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Rational cuspidal curves in a moving family of $\mathbb{P}^2$

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Abstract: In this paper we obtain a formula for the number of rational degree d curves in $\mathbb{P}^3$ having a cusp, whose image lies in a $\mathbb{P}^2$ and that passes through r lines and s points (where $r + 2s = 3d + 1$). This problem can be viewed as a family version of the classical question of counting rational cuspidal curves in $\mathbb{P}^2$, which has been studied earlier by Z. Ran ([13]), R. Pandharipande ([12]) and A. Zinger ([16]). We obtain this number by computing the Euler class of a relevant bundle and then finding out the corresponding degenerate contribution to the Euler class. The method we use is closely based on the method followed by A. Zinger ([16]) and I. Biswas, S. D'Mello, R. Mukherjee and V. Pingali ([1]). We also verify that our answer for the characteristic numbers of rational cuspidal planar cubics and quartics is consistent with the answer obtained by N. Das and the first author ([2]), where they compute the characteristic number of δ-nodal planar curves in $\mathbb{P}^3$ with one cusp (for $\delta \leq 2$).

Keywords: Euler Class; Degenerate Contributions; enumerative geometry

MSC: 14N35, 14J45

1 Introduction

A classical question in enumerative algebraic geometry is:

Question. *What is N_d, the number of rational (genus zero) degree d curves in $\mathbb{P}^2$ that pass through $3d - 1$ generic points?*

Although the computation of N_d is a classical question, a complete solution to the above problem was unknown until the early $90^{'s}$ when Ruan–Tian ([14]) and Kontsevich–Manin ([9]) obtained a formula for N_d. Generalization of this question to enumerate rational curves with higher singularities (such as cusps, tacnodes and higher order cusps) have been studied by Z. Ran ([13]), R. Pandharipande ([12]) and A. Zinger ([16], [17] and [15]). These results have also been generalized to other surfaces (such as $\mathbb{P}^1 \times \mathbb{P}^1$) by J. Kock ([8]) and more generally for del-Pezzo surfaces by I. Biswas, S. D'Mello, R. Mukherjee and V. Pingali ([1]). The problem of enumerating elliptic cuspidal curves has been solved by Z. Ran ([13]), and more recently a solution to this question in any genus has been obtained by Y. Ganor and E. Shustin ([4]) using methods from Tropical Geometry.

A natural generalization of problems in enumerative geometry (where one studies curves inside some fixed ambient surface such as $\mathbb{P}^2$) is to consider a family version of the same problem. This generalization is considered by S. Kleiman and R. Piene ([7]) and more recently by T. Laarakker ([10]) where they study the enumerative geometry of nodal curves in a moving family of surfaces.

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Motivated by this generalization, A. Paul and the authors of this paper studied a family version of computing N_d in ([11]); there the authors find a formula for the characteristic number of rational planar curves in $\mathbb{P}^3$ (i.e. curves in $\mathbb{P}^3$ that lie inside a $\mathbb{P}^2$). In this paper we build up on the results of ([11]) to find the characteristic number of rational planar curves in $\mathbb{P}^3$ having a cusp.

Before stating the main result of our paper, let us explain a few computational things. Let $N_d^{\mathbb{P}^3,\mathrm{Planar}}(r, s, \theta)$ be the number as defined in equation 3.2; this number was computed in [11]. Using this number, we can compute the number $\Phi_d(i, j, r, s, \theta)$ via Lemma 5.1, 5.2 and 5.3 (for $i \leq 2$). Using the values of $\Phi_d(i, j, r, s, \theta)$ (for $i \leq 2$), we can compute the e (Euler class) via equation (4.7). Finally, we can compute the number B (boundary contribution) via equations (4.6) and (5.7). With this computation, we can state the main result of the paper.

Theorem 1.1. *Let $C_d^{\mathbb{P}^3,\mathrm{Planar}}(r, s)$ be the number of genus zero, degree d curves in $\mathbb{P}^3$ having a cusp, whose image lies in a $\mathbb{P}^2$, intersecting r generic lines and s generic points (where $r+2s = 3d+1$) and let e and* B *be as computed above. Then*

$$C_d^{\mathbb{P}^3,\mathrm{Planar}}(r, s) = e - \mathrm{B}.$$

Remark. *Note that when $s \geq 4$, $C_d^{\mathbb{P}^3,\mathrm{Planar}}(r, s)$ is automatically zero, since four generic points do not lie in a plane. We also note that when $s = 3$, $C_d^{\mathbb{P}^3,\mathrm{Planar}}(r, s)$ is equal to the number of rational cuspidal curves in $\mathbb{P}^2$ through $3d - 2$ points; this is because three generic points determine a unique plane. Hence, $s = 3$ reduces to the result of Z. Ran ([13]), R. Pandharipande ([12]) and A. Zinger ([16]).*

We have written a mathematica program to implement our formula; the program is available on our web page

https://www.sites.google.com/site/ritwik371/home.

In section 7, we subject our formula to several low degree checks; in particular, we verify that our numbers are logically consistent with those obtained by N. Das and the first author ([2]).

Let us now give a brief overview of the method we use in this paper; we closely adapt the method applied by A. Zinger ([16]) and I. Biswas, S. D'Mello, R. Mukherjee and V. Pingali ([1]). We express our enumerative number as the number of zeros of a section of an appropriate vector bundle (restricted to an open dense set of an appropriate moduli space). As is usually the case, the Euler class of this vector bundle is our desired enumerative number, *plus* an extra boundary contribution. In ([16]) and ([1]) the method of "dynamic intersections" (cf. Chapter 11 in [3]) is used to compute the degenerate contribution to the relevant Euler class. We argue in section 6 how the multiplicity computation in ([16]) and ([1]) implies the multiplicity of the degenerate locus that occurs in our case. Finally, computation of the Euler class involves the intersection of tautological classes on the moduli space of planar degree d curves; this in turn involves the characteristic number of rational planar curves in $\mathbb{P}^3$, for which we use the result of our paper [11]. Hence, we can compute both the Euler class and the degenerate contribution, which gives us our desired number $C_d^{\mathbb{P}^3,\mathrm{Planar}}(r, s)$.

