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Maurer-Cartan equation in the DGLA of graded derivations

Dedicated to the memory of Pierre Dolbeault

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Abstract: Let M be a smooth manifold and $D = \mathcal{L}_\Psi + \mathcal{I}_\Psi$ a solution of the Maurer-Cartan equation in the DGLA of graded derivations $\mathcal{D}^\star(M)$ of differential forms on M, where Ψ, Ψ are differential 1-form on M with values in the tangent bundle TM and $\mathcal{L}_\Psi$, $\mathcal{I}_\Psi$ are the $d^\star$ and $i^\star$ components of D. Under the hypothesis that $Id_{T(M)} + \Psi$ is invertible we prove that $\Psi = b(\Psi) = -\frac{1}{2}(Id_{TM} + \Psi)^{-1} \circ [\Psi, \Psi]_{\mathcal{FN}}$, where $[\cdot,\cdot]_{\mathcal{FN}}$ is the Frölicher-Nijenhuis bracket. This yields to a classification of the canonical solutions $e_\Psi = \mathcal{L}_\Psi + \mathcal{I}_{b(\Psi)}$ of the Maurer-Cartan equation according to their type: e_Ψ is of finite type r if there exists $r \in \mathbb{N}$ such that $\Psi^r \circ [\Psi, \Psi]_{\mathcal{FN}} = 0$ and r is minimal with this property, where $[\cdot,\cdot]_{\mathcal{FN}}$ is the Frölicher-Nijenhuis bracket. A distribution $\xi \subset TM$ of codimension $k \geqslant 1$ is integrable if and only if the canonical solution e_Ψ associated to the endomorphism Ψ of TM which is trivial on ξ and equal to the identity on a complement of ξ in TM is of finite type $\leqslant 1$, respectively of finite type 0 if $k = 1$.

Keywords: Differential Graded Lie Algebras, Maurer-Cartan equation, Foliations, Graded derivations

MSC: Primary 32G10, 16W25, 53C12

1 Introduction

In [7], one of the last papers of their seminal cycle of works on deformations of differentiable and complex structures, K. Kodaira and D. C. Spencer studied the deformations of multifoliate structures. A $\mathcal{P}$-multifoliate structure on an orientable manifold X of dimension n is an atlas $\left(U_i, \left(x_i^\alpha\right)_{\alpha=1,\cdots,n}\right)$ such that the changes of coordinates verify

$$\frac{\partial x_i^\alpha}{\partial x_k^\beta} = 0 \; for \; \beta \not\gtrsim \alpha,$$

where $(\mathcal{P}, \geqq)$ is a finite partially ordered set, $\{\alpha\}$ the set of integers $\alpha = 1, 2, \cdots, n$ such there is given a map $\{\alpha\} \mapsto [\alpha]$ of α onto $\mathcal{P}$ and the order relation " $\gtrapprox$ " is defined by $\alpha > \beta$ if and only if $[\alpha] > [\beta]$, $\alpha \sim \beta$ if and only if $[\alpha] = [\beta]$. A usual foliation is the particular case when $\mathcal{P} = \{a, b\}$, $a > b$.

They defined a DGLA structure $\left(\mathcal{D}^\star(M), \urcorner, [\cdot,\cdot]\right)$ on the graded algebra of graded derivations introduced by Frölicher and Nijenhuis in [3] and the deformations of the multifoliate structures are related to the solutions of the Maurer-Cartan equation in this algebra. This was done in the spirit of [10], where A. Nijenhuis and R. W. Richardson adapted a theory initiated by M. Gerstenhaber [4] and proved the connection between the deformations of complex analytic structures and the theory of differential graded Lie algebras (DGLA).

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In the paper [1], the authors elaborated a theory of deformations of integrable distributions of codimension 1 in smooth manifolds. Our approach was different of K. Kodaira and D. C. Spencer's in [7] (see remark 14 of [1] for a discussion). We considered in [1] only deformations of codimension 1 foliations, the DGLA algebra $\left(\mathcal{Z}^{\star}(L), \delta, \{\cdot,\cdot\}\right)$ associated to a codimension 1 foliation on a co-oriented manifold L being a subalgebra of the algebra $\left(\Lambda^{\star}(L), \delta, \{\cdot,\cdot\}\right)$ of differential forms on L. Its definition depends on the choice of a DGLA defining couple (γ, X), where γ is a 1-differential form on L and X is a vector field on L such that $\gamma(X) = 1$, but the cohomology classes of the underlying differential vector space structure do not depend on its choice. The deformations are given by forms in $\mathcal{Z}^{1}(L)$ verifying the Maurer-Cartan equation and the moduli space takes in account the diffeomorphic deformations. The infinitesimal deformations along curves are subsets of of the first cohomology group of the DGLA $\left(\mathcal{Z}^{\star}(L), \delta, \{\cdot,\cdot\}\right)$.

