Complex Manifolds 2020; 7:230–240 DE GRUYTER

Research Article Open Access

Indranil Biswas*, Tomás L. Gómez, and André Oliveira

Complex Lagrangians in a hyperKähler manifold and the relative Albanese

https://doi.org/10.1515/coma-2020-0106
Received June 29, 2020; accepted October 18, 2020

Abstract: Let M be the moduli space of complex Lagrangian submanifolds of a hyperKähler manifold X, and let $\varpi : \widehat{\mathcal{A}} \longrightarrow M$ be the relative Albanese over M. We prove that $\widehat{\mathcal{A}}$ has a natural holomorphic symplectic structure. The projection ϖ defines a completely integrable structure on the symplectic manifold $\widehat{\mathcal{A}}$. In particular, the fibers of ϖ are complex Lagrangians with respect to the symplectic form on $\widehat{\mathcal{A}}$. We also prove analogous results for the relative Picard over M.

Keywords: HyperKähler manifold, complex Lagrangian, integrable system, Liouville form, Albanese

MSC: 14J42, 53D12, 37K10, 14D21

1 Introduction

A compact Kähler manifold admits a holomorphic symplectic form if and only if it admits a hyperKähler structure [1], [10]. To explain this, let X be a compact manifold equipped with almost complex structures J_1, J_2, J_3, and let g be a Riemannian metric on X, such that (X, J_1, J_2, J_3, g) is a hyperKähler manifold. Then g defines a C^∞ isomorphism,

$$T^{0,1}X \xrightarrow{g_1} (T^{1,0}X)^*,$$

where $T^{\mathbb{R}}X \otimes \mathbb{C} = T^{1,0}X \oplus T^{0,1}X$ is the type decomposition with respect to the almost complex structure J_1; also J_2 produces a C^∞ isomorphism

$$T^{1,0}X \xrightarrow{J_2'} T^{0,1}X.$$

The composition of homomorphisms

$$T^{1,0}X \xrightarrow{J_2'} T^{0,1}X \xrightarrow{g_1} (T^{1,0}X)^*,$$

which is a section of $(T^{1,0}X)^* \otimes (T^{1,0}X)^*$, is actually is a holomorphic symplectic form on the compact Kähler manifold (X, J_1, g). The compact Kähler manifold (X, J_1, g) is Ricci–flat. Conversely, if a compact Kähler manifold admits a holomorphic symplectic form, then its canonical line bundle is holomorphically trivial and hence it admits a Ricci–flat Kähler metric [10]. Let (X, J_1, g) be a Ricci–flat compact Kähler manifold equipped with a holomorphic symplectic form. Then we may recover J_2 by reversing the above construction. Finally, we have $J_3 = J_1 \circ J_2$.

***Corresponding Author: Indranil Biswas:** School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India, E-mail: indranil@math.tifr.res.in
Tomás L. Gómez: Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Nicolás Cabrera 15, Campus Cantoblanco UAM, 28049 Madrid, Spain, E-mail: tomas.gomez@icmat.es
André Oliveira: Centro de Matemática da Universidade do Porto (CMUP), Faculdade de Ciências, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
On leave from: Departamento de Matemática, Universidade de Trás-os-Montes e Alto Douro, UTAD, Quinta dos Prados, 5000-911 Vila Real, Portugal, E-mail: andre.oliveira@fc.up.pt / agoliv@utad.pt

Let X be a compact Kähler Ricci–flat manifold admitting a holomorphic symplectic form. Fix a Ricci–flat Kähler form ω on X (such a Kähler form exists [10]), and take a holomorphic symplectic form Φ on X. It is known that Φ is parallel (meaning, covariant constant) with respect to the Levi–Civita connection on X associated to ω [1, p. 760, “Principe de Bochner”], [9, p. 142].

Let M denote the moduli space of compact complex submanifolds of X that are Lagrangian with respect to the symplectic form Φ. Consider the corresponding universal family of Lagrangians

$$\mathcal{Z} \longrightarrow M\,. \tag{1.1}$$

Let

$$\varpi : \widehat{\mathcal{A}} \longrightarrow M$$

be the relative Albanese over M. So for any Lagrangian $L \in M$, the fiber of $\widehat{\mathcal{A}}$ over L is $\mathrm{Alb}(L) = H^0(L,\, \Omega^1_L)^*/H_1(L,\, \mathbb{Z})$. This φ is a holomorphic family of compact complex tori over M.

We prove the following (see Theorem 3.3):

The complex manifold $\widehat{\mathcal{A}}$ has a natural holomorphic symplectic form.

The symplectic form on $\widehat{\mathcal{A}}$ is constructed using the canonical Liouville symplectic form on the holomorphic cotangent bundle T^*M of M. The symplectic form Φ on X is implicitly used in the construction of the symplectic form on $\widehat{\mathcal{A}}$. Recall that the Lagrangian submanifolds, and hence M, are defined using Φ.

We prove the following (see Lemma 3.4):

The projection $\varpi : \widehat{\mathcal{A}} \longrightarrow M$ defines a completely integrable structure on the symplectic manifold $\widehat{\mathcal{A}}$. In particular, the fibers of ϖ are Lagrangians with respect to the symplectic form on $\widehat{\mathcal{A}}$.

In Section 4 we consider the relative Picard bundle over M for the family $\mathcal{Z}$ in (1.1). Let

$$\varpi_0 : \mathcal{A} \longrightarrow M$$

be the relative Picard bundle for the family $\mathcal{Z}$.

