

Robert Ream*

The Adjunction Inequality for Weyl-Harmonic Maps

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Abstract: In this paper we study an analog of minimal surfaces called Weyl-minimal surfaces in conformal manifolds with a Weyl connection (M^4, c, D) . We show that there is an Eells-Salamon type correspondence between nonvertical \mathcal{J} -holomorphic curves in the weightless twistor space and branched Weyl-minimal surfaces. When (M, c, J) is conformally almost-Hermitian, there is a canonical Weyl connection. We show that for the canonical Weyl connection, branched Weyl-minimal surfaces satisfy the adjunction inequality

$$\chi(T_f\Sigma) + \chi(N_f\Sigma) \leq \pm c_1(f^*T^{(1,0)}M).$$

The $\pm J$ -holomorphic curves are automatically Weyl-minimal and satisfy the corresponding equality. These results generalize results of Eells-Salamon and Webster for minimal surfaces in Kähler 4-manifolds as well as their extension to almost-Kähler 4-manifolds by Chen-Tian, Ville, and Ma.

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1 Introduction

This paper describes an extension of the notion of minimal surfaces to the setting of conformal manifolds with a Weyl connection. Particular attention is given to almost-Hermitian 4-manifolds endowed with their canonical Weyl connection. We first review the relevant standard theory.

According to Eells and Salamon [5], branched minimal surfaces in an oriented Riemannian 4-manifold M have a one-to-one correspondence with nonvertical \mathcal{J} -holomorphic curves in the twistor space \mathcal{Z} , where \mathcal{J} is the canonical non-integrable almost-complex structure on \mathcal{Z} . Applying twistor techniques, they further show that if M is almost-Kähler with almost-complex structure J , the $\pm J$ -holomorphic curves are minimal. When M is Kähler they prove the adjunction inequality,

$$\chi(T_f\Sigma) + \chi(N_f\Sigma) \leq \pm c_1(f^*T^{(1,0)}M),$$

where $T_f\Sigma$ is the tangent bundle to Σ ramified at the branch points of f , and $N_f\Sigma$ is its normal bundle in f^*TM . Concurrently, Webster [15] obtained his formulas (1), (2) for a minimal surface in a Kähler 4-manifold, which imply the adjunction inequality. The adjunction inequality was extended to minimal surfaces in almost-Kähler 4-manifolds by Chen-Tian [3], Ville [14], and Ma [10].

This leads to the following picture for almost-Kähler manifolds: The adjunction inequality holds for minimal surfaces; every $\pm J$ -holomorphic curve is minimal, and equality holds in (3) with the corresponding sign.

For an almost-Hermitian manifold, in general, the $\pm J$ -holomorphic curves are not minimal, and in [3] they remark that the (3) will not hold for minimal surfaces. In this paper we show that the above scenario for almost-Kähler manifolds can be extended to almost-Hermitian manifolds when considering a conformally

*Corresponding Author: Robert Ream: Clark University Worcester, MA, United States, E-mail: rream@clarku.edu

invariant condition on surfaces related to the minimal condition. We now briefly describe this condition and list our main theorems.

Let M be a manifold with conformal metric c and Weyl connection ∇^D , to be described in detail later. For $i : \Sigma \rightarrow M$ an immersed submanifold, the Weyl second fundamental form B is defined in [13] as follows. Taking $g \in c$, there is a one-form α_g such that $\nabla^D g = -2\alpha_g \otimes g$. Let A_g be the usual second fundamental form, then

$$B = A_g - \left(\alpha_g^{\sharp g} \right)^\perp \otimes g.$$

We say the submanifold is *Weyl-minimal* if $\text{tr}_{i^*g} B = 0$. Branched Weyl-minimal then has the obvious meaning.

We extend the Eells-Salamon twistor correspondence as follows.

Theorem 1.1. *Let (M^4, c, D) be a Weyl manifold, and $(\Sigma, [\eta])$ a Riemann surface. There are complex structures \mathcal{J}_\pm on the weightless twistor spaces \mathcal{Z}_\pm which give a 1-to-1 correspondence between non-vertical \mathcal{J}_\pm -holomorphic curves $\tilde{f}_\pm : \Sigma \rightarrow \mathcal{Z}_\pm$ and non-constant weakly conformal branched Weyl-minimal immersions $f : \Sigma \rightarrow M$.*

If the Weyl derivative is exact, then the Weyl-minimal surfaces are minimal for a preferred metric in c , and this is just the usual correspondence for that metric.

Theorem 1.2. *Let (M^4, c, J, D) be a conformally almost-Hermitian manifold with its canonical Weyl connection. The almost-complex structure gives rise to a \mathcal{J}_+ -holomorphic section of \mathcal{Z}_+ .*

This and the previous theorem imply the following corollary.

Corollary 1.1. *Under the assumptions of Theorem 1.2, a $\pm J$ -holomorphic curve $f : \Sigma \rightarrow M$ is a weakly conformal branched Weyl-minimal immersion.*

Finally we prove that Webster's formulas hold for branched Weyl-minimal immersions.

Theorem 1.3. *For a Riemann surface $(\Sigma, [\eta])$ and a conformally almost-Hermitian manifold with its canonical Weyl-connection (M^4, c, J, D) , if $f : \Sigma \rightarrow M$ is a weakly conformal branched Weyl-minimal immersion with P complex points and Q anti-complex points then*

$$\chi(T_f \Sigma) + \chi(N_f \Sigma) = -P - Q \quad (1)$$

$$c_1(f^* T^{(1,0)} M) = P - Q. \quad (2)$$

The adjunction inequality follows from P and Q being positive.

Corollary 1.2. *For a Riemann surface $(\Sigma, [\eta])$ and a conformally almost-Hermitian manifold with its canonical Weyl-connection (M^4, c, J, D) , if $f : \Sigma \rightarrow M$ is a weakly conformal branched Weyl-minimal immersion then*

$$\chi(T_f \Sigma) + \chi(N_f \Sigma) \leq \pm c_1(f^* T^{(1,0)} M). \quad (3)$$

The corresponding equality holds for $\pm J$ -holomorphic curves.