2 Future directions

In this section, we will explore a few natural questions that would occur to the reader and that we hope to explore in the future. The field of Enumerative Geometry and Gromov-Witten Theory studies the following type of question: we have a fixed surface X (say $\mathbb{P}^2$) and we consider an appropriate class of curves in X of a given homology class β. The basic goal of this field is to study this space of curves (moduli space) and compute intersection numbers.

The aim of the papers [7], [10] and [2] is to study a family version of the classical question of enumerating curves in a fixed linear system. The goal of our earlier paper [11] and the present paper is to develop a family version of the study of intersection theory of Moduli space of stable maps. We feel the most natural example is the study of planar curves in $\mathbb{P}^3$; this is a family version of studying curves in $\mathbb{P}^2$. However, the reader might wonder what other examples one might study with the methods developed in this paper? Let us consider an obvious generalization: find the characteristic number of planar rational curves in $\mathbb{P}^n$ for any n (and a similar question for cuspidal curves). We believe this question is doable, but it is likely to get computationally out of hand very quickly as n increases; even for $n = 4$ this is likely to be a formidable question (from a computational point of view). The reason is as follows. To consider planar curves in $\mathbb{P}^n$, we first of all have to consider the space of planes in $\mathbb{P}^n$; this is the Grassmannian $\mathbb{G}(3, n+1)$. When $n = 3$, this space is much more tractable since it is simply the projective space! For higher n, one can derive formulas (analogous to what we did in our paper [11]) for the characteristic number of rational planar curves. However, computationally it would get out of hand since one would have to compute intersections of Schubert cells in the Grassmannian (something which is doable in principle, but quite laborious in practice). Nevertheless, we do intend to investigate this question in future.

More generally, we believe one can study the following situation: let $\mathcal{S} \longrightarrow M$ be a fibre bundle over M, with fibres F. Consider the space of curves into $\mathcal{S}$, whose image lies inside some F. Study the intersection theory of this space. In our case, $\mathcal{S}$ is as defined in (4.2). But more generally, $\mathcal{S}$ could be the projectivization of some vector bundle. Or $\mathcal{S}$ could be some other fiber bundle (whose fibres are different from the projective space). The methods of our paper [11] and this paper ought to be applicable as long as the intersection theory of the base space M, the fibre F and the total space of the fibre bundle $\mathcal{S}$ is tractable.

A second question that the reader might wonder is if one might be able to extend the result of this paper to other types of singularities that have been worked out for $\mathbb{P}^2$ (for example tacnode, triple point or E_6-singularities[1] as worked out in [15] and [17] for $\mathbb{P}^2$). We believe the answer to this question is yes; we are in fact trying to understand how Zinger obtained the formulas for rational curves with E_6 singularities, tacnodes and triple points and trying to extend it for planar curves in $\mathbb{P}^3$.

We end this section with one last question. The contents of [11] and this paper are about genus zero curves. It is natural to wonder if one might be able to obtain higher genus analogues of this result; for example can one compute the characteristic number of planar elliptic curves in $\mathbb{P}^3$? This is a far more non-trivial question. We expect that the methods employed in [5] can be generalized to enumerate elliptic planar curves in $\mathbb{P}^3$; we intend to pursue this in the future.

3 Notation

Let us define a **planar** curve in $\mathbb{P}^3$ to be a curve, whose image lies inside a $\mathbb{P}^2$. We will now develop some notation to describe the space of planar curves of a given degree d.

Let us denote the dual of $\mathbb{P}^3$ by $\widehat{\mathbb{P}}^3$; this is the space of $\mathbb{P}^2$ inside $\mathbb{P}^3$. An element of $\widehat{\mathbb{P}}^3$ can be thought of as a non zero linear functional $\eta : \mathbb{C}^4 \longrightarrow \mathbb{C}$ upto scaling (i.e., it is the projectivization of the dual of $\mathbb{C}^4$). Given such an η, we define the projectivization of its zero set as $\mathbb{P}^2_\eta$. In other words,

$$\mathbb{P}^2_\eta := \mathbb{P}(\eta^{-1}(0)).$$

Note that this $\mathbb{P}^2_\eta$ is a subset of $\mathbb{P}^3$. Next, we define the moduli space of planar degree d curves into $\mathbb{P}^3$ as a fibre bundle over $\widehat{\mathbb{P}}^3$. More precisely, we define

$$\pi : \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \longrightarrow \widehat{\mathbb{P}}^3$$

to be the fiber bundle, such that

$$\pi^{-1}([\eta]) := \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2_\eta, d).$$

Here we are using the standard notation to denote $\mathcal{M}_{0,k}(X, \beta)$ to be the moduli space of genus zero stable maps, representing the class $\beta \in H_2(X, \mathbb{Z})$ and $\overline{\mathcal{M}}_{0,k}(X, \beta)$ to be its stable map compactification. Since the

1 A curve is said to have an E_6-singularity if in local coordinates it can be written as $y^3 + x^4 = 0$.

dimension of a fiber bundle is the dimension of the base, plus the dimension of the fiber, we conclude that the dimension of $\overline{\mathcal{M}}_{0,k}^{\text{Planar}}(\mathbb{P}^3, d)$ is $3d+2+k$.

Next, we note that there is a natural forgetful map

$$\pi_F : \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \longrightarrow \overline{\mathcal{M}}_{0,1}(\mathbb{P}^3, d)$$

where one forgets the plane $\mathbb{P}^2_\eta$ and simply thinks about the stable map to $\mathbb{P}^3$. When $d \geq 2$, the map π_F is injective when restricted to the open dense subspace of non multiply covered curves (from a smooth domain). This is because every planar degree d degree curve lies in a unique plane, when $d \geq 2$. When $d = 1$, this map is not injective since a line is not contained in a unique plane. Infact we note that the space of lines is 4 dimensional, while the dimension of $\overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, 1)$ is 5.

Let

$$\mathbb{L} \longrightarrow \overline{\mathcal{M}}_{0,1}(\mathbb{P}^3, d)$$

denote the universal tangent line bundle over the marked point (the fiber over each point is the tangent space over the given marked point). This line bundle will pullback to a line bundle over $\overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$ via the map π_F; we will denote it by the same symbol $\mathbb{L}$ (we will in general avoid writing the pullback symbol $\pi_F^\star$ if there is no cause of confusion).

