$$

where

$$\begin{aligned}B(X,Y)Z =&\eta(Z)\Big(\Omega(X,Y) - \Omega(Y,X) + \Omega(X,hY) - \Omega(Y,hX) + \varphi P(X,Y)\Big)\\ &- g\Big(\Omega(X,Y) - \Omega(Y,X) + \Omega(X,hY) - \Omega(Y,hX) + \varphi P(X,Y), Z\Big)\xi\\ &- \eta(Z)\big\{\eta(Y)(X + hX) - \eta(X)(Y + hY)\big\}\\ &- \eta(X)\Omega(Y,Z) + \eta(Y)\Omega(X,Z)\\ &+ \eta(Y)g(X + hX, Z)\xi - \eta(X)g(Y + hY, Z)\xi\\ &+ g(\varphi Y + \varphi hY, Z)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Z)(\varphi Y + \varphi hY)\\ &- 2g(\varphi X, Y)\varphi Z\end{aligned}$$

for all vector fields X, Y, Z in M.

Definition 1. ([16]) *The pseudo-Hermitian* (*or the Tanaka-Webster*) *Ricci curvature tensor $\hat{\rho}$ is defined by*

$$\hat{\rho}(X,Y) = \frac{1}{2}\,\text{trace of}\,\{V \mapsto J\hat{R}(X,JY)V\},$$

where X, Y are vector fields orthogonal to ξ. A strongly pseudo-convex almost CR manifold M is called pseudo-Einstein *if the pseudo-Hermitian Ricci tensor is proportional to the Levi form L.*

We define the pseudo-Hermitian (or the Tanaka-Webster) Ricci operator $\hat{Q}$ by $L(\hat{Q}X, Y) = \hat{\rho}(X, Y)$. The pseudo-Hermitian (or the Tanaka-Webster) scalar curvature $\hat{r}$ is given by

$$\hat{r} = \text{trace of } \{V \mapsto \hat{Q}V\}.$$

4 Contact (k, μ)-spaces

In what follows, an important role will be played by a specific class of contact metric manifolds, namely those for which

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y), \tag{19}$$

where I denotes the identity transformation and $(k, \mu) \in \mathbb{R}^2$. Such spaces are called (k, μ)-*spaces* and were introduced in [2]. As examples, we have Sasakian spaces ($k = 1$ and $h = 0$) and also the unit tangent sphere bundle of spaces of constant curvature b ($k = b(2 - b)$ and $\mu = -2b$). Since the unit tangent sphere bundle is non-Sasakian when $b \neq 1$ [31], this gives us a lot of non-Sasakian examples. Other than the unit tangent bundles, this class contains non-unimodular Lie groups with left-invariant contact metric structure (see [5]). Due to an explicit description of the curvature tensor of (k, μ)-spaces in [5], we have

$$\begin{aligned} g(\hat{R}(X, Y)Z, W) =& g(H(X, Y)Z, W) + (1 - \frac{\mu}{2})\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ &+ g(\varphi Y, Z)g(\varphi X, W) - g(\varphi X, Z)g(\varphi Y, W) - 2g(\varphi X, Y)g(\varphi Z, W)\} \\ &+ \frac{1 - (\mu/2)}{1 - k}\{g(hY, Z)g(hX, W) - g(hX, Z)g(hY, W) \\ &+ g(\varphi hY, Z)g(\varphi hX, W) - g(\varphi hX, Z)g(\varphi hY, W)\} \end{aligned} \tag{20}$$

for all vector fields X, Y, Z and W orthogonal to ξ, where

$$\begin{aligned} g(H(X, Y)Z, W) =& g(Y, Z)g(hX, W) - g(X, Z)g(hY, W) \\ &- g(Y, W)g(hX, Z) + g(X, W)g(hY, Z) \\ &+ g(\varphi Y, Z)g(\varphi hX, W) - g(\varphi X, Z)g(\varphi hY, W) \\ &- g(\varphi Y, W)g(\varphi hX, Z) + g(\varphi X, W)g(\varphi hY, Z). \end{aligned} \tag{21}$$

Furthermore, we have ([16])

Proposition 5. *A non-Sasakian contact (k, μ)-space $(k < 1)$ is pseudo-Einstein with constant pseudo-Hermitian scalar curvature $\hat{r} = 2n^2(2 - \mu)$.*

Corollary 6. *The standard contact metric structure of $T_1M(b)$ of a space of constant curvature b is pseudo-Einstein. Its pseudo-Hermitian scalar curvature $\hat{r} = 4n^2(1 + b)$.*

Let's recall some of the properties of (k, μ)-spaces which we will make use of later on. Firstly, as proved in [2], the class of (k, μ)-spaces is invariant under pseudo-homothetic transformations. More precisely, a pseudo-homothetic transformation with constant a changes (k, μ) into $(\bar{k}, \bar{\mu})$, where

$$\bar{k} = \frac{k + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}. \tag{22}$$

Remark 2. From these formulas, we see that the values $k = 1$ and $\mu = 2$ are preserved under D-homothetic transformations for all $a (\neq 1)$. The case $k = 1$ corresponds to the class of Sasakian manifolds; for the case $\mu = 2$, we will find a geometric interpretation further on (see Theorem 12).