We prove that $\mathcal{A}$ is equipped with a natural holomorphic symplectic structure; see Proposition 4.3.

Let $\Theta_{\mathcal{A}}$ denote the above mentioned holomorphic symplectic structure on $\mathcal{A}$. The following lemma is proved (see Lemma 4.4):

The projection $\varpi_0 : \mathcal{A} \longrightarrow M$ defines a completely integrable structure on $\mathcal{A}$ for the symplectic form $\Theta_{\mathcal{A}}$.

These results are natural generalizations of some known cases of integrable systems, such as the Hitchin system [4] or the Mukai system [7] (see also [2]). One of our motivations has been mirror symmetry; we hope to come back to the study of $\mathcal{A}$ and $\widehat{\mathcal{A}}$ from the point of view of hyperKähler geometry.

2 Cotangent bundle of family of Lagrangians

Let X be a compact Kähler Ricci–flat manifold of complex dimension $2d$ equipped with a Kähler form ω. Let Φ be a holomorphic symplectic form on X which is parallel with respect to the Levi–Civita connection on X given by the Kähler metric on X associated to ω.

A complex Lagrangian submanifold of X is a compact complex submanifold $L \subset X$ of complex dimension d such that $\iota^*\Phi = 0$, where

$$\iota : L \hookrightarrow X \tag{2.1}$$

is the inclusion map.

It is known that the infinitesimal deformations of a complex Lagrangian submanifold of X are unobstructed [6], [8], [5]. Furthermore, the moduli space of complex Lagrangian submanifolds of X is a special Kähler manifold [5, p. 84, Theorem 3].

Let M be the moduli space of complex Lagrangian submanifolds of X. Let

$$L \subset X$$

be a complex Lagrangian submanifold. The point of M representing L will also be denoted by L. Let $N_L \longrightarrow L$ be the normal bundle of $L \subset X$; it is a quotient bundle of $\iota^* TX$ of rank d, where ι is the map in (2.1). The infinitesimal deformations of the complex submanifold L are parametrized by $H^0(L, N_L)$. Since L is complex Lagrangian, the holomorphic symplectic form Φ on X produces a holomorphic isomorphism

$$N_L \xrightarrow{\sim} (TL)^* = \Omega^1_L ,$$

where TL (respectively, Ω^1_L) is the holomorphic tangent (respectively, cotangent) bundle of L. Using this isomorphism we have

$$H^0(L, N_L) = H^0(L, \Omega^1_L) .$$

Among the infinitesimal deformations of the complex submanifold L, there are those which arise from deformations within the category of complex Lagrangian submanifolds, meaning those arise from deformations of complex Lagrangian submanifolds as Lagrangian submanifolds. An infinitesimal deformation

$$\alpha \in H^0(L, N_L) = H^0(L, \Omega^1_L)$$

of L lies in this subclass if and only if the holomorphic 1–form α on L is closed [5, pp. 78–79]. But any holomorphic 1–form on L is closed because L is Kähler. Therefore, M is in fact an open subset of the corresponding Douady space for X.

Consider the Ricci–flat Kähler form ω on X. The form

$$\omega_L := \iota^* \omega \tag{2.2}$$

on L is also Kähler, where ι is the map in (2.1). Therefore, the pairing

$$\phi_L : H^0(L, \Omega^1_L) \otimes H^1(L, \mathcal{O}_L) \longrightarrow \mathbb{C}, \quad w \otimes c \longmapsto \int_L w \wedge c \wedge \omega_L^{d-1} \tag{2.3}$$

is nondegenerate. We shall identify $H^1(L, \mathcal{O}_L)$ with $H^0(L, \Omega^1_L)^*$ using this nondegenerate pairing.

It was noted above that

$$T_L M = H^0(L, N_L) = H^0(L, \Omega^1_L) . \tag{2.4}$$

Using the pairing in (2.3) and (2.4), we have

$$T^*_L M = H^0(L, \Omega^1_L)^* = H^1(L, \mathcal{O}_L) . \tag{2.5}$$

Let

$$\mathcal{Z} \subset X \times M \tag{2.6}$$

be the universal family of complex Lagrangians over M. So $\mathcal{Z}$ is the locus of all $(x, L') \in X \times M$ such that $x \in L' \subset X$. Consider the natural projections $p_X : X \times M \longrightarrow X$ and $p_M : X \times M \longrightarrow M$. Let

$$p : \mathcal{Z} \longrightarrow X \quad \text{and} \quad q : \mathcal{Z} \longrightarrow M \tag{2.7}$$

be the restrictions of p_X and p_M respectively to the submanifold $\mathcal{Z} \subset X \times M$. So $p(q^{-1}(L')) \subset X$ for every $L' \in M$ is the Lagrangian L' itself.

Let $\Omega^1_{\mathcal{Z}/M} \longrightarrow \mathcal{Z}$ be the relative cotangent bundle for the projection q to M in (2.7). It fits in the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow q^* \Omega^1_M \longrightarrow \Omega^1_{\mathcal{Z}} \longrightarrow \Omega^1_{\mathcal{Z}/M} \longrightarrow 0$$

over M. The direct image $R^1 q_* \mathcal{O}_{\mathcal{Z}}$ fits in the short exact sequence of sheaves on M

$$0 \longrightarrow R^0 q_* \Omega^1_{\mathcal{Z}/M} \longrightarrow R^1 q_* \underline{\mathbb{C}} \longrightarrow R^1 q_* \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 , \tag{2.8}$$