2 Preliminaries

2.1 Weyl Geometry

By definition, the *density bundle* on an n -dimensional manifold M is $L := |\Lambda^n TM|^{\frac{1}{n}}$. The tensor bundles $L^w \otimes TM^j \otimes T^* M^k$ are said to have *weight* $w + j - k$. A *Weyl derivative* D is a connection on the density bundle. A *conformal metric* c is a metric on the weightless tangent bundle $L^{-1} TM$ satisfying the normalizing condition $|\det c| = 1$. This can also be considered as a metric on TM with values in L^2 .

Definition 2.1. The triple (M, c, D) is called a *Weyl Manifold*.

The bundle L is trivial, and a nowhere zero section of L , μ , is called a *length scale*. This defines a metric in the conformal class c by $g_\mu = \mu^{-2}c$. The section μ gives a trivialization of L which has a corresponding trivializing connection D^μ . This defines a one form $\alpha_\mu = D - D^\mu$, so that

$$D(h\mu) = (dh + hD)\mu = (dh + h(\alpha_\mu + D^\mu))\mu = (dh + h\alpha_\mu)\mu.$$

There is a unique torsion free connection ∇^D on TM making c parallel,

$$\nabla_X^D Y = \nabla_X^{g_\mu} Y + \alpha_\mu(X)Y + \alpha_\mu(Y)X - g_\mu(X, Y)\alpha_\mu^\sharp,^{\sharp g_\mu},$$

where ∇^{g_μ} is the Levi-Civita connection for the metric g_μ .

2.1.1 Weightless Twistor Space

When M is oriented, c defines a section v_c of the orientation bundle $L^n \Lambda^n T^*M$. This can be used to define the conformal Hodge star

$$* : L^m \Lambda^k T^*M \rightarrow L^{m+n-2k} \Lambda^{n-k} T^*M,$$

where for $\beta, \gamma \in L^k \Lambda^k T^*M$

$$\beta \wedge * \gamma = c(\beta, \gamma)v_c$$

For $n = 4$ and $m = 0$, $* : \Lambda^2 T^*M \rightarrow \Lambda^2 T^*M$ is an involution with ± 1 eigenspaces $\Lambda_\pm^2 T^*M$. The weightless twistor spaces [2] can be constructed as the sphere bundles

$$\mathcal{Z}_\pm = S(L^2 \Lambda_\pm^2 T^*M).$$

We now review the construction of an almost-complex structure \mathcal{J}_\pm on \mathcal{Z}_\pm . This can be seen by working at a point $q_\pm \in \mathcal{Z}_\pm$ which projects to $p \in M$. For U a neighborhood of p , and a local section $s_\pm : U \rightarrow \mathcal{Z}_\pm|_U$ satisfying $s_\pm(p) = q_\pm$, there is a weightless Kähler form σ_\pm given by this section and a corresponding almost-complex structure J_\pm on $T_p M$ given by

$$\sigma_\pm(X, Y) = c(J_\pm X, Y).$$

As the fiber of \mathcal{Z}_\pm at p is a sphere in $L^2 \Lambda_\pm^2 T_p^*M$, the vertical tangent space at q_\pm is the space perpendicular to σ_\pm in $L^2 \Lambda_\pm^2 T_p^*M$. This is the space of weightless J_\pm -anti-invariant 2-forms [4].

Definition 2.2. The space of weightless J_\pm -anti-invariant 2-forms $L^2 \Lambda_\pm^{2, J_\pm} T_p^*M$ is the (-1) -eigenspace for the involution $I_\pm : L^2 \Lambda^2 T_p^*M \rightarrow L^2 \Lambda^2 T_p^*M$ given by $(I_\pm \beta)(X, Y) = \beta(J_\pm X, J_\pm Y)$.

There is an induced almost-complex structure acting on $\beta \in L^2 \Lambda_\pm^{2, J_\pm} T_p^*M$ by

$$(J_\pm \beta)(X, Y) = \beta(J_\pm X, Y).$$

To see that this is an almost-complex structure, first note that $J_\pm \beta$ is a two form as

$$\beta(J_\pm X, Y) = -\beta(Y, J_\pm X) = \beta(J_\pm Y, J_\pm^2 X) = -\beta(J_\pm Y, X).$$

Second,

$$I_\pm J_\pm \beta = J_\pm I_\pm \beta = -J_\pm \beta,$$

so $J_\pm \beta \in L^2 \Lambda_\pm^{2, J_\pm} T_p^*M$. Finally, it is easily seen that $J_\pm^2 \beta = -\beta$.

Extending β to be complex bilinear gives

$$\begin{aligned} \frac{1}{4}\beta(X + iJ_\pm X, Y + iJ_\pm Y) &= \frac{1}{2}(\beta + iJ_\pm \beta)(X, Y), \\ \frac{1}{4}\beta(X - iJ_\pm X, Y + iJ_\pm Y) &= 0 \\ \frac{1}{4}\beta(X - iJ_\pm X, Y - iJ_\pm Y) &= \frac{1}{2}(\beta - iJ_\pm \beta)(X, Y). \end{aligned}$$

Therefore $\beta \in L^2(\Lambda_{\pm}^{2,0}T_p^*M \oplus \Lambda_{\pm}^{0,2}T_p^*M)$. This shows that $L^2\Lambda^{2,J_{\pm}}T_p^*M \perp \sigma_{\pm}$ as there is an orthogonal splitting

$$\Lambda_{\pm}^2T_p^*M \otimes \mathbb{C} = \Lambda_{\pm}^{2,0}T_p^*M \oplus \Lambda_{\pm}^{0,2}T_p^*M \oplus \mathbb{C}\sigma_{\pm}.$$

Furthermore $\frac{1}{2}(\beta - iJ_{\pm}\beta) \in L^2\Lambda_{\pm}^{2,0}T_p^*M$ and $\frac{1}{2}(\beta + iJ_{\pm}\beta) \in L^2\Lambda_{\pm}^{0,2}T_p^*M$.