This theory was adapted to the study of the deformations of Levi-flat hypersurfaces in complex manifolds: we parametrized the Levi-flat hypersurfaces near a Levi-flat hypersurface in a complex manifold and we obtained a second order elliptic partial differential equation for an infinitesimal Levi-flat deformation.

The motivation of this paper was the study of foliations via the properties of the graded algebra of graded derivations defined by Frölicher and Nijenhuis in [3] with the DGLA structure defined by K. Kodaira and D. C. Spencer in [7]. We construct canonical solutions of the Maurer-Cartan equation in this algebra by means of deformations of the d-operator depending on a vector valued differential 1-form Φ and we give a classification of these solutions depending on their type. A canonical solution of the Maurer-Cartan equation associated to an endomorphism Φ is of finite type r if there exists $r \in \mathbb{N}$ such that $\Phi^{r} \circ [\Phi, \Phi]_{\mathcal{FN}} = 0$ and r is minimal with this property, where $[\cdot,\cdot]_{\mathcal{FN}}$ is the Frölicher-Nijenhuis bracket. We show that a distribution ξ of codimension k on a smooth manifold is integrable if and only if the canonical solution of the Maurer-Cartan equation associated to the endomorphism of the tangent space which is the trivial extension of the k-identity on a complement of ξ in TM is of finite type $\leqslant 1$. If ξ is a distribution of dimension s such that there exists an integrable distribution $\xi^{\star}$ of dimension d generated by ξ, we show that there exists locally an endomorphism Φ associated to ξ such that the canonical solution of the Maurer-Cartan equation associated to Φ is of finite type less than $r = \min\left\{m \in \mathbb{N} : m \geqslant \frac{d}{s}\right\}$.

In the case of integrable distributions of codimension 1, we also study the infinitesimal deformations of the canonical solutions of the Maurer-Cartan equation in the algebra of graded derivations by means of the theory of deformations developped in [1]. The authors expected that the differential equation for the infinitesimal deformations of a Levi-flat hypersurface from this section could lead to the solution of the famous conjecture of the nonexistence of Levi-flat hypersurfaces in the complex projective plane, as mentioned at the beginning of the last section.

2 The *DGLA* of graded derivations

It is well-known that deformations are generally governed by Maurer-Cartan equation in a DGLA. As was proved in [7], the defomations of the foliations are determined by the solutions of Maurer-Cartan equation in the DGLA of graded derivations. In this section we recall for the convenience of the reader some definitions and properties of the DGLA of graded derivations from [3], [7] (see [9] &16 and the expository paper of M. Manetti [8] for a complete presentation of the theory).

Notation 1. *Let M be a smooth manifold. We denote by $\Lambda^{\star}M$ the algebra of differential forms on M, by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M and by $\Lambda^{\star}M \otimes TM$ the algebra of TM-valued differential forms on M, where TM is the tangent bundle to M. In the sequel, we will identify $\Lambda^{1}M \otimes TM$ with the algebra $End(TM)$ of endomorphisms of TM by their canonical isomorphism: for $\sigma \in \Lambda^{1}M$, $X, Y \in \mathfrak{X}(M)$, $(\sigma \otimes X)(Y) = \sigma(Y)X$.*

Definition 1. *A differential graded Lie agebra (DGLA) is a triple $\left(V^{\star}, d, [\cdot,\cdot]\right)$ such that:*

1) $V^{\star} = \oplus_{i \in \mathbb{N}} V^{i}$, where $\left(V^{i}\right)_{i \in \mathbb{N}}$ is a family of $\mathbb{C}$-vector spaces and $d : V^{k} \to V^{k+1}$, $k \in \mathbb{N}$, is a graded homomorphism such that $d^{2} = 0$. An element $a \in V^{k}$ is said to be homogeneous of degree $k = \deg a$.