Let us now define a few cycles in $\overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$. Let $\mathcal{H}_L$ and $\mathcal{H}_p$ denote the classes of the cycles in $\overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d)$ that corresponds to the subspace of curves passing through a generic line and a point respectively. We will denote their pullbacks (via the forgetful map that forgets the marked point) to $\overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d)$ by the same symbol $\mathcal{H}_L$ and $\mathcal{H}_p$. We will also denote H and a to be the standard generators of $H^\star(\mathbb{P}^3; \mathbb{Z})$ and $H^\star(\widehat{\mathbb{P}}^3; \mathbb{Z})$ respectively. As π is a projection map from $\overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$ to $\widehat{\mathbb{P}}^3$; we denote the pullback $\pi^\star a$ by the same symbol a. Finally, there is an evaluation map from

$$\text{ev} : \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \longrightarrow \mathbb{P}^3.$$

We will denote the pullback of H via this map to be $\text{ev}^\star H$. This is the one case where we explicitly write the pullback map (since it will avoid confusion; in the remaining cases omitting to write down the pullback map causes no confusion).

We will now define a few numbers by intersecting cycles on $\overline{\mathcal{M}}_{0,k}^{\text{Planar}}(\mathbb{P}^3, d)$. We will use the convention that

$$\langle \alpha, \ [M] \rangle = 0 \qquad \text{if} \qquad \deg(\alpha) \neq \dim(M). \tag{3.1}$$

We now define

$$\begin{aligned} N_d^{\mathbb{P}^3,\text{Planar}}(r, s, \theta) &:= \langle a^\theta, \ \overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \rangle, \\ \Phi_d(i, j, r, s, \theta) &:= \langle c_1(\mathbb{L}^\star)^i (\text{ev}^\star H)^j, \ \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle. \end{aligned} \tag{3.2}$$

We note that using the results of our paper ([11]), the numbers $N_d^{\mathbb{P}^3,\text{Planar}}(r, s, \theta)$ are all computable. In section 5, a formula is given to compute the relevant $\Phi_d(i, j, r, s, \theta)$ necessary to obtain the main result of this paper. We note that using the convention introduced in equation (3.1)

$$\begin{aligned} N_d^{\mathbb{P}^3,\text{Planar}}(r, s, \theta) &= 0 \qquad \text{unless} \qquad r + 2s + \theta = 3d + 2 \qquad \text{and} \\ \Phi_d(i, j, r, s, \theta) &= 0 \qquad \text{unless} \qquad r + 2s + \theta + i + j = 3d + 3. \end{aligned}$$

4 Euler class computation

We will now describe the basic method by which we compute the characteristic number of planar rational cuspidal curves in $\mathbb{P}^3$. We will express this number as the number of zeros of a section of an appropriate bundle restricted to an open dense subspace of the moduli space of planar curves in $\mathbb{P}^3$.

Before we do that, let us make a few abbreviations that we will often use. We denote

$$C_d := C_d^{\mathbb{P}^3,\text{Planar}}(r,s), \qquad \mathcal{M} := \mathcal{M}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d)\cap\mathcal{H}_L^r\cap\mathcal{H}_p^s,$$
$$\overline{\mathcal{M}} := \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3,d)\cap\mathcal{H}_L^r\cap\mathcal{H}_p^s \qquad \text{and} \qquad \partial\overline{\mathcal{M}} := \overline{\mathcal{M}} - \mathcal{M}.$$

Let $\gamma_{\mathbb{P}^3} \longrightarrow \mathbb{P}^3$ and $\gamma_{\widehat{\mathbb{P}}^3} \longrightarrow \widehat{\mathbb{P}}^3$ be the tautological line bundles over $\mathbb{P}^3$ and $\widehat{\mathbb{P}}^3$ respectively. We now note that any linear functional $\eta : \mathbb{C}^4 \longrightarrow \mathbb{C}$ induces a section $\eta : \mathbb{P}^3 \longrightarrow \gamma_{\mathbb{P}^3}^{\star}$ of the dual bundle, given by

$$\{\eta(l)\}(v) := \eta(v) \qquad \forall\ v \in l.$$

Note that we are making a slight abuse of notation here by denoting the linear functional and the induced section by the same letter η.

Let $q \in \mathbb{P}^3$, such that $\eta(q) = 0$ (i.e. $q \in \mathbb{P}^2_\eta$). Let us now consider the vertical derivative, namely

$$\nabla\eta|_q : T_q\mathbb{P}^3 \longrightarrow \gamma_{\mathbb{P}^3}^{\star}.$$

The vertical derivative can be defined by writing down the section η in a trivialization and taking the derivative along the fiber; this operation will be well defined if $\eta(q) = 0$. If η is a non zero functional, then the induced section is transverse to zero. Hence the vertical derivative will be surjective. Let us define

$$W_q := \text{Ker}\nabla\eta|_q.$$

We note that $T\mathbb{P}^2_\eta|_q$ is a subset of the kernel of $\nabla\eta|_q$. Since $\nabla\eta|_q$ is surjective, we conclude that $T\mathbb{P}^2_\eta$ is precisely equal to W_q. Hence, the following sequence

$$0 \longrightarrow W_q \xrightarrow{\ i\ } T_q\mathbb{P}^3 \xrightarrow{\nabla\eta|_q} \gamma_{\mathbb{P}^3}^{\star} \longrightarrow 0 \tag{4.1}$$

is exact, where the first map is the inclusion map. Let us now define

$$\mathcal{S} := \{([\eta], q) \in \widehat{\mathbb{P}}^3 \times \mathbb{P}^3 : \eta(q) = 0\}. \tag{4.2}$$

An element of $\mathcal{S}$ denotes a plane $\mathbb{P}^2_\eta$ in $\mathbb{P}^3$ together with a marked point q that lies in the plane. We will now define $W \longrightarrow \mathcal{S}$ to be the rank two vector bundle, where the fibre over each point $([\eta], q)$ is W_q (i.e. it is $T_q\mathbb{P}^2_\eta$, the tangent space of $\mathbb{P}^2_\eta$ at the point q). Hence, from equation (4.1), we conclude that over $\mathcal{S}$ we have the following exact sequence of vector bundles,

$$0 \longrightarrow W \xrightarrow{\ i\ } \pi_{\mathbb{P}^3}^{\star}\left(T\mathbb{P}^3\right) \xrightarrow{\nabla\eta|_q} \gamma_{\widehat{\mathbb{P}}^3}^{\star} \otimes \gamma_{\mathbb{P}^3}^{\star} \longrightarrow 0, \tag{4.3}$$

where $\pi_{\mathbb{P}^3} : \mathcal{S} \longrightarrow \mathbb{P}^3$ denotes the projection to $\mathbb{P}^3$.