There is an isomorphism, $\beta \mapsto \beta^v$, from $L^2\Lambda^{2,J_{\pm}}T_p^*M$ to the vertical tangent space $V(T_{q_{\pm}}\mathcal{Z}_{\pm})$, so that for $\beta \in L^2\Lambda^{2,J_{\pm}}T_p^*M$ we have $\beta^v \in V(T_{q_{\pm}}\mathcal{Z}_{\pm})$. This isomorphism and the connection induce an isomorphism, $X \mapsto X^h$, from T_pM to the horizontal tangent space $H(T_{q_{\pm}}\mathcal{Z}_{\pm})$ by

$$ds_{\pm}(X) = X^h + (\nabla_X^D\sigma_{\pm})^v. \quad (4)$$

The almost-complex structure on \mathcal{Z}_{\pm} is now given by linearly extending

$$J_{\pm}(X^h) := (J_{\pm}X)^h, \quad (5)$$

$$J_{\pm}(\beta^v) := (J_{\pm}\beta)^v. \quad (6)$$

In [5] Eells and Salamon used a similar complex structure to study weakly conformal harmonic maps. Their complex structure is the same as the complex structure of Penrose, studied by Atiyah, Hitchin and Singer in [1], except that it reverses the orientation of the fibers. The complex structure defined in (5) and (6) differs from that of Eells and Salamon only in the use of a Weyl connection to define the horizontal space rather than the Levi-Civita connection.

2.1.2 Submanifold Geometry

Let (M, c, D) be a Weyl manifold, and $i : \Sigma \rightarrow M$ an immersed submanifold. Then Σ inherits a conformal structure \bar{c} and Weyl derivative \bar{D} . One way to see this is to choose a length scale, $\mu \in \Gamma(L)$. Then the metric g_{μ} and the one form α_{μ} can be pulled back to Σ as $\bar{g}_{\mu} = i^*g_{\mu}$ and $\bar{\alpha}_{\mu} = i^*\alpha_{\mu}$. Hence $\bar{\mu} = |\det \bar{g}_{\mu}|^{-1/(2 \dim \Sigma)}$ is a section of the density bundle of Σ . The inherited conformal metric is $\bar{c} = \bar{\mu}^2\bar{g}_{\mu}$ and the inherited Weyl derivative is

$$\bar{D}(h\bar{\mu}) = (dh + h\bar{\alpha}_{\mu})\bar{\mu}.$$

The connection $\bar{\nabla}^{\bar{D}}$ on Σ is defined so that \bar{c} is parallel,

$$\bar{\nabla}_X^{\bar{D}}Y = \bar{\nabla}_X^{\bar{g}_{\mu}}Y + \bar{\alpha}_{\mu}(X)Y + \bar{\alpha}_{\mu}(Y)X - \langle X, Y \rangle_{\bar{g}_{\mu}}\bar{\alpha}_{\mu}^{\sharp_{\bar{g}_{\mu}}}.$$

The Weyl second fundamental form [13] is given by

$$B^D(X, Y) = \nabla_X^D Y - \bar{\nabla}_X^{\bar{D}} Y.$$

Equivalently, for $A_{g_{\mu}} = \nabla^{g_{\mu}} - \bar{\nabla}^{\bar{g}_{\mu}}$

$$B^D(X, Y) = A_{g_{\mu}}(X, Y) - \langle X, Y \rangle_{g_{\mu}} \left(\alpha_{\mu}^{\sharp_{g_{\mu}}} \right)^{\perp}.$$

The Weyl mean curvature is

$$\mathbf{H}^D = \frac{1}{\dim \Sigma} \operatorname{tr}_{\bar{g}_{\mu}} B^D = \mathbf{H}_{g_{\mu}} - \left(\alpha_{\mu}^{\sharp_{g_{\mu}}} \right)^{\perp}, \quad (7)$$

where $\mathbf{H}_{g_{\mu}}$ is the usual mean curvature of Σ with respect to the metric g_{μ} .

Definition 2.3. The immersion $i : \Sigma \rightarrow M$ is *Weyl-minimal* if $\mathbf{H}^D = 0$.

Example 2.1. Exact and Closed Weyl Derivatives

If (M, c, D) is a Weyl manifold and there is a length scale μ so that α_{μ} is exact, then D is called exact. If $\alpha_{\mu} = du$, then $D = D^{e^{-u}\mu}$ and the Weyl-minimal surfaces are just the minimal surfaces for the metric $g^{e^{-u}\mu} = e^{2u}g_{\mu}$. Similarly, if α_{μ} is closed then D is called closed. In this case, if $f : \Sigma \rightarrow M$ is a Weyl-minimal branched immersion then there is a lift to the universal cover $\tilde{f} : \tilde{\Sigma} \rightarrow \tilde{M}$. The conformal metric and Weyl Derivative can be lifted to \tilde{M} and the closed Weyl derivative becomes exact. Thus \tilde{f} is a minimal surface for a metric in the lifted conformal class.

The harmonic map equation can also be generalized to this setting. In [8] the second fundamental form of a map $f : \Sigma \rightarrow M$ is defined for manifolds Σ and M with torsion-free connections ∇^Σ and ∇^M . If ∇ is the induced connection on $T^*\Sigma \otimes f^*TM$, then the second fundamental form is just ∇df . If η is a metric on Σ then the *tension* of the map can be defined as

$$\tau(\eta, \nabla^\Sigma, \nabla^M) = \text{tr}_\eta \nabla df.$$

A map is *psuedo-harmonic* if the tension field is zero. We study the case where the domain (Σ, η) is a Riemannian manifold with its Levi-Civita connection ∇^η and the target manifold (M, c, D) is a Weyl manifold. This is opposite of the case studied in [8], where the domain is Weyl and the target is Riemannian.

Definition 2.4. A map $f : \Sigma \rightarrow M$ is *Weyl-harmonic* if $\tau(\eta, \nabla^\eta, \nabla^D) = 0$.