Secondly, the associated pseudo-Hermitian structure of a (k, μ)-space is integrable, i.e., these spaces are strongly pseudo-convex CR manifolds. This gives us an expression for $\nabla\varphi$ via (10). Moreover, also for ∇h, we have an explicit formula [2]:

$$(\nabla_X h)Y = \{(1 - k)g(X, \varphi Y) - g(X, \varphi hY)\}\xi - \eta(Y)\{(1 - k)\varphi X + \varphi hX\} - \mu\eta(X)\varphi hY. \tag{23}$$

From (23), it follows immediately that a (k,μ)-space satisfies $g((\nabla_X h)Y, Z) = 0$ for all vector fields X, Y, Z orthogonal to ξ, i.e., it is an *η-parallel contact metric space*. Boeckx and the present author [7] proved that also the converse holds:

Theorem 7. *An η-parallel contact metric space is a K-contact space or a (k,μ)-space.*

By (17), we also find that an η-parallel contact metric space is characterized by the condition: $L((\hat{\nabla}_X h)Y, Z) = 0$ for all vector fields X, Y, Z orthogonal to ξ.

We finish this section by noticing a very recent result about contact (k,μ)-spaces in the topic of Ricci solitons. Hamilton's Ricci flow ([22]) is given by

$$\frac{\partial}{\partial t} g_t = -2\rho(g_t),$$

where $\rho(g_t)$ denotes the (Riemannian) Ricci curvature tensor for a Riemannian metric g_t. Then representing the self-similar solution of it, the initial data (g, V) is called a Ricci soliton:

$$\frac{1}{2}\mathcal{L}_V g + \rho - \lambda g = 0,$$

where λ is a constant. In [20], Ricci soliton contact (k,μ)-spaces have been studied. The gradient case is well-understood, whereas they provided a list of candidates for the non-gradient case. These candidates can be realized as Lie groups, but one only knows the structures of the underlying Lie algebras, which are hard to be analyzed apart from the three-dimensional case. In this circumstance, we proved

Theorem 8. ([18]) *A simply connected and complete non-Sasakian (k,μ)-space is a Ricci soliton if and only if $(k,\mu) = (0,0)$ or $(0,4)$.*

This result comes from the following realization of contact $(0,4)$-spaces.

Theorem 9. ([19]) *Let M be the simply connected and complete $(0,4)$-space of dimension $2n+1$ with $n \geq 2$. Then M is isomorphic to some homogeneous real hypersurface of the non-compact real two-plane Grassmannians $G_2^\star(\mathbb{R}^{n+3})$ as a contact metric manifold.*

5 Strongly pseudo-convex CR space forms

A holomorphic section is a plane in $D_pM \subset T_pM$ of M which is invariant by J. Then we may consider two kinds of holomorphic sectional curvature with respect to $\hat{\nabla}$ and ∇, that is, $\hat{K}(X, JX) = L(\hat{R}(X, JX)JX, X)$ and $K(X, JX) = g(R(X, JX)JX, X)$, respectively for any unit vector field X orthogonal to ξ.

Definition 2. ([15]) *Let $(M; \eta, J)$ be a contact strongly pseudo-convex almost CR manifold. Then M is said to be of constant holomorphic sectional curvature c with respect to the generalized Tanaka-Webster connection if M satisfies $\hat{K}(X, JX) = c$ for any unit vector field X orthogonal to ξ. In particular, for the CR-integrable case we call M a strongly pseudo-convex CR space form.*

Then for a strongly pseudo-convex almost CR manifold M, from (18) we get

$$g(\hat{R}(X,\varphi X)\varphi X, X) = g(R(X,\varphi X)\varphi X, X) + 3g(X,X)^2 - g(hX,X)^2 - g(\varphi hX, X)^2 \tag{24}$$

or

$$\hat{K}(X,\varphi X) = K(X,\varphi X) + 3g(X,X)^2 - g(hX,X)^2 - g(\varphi hX,X)^2.$$

From this, we easily see that a Sasakian space form $M^{2n+1}(c_0)$ of constant holomorphic sectional curvature c_0 with respect to the Levi-Civita connection is a strongly pseudo-convex CR space form of constant holomorphic sectional curvature $c = c_0 + 3$ with respect to the Tanaka-Webster connection.

We proved

Proposition 10. ([15]) *The strongly pseudo-convex CR space form is invariant under a pseudo-homothetic transformation.*

We remark that for a regular (i.e., the foliation defined by the vector field ξ is regular) Sasakian space form $M^{2n+1}(c_0)$, the quotient $M^{2n+1}(c_0)/\xi$ with the induced metric and the complex structure J given by $J\pi_\star X = \pi_\star\varphi X$ is a complex space form $\widetilde{M}^n((c_0+3)/4)$, where $\pi : M \to M/\xi$ is the Riemannian submersion. (For a typical example, we have a Hopf-fibration $\pi : S^{2n+1} \to P_nC$.)