Hence, from equation (4.3), we conclude that

$$c(W)c(\gamma_{\widehat{\mathbb{P}}^3}^{\star} \otimes \gamma_{\mathbb{P}^3}^{\star}) = c(\pi_{\mathbb{P}^3}^{\star}T\mathbb{P}^3), \quad \Longrightarrow \quad c_1(W) = 3H - a \qquad \text{and}$$
$$c_2(W) = a^2 - 2aH + 3H^2.$$

Next, we note that C_d is the cardinality of the set

$$\{[u,y] \in \mathcal{M} : du|_y = 0\}.$$

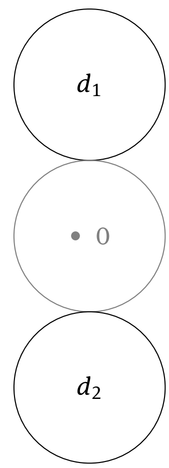
The process of taking the derivative of u at the marked point induces a section of the rank two vector bundle

$$\mathbb{L}^{\star} \otimes \text{ev}^{\star}W \longrightarrow \overline{\mathcal{M}}, \qquad \text{given by} \qquad [u,y] \longrightarrow du|_y. \tag{4.4}$$

Let us denote this section as ψ. In section 6 we show that when restricted to $\mathcal{M}$, ψ is transverse to zero. However, the section will also vanish on the boundary $\partial\overline{\mathcal{M}}$. The following Lemma precisely identifies the part of the boundary stratum on which the section vanishes.

Lemma 4.1. *Let $\mathcal{Y}$ denote the following subset of $\partial\overline{\mathcal{M}}$: stable map from a wedge of three spheres of degree d_1, 0 and d_2, where the marked point lies on the degree zero component (which is also called a ghost bubble). Restricted to $\partial\overline{\mathcal{M}}$, the section ψ vanishes* ***precisely*** *on $\mathcal{Y}$, i.e.*

$$\psi^{-1}(0) \cap \partial\overline{\mathcal{M}} = \mathcal{Y}. \tag{4.5}$$



Proof: An element of $\mathcal{Y}$ is of the above type (see the picture). We first note that the section ψ vanishes identically on $\mathcal{Y}$ since taking the derivative of a constant map is zero. Hence, the right hand side of equation (4.5) is a subset of its left hand side. Let us now show that the left hand side is a subset of its right hand side (i.e. we need to show that there are no other components of the boundary where the section vanishes).

To see why the section does not vanish on any other configuration, we consider all the remaining possible cases. Suppose the curve splits as $d = d_1 + d_2 + \ldots + d_k$, with $k \geq 3$ and $d_i \neq 0$ for all i. Let us compute the dimension of the space of such curves. If the plane was fixed, then the dimension of the space of such curves is $\big((3d_1 - 1) + \ldots + (3d_k - 1)\big)$. However, since the space of planes is three dimensional, we conclude that the space of such curves is $\big((3d_1 - 1) + \ldots + (3d_k - 1)\big) + 3$. This number is strictly less than $\big(3d - 2\big) + 3$ if $k \geq 3$; hence this configuration will not intersect r generic lines and s generic points (where $r + 2s = 3d + 1$).

Next, suppose the curve splits as $d = d_1 + d_2$, with $d_1\,, d_2 \neq 0$ and the marked point lying on say the d_1 component. Then the d_1 curve is cuspidal. The dimension of this space of curves is $\big(3d_1 - 2 + 3d_2 - 1\big) + 3$. This number is strictly less than $\big(3d - 2\big) + 3$; hence this configuration can not pass through r lines and s points (where $r + 2s = 3d + 1$).

Finally, we also observe that although the section vanishes on multiply covered curves, such curves will not pass through r lines and s points; this can be seen by considering the dimensions of the space of the underlying reduced curve and observing that the dimension is strictly less than $3d + 1$. Hence, the only configurations on which the section vanishes are those that split as a d_1 curve, a ghost bubble and a d_2 curve, with the last marked point lying on a ghost bubble (which is $\mathcal{Y}$). □

Next, we need to compute the cardinality of the set $\mathcal{Y}$. This cardinality is computed in ([11]) (in the proof of Theorem 3.3); it is given by

$$\mathrm{B} := |\mathcal{Y}| = \frac{1}{2} \sum_{\substack{d_1+d_2=d,\\ s_1+s_2=s,\\ r_1+r_2=r}} d_1 d_2 B_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta) \binom{r}{r_1} \binom{s}{s_1}, \tag{4.6}$$

where $B_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta)$ is as defined in equation (5.7). We claim that this boundary contributes with a multiplicity of one to the Euler class; this is justified in section 6.

It remains to compute the Euler class. By the splitting principle, we note that

$$\begin{aligned} e := \langle e(\mathbb{L}^* \otimes \mathrm{ev}^* W), \overline{\mathcal{M}}\rangle &= \langle c_1(\mathbb{L}^*)^2 + c_1(\mathbb{L}^*)c_1(\mathrm{ev}^* W) + c_2(\mathrm{ev}^* W), \overline{\mathcal{M}}\rangle \\ &= \Phi_d(2,0,r,s,0) - \Phi_d(1,0,r,s,1) + 3\Phi_d(1,1,r,s,0) \\ &+ \Phi_d(0,0,r,s,2) - 2\Phi_d(0,1,r,s,1) \\ &+ 3\Phi_d(0,2,r,s,0). \end{aligned} \tag{4.7}$$

The numbers $\Phi_d(i,j,r,s,\theta)$ that arise in the right hand side of equation (4.7) can be computed using the results of section 5 (namely, Lemmas 5.1, 5.2 and 5.3). Since the boundary contributes with a multiplicity of one, we conclude that

$$e = C_d + \mathrm{B}. \tag{4.8}$$

Using equations (4.8), (4.7), the values of $\Phi_d(i,j,r,s,\theta)$ from the results of section 5, equation (4.6) and the values of $N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s,\theta)$ from the paper ([11]), we obtain the value of $C_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s,\theta)$. In section 7, we present the values of C_d for a few values of d.