From this point we only consider the case where Σ has dimension two. Using local isothermal coordinates on Σ so that $\eta = e^{2\lambda}(dx^2 + dy^2)$, the tension field is

$$\tau(\eta, \nabla^\eta, \nabla^D) = e^{-2\lambda}(\nabla_{\partial_x}^D f_x + \nabla_{\partial_y}^D f_y), \quad (8)$$

where $f_x = df(\partial_x)$. In terms of the complex coordinate $z = x + iy$ this is just

$$\tau(\eta, \nabla^\eta, \nabla^D) = 4e^{-2\lambda}\nabla_{\partial_z}^D f_z. \quad (9)$$

This can also be written more explicitly as

$$\tau(\eta, \nabla^\eta, \nabla^D) = e^{-2\lambda}(\nabla_{\partial_x}^{g_\mu} f_x + \nabla_{\partial_y}^{g_\mu} f_y + 2\alpha_\mu(f_x)f_x + 2\alpha_\mu(f_y)f_y - (|f_x|_{g_\mu}^2 + |f_y|_{g_\mu}^2)\alpha_\mu^{\sharp g_\mu}). \quad (10)$$

From this we see that when Σ has dimension 2

$$\tau(e^{2u}\eta, \nabla^{e^{2u}\eta}, \nabla^D) = e^{-2u}\tau(\eta, \nabla^\eta, \nabla^D).$$

Thus, in this case, the definition of Weyl-harmonic depends only on the conformal class of η . We are also interested in the case where f is weakly conformal.

Definition 2.5. A map $f : \Sigma \rightarrow M$ is *weakly conformal* if it is conformal whenever $df \neq 0$.

If $z = x + iy$ is a complex coordinate on Σ so that $\eta = e^{2\lambda}(dx^2 + dy^2)$, then the equations

$$\eta(\partial_x, \partial_y) = 0 \quad \text{and} \quad \eta(\partial_x, \partial_x) = \eta(\partial_y, \partial_y)$$

are conformally invariant. If f is weakly conformal then this implies that

$$c(f_x, f_y) = 0 \quad \text{and} \quad c(f_x, f_x) = c(f_y, f_y).$$

Extending the conformal inner product to be complex bilinear, these are equivalent to the equation

$$c(f_z, f_z) = 0, \quad (11)$$

where $f_z = \frac{1}{2}(f_x - if_y)$. Any point where df is not full rank is called a singular point. A branch point p is a singular point where in some neighborhood of p , $f_z = z^k Z$ and $Z_p \neq 0$.

Proposition 2.1. *If $f : \Sigma \rightarrow M$ is weakly conformal, Weyl-harmonic, and non-constant, then df is rank 2 except at an isolated set of branch points.*

Proof. Like the harmonic map equation, the Weyl-harmonic map equation can be written as the Laplace equation plus terms quadratic in df . Thus the hypotheses of Aronzajn's unique continuation theorem and the Hartman-Wintner theorem [11] are still satisfied in the Weyl-harmonic case. As f is non-constant, Aronzajn's

theorem implies that df is not zero on an open set. By the Hartman-Wintner Theorem, at every point there is an integer $m \geq 1$ so that in an isothermal coordinate chart,

$$f(x, y) = h(x, y) + o(|(x, y)|^m) \quad \text{and} \quad df(x, y) = dh(x, y) + o(|(x, y)|^{m-1}).$$

for some non-zero homogeneous degree m polynomial h . The zeros of dh are isolated, so the zeros of df must be as well. In fact, h must also be weakly conformal and harmonic, thus the only zeros of dh are branch point singularities. Details and further analysis of the structure of these branch points is contained in [12]. \square

If $f : \Sigma \rightarrow M$ is an immersion with only branch point singularities, then it is called a *branched immersion*. When f is a branched immersion the conformal class f^*c on Σ can be defined across the branch points and f is weakly conformal when $f^*c = [\eta]$. A branched immersion which is Weyl-minimal away from the branch points is called a *branched Weyl-minimal immersion*.

Proposition 2.2. *If $f : \Sigma \rightarrow M$ is weakly conformal then f is Weyl-harmonic if and only if it is a branched Weyl-minimal immersion.*

Proof. In this case, away from the branch points,

$$\tau(\eta, \nabla^\eta, \nabla^D) = e^{-2\lambda}(|f_x|_{g_\mu}^2 + |f_y|_{g_\mu}^2) \left(\mathbf{H}_{g_\mu} - \left(\alpha_\mu^{\#g_\mu} \right)^\perp \right).$$

Comparing this with equation (7) we see that $\tau(\eta, \nabla^\eta, \nabla^D) = 0$ if and only if $\mathbf{H}^D = 0$. \square

There is a splitting $TM = T_f\Sigma \oplus N_f\Sigma$, where $T_f\Sigma = df(T\Sigma)$ away from the branch points. At a branch point p , $f_z = z^k Z$, $Z_p \neq 0$, and $T_f\Sigma = \text{span}\{\text{Re}(Z_p), \text{Im}(Z_p)\}$. In both cases $N_f\Sigma$ is the orthogonal complement of $T_f\Sigma$.

Definition 2.6. For M oriented, the *twistor lifts* of f , $\tilde{f}_\pm : \Sigma \rightarrow \mathcal{Z}_\pm$ are determined by two complex structures on f^*TM . There are two orthogonal complex structures J_\pm which agree with the complex structure of Σ on $T_f\Sigma$. The complex structure J_+ preserves the orientation while J_- is orientation reversing. The corresponding weightless Kähler form is then

$$\tilde{f}_\pm = c(J_\pm \cdot, \cdot)$$

The complex structures J_\pm determine a splitting of $f^*TM \otimes \mathbb{C} = f^*T_\pm^{(1,0)}M \oplus f^*T_\pm^{(0,1)}M$.