where $\underline{\mathbb{C}}$ is the constant sheaf on $\mathcal{Z}$ with stalk $\mathbb{C}$. The direct image $R^1q_\star\underline{\mathbb{C}}$ is a flat complex vector bundle, equipped with the Gauss–Manin connection. We briefly recall the construction of the Gauss–Manin connection. For any point $L' = y \in M$, let $U_y \subset M$ be a contractible open neighborhood of y. Since U_y is contractible, the inverse image $q^{-1}(U_y)$ is diffeomorphic to $U_y \times L'$ such that the diffeomorphism between $q^{-1}(U_y)$ and $U_y \times L'$ takes q to the natural projection from $U_y \times L'$ to U_y. Using this diffeomorphism, the restriction of $R^1q_\star\underline{\mathbb{C}}$ to U_y coincides with the trivial vector bundle

$$U_y \times H^1(L', \mathbb{C}) \longrightarrow U_y \tag{2.9}$$

with fiber $H^1(L', \mathbb{C})$. Using this isomorphism between $(R^1q_\star\underline{\mathbb{C}})|_{U_y}$ and the trivial vector bundle $U_y \times H^1(L', \mathbb{C})$ in (2.9), the trivial connection on the trivial vector bundle in (2.9) produces a flat connection on $(R^1q_\star\underline{\mathbb{C}})|_{U_y}$. This connection on $(R^1q_\star\underline{\mathbb{C}})|_{U_y}$ does not depend on the choice of the diffeomorphism between $q^{-1}(U_y)$ and $U_y \times L'$. Consequently, these locally defined flat connections on $R^1q_\star\underline{\mathbb{C}}$ patch together compatibly to define a flat connection on $R^1q_\star\underline{\mathbb{C}}$. This flat connection is the Gauss–Manin connection mentioned above.

Since the Gauss–Manin connection on $R^1q_\star\underline{\mathbb{C}}$ is flat, and M is a complex manifold, the Gauss–Manin connection produces a natural holomorphic structure on the C^∞ vector bundle $R^1q_\star\underline{\mathbb{C}}$. The direct image $R^0q_\star\Omega^1_{\mathcal{Z}/M}$ is a holomorphic subbundle of $R^1q_\star\underline{\mathbb{C}}$, but it is not preserved by the flat connection in general. The holomorphic structure on the quotient $R^1q_\star\mathcal{O}_{\mathcal{Z}}$ induced by that of $R^1q_\star\underline{\mathbb{C}}$ coincides with its own holomorphic structure; the fiber of $R^1q_\star\mathcal{O}_{\mathcal{Z}}$ over any $L' \in M$ is $H^1(L', \mathcal{O}_{L'})$.

Since ω_L in (2.2) is the restriction of a global Kähler form on X, the section of $R^2q_\star\underline{\mathbb{C}}$

$$M \longrightarrow R^2q_\star\underline{\mathbb{C}}\,, \;\; L' \longmapsto [\omega_{L'}] = [\omega|_{L'}] \in H^2(L', \mathbb{C})$$

is covariant constant with respect to the Gauss–Manin connection on $R^2q_\star\underline{\mathbb{C}}$. Consequently, the homomorphism

$$(R^1q_\star\underline{\mathbb{C}}) \otimes (R^1q_\star\underline{\mathbb{C}}) \longrightarrow \underline{\mathbb{C}}\,, \tag{2.10}$$

that sends any $v \otimes w \in (R^1q_\star\underline{\mathbb{C}})_t \otimes (R^1q_\star\underline{\mathbb{C}})_t$, $t \in M$, to

$$\int_{q^{-1}(t)} v \wedge w \wedge (\omega|_{q^{-1}(t)})^{d-1} \in \mathbb{C}$$

is also covariant constant with respect to the connection on $(R^1q_\star\underline{\mathbb{C}}) \otimes (R^1q_\star\underline{\mathbb{C}})$ induced by the Gauss–Manin connection on $R^1q_\star\underline{\mathbb{C}}$. Hence the pairing in (2.10) produces a holomorphic isomorphism of vector bundles

$$(q_\star\Omega^1_{\mathcal{Z}/M})^\star \xrightarrow{\sim} R^1q_\star\mathcal{O}_{\mathcal{Z}} \tag{2.11}$$

on M. The restriction of this isomorphism to any point $L \in M$ coincides with the isomorphism $H^0(L, \Omega^1_L)^\star = H^1(L, \mathcal{O}_L)$ in (2.5).

On the other hand, the pointwise isomorphisms in (2.4) combine together to produce a holomorphic isomorphism of vector bundles

$$\Omega^1_M \xrightarrow{\sim} (q_\star\Omega^1_{\mathcal{Z}/M})^\star \tag{2.12}$$

on M. Composing the isomorphisms in (2.11) and (2.12), we obtain a holomorphic isomorphism of vector bundles

$$\chi : \Omega^1_M \longrightarrow R^1q_\star\mathcal{O}_{\mathcal{Z}} \tag{2.13}$$

over M.