In [15] we determined the Riemannian curvature tensor explicitly for a strongly pseudo-convex CR space of pointwise constant holomorphic sectional curvature $c = c(p)$ ($p \in M$). Then from (18) we have

$$\begin{aligned} g(\hat{R}(X,Y)Z,W) = g(H(X,Y)Z,W) + \frac{c}{4}\Big\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+ g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W)\Big\} \end{aligned} \tag{25}$$

for all vector fields $X, Y, Z, W \perp \xi$. From (25), we obtain

$$\hat{\rho}(X,Y) = \frac{1}{2}(n+1)cg(X,Y) \tag{26}$$

for any vector fields X, Y orthogonal to ξ. Then we have

Proposition 11. ([16]) *A strongly pseudo-convex CR space form of constant holomorphic sectional curvature c is pseudo-Einstein with constant pseudo-Hermitian scalar curvature $\hat{r} = n(n+1)c$.*

For a class of the contact (k,μ)-spaces, whose associated pseudo-Hermitian structures are integrable as stated in Section 4, we prove

Theorem 12. *Let M be a contact (k,μ)-space. Then M is of constant holomorphic sectional curvature c for the Tanaka-Webster connection if and only if* (1) *M is a Sasakian space of constant holomorphic sectional curvature $c_0 = c-3$,* (2) *$\mu = 2$ and $c = 0$, or* (3) *dim M=3 and $\mu = 2-c$.*

Proof. If $k = 1$, then we see that M is Sasakian and of constant holomorphic sectional curvature $c_0 = c - 3$. Now, we assume that $k < 1$. Then from (20) and (25) we have

$$\begin{aligned} g(\hat{R}(X,Y)Z,W) = g(\hat{R}_{c'}(X,Y)Z,W) + \frac{c'}{4(1-k)}\{&g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W) \\ &+ g(\varphi hY,Z)g(\varphi hX,W) - g(\varphi hX,Z)g(\varphi hY,W)\}, \end{aligned} \tag{27}$$

where $g(\hat{R}_{c'}(X,Y)Z,W)$ denotes the curvature form of constant holomorphic sectional curvature c' (for $\hat{\nabla}$) and we have put $c' = 4(1-\mu/2)$. Suppose that M has constant holomorphic sectional curvature c (for $\hat{\nabla}$). Then from (25) and (27) we have

$$\begin{aligned} &(c-c')\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W)\} \\ &\quad - \frac{c'}{1-k}\{g(hY,Z)g(hX,W) - g(X,Z)g(Y,W) \\ &\quad\quad + g(\varphi hY,Z)g(\varphi hX,W) - g(\varphi hX,Z)g(\varphi hY,W)\} = 0 \end{aligned} \tag{28}$$

for any vector fields X, Y, Z, W orthogonal to ξ. We put $Y = Z = e_i$ and summing for $i = 1, 2, \cdots, 2n$, where $\{e_i, e_{2n+1} = \xi\}$ is an adapted orthonormal basis. Then we have

$$(c-c')(n+1)g(X,W) + \frac{c'}{1-k}g(h^2X,W) = 0.$$

From this, we have

$$(c - c')(n + 1) + c' = 0. \tag{29}$$

On the other hand, from (20) we have

$$\hat{K}(X, \varphi X) = (4 - 2\mu)g(X, X)^2 + \frac{2 - \mu}{k - 1}\{g(hX, X)^2 + g(\varphi hX, X)^2\} \tag{30}$$

for $\|X\| = 1$. We assume $hX = \lambda X$ in (30). Then $\lambda^2 = 1 - k$ and (30) yields $c(= 2 - \mu) = c'/2$. Thus, together with (29) we have

$$c(n - 1) = 0.$$

Hence, we have either $c = c' = 0$ ($\mu = 2$) or $\dim M = 3$. If $\dim M = 3$, then we see that $g(hX, X)^2 + g(\varphi hX, X)^2 = 1/2(\text{trace of } h^2)$. But, since $h^2 = (k - 1)\varphi^2$, from (30) we have $\mu = 2 - c$. Conversely, from (30) we easily find that the cases: (i) $k < 1$ and $\mu = 2$, (ii) $\dim M = 3$ and $\mu = 2 - c$ have constant holomorphic sectional curvature $c = 0$ and $c = 2 - \mu$, respectively. Therefore, we have completed the proof. □

Corollary 13. ([15]) *The standard contact metric structure on a unit tangent sphere bundle $T_1M(b)$ of $(n + 1)$-dimensional space of constant curvature b has constant holomorphic sectional curvature c for $\hat{\nabla}$ if and only if $b = -1$ and $c = 0$, or $n = 1$ and $b = (c - 2)/2$.*

Remark 3. The unit tangent sphere bundle $T_1\mathbb{H}^{n+1}(-1)$ of a hyperbolic space $\mathbb{H}^{n+1}(-1)$ is a non-Sasakian example of constant holomorphic sectional curvature for Tanaka-Webster connection, whereas it has not constant holomorphic sectional curvature for Levi-Civita connection.