2.2 Conformally Almost-Hermitian Manifolds

A conformally almost-Hermitian manifold (M^4, c, J) is an almost-complex manifold with a conformal structure satisfying

$$c(X, Y) = c(JX, JY).$$

The conformal Kähler form $\omega_c = c(J\cdot, \cdot)$ can be viewed as a 2-form with values in L^2 . Then there is a unique Weyl derivative satisfying $d^D\omega_c = 0$. Fixing $\mu \in \Gamma(L)$ we can define this Weyl derivative using the *Lee form*

$$\theta_\mu = J\delta_{g_\mu}\omega_\mu = -(\delta_{g_\mu}\omega_\mu)J,$$

with δ_{g_μ} denoting the divergence and $\omega_c = \mu^2\omega_\mu$. In terms of an orthonormal coframe $\{e^i\}$ of g_μ satisfying

$$\omega_\mu = e^1 \wedge e^2 + e^3 \wedge e^4,$$

and $a_i = \langle e_i, \delta_{g_\mu}\omega_\mu \rangle_{g_\mu}$ one can check that

$$d\omega_\mu = a_1 e^2 \wedge e^3 \wedge e^4 - a_2 e^1 \wedge e^3 \wedge e^4 + a_3 e^1 \wedge e^2 \wedge e^4 - a_4 e^1 \wedge e^2 \wedge e^3.$$

Then the Lee form is

$$\theta_\mu = -J^* d\omega_\mu = J(a_1 e^1 + a_2 e^2 + a_3 e^3 + a_4 e^4) = a_1 e^2 - a_2 e^1 + a_3 e^4 - a_4 e^3,$$

furthermore

$$\theta_\mu \wedge \omega_\mu = d\omega_\mu.$$

Then for Weyl derivative $D = d + \alpha_\mu$,

$$d^D \omega = d^D \mu^2 \omega_\mu = 2\mu D(\mu)\omega + \mu^2 d\omega = 2\mu^2 \alpha_\mu \wedge \omega_\mu + \mu^2 \theta_\mu \wedge \omega_\mu.$$

The canonical Weyl derivative of (M, c, J) is then determined by setting $\alpha_\mu = -\frac{1}{2}\theta_\mu$. The induced connection on TM is given by

$$\nabla_X^D Y = \nabla_X^{g_\mu} Y - \frac{1}{2}\theta_\mu(X)Y - \frac{1}{2}\theta_\mu(Y)X + \frac{1}{2}\langle X, Y \rangle_{g_\mu} \theta_\mu^\sharp.$$

The Nijenhuis Tensor of J is given by

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

This is J antilinear in both slots, that is $N(JX, Y) = -JN(X, Y) = N(X, JY)$. For any vector field, X , $\nabla_X^D J$ is also J antilinear.

Proposition 2.3. *For any almost-Hermitian manifold*

$$\langle N(X, Y), JZ \rangle_{g_\mu} = d\omega_\mu(X, Y, Z) - d\omega_\mu(JX, JY, Z) - 2\langle (\nabla_Z^{g_\mu} J)X, Y \rangle_{g_\mu}.$$

This is proposition 4.2 in [7] with different conventions. The corresponding formula for conformally almost-Hermitian manifolds with a Weyl derivative is

$$c(N(X, Y), JZ) = d^D \omega_c(X, Y, Z) - d^D \omega_c(JX, JY, Z) - 2c((\nabla_Z^D J)X, Y).$$

Proposition 2.4. *For any conformally almost-Hermitian manifold with canonical Weyl derivative D ,*

$$c(N(X, Y), JZ) = -2c((\nabla_Z^D J)X, Y).$$

Corollary 2.1. *The global section $s : (M, J) \rightarrow (\mathcal{Z}_+, \mathcal{J}_+)$ defined by ω_c is holomorphic.*

Proof. Letting $X, Y, Z \in TM$, by the proposition,

$$(\nabla_Z^D \omega_c)(X, Y) = -\frac{1}{2}c(N(X, Y), JZ),$$

and by the symmetries of the Nijenhuis tensor,

$$(\nabla_{JZ}^D \omega_c)(X, Y) = \frac{1}{2}c(N(X, Y), Z) = \frac{1}{2}c(JN(X, Y), JZ) = -\frac{1}{2}c(N(JX, Y), JZ).$$

Therefore $(\nabla_{JZ}^D \omega_c) = \mathcal{J}_+(\nabla_Z^D \omega_c)$, and by equations (4), (5), and (6)

$$ds(JZ) = (JZ)^h + (\nabla_{JZ}^D \omega_c)^v = \mathcal{J}_+ Z^h + \mathcal{J}_+(\nabla_Z^D \omega_c)^v = \mathcal{J}_+ ds(Z).$$

□

3 Main Theorems

3.1 Twistor Correspondence

Following [5] we now show there is a correspondence between weakly conformal Weyl-harmonic maps and non-vertical \mathcal{J}_\pm -holomorphic maps into the weightless twistor space with complex structure given by (5) and (6).

The twistor lifts of a weakly conformal map $f : \Sigma \rightarrow M$ are given by

$$\tilde{f}_{\pm} = \left(\frac{(1 \pm *)f_z \wedge f_{\bar{z}}}{ic(f_z, f_{\bar{z}})} \right)^{b_c},$$

where $i = \sqrt{-1}$. The natural isomorphism $b_c : L^{-1}TM \rightarrow LT^*M$ preserves weights, but interchanges the holomorphic and anti-holomorphic spaces.