2) $[\cdot,\cdot]: V^\star \times V^\star \to V^\star$ *defines a structure of graded Lie algebra i.e. for homogeneous elements we have*

$$[a,b] = -(-1)^{\deg a \deg b}[b,a]$$

and

$$[a,[b,c]] = [[a,b],c] + (-1)^{\deg a \deg b}[b,[a,c]]$$

3) d is compatible with the graded Lie algebra structure i.e.

$$d[a,b] = [da,b] + (-1)^{\deg a}[a,db].$$

Definition 2. *Let* $(V^\star, d, [\cdot,\cdot])$ *be a DGLA and* $a \in V^1$. *We say that a verifies the Maurer-Cartan equation in* $(V^\star, d, [\cdot,\cdot])$ *if*

$$da + \frac{1}{2}[a,a] = 0. \tag{2.1}$$

Definition 3. *Let* $A = \oplus_{k\in\mathbb{Z}} A_k$ *be a graded algebra. A linear mapping* $D: A \to A$ *is called a graded derivation of degree* $p = |D|$ *if* $D: A_k \to A_{k+p}$ *and* $D(ab) = D(a)b + (-1)^{p \deg a} aD(b)$.

Definition 4. *Let M be a smooth manifold. We denote by* $\mathcal{D}^\star(M)$ *the graded algebra of graded derivations of* $\Lambda^\star M$.

Definition 5. *Let* P, Q *be homogeneous elements of degree* $|P|$, $|Q|$ *of* $\mathcal{D}^\star(M)$. *We define*

$$[P,Q] = PQ - (-1)^{|P||Q|}QP,$$

$$\daleth P = [d,P].$$

Lemma 1. *Let M be a smooth manifold. Then* $(\mathcal{D}^\star(M), [\cdot,\cdot], \daleth)$ *is a DGLA.*

Definition 6. *Let* $\alpha \in \Lambda^\star M$, $\Phi \in \Lambda^\star M \otimes TM$ *and* $X \in \mathfrak{X}(M)$. *We define* $\mathcal{I}_{\alpha\otimes X}$ *by:*

i)

$$\mathcal{I}_{\alpha\otimes X}\sigma = \alpha \wedge \iota_X \sigma,\ \ \sigma \in \Lambda^\star(M) \tag{2.2}$$

and $\mathcal{I}_\Phi$ *as the extension by linearity of (2.2).*

ii)

$$\mathcal{L}_\Phi = [\mathcal{I}_\Phi, d]. \tag{2.3}$$

Lemma 2. *For every* $\Phi \in \Lambda^k M \otimes TM$, $\mathcal{L}_\Phi, \mathcal{I}_\Phi \in \mathcal{D}^\star(M)$, $|\mathcal{L}_\Phi| = k$, $|\mathcal{I}_\Phi| = k-1$.

Notation 2.

$$\mathcal{L}(M) = \left\{\mathcal{L}_\Phi :\ \Phi \in \Lambda^\star M \otimes TM\right\},\ \mathcal{I}(M) = \left\{\mathcal{I}_\Phi :\ \Phi \in \Lambda^\star M \otimes TM\right\}.$$

In [3] the graded derivations of $\mathcal{L}(M)$ *(respectively of* $\mathcal{I}(M)$*) are called of type* $d_\star$ *(respectively of type* $\iota_\star$*).*

Lemma 3. *1. For every* $D \in \mathcal{D}^k(M)$ *there exist unique forms* $\Phi \in \Lambda^k M \otimes TM$, $\Psi \in \Lambda^{k+1} M \otimes TM$ *such that*

$$D = \mathcal{L}_\Phi + \mathcal{I}_\Psi,$$

so

$$\mathcal{D}^\star(M) = \mathcal{L}(M) \oplus \mathcal{I}(M).$$

We denote $\mathcal{L}_\Phi = \mathfrak{L}(D)$ *and* $\mathcal{I}_\Psi = \mathfrak{I}(D)$. *Moreover* $\mathfrak{L}(D)$ *(respectively* $\mathfrak{I}(D)$*)is uniquely determined by the restriction of D to* $\Lambda^0 M$ *(rspectively* $\Lambda^1 M$*).*

2. The mapping $\mathcal{L}: \Lambda^\star M \otimes TM \to \mathcal{D}^\star(M)$ *defined by* $\mathcal{L}(\Phi) = \mathcal{L}_\Phi$ *is an injective morphism of graded Lie algebras.*