As mentioned in Remark 2, the unit tangent sphere bundle $T_1(\mathbb{H}^{n+1})$ of a hyperbolic space $\mathbb{H}^{n+1}$ of constant curvature -1 is the same under all D-homothetic transformations. Due to Theorem 7 and Theorem 12, we have

Theorem 14. ([15]) *The complete simply connected strongly pseudo-convex η-parallel CR space forms are equivalent (up to pseudo-homothetic transform) to the following:*

(i) Sasakian space forms: the unit sphere S^{2n+1} with $c_0 = 1$ for $c > 0$, the Heisenberg group H^{2n+1} with $c_0 = -3$ for $c = 0$, or $B^n \times R$ with $c_0 = -7$ for $c < 0$, where B^n is a simply connected bounded domain in C^n with the Bergman metric,

(ii) the unit tangent sphere bundle $T_1(\mathbb{H}^{n+1})$ of a hyperbolic space $\mathbb{H}^{n+1}$ of constant curvature -1, or

(iii) a (non-Sasakian) unimodular Lie group with a special left-invariant contact metric, SU(2), $\widetilde{\mathrm{SL}}(2, \mathrm{R})$, *the universal covering $\widetilde{E}(2)$ of the group $E(2)$ of rigid motions of Euclidean 2-space, the group $E(1, 1)$ of rigid motions of the Minkowski 2-space.*

6 Pseudo-Hermitian symmetric spaces

Boeckx and the present author [8] proved that a contact Riemannian manifold with Cartan's local symmetry ($\nabla R = 0$) is locally isometric to either the unit sphere $S^{2n+1}(1)$ or the unit tangent sphere bundle $T_1M(0)$ of Euclidean space. As a pseudo-Hermitian analogue of it, we introduced the so-called Tanaka-Webster parallel space.

Definition 3. ([9]) Let $(M; \eta, J)$ be a contact strongly pseudo-convex almost CR manifold. Then M is said to be a *Tanaka-Webster parallel space* (in short, T.-W. parallel space) if its generalized Tanaka-Webster torsion tensor $\hat{T}$ and its curvature tensor $\hat{R}$ satisfy $\hat{\nabla}\hat{T} = 0$ and $\hat{\nabla}\hat{R} = 0$.

Kobayashi and Nomizu [21] call a connection *invariant by parallelism* if for any points p and q in M and for any curve γ from p to q, there exists a (unique) local affine isomorphism f such that $f(p) = q$ and such that the differential of f at p coincides with the parallel displacement $\tau_\gamma : T_pM \to T_qM$ along γ. By [21, Corollary 7.6], this is equivalent to the connection having parallel torsion and curvature tensor. In other words, a T.-W.

parallel space is one for which the Tanaka-Webster connection $\hat{\nabla}$ is an invariant connection by parallelism. Then we have

Theorem 15. ([9]) *A contact metric space M is a Tanaka-Webster parallel space if and only if M is a Sasakian locally φ-symmetric space or a non-Sasakian $(k, 2)$-space.*

As an odd-dimensional analogue of Hermitian locally symmetric spaces, Takahashi [27] introduced the Sasakian local φ-symmetry.

Definition 4. A Sasakian manifold is said to be *locally φ-symmetric space* if M satisfies

$$g((\nabla_X R)(Y, Z)U, V) = 0 \tag{31}$$

for all X, Y, Z, U, V orthogonal to ξ.

Geometrically, the above condition (31) corresponds to the fact that the characteristic reflections (i.e., reflections with respect to the integral curves of ξ) are local automorphisms of the Sasakian structure. In fact, it is already sufficient that the reflections are local isometries [4]. From this context, we may consider two generalizations of the notion of local φ-symmetry to the larger class of contact Riemannian spaces. The first one, in [3], defines locally φ-symmetric contact metric space to the one for which the curvature property (31) holds. A second generalization was proposed by Boeckx and Vanhecke [11]: a contact Riemannian manifold is called locally φ-symmetric in the strong sense if its characteristic reflections are local isometries. Indeed, we find that the second generalization is a priori more restrictive than the first (cf. [10]). Now, in the pseudo-Hermitian view point, we define strongly or weakly locally pseudo-Hermitian symmetric spaces. Namely,

Definition 5. Let $(M; \eta, J)$ be a contact strongly pseudo-convex almost CR manifold. Then M is said to be a *strongly locally pseudo-Hermitian symmetric space* if all characteristic $\hat{\nabla}$-reflections are affine mappings, i.e., they preserve the Tanaka-Webster connection $\hat{\nabla}$.

Then we have

Theorem 16. ([9]) *A contact metric manifold M is locally pseudo-Hermitian symmetric in the strong sense if and only if M is either a Sasakian locally φ-symmetric space or a non-Sasakian (k, μ)-space.*

Definition 6. ([9]) Let $(M; \eta, J)$ be a contact strongly pseudo-convex almost CR manifold. Then M is said to be a *weakly locally pseudo-Hermitian symmetric space* if M satisfies

$$L((\hat{\nabla}_X \hat{R})(Y, Z)U, V) = 0$$

for all X, Y, Z, U, V orthogonal to ξ.