For notational convenience, the total space of $R^1q_\star\mathcal{O}_{\mathcal{Z}}$ will be denoted by $\mathcal{Y}$. Let

$$\gamma : \mathcal{Y} \longrightarrow M \tag{2.14}$$

be the natural projection. Consider the canonical Liouville 1-form on Ω^1_M. Using the isomorphism χ in (2.13) this Liouville 1-form on Ω^1_M gives a holomorphic 1-form on $\mathcal{Y}$. Let

$$\theta \in H^0(\mathcal{Y}, \Omega^1_{\mathcal{Y}}) \tag{2.15}$$

be the holomorphic 1-form given by the Liouville 1-form on Ω^1_M. Note that for any $L \in M$, and any $v \in \gamma^{-1}(L)$, the form

$$\theta(v) : T_v\mathcal{Y} \longrightarrow \mathbb{C}$$

coincides with the composition of homomorphisms

$$T_v\mathcal{Y} \xrightarrow{d\gamma} T_LM = H^0(L, \Omega^1_L) \xrightarrow{\phi_L(-,v)} \mathbb{C},$$

where ϕ_L is the bilinear pairing constructed in (2.3), and $d\gamma$ is the differential of the projection γ in (2.14); the above identification

$$T_LM = H^0(L, \Omega^1_L)$$

is the one constructed in (2.4).

It is straight-forward to check that the 2-form

$$d\theta \in H^0(\mathcal{Y}, \Omega^2_{\mathcal{Y}}) \tag{2.16}$$

is a holomorphic symplectic form on the manifold $\mathcal{Y}$ in (2.14). Indeed, $d\theta$ evidently coincides with the 2-form on $\mathcal{Y}$ given by the Liouville symplectic form on Ω^1_M via the isomorphism χ in (2.13).

3 The family of Albanese tori

Take a compact complex Lagrangian submanifold $L \subset X$ represented by a point of M. We know that $T^*_LM \cong H^0(L, \Omega^1_L)^*$ (see (2.4)). Note that the non-degenerate pairing

$$H^{1,0}(L) \otimes H^{d-1,d}(L) \longrightarrow \mathbb{C}, \quad \alpha \otimes \beta \longmapsto \int_L \alpha \wedge \beta,$$

which is also the Serre duality pairing, yields an isomorphism

$$H^0(L, \Omega^1_L)^* \cong H^d(L, \Omega^{d-1}_L).$$

For a fixed L, we have the Hodge decomposition

$$H^{2d-1}(L, \mathbb{C}) = H^{d,d-1}(L) \oplus H^{d-1,d}(L);$$

but if we move L in the family M, meaning if we consider the universal family

$$q : \mathcal{Z} \longrightarrow M$$

in (2.6), then only $R^{d-1}q_*\Omega^d_{\mathcal{Z}/M}$ is a holomorphic subbundle of $R^{2d-1}q_*\underline{\mathbb{C}}$, and we have the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow R^{d-1}q_*\Omega^d_{\mathcal{Z}/M} \longrightarrow R^{2d-1}q_*\underline{\mathbb{C}} \longrightarrow R^dq_*\Omega^{d-1}_{\mathcal{Z}/M} \longrightarrow 0, \tag{3.1}$$

on M.

The holomorphic vector bundle $R^{2d-1}q_*\underline{\mathbb{C}}$ is equipped with the Gauss–Manin connection, which is an integrable connection. The quotient $R^dq_*\Omega^{d-1}_{\mathcal{Z}/M}$, in (3.1), of $R^{2d-1}q_*\underline{\mathbb{C}}$ is a holomorphic vector bundle on M with fiber $H^d(L', \Omega^{d-1}_{L'})$ over any $L' \in M$.

The homomorphism

$$(R^1q_*\underline{\mathbb{C}}) \otimes (R^{2d-1}q_*\underline{\mathbb{C}}) \longrightarrow \underline{\mathbb{C}}, \tag{3.2}$$

that sends any $\alpha \otimes \beta \in (R^1q_*\underline{\mathbb{C}})_t \otimes (R^{2d-1}q_*\underline{\mathbb{C}})_t$, $t \in M$, to

$$\int_{q^{-1}(t)} v \wedge w \in \mathbb{C}$$

is covariant constant with respect to the connection on $(R^1 q_\star \underline{\mathbb{C}}) \otimes (R^{2d-1} q_\star \underline{\mathbb{C}})$ induced by the Gauss–Manin connections on $R^1 q_\star \underline{\mathbb{C}}$ and $R^{2d-1} q_\star \underline{\mathbb{C}}$. Consequently, the pairing in (3.2) yields a holomorphic isomorphism of vector bundles

$$(q_\star \Omega^1_{\mathcal{Z}/M})^\star \xrightarrow{\sim} R^d q_\star \Omega^{d-1}_{\mathcal{Z}/M}$$

over M. Combining this isomorphism with the isomorphism in (2.12) we get a holomorphic isomorphism of vector bundles

$$\widehat{\chi} : \Omega^1_M \xrightarrow{\sim} R^d q_\star \Omega^{d-1}_{\mathcal{Z}/M} \tag{3.3}$$

over M.

We shall denote the total space of the holomorphic vector bundle $R^d q_\star \Omega^{d-1}_{\mathcal{Z}/M}$ by $\mathcal{W}$, so

$$\mathcal{W} := R^d q_\star \Omega^{d-1}_{\mathcal{Z}/M} \xrightarrow{\widehat{\gamma}} M \tag{3.4}$$

is a holomorphic fiber bundle.

As in Section 2, consider the canonical Liouville holomorphic 1-form on the total space of Ω^1_M. Using the isomorphism in (3.3), this Liouville 1-form on the total space of Ω^1_M produces a holomorphic 1-form on $\mathcal{W}$ in (3.4). Let

$$\theta' \in H^0(\mathcal{W}, \Omega^1_{\mathcal{W}})$$

be this holomorphic 1-form on $\mathcal{W}$. We note that

$$d\theta' \in H^0(\mathcal{W}, \Omega^2_{\mathcal{W}}) \tag{3.5}$$

is a holomorphic symplectic form on $\mathcal{W}$. Indeed, the isomorphism in (3.3) takes $d\theta'$ to the Liouville symplectic form on the total space of Ω^1_M.