Theorem 3.1. *For any Weyl manifold (M, c, D) , a weakly conformal map $f : \Sigma \rightarrow M$ is Weyl-harmonic if and only if the twistor lifts $\tilde{f}_{\pm} : \Sigma \rightarrow \mathcal{Z}_{\pm}$ are \mathcal{J}_{\pm} -holomorphic.*

Proof. The twistor lifts \tilde{f}_{\pm} are \mathcal{J}_{\pm} -holomorphic provided $d\tilde{f}_{\pm}(\partial_z) \in T^{(1,0)}\mathcal{Z}_{\pm}$. We have

$$d\tilde{f}_{\pm}(\partial_z) = (f_z)^h + \left(\nabla_{\partial_z}^D \tilde{f}_{\pm} \right)^v,$$

and since $f_z \in T_{\pm}^{(1,0)}M$, we have $(f_z)^h \in H(T^{(1,0)}\mathcal{Z}_{\pm})$. Thus all that is required is $\nabla_{\partial_z}^D \tilde{f}_{\pm} \in L^2\Lambda^{(2,0)}T^*M$. We find that

$$\begin{aligned} (\nabla_{\partial_z}^D \tilde{f}_{\pm})_{\pm}^{(0,2)} &= \left(\frac{\sqrt{2}f_z \wedge (\nabla_{\partial_z}^D f_{\bar{z}})^{(1,0)}}{ic(f_z, f_{\bar{z}})} \right)^{b_c}, \\ (\nabla_{\partial_z}^D \tilde{f}_{\pm})_{\pm}^{(1,1)} &= 0, \\ (\nabla_{\partial_z}^D \tilde{f}_{\pm})_{\pm}^{(2,0)} &= \left(\frac{\sqrt{2}(\nabla_{\partial_z}^D f_z)^{(0,1)} \wedge f_{\bar{z}}}{ic(f_z, f_{\bar{z}})} \right)^{b_c}. \end{aligned}$$

It follows that \tilde{f}_{\pm} is pseudo-holomorphic map if and only if $(\nabla_{\partial_z}^D f_{\bar{z}})^{(1,0)}_{\pm} = kf_z$, for some function k . Taking the conformal inner-product with $f_{\bar{z}}$ gives

$$c((\nabla_{\partial_z}^D f_{\bar{z}})^{(1,0)}_{\pm}, f_{\bar{z}}) = kc(f_z, f_{\bar{z}}).$$

Since $f_{\bar{z}} \in T_{\pm}^{(0,1)}M$ this is just

$$c(\nabla_{\partial_z}^D f_{\bar{z}}, f_{\bar{z}}) = kc(f_z, f_{\bar{z}}),$$

and since c is ∇^D parallel we have

$$\nabla_{\partial_z}^D c(f_{\bar{z}}, f_{\bar{z}}) = 2kc(f_z, f_{\bar{z}}).$$

For f weakly conformal, this shows that $k = 0$. Therefore \tilde{f}_{\pm} is \mathcal{J} -holomorphic if and only if $(\nabla_{\partial_z}^D f_{\bar{z}})^{(1,0)}_{\pm} = 0$, but since $\nabla_{\partial_z}^D f_{\bar{z}}$ is real, this can only be true when it is zero. \square

Corollary 3.1. *There is a one-to-one correspondence between weakly conformal Weyl-harmonic maps to (M, c, D) and non-vertical \mathcal{J}_{\pm} -holomorphic maps to the twistor space.*

Proof. It only remains to show that for a non-vertical \mathcal{J} -holomorphic curve, $\phi : \Sigma \rightarrow \mathcal{Z}_{\pm}$ the projection $\bar{\phi} : \Sigma \rightarrow M$ is weakly conformal and Weyl-harmonic. It is clearly weakly conformal as $\bar{\phi}_z$ is holomorphic with respect to the complex structure defined by ϕ , which implies $c(\bar{\phi}_z, \bar{\phi}_z) = 0$. It is Weyl-harmonic as ϕ is its twistor lift and is \mathcal{J} -holomorphic. \square

Corollary 3.2. *The \mathcal{J} -holomorphic curves $f : \Sigma \rightarrow M$ are weakly conformal and Weyl-harmonic.*

Proof. The composition with the section $s : M \rightarrow \mathcal{Z}_{\pm}$ determined by J is a \mathcal{J}_{\pm} -holomorphic curve of \mathcal{Z}_{\pm} . \square

3.2 Adjunction Inequality

In this section we prove theorem 1.3

Proof. Fix a metric $\langle \cdot, \cdot \rangle$ in the conformal class. For a holomorphic normal coordinate z on Σ , split f_z into its holomorphic and antiholomorphic parts

$$\alpha = \frac{1}{2}(f_z - iJf_z) \quad \bar{\beta} = \frac{1}{2}(f_z + iJf_z),$$

we have

$$\langle \alpha, \alpha \rangle = 0 = \langle \beta, \beta \rangle.$$

When f is weakly conformal, this implies that

$$\langle \alpha, \bar{\beta} \rangle = 0.$$

Thus α and β are Hermitian orthogonal and away from their zeros span the holomorphic tangent bundle $f^*T^{(1,0)}M$. The Weyl-harmonic map equation in coordinates is

$$\nabla_{\partial_z}^D f_z = 0.$$

Since ∇^D does not preserve the almost-complex structure, we write the equation using the connection $\nabla^{D,J}$, given by

$$\nabla_X^{D,J} Y = \nabla_X^D Y - \frac{1}{2}J(\nabla_X^D J)Y.$$

This connection preserves the complex structure, and thus preserves the holomorphic and anti-holomorphic tangent spaces. In terms of this connection the Weyl-harmonic map equation is

$$\nabla_{\partial_z}^{D,J} \frac{\partial \phi}{\partial z} = -\frac{1}{2}J(\nabla_{\partial_z}^D J) \frac{\partial \phi}{\partial z}.$$

Since $\nabla_{\partial_z}^D J$ is J anti-linear, it maps from $T^{(1,0)}M$ to $T^{(0,1)}M$ and from $T^{(0,1)}M$ to $T^{(1,0)}M$. The Weyl-harmonic map equation can then be written in terms of α and $\bar{\beta}$ as

$$\nabla_{\partial_z}^{D,J} \alpha = -\frac{i}{2}(\nabla_{\partial_z}^D J)\bar{\beta}, \quad (12)$$

$$\nabla_{\partial_z}^{D,J} \bar{\beta} = \frac{i}{2}(\nabla_{\partial_z}^D J)\alpha. \quad (13)$$