3. *For every* $\Phi \in \Lambda^* M \otimes TM$
$$\urcorner (-1)^{|\Phi|} \mathcal{I}_\Phi = \mathcal{L}_\Phi.$$
4.
$$\mathcal{L}(M) = \ker \urcorner.$$

Remark 1. $d \in \mathcal{D}^1(M)$ *and*
$$d = \mathcal{L}_{Id_{T(M)}} = -\urcorner \mathcal{I}_{Id_{T(M)}}.$$

By Lemma 1 and the Jacobi identity, for every $\Phi \in \Lambda^k M \otimes TM$, $\Psi \in \Lambda^l M \otimes TM$ we have
$$\begin{aligned}\urcorner([\mathcal{L}_\Phi, \mathcal{L}_\Psi]) &= [d, [\mathcal{L}_\Phi, \mathcal{L}_\Psi]] = [[d, \mathcal{L}_\Phi], \mathcal{L}_\Psi] + (-1)^{|\Phi|} [\mathcal{L}_\Phi, [d, \mathcal{L}_\Psi]] \\ &= [\urcorner \mathcal{L}_\Phi, \mathcal{L}_\Psi] + (-1)^{|\Phi|} [\mathcal{L}_\Phi, \urcorner \mathcal{L}_\Phi] = 0,\end{aligned}$$
so there exists a unique form $[\Phi, \Psi] \in \Lambda^{k+l} M \otimes TM$ such that
$$[\mathcal{L}_\Phi, \mathcal{L}_\Psi] = \mathcal{L}_{[\Phi,\Psi]}. \tag{2.4}$$

This gives the following

Definition 7. *Let* $\Phi, \Psi \in \Lambda^* M \otimes TM$. *The Frölicher-Nijenhuis bracket of* Φ *and* Ψ *is the unique form* $[\Phi, \Psi]_{\mathcal{FN}} \in \Lambda^* M \otimes TM$ *verifying (2.4).*

Proposition 1. *Let* $\alpha \in \Lambda^k(M)$, $\beta \in \Lambda^l(M)$, $X, Y \in \mathfrak{X}(M)$.*Then:*
$$\begin{aligned}[\alpha \otimes X, \beta \otimes Y]_{\mathcal{FN}} &= \alpha \wedge \beta \otimes [X, Y] + \alpha \wedge \mathcal{L}_X \beta \otimes Y - \mathcal{L}_Y \alpha \wedge \beta \otimes X \\ &\quad + (-1)^k (d\alpha \wedge \iota_X \beta \otimes Y + \iota_Y \alpha \wedge d\beta \otimes X).\end{aligned}$$

Lemma 4. *Let* $\Phi_1 \in \Lambda^{k_1} M \otimes TM$, $\Phi_2 \in \Lambda^{k_2} M \otimes TM$, $\Psi_1 \in \Lambda^{k_1+1} M \otimes TM$, $\Psi_2 \in \Lambda^{k_2+1} M \otimes TM$. *Then*
$$\begin{aligned}\left[\mathcal{L}_{\Phi_1} + \mathcal{I}_{\Psi_1}, \mathcal{L}_{\Phi_2} + \mathcal{I}_{\Psi_2}\right] &= \mathcal{L}_{[\Phi_1,\Phi_2]_{\mathcal{FN}} + \mathcal{I}_{\Psi_1}\Phi_2 - (-1)^{k_1k_2}\mathcal{I}_{\Psi_2}\Phi_1} \\ &\quad + \mathcal{I}_{\mathcal{I}_{\Psi_1}\Psi_2 - (-1)^{k_1k_2}\mathcal{I}_{\Psi_2}\Psi_1 + [\Phi_1,\Psi_2]_{\mathcal{FN}} - (-1)^{k_1k_2}[\Phi_2,\Psi_1]_{\mathcal{FN}}}.\end{aligned}$$

In particular
$$[\mathcal{I}_\Phi, \mathcal{I}_\Psi] = \mathcal{I}_{\mathcal{I}_\Phi \Psi} - (-1)^{(|\Phi|+1)(|\Psi|+1)} \mathcal{I}_{\mathcal{I}_\Psi \Phi};$$
$$[\mathcal{L}_\Phi, \mathcal{I}_\Psi] = \mathcal{I}_{[\Phi,\Psi]_{\mathcal{FN}}} - (-1)^{|\Phi|(|\Psi|+1)} \mathcal{L}_{\mathcal{I}_\Psi \Phi};$$
$$[\mathcal{I}_\Psi, \mathcal{L}_\Phi] = \mathcal{L}_{\mathcal{I}_\Psi \Phi} - (-1)^{|\Phi|} \mathcal{I}_{[\Psi,\Phi]_{\mathcal{FN}}}.$$