Remark 3.1. Note that while the Kähler form ω on X was used in the construction of the symplectic form $d\theta$ on $\mathcal{Y}$ in (2.16) (see the pairing in (2.10)), the construction of $d\theta'$ in (3.5) does not use the Kähler form ω on X. We recall that the isomorphism in (2.12) is constructed from the the pointwise isomorphisms in (2.4). Note that the isomorphism in (2.4) does not depend on the Kähler form ω.

The Albanese $\mathrm{Alb}(Y)$ of a compact Kähler manifold Y is defined to be

$$\mathrm{Alb}(Y) = H^0(Y, \Omega^1_Y)^\star / H_1(Y, \mathbb{Z}) = H^n(Y, \Omega^{n-1}_Y)/H_1(Y, \mathbb{Z}),$$

where $n = \dim_{\mathbb{C}} Y$ (see [3, p. 331]). It is a compact complex torus.

For each point $L \in M$, consider the composition of homomorphisms

$$H^{2d-1}(L, \mathbb{Z}) \longrightarrow H^{2d-1}(L, \mathbb{C}) \longrightarrow H^{2d-1}(L, \mathbb{C})/H^{d-1}(L, \Omega^d_L) = H^d(L, \Omega^{d-1}_L)$$

(see (3.1)). It produces a homomorphism

$$R^{2d-1} q_\star \underline{\mathbb{Z}} \longrightarrow \mathcal{W}, \tag{3.6}$$

where $\mathcal{W}$ is defined in (3.4).

Remark 3.2. The Gauss–Manin connection on $R^{2d-1} q_\star \underline{\mathbb{C}} \longrightarrow M$ evidently preserves the subbundle of lattices

$$R^{2d-1} q_\star \underline{\mathbb{Z}} \subset R^{2d-1} q_\star \underline{\mathbb{C}}.$$

From this it follows immediately that the C^∞ submanifold $R^{2d-1} q_\star \underline{\mathbb{Z}} \subset \mathcal{W}$ in (3.6) is in fact a complex submanifold.

The quotient

$$\widehat{\mathcal{A}} := \mathcal{W}/(R^{2d-1} q_\star \underline{\mathbb{Z}}) \longrightarrow M \tag{3.7}$$

for the homomorphism in (3.6) is in fact a holomorphic family of compact complex tori over M. Note that the fiber of $\widehat{\mathcal{A}}$ over each $L \in M$ is the Albanese torus

$$\mathrm{Alb}(L) = H^d(L, \Omega_L^{d-1})/H^{2d-1}(L, \mathbb{Z}).$$

For any complex Lagrangian $L \in M$, using Serre duality,

$$H^d(L, \Omega_L^{d-1}) = H^0(L, \Omega_L^1)^\star,$$

and the underlying real vector space for $H^0(L, \Omega_L^1)^\star$ is identified with

$$H^1(L, \mathbb{R})^\star = H_1(L, \mathbb{R}).$$

Using Poincaré duality for L, we have $H^{2d-1}(L, \mathbb{Z})/\mathrm{Torsion} = H_1(L, \mathbb{Z})/\mathrm{Torsion}$. Consequently, $\widehat{\mathcal{A}}$ in (3.7) admits the following isomorphism:

$$\widehat{\mathcal{A}} = \mathcal{W}/(R^{2d-1}q_\star\underline{\mathbb{Z}}) = \mathcal{W}/\widetilde{H}_1(\mathbb{Z}) = (q_\star\Omega^1_{\mathcal{Z}/M})^\star/\widetilde{H}_1(\mathbb{Z}), \tag{3.8}$$

where $\widetilde{H}_1(\mathbb{Z})$ is the local system on M whose stalk over any $L \in M$ is $H_1(L, \mathbb{Z})/\mathrm{Torsion}$. Also we have the isomorphism of real tori

$$\widehat{\mathcal{A}} = (R^1 q_\star\underline{\mathbb{R}})^\star/\widetilde{H}_1(\mathbb{Z}) = \widetilde{H}_1(\mathbb{R})/\widetilde{H}_1(\mathbb{Z}), \tag{3.9}$$

where $\widetilde{H}_1(\mathbb{R})$ is the local system on M whose stalk over any $L \in M$ is $H_1(L, \mathbb{R})$.

Theorem 3.3. *The 2-form $d\theta'$ in (3.5) on $\mathcal{W}$ descends to the quotient torus $\widehat{\mathcal{A}}$ in (3.7).*

Proof. Take a point $L_0 \in M$. In [5] Hitchin constructed a C^∞ coordinate function on M defined around the point $L_0 \in M$ that takes values in $H^1(L_0, \mathbb{R})$ [5, p. 79, Theorem 2]; we will briefly recall this construction.

Take any

$$c \in H_1(L_0, \mathbb{Z})/\mathrm{Torsion}. \tag{3.10}$$

Let $U \subset M$ be a contractible neighborhood of the point L_0. Choose a S^1-subbundle of the fiber bundle $\mathcal{Z}$ (see (2.6)) over U

$$B \xhookrightarrow{\iota_U} \mathcal{Z}|_U \xrightarrow{q} U \tag{3.11}$$

such that the fiber of the S^1-bundle B over L_0 represents the homology class c in (3.10); recall that the fiber of $\mathcal{Z}$ over the point $L_0 \in M$ is the Lagrangian L_0 itself.