5 Intersection of Tautological Classes

We will now give a formula for the relevant $\Phi_d(i,j,r,s,\theta)$ that are necessary to compute the Euler class. We will often refer to $\Phi_d(i,j,r,s,\theta)$ as a level i number.

Lemma 5.1. *The level zero numbers $\Phi_d(0,j,r,s,\theta)$ are given by*

$$\Phi_d(0,j,r,s,\theta) = \begin{cases} 0 & \text{if } j=0, \\ dN_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s,\theta) & \text{if } j=1, \\ N_d^{\mathbb{P}^3,\mathrm{Planar}}(r+1,s,\theta) & \text{if } j=2, \\ N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s+1,\theta) & \text{if } j=3, \\ 0 & \text{if } j>3. \end{cases} \tag{5.1}$$

Lemma 5.2. *The level one numbers $\Phi_d(1,j,r,s,\theta)$ are given by*

$$\Phi_d(1,j,r,s,\theta) = \begin{cases} -2N_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s,\theta) & \text{if } j=0, \\ \frac{1}{d^2}\Phi_d(0,1,r+1,s,\theta) - \frac{2}{d}\Phi_d(0,2,r,s,\theta) & \\ +\frac{1}{d^2}\sum\limits_{\substack{r_1+r_2=r\\ s_1+s_2=s\\ d_1+d_2=d\\ d_1,d_2>0}} d_1^2 d_2^3 \binom{r}{r_1}\binom{s}{s_1} B_{d_1,d_2}(r_1,s_1,r_2,s_2,\theta) & \text{if } j=1, \end{cases} \tag{5.2}$$

where $B_{d_1,d_2}(r_1,s_1,r_2,s_2,\theta)$ is as defined in equation (5.7).

Lemma 5.3. *The level two number $\Phi_d(2,0,r,s,\theta)$ is given by*

$$\begin{aligned} \Phi_d(2,0,r,s,\theta) &= \frac{1}{d^2}\Phi_d(1,0,r+1,s,\theta) \\ &- \frac{2}{d}\Phi_d(1,1,r,s,\theta) + \frac{1}{d^2}(T_1(r,s,\theta) + T_2(r,s,\theta)), \end{aligned} \tag{5.3}$$

where $B_{d_1,d_2}(r_1,s_1,r_2,s_2,\theta)$ is as defined in equation (5.7),

$$T_1(r,s,\theta) := \sum_{\substack{r_1+r_2=r\\ s_1+s_2=s\\ d_1+d_2=d\\ d_1,d_2>0}} \binom{r}{r_1}\binom{s}{s_1} d_1 d_2^3 B_{d_1,d_2}(r_1,s_1,r_2,s_2,\theta),$$

$$T_2(r,s,\theta) := \sum_{\substack{r_1+r_2=r\\ s_1+s_2=s\\ d_1+d_2=d\\ d_1,d_2>0}} \binom{r}{r_1}\binom{s}{s_1} d_1 d_2^3 \widetilde{B}_{d_1,d_2}(r_1,s_1,r_2,s_2,\theta),$$

and

$$\widetilde{B}_{d_1,d_2}(r_1,s_1,r_2,s_2,\theta) := \sum_{i=0}^{3} \Phi_{d_1}(1,0,r_1,s_1,i)\times N_{d_2}^{\mathbb{P}^3,\mathrm{Planar}}(r_2,s_2,\theta+3-i).$$

Remark 5.1. *The number $\Phi_d(1,j,r,s,\theta)$ for $j>1$ and $\Phi_d(2,j,r,s,\theta)$ for $j>0$ can be computed without any further effort; we have not presented the formulas since they are not needed for the Euler class computation.*

Before we start proving these Lemmas, let us first recall an important result about $c_1(\mathbb{L}^*)$.

Lemma 5.4. *On $\overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3,d)$, the following equality of divisors holds:*

$$c_1(\mathbb{L}^*) = \frac{\mathcal{H}_L}{d^2} - \frac{2}{d}\mathrm{ev}^*(H) + \frac{1}{d^2}\sum_{\substack{d_1+d_2=d,\\ d_1,d_2\neq 0}} d_2^2 \mathcal{B}_{d_1,d_2},$$

where $\mathcal{B}_{d_1,d_2}$ denotes the boundary stratum corresponding to a bubble map of degree d_1 curve and degree d_2 curve with the marked point lying on the degree d_1 component.

Proof: This lemma is proved in ([6], Lemma 2.3) for $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^3,d)$. The corresponding statement for $\overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3,d)$ follows immediately by pulling pack the relationship via the natural map π_F from $\overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3,d)$ to $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^3,d)$ (the map that forgets the plane). □

We are now ready to prove the Lemmas that involve the computation of $\Phi_d(i,j,r,s,\theta)$.

Proofs of Lemmas 5.1, 5.2 and 5.3: Let us start by proving Lemma 5.1. We recall that

$$\Phi_d(0,j,r,s,\theta) = \langle (\mathrm{ev}^*H)^j,\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3,d)\cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta\rangle.$$

This number is zero unless $r+2s+\theta+j=3d+3$. Let us start by considering the case when $j=0$. Let us assume $r+2s+\theta=3d+3$ (otherwise the number is zero). In that case, we note that

$$\begin{aligned}\Phi_d(0,0,r,s,\theta) &= \langle (\mathrm{ev}^*H)^0,\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3,d)\cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta\rangle\\ &= \langle \mathbf{1},\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3,d)\cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta\rangle\\ &= 0.\end{aligned}$$

The last equality holds because the intersection is occurring inside $\overline{\mathcal{M}}_{0,0}^{\mathrm{Planar}}(\mathbb{P}^3,d)$ and $r+2s+\theta=3d+3$.