Proposition 2. *Let* $\Phi, \Psi \in \Lambda^1 M \otimes TM$, $X, Y \in \mathfrak{X}(M)$. *Then*
$$\begin{aligned}[\Phi, \Psi]_{\mathcal{FN}}(X, Y) &= [\Phi X, \Psi Y] - [\Phi Y, \Psi X] - \Psi([\Phi X, Y] - [\Phi Y, X]) \\ &\quad - \Phi([\Psi X, Y] - [\Psi Y, X]) + (\Phi\Psi + \Psi\Phi)[X, Y].\end{aligned}$$

In particular,
$$[\Phi, \Phi]_{\mathcal{FN}}(X, Y) = 2\left([\Phi X, \Phi Y] + \Phi^2 [X, Y] - \Phi [\Phi X, Y] - \Phi [X, \Phi Y]\right).$$

Definition 8. *Let* $\Phi \in \Lambda^1 M \otimes TM$. *The Nijenhuis tensor of* Φ *is* $N_\Phi \in \Lambda^2 M \otimes TM$ *defined by*
$$N_\Phi = \frac{1}{2} [\Phi, \Phi]_{\mathcal{FN}}.$$

Remark 2. *Let* $\Phi \in \Lambda^1 M \otimes TM$ *such that* $\Phi^2 = \Phi$. *We have*
$$[\Phi, \Phi]_{\mathcal{FN}} = 2R + 2\overline{R}$$

where $R, \overline{R} \in \Lambda^2(M, TM)$,

$$R(X, Y) = \Phi[(Id - \Phi)X, (Id - \Phi)Y], \ \overline{R}(X, Y) = (Id - \Phi)[\Phi X, \Phi Y].$$

Moreover, suppose that Φ *has constant rank. Then* $\ker \Phi$ *and* $\operatorname{Im} \Phi$ *are subbundles of* TM *and* R *and (respectively* $\overline{R}$*) are the obstruction against integrability of* $\ker \Phi$ *(respectively* $\operatorname{Im} \Phi$*).*

3 Canonical solutions of Maurer-Cartan equation

The purpose of this section is to characterize the solutions e_Φ of the Maurer-Cartan equation (2.1) in the DGLA $\left(\mathcal{D}^*(M), [\cdot,\cdot], \urcorner\right)$ with the property $\mathfrak{L}(e_\Phi) = \mathcal{L}_\Phi$, for every $\Phi \in \Lambda^1 M \otimes TM$ such that $Id_{T(M)} + \Phi$ is invertible.

The following definitions for a pull-back and for a composition involving vector valued forms, will be used in the sequel:

Definition 9. *Let* $\Phi \in \Lambda^1 M \otimes TM$.

a) We define a homomorphism $\Phi^* : \Lambda^* M \to \Lambda^* M$ *by* $\Phi^* = Id_{\Lambda^0(M)}$ *if* $p = 0$ *and*

$$\Phi^*(\sigma)(V_1, \cdots, V_p) = \sigma(\Phi V_1, \cdots, \Phi V_p) \text{ if } p \geqslant 1, \ V_1, \cdots, V_p \in \mathfrak{X}(M).$$

b) Let $\Psi \in \Lambda^p M \otimes TM$. *We define* $\Phi \circ \Psi \in \Lambda^p M \otimes TM$ *by* $\Phi \circ \Psi = \Psi$ *if* $p = 0$ *and*

$$\Phi \circ \Psi(V_1, \ldots, V_p) = \Phi(\Psi(V_1, \ldots, V_p)), \ V_1, \ldots, V_p \in \mathfrak{X}(M) \text{ if } p \geqslant 1.$$