Using proposition 2.4, a weakly conformal Weyl-harmonic map must satisfy

$$\begin{aligned} \langle \nabla_{\partial_z}^{D,J} \alpha, \bar{\alpha} \rangle &= -\frac{i}{2} \langle (\nabla_{\partial_z}^D J)\bar{\beta}, \bar{\alpha} \rangle, & \langle \nabla_{\partial_z}^{D,J} \alpha, \bar{\beta} \rangle &= -\frac{i}{2} \langle (\nabla_{\partial_z}^D J)\bar{\beta}, \bar{\beta} \rangle, \\ &= \frac{i}{4} \langle N(\bar{\beta}, \bar{\alpha}), Jf_{\bar{z}} \rangle, & &= \frac{i}{4} \langle N(\bar{\beta}, \bar{\beta}), Jf_{\bar{z}} \rangle, \\ &= \frac{1}{4} \langle N(\bar{\alpha}, \bar{\beta}), \bar{\alpha} \rangle, & &= 0, \end{aligned}$$

where the last line follows from $\bar{\alpha}, \bar{\beta} \in f^*T^{(0,1)}M$, which implies $N(\bar{\alpha}, \bar{\beta}) \in f^*T^{(1,0)}M$. This implies that away from the zeros of α and β ,

$$\nabla_{\partial_z}^{D,J} \alpha = \frac{\langle N(\bar{\alpha}, \bar{\beta}), \bar{\alpha} \rangle}{4\|\alpha\|^2} \alpha. \quad (14)$$

Similarly

$$\begin{aligned} \langle \nabla_{\partial_z}^{D,J} \bar{\beta}, \beta \rangle &= \frac{i}{2} \langle (\nabla_{\partial_z}^D J)\alpha, \beta \rangle, & \langle \nabla_{\partial_z}^{D,J} \bar{\beta}, \alpha \rangle &= \frac{i}{2} \langle (\nabla_{\partial_z}^D J)\alpha, \alpha \rangle, \\ &= -\frac{i}{4} \langle N(\alpha, \beta), Jf_{\bar{z}} \rangle, & &= -\frac{i}{4} \langle N(\alpha, \alpha), Jf_{\bar{z}} \rangle, \\ &= \frac{1}{4} \langle N(\alpha, \beta), \beta \rangle, & &= 0. \end{aligned}$$

This gives

$$\nabla_{\partial_z}^{D,J} \bar{\beta} = \frac{\langle N(\alpha, \beta), \beta \rangle}{4\|\beta\|^2} \bar{\beta}. \quad (15)$$

By the Koszul-Malgrange theorem [9], there are holomorphic structures on $f^*T^{(1,0)}M$ and $f^*T^{(0,1)}M$ so that

$$\bar{\partial}X = \nabla_{\frac{D}{\partial \bar{z}}}^J X \otimes d\bar{z}.$$

Then for a Weyl-harmonic map

$$\bar{\partial}\alpha = \frac{\langle N(\bar{\alpha}, \bar{\beta}), \bar{\alpha} \rangle}{4\|\alpha\|^2} \alpha \otimes d\bar{z}, \quad (16)$$

$$\bar{\partial}\bar{\beta} = \frac{\langle N(\alpha, \beta), \beta \rangle}{4\|\beta\|^2} \bar{\beta} \otimes d\bar{z}. \quad (17)$$

The Bers-Vekua similarity principle (see [6]) implies that near any point $p \in \Sigma$ we have

$$\alpha = \gamma_p e^{\sigma_p}, \quad \bar{\beta} = \delta_p e^{\tau_p},$$

for some local holomorphic sections γ_p of $f^*T^{(1,0)}M$, δ_p of $f^*T^{(0,1)}M$, and some bounded functions σ_p, τ_p . This can be used to define the indices

$$R = \sum_{f_z(p)=0} \text{ord}_p(f_z) \geq 0, \quad (18)$$

$$Q = \sum_{\alpha(p)=0} \text{ord}_p(\gamma_p) - R \geq 0, \quad (19)$$

$$P = \sum_{\bar{\beta}(p)=0} \text{ord}_p(\delta_p) - R \geq 0. \quad (20)$$

These are the total ramification index R , the number of anti-complex points Q , and the number of complex points P . Following [5], these determine the degrees of the line bundles spanned by the vector valued one forms $f_z dz$, αdz and $\bar{\beta} dz$ respectively. If $[f_z]$, $[\alpha]$, and $[\beta]$ are line bundles generated by the locally defined sections then we have

$$R = -\chi(\Sigma) + c_1([f_z]),$$

$$Q + R = -\chi(\Sigma) + c_1([\alpha]),$$

$$P + R = -\chi(\Sigma) - c_1([\beta]).$$

We also have $f^*T^{(1,0)}M = [\alpha] \oplus [\beta]$, and since α and $\bar{\beta}$ span a negatively oriented, maximal isotropic subspace of $f^*TM \otimes \mathbb{C}$ which contains f_z , it must be that $f^*T^{(1,0)}M = [\alpha] \oplus [\bar{\beta}]$. Therefore we have

$$c_1(f^*T^{(1,0)}M) = Q - P,$$

$$c_1(f^*T^{(1,0)}M) = Q + P + 2R + 2\chi(\Sigma),$$

$$= Q + P + 2c_1([f_z]),$$

$$= Q + P + 2\chi(T_f\Sigma).$$

Since $c_1(f^*T^{(1,0)}M) = \chi(T_f\Sigma) - \chi(N_f\Sigma)$ we now have the Webster's formulas

$$c_1(f^*T^{(1,0)}M) = Q - P, \quad (21)$$

$$\chi(T_f\Sigma) + \chi(N_f\Sigma) = -P - Q. \quad (22)$$

□

Since P and Q are both non-negative, this implies the adjunction inequality (3) of corollary 1.2

$$\chi(T_f\Sigma) + \chi(N_f\Sigma) + c_1(f^*T^{(1,0)}M) = -2P \leq 0, \quad (23)$$

$$\chi(T_f\Sigma) + \chi(N_f\Sigma) - c_1(f^*T^{(1,0)}M) = -2Q \leq 0. \quad (24)$$

4 Examples

4.1 Hopf Surfaces

The primary Hopf surface $M = S^1 \times S^3$ is fibered over S^2 with fiber T^2 . The bundle projection is just the projection to the S^3 component followed by the Hopf map. There is a Hermitian structure on M induced by the standard Hermitian structures on the base and fiber. The Lee form is just $d\phi$, where ϕ is the angle along S^1 . Every fiber is J -holomorphic and is therefore Weyl-minimal. It is also minimal as θ^\sharp is tangent to the fiber.