Next, let us consider the case when $j=1$. Let us assume $r+2s+\theta+1=3d+3$ (otherwise the number is zero). Let us consider the forgetful map

$$\delta : \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3,d)\cap \mathrm{ev}^*(H) \longrightarrow \overline{\mathcal{M}}_{0,0}^{\mathrm{Planar}}(\mathbb{P}^3,d).$$

We note that the degree of this map is d (since a degree d curve intersects a plane at d points). Hence

$$\begin{aligned}\Phi_d(0,1,r,s,\theta) &= \langle (\mathrm{ev}^*H),\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3,d)\cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta\rangle\\ &= \langle \mathbf{1},\ \Big(\overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3,d)\cap \mathrm{ev}^*(H)\Big)\cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta\rangle\\ &= \deg(\delta)|\overline{\mathcal{M}}_{0,0}^{\mathrm{Planar}}(\mathbb{P}^3,d)\cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta|\\ &= dN_d^{\mathbb{P}^3,\mathrm{Planar}}(r,s,\theta).\end{aligned}$$

Next, let us consider the case when $j = 2$. Let us assume $r + 2s + \theta + 2 = 3d + 3$ (otherwise the number is zero). Let us consider the forgetful map

$$\delta : \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \text{ev}^{\star}(H^2) \longrightarrow \overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L.$$

We note that the degree of this map is one since in the first case we are considering a curve and a marked point that goes to a line (H^2), while in the second case we are considering a curve whose image intersects a line. Hence

$$\begin{aligned} \Phi_d(0, 2, r, s, \theta) &= \langle (\text{ev}^{\star} H^2),\ \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^{\theta} \rangle \\ &= \langle \mathbf{1},\ \left(\overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \text{ev}^{\star}(H^2)\right) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^{\theta} \rangle \\ &= \deg(\delta) |\overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^{r+1} \cap \mathcal{H}_p^s \cap a^{\theta}| \\ &= N_d^{\mathbb{P}^3, \text{Planar}}(r+1, s, \theta). \end{aligned}$$

Finally, let us consider the case when $j = 3$. Let us assume $r + 2s + \theta + 3 = 3d + 3$ (otherwise the number is zero). Let us consider the forgetful map

$$\delta : \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \text{ev}^{\star}(H^3) \longrightarrow \overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_p.$$

We note that the degree of this map is one since in the first case we are considering a curve and a marked point that goes to a point (H^3), while in the second case we are considering a curve whose image passes through a point. Hence

$$\begin{aligned} \Phi_d(0, 3, r, s, \theta) &= \langle (\text{ev}^{\star} H^3),\ \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^{\theta} \rangle \\ &= \langle \mathbf{1},\ \left(\overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \text{ev}^{\star}(H^3)\right) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^{\theta} \rangle \\ &= \deg(\delta) |\overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^{r} \cap \mathcal{H}_p^{s+1} \cap a^{\theta}| \\ &= N_d^{\mathbb{P}^3, \text{Planar}}(r, s+1, \theta). \end{aligned}$$

For $j > 3$, let us assume $r+2s+\theta+j = 3d+3$, since $ev^{\star} H^j = 0$ for all $j > 3$. Therefore, we have $\Phi_d(0, j, r, s, \theta) = 0$.

Let us now prove Lemma 5.2. We recall that

$$\Phi_d(1, j, r, s, \theta) = \langle c_1(\mathbb{L}^{\star})(\text{ev}^{\star} H)^j,\ \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^{\theta} \rangle.$$

This number is zero unless $r + 2s + \theta + 1 + j = 3d + 3$. Let us start by considering the case when $j = 0$. Let us assume $r + 2s + \theta + 1 = 3d + 3$ (otherwise the number is zero). We note that

$$\langle \mathcal{H}_L,\ \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^{\theta} \rangle = 0, \tag{5.4}$$

$$\begin{aligned} \langle \text{ev}^{\star}(H),\ \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^{\theta} \rangle &= \Phi_d(0, 1, r, s, \theta) \\ &= d N_d^{\mathbb{P}^3, \text{Planar}}(r, s, \theta). \end{aligned} \tag{5.5}$$

Next, let us consider the boundary divisor $\mathcal{B}_{d_1, d_2}$, the boundary stratum of $\overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$ corresponding to a bubble map of degree d_1 curve and degree d_2 curve, with the marked point lying on the degree d_1 component. Let us now compute the degree of the divisor

$$B_{d_1, d_2}(r, s, \theta) := \deg\left(\mathcal{B}_{d_1, d_2} \cap \text{ev}^{\star}(H^j) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^{\theta}\right). \tag{5.6}$$

As per the convention decided in equation (3.1), we formally declare the degree to be zero unless $r + 2s + \theta + 1 + j = 3d + 3$. First, let us compute the number of (ordered) two component rational curves of type (d_1, d_2) that lies inside $\overline{\mathcal{M}}_{0,0}^{\text{Planar}}(\mathbb{P}^3, d) \cap a^{\theta}$. This is computed in ([11]), in the proof of Theorem 3.3 (and in equation (2.2)), given by

$$B_{d_1, d_2}(r_1, s_1, r_2, s_2, \theta) = \sum_{i=0}^{3} N_{d_1}^{\mathbb{P}^3, \text{Planar}}(r_1, s_1, i) \times N_{d_2}^{\mathbb{P}^3, \text{Planar}}(r_2, s_2, \theta + 3 - i). \tag{5.7}$$

We note there that each element of $B_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta)$ corresponds to $d_1 d_2$ bubble maps in $\overline{\mathcal{M}}_{0,0}^{\mathrm{Planar}}(\mathbb{P}^3, d)$, because there are $d_1 d_2$ choices for the nodal point of the domain. Furthermore, each such bubble map corresponds to d_1 bubble maps in $\overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \cap \mathrm{ev}^\star(H)$ since each degree d_1 curve intersects a hyperplane in d_1 points. Hence,

$$B_{d_1,d_2}(r, s, \theta) = \sum_{\substack{r_1+r_2=r,\\ s_1+s_2=s}} \binom{r}{r_1}\binom{s}{s_1} d_1^2 d_2 B_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta). \tag{5.8}$$

By equation (5.7) and (5.8), we conclude that $B_{d_1,d_2}(r, s, \theta)$ is zero if $1 + r + 2s + \theta = 3d + 3$. Hence, equations (5.4), (5.5), (5.8) and Lemma 5.4 imply the result of Lemma 5.2 for $j = 0$.