Remark 3. *i) By* $\circ : \left(\Lambda^1 M \otimes TM\right) \times \left(\Lambda^p M \otimes TM\right) \to \Lambda^p M \otimes TM$, $\Lambda^p M \otimes TM$ *becomes a* $\Lambda^1 M \otimes TM$*-module.*

ii) Let $\omega \in \Lambda^2(M)$, $Z \in \mathfrak{X}(M)$ *and* $\sigma \in \Lambda^1(M)$. *Then for every* $X, Y \in \mathfrak{X}(M)$

$$\mathfrak{I}_{\omega \otimes Z}\sigma(X, Y) = (\omega \wedge \iota_Z \sigma)(X, Y) = \sigma(Z)\,\omega(X, Y) = \sigma((\omega \otimes Z)(X, Y)).$$

By linearity, for every $\Phi \in \Lambda^2 M \otimes TM$, $\sigma \in \Lambda^1(M)$, $X, Y \in \mathfrak{X}(M)$ *we have*

$$\mathfrak{I}_\Phi \sigma = \sigma \circ \Phi. \tag{3.1}$$

(3.1) is valid for every $\Phi \in \Lambda^* M \otimes TM$ *(see [9]).*

Notation 3. *Let* $D \in \mathcal{D}^k(M)$ *and* $\Phi \in \Lambda^1 M \otimes TM$ *be invertible. Set* $\Phi^{-1} D \Phi = \left(\Phi^*\right)^{-1} D \Phi^* : \Lambda^* M \to \Lambda^* M$.

Remark 4. *Let* $D \in \mathcal{D}^k(M)$ *and* $\Phi \in \Lambda^1 M \otimes TM$ *be invertible. Then* Φ^* *is an automorphism and it follows that* $\Phi^{-1} D \Phi \in \mathcal{D}^k(M)$.

Lemma 5. *Let* $D \in \mathcal{D}^1(M)$ *be such that* $[D, D] = 0$. *Then* $D - d$ *is solution of the Maurer-Cartan equation in* $\left(\mathcal{D}^*(M), [\cdot,\cdot], \urcorner\right)$.

Proof. Since $\urcorner D = [D, d]$, it follows that

$$\urcorner(D - d) + \frac{1}{2}[D - d, D - d] = [D, d] - [D, d] = 0.$$

∎

Notation 4. *Let* $\Phi \in \Lambda^1 M \otimes TM$ *be such that* $Id_{TM} + \Phi$ *is invertible. Set*

$$d_\Phi = (Id_{TM} + \Phi)\, d\, (Id_{TM} + \Phi)^{-1},$$

$$e_\Phi = d_\Phi - d.$$

Since $[d_\Phi, d_\Phi] = 0$, by Lemma 5 we obtain:

Corollary 1. *e_Φ is solution of the Maurer-Cartan equation in $\left(\mathcal{D}^*(M), [\cdot,\cdot], \urcorner\right)$.*

Theorem 1. *Let $\Phi \in \Lambda^1 M \otimes TM$ be such that $Id_{T(M)} + \Phi$ is invertible. Let $\Psi \in \Lambda^2 M \otimes TM$ be such that $D = \mathcal{L}_\Phi + \mathcal{I}_\Psi$ is a solution of the Maurer-Cartan equation in $\left(\mathcal{D}^*(M), [\cdot,\cdot], \urcorner\right)$. Then $\Psi = b(\Phi)$, where*

$$b(\Phi) = -\frac{1}{2}\left(Id_{T(M)} + \Phi\right)^{-1} \circ [\Phi, \Phi]_{\mathcal{FN}}.$$

Proof. Let $D = \mathcal{L}_\Phi + \mathcal{I}_\Psi$, $\Phi \in \Lambda^1 M \otimes TM$, $\Psi \in \Lambda^2 M \otimes TM$. By using Lemma 1 and Lemma 4 we have

$$\urcorner D = \urcorner \mathcal{I}_\Psi = \mathcal{L}_\Psi$$

and

$$\begin{aligned} [D, D] &= [\mathcal{L}_\Phi + \mathcal{I}_\Psi, \mathcal{L}_\Phi + \mathcal{I}_\Psi] = \mathcal{L}_{[\Phi,\Phi]_{\mathcal{FN}}} + 2[\mathcal{L}_\Phi, \mathcal{I}_\Psi] + [\mathcal{I}_\Psi, \mathcal{I}_\Psi] \\ &= \mathcal{L}_{[\Phi,\Phi]_{\mathcal{FN}}} + 2\left(\mathcal{I}_{[\Phi,\Psi]_{\mathcal{FN}}} + \mathcal{L}_{\mathcal{I}_\Psi \Phi}\right) + [\mathcal{I}_\Psi, \mathcal{I}_\Psi], \end{aligned}$$

so

$$\urcorner D + \frac{1}{2}[D, D] = \mathcal{L}_\Psi + \frac{1}{2}\mathcal{L}_{[\Phi,\Phi]_{\mathcal{FN}}} + \left(\mathcal{I}_{[\Phi,\Psi]_{\mathcal{FN}}} + \mathcal{L}_{\mathcal{I}_\Psi \Phi}\right) + \frac{1}{2}[\mathcal{I}_\Psi, \mathcal{I}_\Psi].$$