Proposition 17. ([9]) *A Sasakian manifold is a weakly locally pseudo-Hermitian symmetric space if and only if it is locally φ-symmetric.*

Proposition 18. ([9]) *A non-Sasakian (k, μ)-space is a weakly locally pseudo-Hermitian symmetric space.*

Proposition 19. *A locally pseudo-Hermitian symmetric space has constant Tanaka-Webster scalar curvature $\hat{r}$.*

Proof. First, we note that $g(\hat{\nabla}_V X, \xi) = 0$ for any vector fields $V, X, Y \perp \xi$. Suppose that $(M; \eta, J)$ is locally pseudo-Hermitian symmetric. Then

$$(\hat{\nabla}_V \hat{\rho})(X, Y) = \frac{1}{2} \sum_{i=1}^{2n} L((\hat{\nabla}_V \hat{R})(X, JY)Je_i, e_i) = 0,$$

$i = 1, \cdots, 2n$. And $V(\hat{r}) = \hat{\nabla}_V \hat{r} = \sum_{i=1}^{2n} (\hat{\nabla}_V \hat{\rho})(e_i, e_i) = 0$. Hence we have $e_i(\hat{r}) = \varphi e_i(\hat{r}) = 0$, and then $[e_i, \varphi e_i](\hat{r}) = 0$. But $[e_i, \varphi e_i] = \sum_{j=1}^{n} g([e_i, \varphi e_i], e_j)e_j + g([e_i, \varphi e_i], \xi)\xi$. Thus, we have $\eta([e_i, \varphi e_i])\xi(\hat{r}) = -2d\eta(e_i, \varphi e_i)\xi(\hat{r}) = 2\xi(\hat{r}) = 0$, where we have used the second condition of (1). □

Then we prove

Theorem 20. *Let $(M;\eta,J)$ be a strongly pseudo-convex CR space form of constant holomorphic sectional curvature c. Then M is weakly locally pseudo-Hermitian symmetric if and only if (i) dim $M=3$, (ii) M is a Sasakian space form, or (iii) M is locally isometric to the unit tangent sphere bundle $T_1(\mathbb{H}^{n+1})$ of a hyperbolic space $\mathbb{H}^{n+1}$ of constant curvature -1.*

Proof. Suppose that a strongly pseudo-convex CR space form $(M^{2n+1};\eta,J)$ is locally pseudo-Hermitian symmetric in the weak sense. Then from (25) we have $(n-1)g((\hat\nabla_X h)Y,Z)=0$ for any vector fields X,Y,Z orthogonal to ξ. By using Theorem 14 we have that M is a Sasakian space form, or locally isometric to the unit tangent sphere bundle $T_1(\mathbb{H}^{n+1})$ of a hyperbolic space $\mathbb{H}^{n+1}$ of constant curvature -1, if $n>1$. Conversely, by Proposition 17 and 18 a Sasakian space form and a non-Sasakian contact (k,μ)-space are weakly locally pseudo-Hermitian symmetric. For $n=1$, we see that every strongly pseudo-convex CR space form is weakly locally pseudo-Hermitian symmetric (see, Theorem 25 in Section 7). This completes the proof. □

7 The three-dimensional case

In this section, we study three-dimensional strongly pseudo-convex CR space forms and weakly pseudo-Hermitian symmetric spaces.

Homogeneous contact Riemannian 3-manifolds

A contact Riemannian manifold $(M;\eta,g)$ is said to be *homogeneous* if there exists a connected Lie group acting transitively as a group of isometries on it which preserves η. Perrone proved that 3-dimensional simply connected homogeneous contact Riemannian manifolds are Lie groups together with left invariant contact metric structures. Moreover, such homogeneous spaces are classified by the pseudo-Hermitian scalar curvature $\hat r$ and the torsion invariant $|\tau|$, where $\tau=\mathcal{L}_\xi g$, as follows:

Proposition 21. ([25]) *Let $(M;\eta,g)$ be a 3-dimensional simply connected homogeneous contact Riemannian manifold. Then M is a Lie group G together with a left invariant contact metric structure (η,g).*

If G is unimodular, then G is one of the following:
- *the Heisenberg group H^3 if $\hat r=|\tau|=0$;*
- $\mathrm{SU}(2)$ *if* $4\sqrt{2}\hat r>|\tau|$;
- $\widetilde{E}(2)$ *if* $4\sqrt{2}\hat r=|\tau|>0$;
- $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *if* $-|\tau|\neq 4\sqrt{2}\hat r<|\tau|$;
- $E(1,1)$ *if* $4\sqrt{2}\hat r=-|\tau|<0$.

The Lie algebra $\mathfrak{g}$ of G is generated by an orthonormal basis $\{e_1,e_2=\varphi e_1,e_3=\xi\}$ with commutation relation:

$$[e_1,e_2]=2e_3,\ [e_2,e_3]=c_2e_1,\ [e_3,e_1]=c_3e_2.$$

If G is non-unimodular, then the Lie algebra $\mathfrak{g}$ of G satisfies the commutation relations:

$$[e_1,e_2]=\alpha e_2+2e_3,\ [e_2,e_3]=0,\ [e_3,e_1]=\gamma e_2, \tag{32}$$

where $e_3=\xi$, $e_1,e_2\in\operatorname{Ker}\eta$, $e_2=\varphi e_1$, $\alpha\neq 0$ and $4\sqrt{2}\hat r<|\tau|$.