Next, let us prove Lemma 5.2 for the case when $j = 1$. Let us assume $r + 2s + \theta + 1 + 1 = 3d + 3$ (otherwise the number is zero). We note that

$$\langle \mathcal{H}_L \cdot \mathrm{ev}^\star(H),\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = \Phi_d(0, 1, r + 1, s, \theta), \tag{5.9}$$

$$\langle \mathrm{ev}^\star(H^2),\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = \Phi_d(0, 2, r, s, \theta). \tag{5.10}$$

We note that equation (5.8) is true irrespective of the value of r and s. Hence, using equations (5.9), (5.10), (5.8) and Lemma 5.4, we obtain the result of Lemma 5.2 for the case when $j = 1$.

Finally, let us prove Lemma 5.3. We recall that

$$\Phi_d(2, j, r, s, \theta) = \langle c_1(\mathbb{L}^\star)^2 (\mathrm{ev}^\star H)^j,\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle.$$

This number is zero unless $r + 2s + \theta + 2 + j = 3d + 3$. We will only consider the case when $j = 0$. Let us assume $r + 2s + \theta + 2 = 3d + 3$ (otherwise the number is zero). We note that

$$\langle c_1(\mathbb{L}^\star) \cdot \mathcal{H}_L,\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = \Phi_d(1, 0, r + 1, s, \theta), \tag{5.11}$$

$$\langle c_1(\mathbb{L}^\star) \cdot \mathrm{ev}^\star(H),\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \rangle = \Phi_d(1, 1, r, s, \theta). \tag{5.12}$$

Next we will show that

$$\sum_{\substack{d_1+d_2=d,\\ d_1,d_2\neq 0}} d_2^2 \Big\langle c_1(\mathbb{L}^\star) \cdot \mathcal{B}_{d_1,d_2},\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \Big\rangle = T_1(r, s, \theta) + T_2(r, s, \theta). \tag{5.13}$$

We note that equations (5.11), (5.12), (5.13) and Lemma 5.4 implies Lemma 5.3.

Let us now prove equation (5.13). Let us consider the map

$$\pi : \left(\mathcal{B}_{d_1,d_2} \cap \mathcal{H}_L^{r_1} \cap \mathcal{H}_p^{s_1}\right) \longrightarrow \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d_1) \cap \mathcal{H}_L^{r_1} \cap \mathcal{H}_p^{s_1},$$

that maps to the degree d_1 component. Let $\mathbb{L}_1$ denotes the pullback of the universal tangent bundle over $\overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d_1)$ to $\mathcal{B}_{d_1,d_2}$ via the map π. By ([6]) (equation (2.10), Page 29), we conclude that on $\left(\mathcal{B}_{d_1,d_2} \cap \mathcal{H}_L^{r_1} \cap \mathcal{H}_p^{s_1}\right)$, we have the equality of divisors

$$c_1(\mathbb{L}^\star)|_{\mathcal{B}_{d_1,d_2}} = c_1(\mathbb{L}_1^\star) + \mathcal{G}.$$

Now let us compute each of the terms. Let us now consider the space

$$\overline{\mathcal{M}}_{0,0}^{\mathrm{Planar}}(\mathbb{P}^3, d_2) \cap \mathcal{H}_L^{r_2} \cap \mathcal{H}_p^{s_2} \cap a^\theta.$$

Hence, we conclude that

$$\begin{aligned}
&\Big\langle c_1(\mathbb{L}_1^*),\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \Big\rangle \\
&= \sum_{\substack{r_1+r_2=r,\\ s_1+s_2=s}} \binom{r}{r_1}\binom{s}{s_1} d_1 d_2 \Big\langle c_1(\mathbb{L}_1^*) \cdot \Delta_{\widehat{\mathbb{P}}^3}, \\
&\qquad \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d_1) \cap \mathcal{H}_L^{r_1} \cap \mathcal{H}_p^{s_1} \times \overline{\mathcal{M}}_{0,0}^{\mathrm{Planar}}(\mathbb{P}^3, d_2) \cap \mathcal{H}_L^{r_2} \cap \mathcal{H}_p^{s_2} \cap a^\theta \Big\rangle \\
&= \sum_{\substack{r_1+r_2=r,\\ s_1+s_2=s}} \binom{r}{r_1}\binom{s}{s_1} d_1 d_2 \widetilde{B}_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta).
\end{aligned}$$

Here $\mathbb{L}_1^*$ denotes the universal cotangent bundle over the first marked point (over the moduli space of degree d_1 curves) and $\Delta_{\widehat{\mathbb{P}}^3}$ denotes the diagonal of $\widehat{\mathbb{P}}^3 \times \widehat{\mathbb{P}}^3$. We note that the class of the diagonal is given by $\sum_{i=0}^{3} \pi_1^* a^i \pi_2^* a^{3-i}$, where $\pi_i : \widehat{\mathbb{P}}^2 \times \widehat{\mathbb{P}}^2 \longrightarrow \widehat{\mathbb{P}}^2$ is the projection map to the i^{th} factor.