Consider the symplectic form Φ on X. Integrating $\iota_U^\star p^\star\mathrm{Re}(\Phi)$ along the fibers of B, where ι_U and p are the maps in (3.11) and (2.7) respectively, we get a closed 1-form ξ_c on U. Let f_c be the unique function on U such that $f_c(L_0) = 0$ and $df_c = \xi_c$. Now, let

$$\mu : U \longrightarrow H^1(L_0, \mathbb{R})$$

be the function uniquely determined by the condition that $\phi_{L_x}(c \otimes \mu(x)) = f_c(x)$ for all $x \in U$ and $c \in H_1(L_0, \mathbb{Z})/\mathrm{Torsion}$, where ϕ_{L_x} is the pairing constructed as in (2.3) for the complex Lagrangian $L_x = q^{-1}(x) \subset X$, where q is the projection in (2.7). This μ is a local diffeomorphism [5, p. 79, Theorem 2].

Using the Kähler form $\omega_{L_0} := \omega|_{L_0}$ on L_0 (see (2.2)), we identify $H^1(L_0, \mathbb{R})$ with $H_1(L_0, \mathbb{R})$ as follows. Since the pairing

$$H^1(L_0, \mathbb{R}) \otimes H^1(L_0, \mathbb{R}) \longrightarrow \mathbb{R}, \quad v \otimes w \longmapsto \int_L v \wedge w \wedge \omega_{L_0}^{d-1}$$

is nondegenerate, it produces an isomorphism

$$H^1(L_0, \mathbb{R}) \xrightarrow{\sim} H^1(L_0, \mathbb{R})^\star = H_1(L_0, \mathbb{R}). \tag{3.12}$$

On the other hand, there is the natural homomorphism $H^1(L_0, \mathbb{Z}) \longrightarrow H^1(L_0, \mathbb{R})$. Let

$$\Gamma \subset H_1(L_0, \mathbb{R}) \tag{3.13}$$

be the subgroup that corresponds to $H^1(L_0, \mathbb{Z})$ by the isomorphism in (3.12).

Using the above coordinate function μ on U, we have

$$\mathcal{T}^\star U \xrightarrow{\sim} \mathcal{T}^\star \mu(U) = \mu(U) \times H^1(L_0, \mathbb{R})^\star = \mu(U) \times H_1(L_0, \mathbb{R}),$$

where $\mathcal{T}^\star$ denotes the real cotangent bundle.

The Liouville symplectic form on $\mathcal{T}^\star \mu(U)$ is clearly the constant 2-form on

$$H^1(L_0, \mathbb{R}) \times H_1(L_0, \mathbb{R})$$

given by the natural isomorphism of $H_1(L_0, \mathbb{R})$ with $H^1(L_0, \mathbb{R})^\star$. From this it follows immediately that

$$\mu(U) \times \Gamma \subset \mu(U) \times H_1(L_0, \mathbb{R})$$

is a Lagrangian submanifold with respect to the Liouville symplectic form on $\mathcal{T}^\star \mu(U)$, where Γ defined in (3.13).

Since $\mu(U) \times \Gamma \subset \mu(U) \times H_1(L_0, \mathbb{R})$ is a Lagrangian submanifold, it follows that for $\mathcal{W} = (R^1 q_\star \underline{\mathbb{R}})^\star$ (see (3.9) and (3.4)), the image of the natural map

$$\widetilde{H}_1(\mathbb{Z}) \longrightarrow (R^1 q_\star \underline{\mathbb{R}})^\star = \mathcal{W}$$

(see (3.8)) is Lagrangian with respect to the real symplectic form $\mathrm{Re}(d\theta')$ on $\mathcal{W}$, where $d\theta'$ is constructed in (3.5).

It was noted in Remark 3.2 that $R^{2d-1} q_\star \underline{\mathbb{Z}}$ is a complex submanifold of $\mathcal{W}$. Consequently, the 2-form on $R^{2d-1} q_\star \underline{\mathbb{Z}}$ obtained by restricting the holomorphic 2-form $d\theta'$ on $\mathcal{W}$ is also holomorphic. Since the real part of the holomorphic 2-from on $R^{2d-1} q_\star \underline{\mathbb{Z}}$ given by $d\theta'$ vanishes identically, we conclude that the holomorphic 2-from on $R^{2d-1} q_\star \underline{\mathbb{Z}}$, given by $d\theta'$, itself vanishes identically. Therefore, $R^{2d-1} q_\star \underline{\mathbb{Z}}$ is a Lagrangian submanifold of the holomorphic symplectic manifold $\mathcal{W}$ equipped with the holomorphic symplectic form $d\theta'$.

To complete the proof we recall a general property of the Liouville symplectic form.

Let N be a manifold and α a 1-form on N. Let

$$t : T^\star N \longrightarrow T^\star N$$

be the diffeomorphism that sends any $v \in T^\star_n N$ to $v + \alpha(n)$. If ψ is the Liouville symplectic form on $T^\star N$, the

$$t^\star \psi = \psi + d\alpha.$$

In particular, the map t preserves ψ if and only if the form α is closed. Also, the image of t is Lagrangian submanifold of $T^\star N$ for ψ if and only α is closed.