For a unimodular Lie group G we compute the holomorphic sectional curvature for $\hat\nabla$: $\hat K=c_3+c_2$, and for a non-unimodular Lie group G we have $\hat K=\gamma-\alpha^2$ (cf. [16]). Then we have

Proposition 22. *A homogeneous contact Riemannian 3-manifold is weakly locally pseudo-Hermitian symmetric.*

Here, we note that a non-unimodular Lie group G is weakly locally φ-symmetric (that is, $g((\nabla_X R)(Y,Z)V,W) = 0$ for all X,Y,Z,V,W orthogonal to ξ) if and only if $\gamma = 0$ (the Sasakian case) or $\gamma = 2$ ([6]).

Unit tangent sphere bundles of surfaces

It is known that the tangent bundle of a Riemannian manifold admits an almost complex structure J which is compatible with the Sasaki metric $\bar{g}$ and further $(J,\bar{g})$ is an almost Kähler structure (cf. Chapter 9 in [1]). Let M be a 2-dimensional Riemannian manifold, TM its tangent bundle equipped with $(J,\bar{g})$ and T_1M its unit tangent sphere bundle of M(i.e., the set of all unit tangent vectors of M) with the projection map $\pi : T_1M \to M$. Let (x^1,x^2) be local isothermal coordinates on M such that the Riemannian metric G is given by

$$G = f^2((dx^1)^2 + (dx^2)^2),$$

where f is a positive-valued function on M. Then, we compute the Gaussian curvature κ of M:

$$\kappa = -\frac{\Delta_0 \log f}{f^2}$$

where Δ_0 is the Laplacian with respect to Euclidean metric. For any point $Z \in T_1M$ in TM, we let (x^1,x^2,u^1,u^2) be a local coordinate system around Z such that (u^1,u^2) is the fiber coordinate with $f^2((u^1)^2 + (u^2)^2) = 1$. Then $N = u^1\frac{\partial}{\partial u^1} + u^2\frac{\partial}{\partial u^2}$ is a unit normal and position vector for the point Z of T_1M. Denote by g' the metric of T_1M induced from $\bar{g}$(Sasaki metric) on TM. Define φ', ξ', η' by

$$JN = -\xi',\ JX = \varphi' X + \eta'(X)N \tag{33}$$

for any vector field X on T_1M. Then we get $g'(X,\varphi' Y) = 2d\eta'(X,Y)$. By a simple rectification, namely, $\eta = \frac{1}{2}\eta'$, $\xi = 2\xi'$, $\varphi' = \varphi$ and $g = \frac{1}{4}g'$, we have a contact metric tangent sphere bundle $T_1M = (T_1M;\eta,g)$. (cf. [1, p.177]). Also, taking account of (33) and the definitions of J and g, we have a local orthonormal frame field $\{e_1,e_2,e_3\}$ such that

$$\begin{aligned} e_3 &= \xi = 2\sum_{ijk}(u^i\frac{\partial}{\partial x^i} - \{ijk\}u^ju^k\frac{\partial}{\partial u^i}),\\ e_1 &= 2\sum_i v^i\frac{\partial}{\partial u^i},\\ e_2 &= \varphi e_1 = -2\sum_{ijk}\left(v^i\frac{\partial}{\partial u^i} - \{ijk\}u^jv^k\frac{\partial}{\partial u^i}\right) \end{aligned} \tag{34}$$

for $i,j,k = 1,2$, where $(v^1,v^2) = (-u^2,u^1)$ and $\{ijk\}$ denote the Christoffel symbols of Riemannian connection of M. Then, we have

$$[e_1,e_2] = 2e_3,\ [e_2,e_3] = 2\kappa e_1,\ [e_3,e_1] = 2e_2. \tag{35}$$

Put

$$\Gamma_{ijk} = g(\nabla_{e_i}e_j,e_k) \quad \text{for} \quad i,j,k = 1,2,3.$$

Then we have $\Gamma_{ijk} = -\Gamma_{ikj}$. Using the Koszul formula, we have

$$\Gamma_{123} = 2-\kappa,\ \Gamma_{213} = \Gamma_{321} = -\kappa,\ \text{all other } \Gamma_{ijk} \text{ being zero.} \tag{36}$$

From (36) we see that e_1,e_2,e_3 are all geodesic vector fields, i.e., self-parallel vector fields and div $e_i = 0$ $(i = 1,2,3)$. We compute

$$he_1 = -(\kappa-1)e_1 \quad he_2 = (\kappa-1)e_2. \tag{37}$$

Then, from (35) and (36), we have $K(p) = \kappa^2$, $p \in M$. Moreover, from (24) and (37) we have $\hat{K}(p) = 2(\kappa+1)$. Thus, we have