In addition, there is a Lagrangian Weyl-minimal surface for every great circle γ in the base S^2 . To see this consider the Clifford torus in S^3 which maps to γ under the Hopf map. This torus contains two great circles on S^3 , one tangent to the fiber and one perpendicular to the fiber. The great circle perpendicular to the fiber times the product S^1 gives a Lagrangian totally geodesic T^2 to which θ^\sharp is tangent, and is therefore Weyl-minimal.

Since $\theta = d\phi$ is closed we can look at the universal cover $\tilde{M} = \mathbb{R} \times S^3$. Using ϕ as the coordinate on \mathbb{R} the metric is just

$$g_{\tilde{M}} = d\phi^2 + g_{S^3}.$$

Therefore the Weyl-minimal surfaces will lift to minimal surfaces of the conformal metric

$$e^{2\phi} g_{\tilde{M}} = e^{2\phi} d\phi^2 + e^{2\phi} g_{S^3} = (de^\phi)^2 + e^{2\phi} g_{S^3}.$$

Using the new coordinate $r = e^\phi$ this is just the (incomplete) flat metric on $\mathbb{R}^4 \setminus 0 \cong \tilde{M}$.

$$dr^2 + r^2 g_{S^3}.$$

Any surface lifted from M will be invariant under deck transformation $\phi \mapsto \phi + 2\pi$ or $r \mapsto e^{2\pi} r$. The Weyl-minimal surfaces described above correspond to the planes through the origin in \mathbb{R}^4 .

4.2 $U(1) \times U(1)$ Principal Bundles over a Riemann Surface

Let $p : M \rightarrow \Sigma$ be a $U(1) \times U(1)$ principal bundle over a Riemann surface Σ with volume form ω_Σ . If $i\beta$ is a connection form then β is an \mathbb{R}^2 -valued form with components β_1 and β_2 . If $\tilde{\omega}_\Sigma = p^* \omega_\Sigma$ then $d\beta = F\tilde{\omega}_\Sigma$ where $F = (F_1, F_2) : \Sigma \rightarrow \mathbb{R}^2$. The Kähler form $\omega = \beta_1 \wedge \beta_2 + \tilde{\omega}_\Sigma$ has exterior derivative

$$d\omega = (F_1\beta_2 - F_2\beta_1) \wedge \tilde{\omega}_\Sigma = (F_1\beta_2 - F_2\beta_1) \wedge \omega.$$

Therefore the Lee form is $\theta = F_1\beta_2 - F_2\beta_1$. For a constant curvature connection, this will be closed. The Hopf surface is a special case for this example where $\Sigma = S^2$ with the round metric, and M has associated bundle $M \times \mathbb{C}^2 / U(1) \times U(1) = \mathbb{C} \oplus \mathbf{K}$. As in that case, the fiber is always a J -holomorphic curve and therefore Weyl-minimal. If a closed geodesic on $\gamma : S^1 \rightarrow \Sigma$ has a closed horizontal lift $\tilde{\gamma}$ and the connection has constant curvature then $\tilde{\gamma}(s) \cdot (e^{-iF_2 t}, e^{iF_1 t})$ parametrizes a Lagrangian minimal torus on M to which θ^\sharp is tangent, and thus Weyl-minimal.

References

- [1] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional riemannian geometry. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 362(1711):425–461, 1978.
- [2] David M. J. Calderbank and Henrik Pedersen. Einstein-Weyl geometry. In *Surveys in differential geometry: essays on Einstein manifolds*, volume 6 of *Surv. Differ. Geom.*, pages 387–423. Int. Press, Boston, MA, 1999.
- [3] J. Chen and G. Tian. Minimal surfaces in Riemannian 4-manifolds. *Geometric & Functional Analysis GAFA*, 7(5):873–916, Oct 1997.
- [4] Tedi Draghici, Tian-Jun Li, and Weiyi Zhang. On the J -anti-invariant cohomology of almost complex 4-manifolds. *The Quarterly Journal of Mathematics*, 64(1):83–111, 12 2011.

- [5] James Eells and Simon Salamon. Twistorial construction of harmonic maps of surfaces into four-manifolds. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 4, 12(4):589–640, 1985.
- [6] Jürgen Jost. *Two-dimensional geometric variational problems*. Pure and Applied Mathematics (New York). John Wiley & Sons, Ltd., Chichester, 1991. A Wiley-Interscience Publication.
- [7] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. II*. Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II. Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.
- [8] Gerasim Kokarev. On pseudo-harmonic maps in conformal geometry. *Proc. Lond. Math. Soc. (3)*, 99(1):168–194, 2009.
- [9] J. L. Koszul and B. Malgrange. Sur certaines structures fibrées complexes. *Archiv der Mathematik*, 9(1):102–109, Apr 1958.
- [10] Renyi Ma. Complex points of minimal surfaces in almost Kähler manifolds. *manuscripta mathematica*, 95(2):159–168, Feb 1998.
- [11] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, second edition, 2012.
- [12] Mario J. Micallef and Brian White. The structure of branch points in minimal surfaces and in pseudoholomorphic curves. *Annals of Mathematics*, 141(1):35–85, 1995.
- [13] Henrik Pedersen, Yat Sun Poon, and Andrew Swann. Einstein-Weyl deformations and submanifolds. *INT. J. MATH*, 7:705–719, 1995.
- [14] Marina Ville. On the normal bundle of minimal surfaces in almost Kähler 4-manifolds. In *Harmonic morphisms, harmonic maps, and related topics (Brest, 1997)*, volume 413 of *Chapman & Hall/CRC Res. Notes Math.*, pages 159–173. Chapman & Hall/CRC, Boca Raton, FL, 2000.
- [15] S. M. Webster. Minimal surfaces in a Kähler surface. *J. Differential Geom.*, 20(2):463–470, 1984.