Since

$$R^{2d-1} q_\star \underline{\mathbb{Z}}$$

is a Lagrangian submanifold of the symplectic manifold $(\mathcal{W}, d\theta')$, from the above property of the Liouville symplectic form it follows immediately that the 2-form $d\theta'$ on $\mathcal{W}$ descends to the quotient space $\widehat{\mathcal{A}}$ in (3.7). □

Let

$$q_{\widehat{\mathcal{A}}} : \mathcal{W} \longrightarrow \widehat{\mathcal{A}} := \mathcal{W}/(R^{2d-1} q_\star \underline{\mathbb{Z}}) \tag{3.14}$$

be the quotient map (see (3.7)). From Theorem 3.3 we know that there is a unique 2-form

$$\Theta_{\widehat{\mathcal{A}}} \in H^0(\widehat{\mathcal{A}}, \Omega^2_{\widehat{\mathcal{A}}}) \tag{3.15}$$

such that

$$q^\star_{\widehat{\mathcal{A}}} \Theta_{\widehat{\mathcal{A}}} = d\theta', \tag{3.16}$$

where $q_{\mathcal{A}}$ is the map in (3.14). Since $d\theta'$ is a holomorphic symplectic form, it follows immediately that $\Theta_{\widehat{\mathcal{A}}}$ is a holomorphic symplectic form on $\widehat{\mathcal{A}}$.

The projection $\widehat{\gamma} : \mathcal{W} \longrightarrow M$ in (3.4) clearly descends to a map from $\mathcal{A}$ to M. Let

$$\varpi : \widehat{\mathcal{A}} \longrightarrow M \tag{3.17}$$

be the map given by $\widehat{\gamma}$; so we have

$$\widehat{\gamma} = \varpi \circ q_{\widehat{\mathcal{A}}},$$

where $q_{\widehat{\mathcal{A}}}$ is constructed in (3.14).

Lemma 3.4. *The projection ϖ in* (3.17) *defines a completely integrable structure on $\widehat{\mathcal{A}}$ for the symplectic form $\Theta_{\widehat{\mathcal{A}}}$ constructed in* (3.15). *In particular, the fibers of ϖ are Lagrangians with respect to the symplectic form $\Theta_{\widehat{\mathcal{A}}}$.*

Proof. Recall that $\mathcal{W}$ is holomorphically identified with Ω^1_M by the map $\widehat{\chi}$ in (3.3) (see (3.4)). This map $\widehat{\chi}$ takes the Liouville symplectic form on Ω^1_M to the symplectic form $d\theta'$ on $\mathcal{W}$. Therefore, from (3.14) and (3.16) we conclude that $\widehat{\mathcal{A}}$ is locally isomorphic to Ω^1_M such that the projection ϖ is taken to the natural projection $\Omega^1_M \longrightarrow M$, and the symplectic form $\Theta_{\widehat{\mathcal{A}}}$ on $\widehat{\mathcal{A}}$ is taken to the Liouville symplectic form on Ω^1_M. The lemma follows immediately from these, because the natural projection $\Omega^1_M \longrightarrow M$ defines a completely integrable structure on Ω^1_M for the Liouville symplectic form. □

4 The relative Picard group

For any $L \in M$, consider the homomorphisms

$$H^1(L, \mathbb{Z}) \longrightarrow H^1(L, \mathbb{C}) \longrightarrow H^1(L, \mathcal{O}_L), \tag{4.1}$$

where $H^1(L, \mathbb{Z}) \longrightarrow H^1(L, \mathbb{C})$ is the natural homomorphism given by the inclusion of $\mathbb{Z}$ in $\mathbb{C}$, and the projection $H^1(L, \mathbb{C}) \longrightarrow H^1(L, \mathcal{O}_L)$ corresponds to the isomorphism

$$H^1(L, \mathbb{C})/H^0(L, \Omega^1_L) = H^1(L, \mathcal{O}_L)$$

(see (2.8)). The image of the composition of homomorphisms in (4.1) is actually a cocompact lattice in $H^1(L, \mathcal{O}_L)$; so $H^1(L, \mathcal{O}_L)/H^1(L, \mathbb{Z})$ is a compact complex torus. We note that the composition of homomorphisms in (4.1) is in fact injective. When the compact complex manifold L is a complex projective variety, then $H^1(L, \mathcal{O}_L)/H^1(L, \mathbb{Z})$ is in fact an abelian variety.

As L moves over the family M, these cocompact lattices fit together to produce a C^∞ submanifold of the complex manifold $\mathcal{Y}$ in (2.14).

The Gauss–Manin connection on $R^1q_\star\underline{\mathbb{C}} \longrightarrow M$ evidently preserves the above bundle of cocompact lattices $R^1q_\star\underline{\mathbb{Z}} \subset R^1q_\star\underline{\mathbb{C}}$. From this it follows immediately that the above C^∞ submanifold $R^1q_\star\underline{\mathbb{Z}} \subset \mathcal{Y}$ is in fact a complex submanifold.

Taking fiber-wise quotients, we conclude that

$$\mathcal{A} := \mathcal{Y}/(R^1q_\star\underline{\mathbb{Z}}) \longrightarrow M \tag{4.2}$$

is a holomorphic family of compact complex tori over M.

Remark 4.1. Let Y be a compact Kähler manifold. Consider the short exact sequence of sheaves on Y given by the exponential map

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_Y \xrightarrow{\lambda \mapsto \exp(2\pi\sqrt{-1}\lambda)} \mathcal{O}^\star_Y \longrightarrow 0,$$

where $\mathcal{O}^\star_Y$ is a multiplicative sheaf of holomorphic functions with values in $\mathbb{C} \setminus \{0\}$. For the corresponding long exact sequence of cohomologies

$$H^1(Y, \underline{\mathbb{Z}}) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^1(Y, \mathcal{O}^\star_Y) \xrightarrow{c_1} H^2(Y, \underline{\mathbb{Z}}),$$

where the connecting homomorphism c_1 sends any holomorphic line bundle $\xi \in H^1(Y, \mathcal{O}_Y^\star)$ to $c_1(\xi)$, the quotient

$$H^1(Y, \mathcal{O}_Y)/H^1(Y, \mathbb{Z})$$

gets identified with the Picard group $\mathrm{Pic}^0(Y)$ that parametrizes the topologically trivial holomorphic line bundles on Y.