Similarly, we note that

$$\begin{aligned}
&\Big\langle \mathcal{G},\ \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \cap \mathcal{H}_L^r \cap \mathcal{H}_p^s \cap a^\theta \Big\rangle \\
&= \sum_{\substack{r_1+r_2=r,\\ s_1+s_2=s}} \binom{r}{r_1}\binom{s}{s_1} d_1 d_2 \Big\langle \Delta_{\widehat{\mathbb{P}}^3}, \\
&\qquad \overline{\mathcal{M}}_{0,0}^{\mathrm{Planar}}(\mathbb{P}^3, d_1) \cap \mathcal{H}_L^{r_1} \cap \mathcal{H}_p^{s_1} \times \overline{\mathcal{M}}_{0,0}^{\mathrm{Planar}}(\mathbb{P}^3, d_2) \cap \mathcal{H}_L^{r_2} \cap \mathcal{H}_p^{s_2} \cap a^\theta \Big\rangle \\
&= \sum_{\substack{r_1+r_2=r,\\ s_1+s_2=s}} d_1 d_2 \binom{r}{r_1}\binom{s}{s_1} B_{d_1,d_2}(r_1, s_1, r_2, s_2, \theta).
\end{aligned}$$

This proves the claim. □

6 Transversality and degenerate contribution to the Euler class

Let us start by showing that the section of the bundle, considered in equation (4.4) is transverse to zero (restricted to $\mathcal{M}$). This follows from the fact that restricted to each fibre, the section is transverse to zero. Fibre wise transversality is proved in [1, Lemma 8.2, Page 105] and it is also used in [16]. We will now justify the multiplicity.

Let us recapitulate the notation of section 3. We have defined

$$\pi : \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d) \longrightarrow \widehat{\mathbb{P}}^3$$

to be the fiber bundle, such that the fiber over each point is given by

$$\pi^{-1}([\eta]) := \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2_\eta, d).$$

Let us abbreviate the Euler class of the rank two bundle considered in section 4 as E, i.e.

$$E := e(\mathbb{L}^* \otimes \mathrm{ev}^* W).$$

Note that this is a (complex) degree two cohomology class (codimension two cycle) in $\overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d)$; it is not a number.

Next, let us denote $B \subset \overline{\mathcal{M}}_{0,1}^{\mathrm{Planar}}(\mathbb{P}^3, d)$ to be the boundary class; i.e. it is the closure of the space of bubble maps of type (d_1, d_2) such that the marked point is the nodal point (or more precisely, it is a map from a wedge of three spheres, where the middle component is constant and the marked point lies on it while the other two components are of degree d_1 and d_2).

Finally, let $C \subset \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$ denote the class determined by the closure of the space of cuspidal curves (the cusp being on the marked point). Let

$$\psi : \overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d) \longrightarrow \mathbb{L}^\star \otimes \text{ev}^\star W$$

denote the section that corresponds to taking the derivative at the marked point (i.e. the section defined in equation (4.4)). We note that the section vanishes on C and B. Hence, set theoretically, $\psi^{-1}(0)$ is the union of C and B. Hence, as cycles, we conclude that

$$E = m_1[C] + m_2[B], \tag{6.1}$$

where m_1 and m_2 are integers. Since the section ψ vanishes transversally on the cuspidal curves, we conclude that $m_1 = 1$. Let us now show that $m_2 = 1$. Consider the cycle

$$[\mathcal{Z}] := \mathcal{H}_l^{3d-2} a^3.$$

Now intersect the left hand side and right hand side of equation (6.1) with $\mathcal{Z}$ to conclude that

$$E \cdot [\mathcal{Z}] = [C] \cdot [\mathcal{Z}] + m_2[B] \cdot [\mathcal{Z}].$$

We now note that intersecting with a^3 corresponds to fixing a $\mathbb{P}^2$. Hence we are in the situation considered in [1], where we show that

$$E \cdot [\mathcal{Z}] = [C] \cdot [\mathcal{Z}] + 1 \times [B] \cdot [\mathcal{Z}].$$

Hence, $m_2 = 1$. Let us now consider the following cycle in $\overline{\mathcal{M}}_{0,1}^{\text{Planar}}(\mathbb{P}^3, d)$

$$[\mathcal{Z}] := \mathcal{H}_L^r \cdot \mathcal{H}_p^s \cdot a^\theta.$$

Choose r, s and θ such that dimension of $\mathcal{Z}$ is two (i.e. $r + 2s + \theta = 3d + 1$). Intersecting the left hand side and right hand side of equation (6.1) with $\mathcal{Z}$, we get the following equality of numbers

$$E \cdot [\mathcal{Z}] = [C] \cdot [\mathcal{Z}] + [B] \cdot [\mathcal{Z}].$$

This is precisely equation (4.8).

7 Low degree checks

In this section we subject our formula to certain low degree checks. All these numbers have been computed using our mathematica program. We will abbreviate $C_d^{\mathbb{P}^3,\text{Planar}}(r, s)$ by $C_d(r, s)$.

First of all our formula gives us the value of zero for $C_d(r, s)$ when $d = 2$. This is as geometrically as expected since there are no conics with a cusp.

Next, in ([2]), N. Das and the first author compute the following numbers: what is $N_d(A_1^\delta A_2, r, s)$, the number of planar degree d curves in $\mathbb{P}^3$, passing through r lines and s points, that have δ (ordered) nodes and one cusp, for all $\delta \leq 2$. Note that here $r + 2s = \delta + 2$. For $d = 3$, and $\delta = 0$, this number should be the same as the characteristic number of genus zero planar cubics in $\mathbb{P}^3$ with a cusp, i.e. $C_d(r, s)$. We have verified that is indeed the case. We tabulate the numbers for the readers convenience:

$$C_3(10, 0) = 17760, \qquad C_3(8, 1) = 2064,$$
$$C_3(6, 2) = 240 \qquad \text{and} \qquad C_3(4, 3) = 24.$$

These numbers are the same as $N_d(A_1^\delta A_2, r, s)$ for $d = 3$ and $\delta = 0$.

Next, we note that when $d = 4$ and $\delta = 2$, the number $\frac{1}{\delta!}N_d(A_1^\delta A_2, r, s)$ is same as the characteristic number of genus zero planar quartics in $\mathbb{P}^3$ with a cusp, i.e. $C_d(r, s)$. We have verified that fact. The numbers are

$$C_4(13, 0) = 10613184, \qquad C_4(11, 1) = 760368,$$
$$C_4(9, 2) = 49152 \qquad \text{and} \qquad C_4(7, 3) = 2304.$$

These numbers are the same as $\frac{1}{2!}N_d(A_1^\delta A_2, r, s)$ for $d = 4$ and $\delta = 2$. We have to divide out by a factor of $\delta!$ because in the definition of $N_d(A_1^\delta A_2, r, s)$, the nodes are ordered.

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