Proposition 23. *The standard contact metric structure of unit tangent sphere bundle T_1M of a 2-dimensional Riemannian manifold M is a strongly pseudo-convex CR space form if and only if the Gaussian curvature of M is constant. In particular, $\hat{K} = 0$ if and only if M is a hyperbolic surface of constant Gaussian curvature -1.*

For a general study, let $(M; \eta, g)$ be a contact Riemannian 3-manifold. We consider on M the maximal open set U_1 on which $h \neq 0$ and the maximal open subset U_2 on which h is identically zero. Suppose that M is non-Sasakian. Then U_1 is non-empty and there is a local orthonormal frame field $\{\xi, e, \varphi e\}$ on U_1 such that $h(e) = \lambda e$, $h(\varphi e) = -\lambda \varphi e$ for some positive function λ. The covariant derivative is then of the following form (see [12, Lemma 2.1]):

$$\begin{aligned}
&\nabla_\xi \xi = 0, \qquad \nabla_\xi e = -a\varphi e, \qquad \nabla_\xi \varphi e = ae,\\
&\nabla_e \xi = -(\lambda+1)\varphi e, \qquad \nabla_{\varphi e}\xi = (1-\lambda)e,\\
&\nabla_e e = \frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}\varphi e, \qquad \nabla_{\varphi e}\varphi e = \frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}e,\\
&\nabla_e \varphi e = -\frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}e + (\lambda+1)\xi,\\
&\nabla_{\varphi e}\varphi e = \frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}\varphi e + (\lambda-1)\xi.
\end{aligned} \tag{38}$$

Here, a is a smooth function and $\sigma = \rho(\xi, \cdot)$ where ρ denotes the Ricci tensor. By using (14) we also calculate the (1,2)-tensor field A:

$$\begin{aligned}
&A(\xi,\xi) = 0, \qquad A(e,\xi) = (1+\lambda)\varphi e, \qquad A(\varphi e, \xi) = -(1-\lambda)e,\\
&A(\xi, e) = \varphi e, \qquad A(e,e) = 0, \qquad A(\varphi e, e) = (1-\lambda)\xi,\\
&A(\xi, \varphi e) = -e, \qquad A(e, \varphi e) = \xi, \qquad A(\varphi e, \varphi e) = 0.
\end{aligned} \tag{39}$$

Using (38) and (39), we get the following expressions for $\hat{\nabla}$ from (13):

$$\begin{aligned}
&\hat{\nabla}_\xi \xi = 0, \qquad \hat{\nabla}_\xi e = (1-a)\varphi e, \qquad \hat{\nabla}_\xi \varphi e = -(1-a)e,\\
&\hat{\nabla}_e \xi = 0, \quad \hat{\nabla}_e e = \frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}\varphi e, \quad \hat{\nabla}_e \varphi e = -\frac{1}{2\lambda}\{(\varphi e)(\lambda) + \sigma(e)\}e,\\
&\hat{\nabla}_{\varphi e}\xi = 0, \quad \hat{\nabla}_{\varphi e} e = -\frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}\varphi e, \quad \hat{\nabla}_{\varphi e}\varphi e = \frac{1}{2\lambda}\{e(\lambda) + \sigma(\varphi e)\}e.
\end{aligned} \tag{40}$$

From this, we calculate the Tanaka-Webster curvature tensor:

$$\begin{aligned}
\hat{R}(\xi, e)e &= \hat{\nabla}_\xi \hat{\nabla}_e e - \hat{\nabla}_e \hat{\nabla}_\xi e - \hat{\nabla}_{[\xi,e]} e\\
&= \big(\xi(p) + e(a) + (1+\lambda-a)q\big)\varphi e,\\
\hat{R}(\xi, e)\varphi e &= -\big(\xi(p) + e(a) + (1+\lambda-a)q\big)e,\\
\hat{R}(\xi, \varphi e)e &= -\big(\xi(q) - \varphi e(a) - (1-\lambda-a)p\big)\varphi e,\\
\hat{R}(\xi, \varphi e)\varphi e &= \big(\xi(q) - \varphi e(a) - (1-\lambda-a)p\big)e,\\
\hat{R}(e, \varphi e)e &= -\big(e(q) + \varphi e(p) - (p^2+q^2) + 2(1-a)\big)\varphi e,\\
\hat{R}(e, \varphi e)\varphi e &= \big(e(q) + \varphi e(p) - (p^2+q^2) + 2(1-a)\big)e,\\
\hat{R}(\cdot,\cdot)\xi &= 0,
\end{aligned} \tag{41}$$

where we have put $p = \frac{1}{2\lambda}(\varphi e(\lambda) + \sigma(e))$, $q = \frac{1}{2\lambda}(e(\lambda) + \sigma(\varphi e))$. Thus, we have

Proposition 24. *A three-dimensional contact metric space $(M; \eta, g)$ is a strongly pseudo-convex CR space form if and only if it is a Sasakian space form on U_2 and $\hat{K} := e(q) + \varphi e(p) - (p^2+q^2) + 2(1-a)$ is constant on U_1.*