Consequently, the quotient $\mathcal{A}$ in (4.2) is naturally identified with the moduli space $\mathrm{Pic}^0_{\mathcal{Z}/M}$ of topologically trivial holomorphic line bundles on the fibers of $q : \mathcal{Z} \longrightarrow M$. In other words, $\mathcal{A}$ parametrizes all pairs of the form (L, ξ), where $L \in M$, and ξ is a topologically trivial holomorphic line bundle on L.

Recall that we have the holomorphic symplectic form $d\theta$ on $\mathcal{Y}$, where θ is constructed in (2.15).

Remark 4.2. Recall from Remark 3.1 that $d\theta$ does depend on the Kähler form ω on X.

Proposition 4.3. *The* 2*-form $d\theta$ on $\mathcal{Y}$ descends to the quotient space $\mathcal{A}$ in* (4.2)*, or in other words, $d\theta$ is the pullback of a* 2*-form on $\mathcal{A}$.*

Proof. The proof of the proposition is very similar to the proof of Theorem 3.3. As before, the local coordinate functions on M constructed in [5] play a crucial role. We omit the details of the proof. □

Let

$$q_{\mathcal{A}} : \mathcal{Y} \longrightarrow \mathcal{A} := \mathcal{Y}/(R^1 q_\star \mathbb{Z}) \tag{4.3}$$

be the quotient map (see (4.2)). Let $\Theta_{\mathcal{A}} \in H^0(\mathcal{A}, \Omega^2_{\mathcal{A}})$ be the holomorphic symplectic form given by Proposition 4.3, so

$$q^\star_{\mathcal{A}} \Theta_{\mathcal{A}} = d\theta\,, \tag{4.4}$$

where $q_{\mathcal{A}}$ is the map in (4.3).

The projection γ in (2.14) clearly descends to a map

$$\varpi_0 : \mathcal{A} \longrightarrow M\,. \tag{4.5}$$

Note that the isomorphism between $\mathcal{A}$ and $\mathrm{Pic}^0_{\mathcal{Z}/M}$ in Remark 4.1 takes ϖ_0 to the forgetful map

$$\mathrm{Pic}^0_{\mathcal{Z}/M} \longrightarrow M$$

that forgets the line bundle, or in other words, it sends any $(L, \xi) \in \mathrm{Pic}^0_{\mathcal{Z}/M}$ to the complex Lagrangian L forgetting the line bundle ξ.

Lemma 4.4. *The projection ϖ_0 in* (4.5) *defines a completely integrable structure on $\mathcal{A}$ for the symplectic form $\Theta_{\mathcal{A}}$.*

Proof. Recall that $\mathcal{Y}$ is holomorphically identified with Ω^1_M by the map χ in (2.13), and χ takes the Liouville symplectic form on Ω^1_M to the symplectic form $d\theta$ on $\mathcal{Y}$. Therefore, from (3.14) and (4.4) we conclude that $\mathcal{A}$ is locally isomorphic to Ω^1_M such that the projection ϖ_0 is taken to the natural projection $\Omega^1_M \longrightarrow M$, and the symplectic form $\Theta_{\mathcal{A}}$ on $\mathcal{A}$ is taken to the Liouville symplectic form on Ω^1_M. The lemma follows immediately from these. □

Acknowledgements: We thank the referee for helpful comments. The first author is partially supported by a J. C. Bose fellowship. The second author is supported by Ministerio de Ciencia e Innovación of Spain (grants MTM2016-79400-P, PID2019-108936GB-C21, and ICMAT Severo Ochoa project SEV-2015-0554) and CSIC (2019AEP151 and *Ayuda extraordinaria a Centros de Excelencia Severo Ochoa* 20205CEX001) The third author is supported by CMUP, financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the project with reference UIDB/00144/2020. The second and third authors wish to thank TIFR Mumbai for hospitality and for the excellent conditions provided.

References

[1] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, *Jour. Differ. Geom.* **18** (1983), 755–782.

[2] R. Donagi, L. Ein and R. K. Lazarsfeld, Nilpotent cones and sheaves on $K3$ surfaces, *Birational algebraic geometry (Baltimore, MD, 1996)*, 51–61, Contemp. Math., 207, Amer. Math. Soc., Providence, RI, 1997.

[3] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.

[4] N. J. Hitchin, Stable bundles and integrable systems, *Duke Math. J.* **54** (1987), 91–114.

[5] N. J. Hitchin, The moduli space of complex Lagrangian submanifolds, *Asian Jour. Math.* **3** (1999), 77–91.

[6] R. C. McLean, Deformations of calibrated submanifolds, *Comm. Anal. Geom.* **6** (1998), 705–747.

[7] S. Mukai, Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, *Invent. Math.* **77** (1984), 101–116.

[8] C. Voisin, Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes, *Complex projective geometry* (Trieste, 1989/Bergen, 1989), 294–303, London Math. Soc. Lecture Note Ser., 179, Cambridge Univ. Press, Cambridge, 1992.

[9] K. Yano and S, Bochner, *Curvature and Betti numbers*, Annals of Mathematics Studies, No. 32, Princeton University Press, Princeton, N. J., 1953.

[10] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.