From (40) and (41) we compute

$$\begin{aligned}
(\hat{\nabla}_e \hat{R})(e, \varphi e)e &= \hat{\nabla}_e(\hat{R}(e,\varphi e)e) - \hat{R}(\hat{\nabla}_e e, \varphi e)e\\
&\quad - \hat{R}(e, \hat{\nabla}_e \varphi e)e - \hat{R}(e,\varphi e)\hat{\nabla}_e e\\
&= -e\big(e(q) + \varphi e(p) - (p^2+q^2) + 2(1-a)\big)\varphi e,\\
(\hat{\nabla}_{\varphi e}\hat{R})(e,\varphi e)e &= -\varphi e\big(e(q) + \varphi e(p) - (p^2+q^2) + 2(1-a)\big)\varphi e.
\end{aligned}$$

Now, suppose that M is weakly locally pseudo-Hermitian symmetric, then

$$\begin{aligned} 0 &= e\big(e(q) + \varphi e(p) - p^2 - q^2 + 2(1-a)\big), \\ 0 &= \varphi e\big(e(q) + \varphi e(p) - p^2 - q^2 + 2(1-a)\big). \end{aligned}$$

In that case, we also have, using the second condition of (1):

$$\begin{aligned} 0 &= [e, \varphi e]\big(e(q) + \varphi e(p) - p^2 - q^2 + 2(1-a)\big) \\ &= \eta([e, \varphi e])\xi\big(e(q) + \varphi e(p) - p^2 - q^2 + 2(1-a)\big) \\ &= -2d\eta(e, \varphi e)\xi\big(e(q) + \varphi e(p) - p^2 - q^2 + 2(1-a)\big) \\ &= 2\xi\big(e(q) + \varphi e(p) - p^2 - q^2 + 2(1-a)\big). \end{aligned}$$

Since, in dimension 3, a Sasakian space form coincides with a Sasakian locally φ-symmetric space ([4]), we have

Theorem 25. *A three-dimensional contact metric space $(M; \eta, g)$ is a strongly pseudo-convex CR space form if and only if it is a weakly locally pseudo-Hermitian symmetric space.*

EXAMPLE 1. ([26]) Let $M_1 = \{(x, y, z) \in \mathbf{R}^3(x, y, z) \mid x \neq 0\}$ be a contact three-manifold endowed with the contact form $\eta = xydx + dz$. Its characteristic vector field is given by $\xi = \partial/\partial z$. Take a frame field

$$e_1 = -\frac{2}{x}\frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial x} - \frac{4z}{x}\frac{\partial}{\partial y} - xy\frac{\partial}{\partial z}, \quad e_3 = \xi$$

and define a Riemannian metric g such that $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. Moreover, we define φ by $\varphi e_1 = e_2,\ \varphi e_2 = -e_1$ and $\varphi\xi = 0$. Then (η, g) is a contact metric structure. The structure operator h satisfies $he_1 = e_1,\ he_2 = -e_2$. The components of the Ricci operator are $\rho_{13} = \rho_{31} = -\frac{2}{x}, \rho_{22} = -8$ and the other components are zero. Then we obtain: $a = 2$, $\lambda = 1$, $p = -\frac{1}{x}$, $q = 0$. This yields $\hat{K} = e_1(q) + e_2(p) - p^2 - q^2 + 2(1-a) = -2(< 0)$, that is, M_1 is a strongly pseudo-convex CR space form with a negative holomorphic sectional curvature. Note that M_1 is locally φ-symmetric in the weak sense.

EXAMPLE 2. ([23]) On Cartesian 3-space $\mathbb{R}^3(x, y, z)$, we define a contact 1-form η by

$$\eta = dx + 2ye^{-z}dz.$$

And define a frame field $\{e_1, e_2, e_3 = \xi\}$ by

$$e_1 = -2y\frac{\partial}{\partial x} + (2x - ye^z)\frac{\partial}{\partial y} + e^z\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad \xi = \frac{\partial}{\partial x}.$$

Then we define a Riemannian metric g by the condition that $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. One can see that g is an associated metric to η. As usual, the endomorphism field φ is defined by $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi e_3 = 0$. A direct computation shows that $h = \omega^1 \otimes e_1 - \omega^2 \otimes e_2$. The components of the Ricci operator are $\rho_{23} = \rho_{32} = 2e^z$. Other components are zero. Then we obtain: $a = 0$, $\lambda = 1$, $p = 0$, $q = e^z$. This yields $\hat{K} = e_1(q) + e_2(p) - p^2 - q^2 + 2(1-a) = 2(> 0)$, that is, $M_2 = (\mathbb{R}^3; \eta, g)$ is a strongly pseudo-convex CR space form with a positive holomorphic sectional curvature and also a weakly locally pseudo-Hermitian symmetric space. But, it is not locally φ-symmetric (in the weak sense). Indeed we compute $g((\nabla_{e_1}R)(e_1, e_2)e_1, e_2) = 8e^z \neq 0$.

Remark 4. Example 1 and Example 2 are non-homogeneous and weakly locally pseudo-Hermitian symmetric spaces.

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