It follows that

$$\mathfrak{L}\left(\urcorner D + \frac{1}{2}[D, D]\right) = \mathcal{L}_\Psi + \frac{1}{2}\mathcal{L}_{[\Phi,\Phi]_{\mathcal{FN}}} + \mathcal{L}_{\mathcal{I}_\Psi \Phi}$$

and

$$\mathfrak{I}\left(\urcorner D + \frac{1}{2}[D, D]\right) = \mathcal{I}_{[\Phi,\Psi]_{\mathcal{FN}}} + \frac{1}{2}[\mathcal{I}_\Psi, \mathcal{I}_\Psi].$$

Suppose that $D = \mathcal{L}_\Phi + \mathcal{I}_\Psi$ verifies the Maurer-Cartan equation. Then

$$0 = \mathfrak{L}\left(\urcorner D + \frac{1}{2}[D, D]\right) = \mathcal{L}\left(\Psi + \frac{1}{2}[\Phi, \Phi]_{\mathcal{FN}} + \mathcal{I}_\Psi \Phi\right).$$

Since $\mathcal{L}$ is injective, this implies

$$\Psi + \frac{1}{2}[\Phi, \Phi]_{\mathcal{FN}} + \mathcal{I}_\Psi \Phi = 0.$$

By (3.1) we obtain

$$\Psi + \frac{1}{2}[\Phi, \Phi]_{\mathcal{FN}} + \Phi \circ \Psi = 0,$$

which is equivalent to

$$\Psi = -\frac{1}{2}(Id_{TM} + \Phi)^{-1} \circ [\Phi, \Phi]_{\mathcal{FN}} = b(\Phi).$$

∎

Corollary 2. *Let $\Phi \in \Lambda^1 M \otimes TM$ be such that $Id_{T(M)} + \Phi$ is invertible. Then*

$$e_\Phi = \mathcal{L}_\Phi + \mathcal{I}_{b(\Phi)}. \tag{3.2}$$

Proof. By Corollary 1 e_Φ is solution of the Maurer-Cartan equation in $\left(\mathcal{D}^*(M), [\cdot,\cdot], \urcorner\right)$, so by Theorem 1 $\mathfrak{I}(e_\Phi) = \mathcal{I}_{b(\Phi)}$. By Lemma 1 $\mathfrak{L}(e_\Phi)$ is uniquely determined by the restriction of e_Φ to $\Lambda^0(M)$. Since each derivation of type $i_\star$ vanishes on $\Lambda^0(M)$, in order to prove (3.2) it is enough to show that $e_\Phi = \mathcal{L}_\Phi$ on $\Lambda^0(M)$.

Set $R_\Phi = Id_{T(M)} + \Phi$ and let $f \in \Lambda^0(M)$, $X \in \mathfrak{X}(M)$. Then

$$d_\Phi f(X) = \left(R_\Phi d R_\Phi^{-1} f\right)(X) = (R_\Phi df)(X) = df(Id_{TM} + \Phi)(X) = df(X) + df(\Phi(X)). \tag{3.3}$$

If $\alpha \in \Lambda^1(M)$ and $Y \in \mathfrak{X}(M)$, by (2.2),

$$\mathcal{I}_{\alpha \otimes Y}(df)(X) = (\alpha \otimes \iota_Y df)(X) = df(Y)(\alpha(X)) = df(\alpha \otimes Y)X$$

and by linearity we obtain

$$\mathcal{I}_{\Phi}(df)(X) = df(\Phi(X)). \tag{3.4}$$

So, by (3.3) and (3.4) it follows that

$$e_{\Phi} f(X) = d_{\Phi} f(X) - df(X) = df(\Phi(X)) = \mathcal{I}_{\Phi}(df)(X) = [\mathcal{I}_{\Phi}, d] f(X) = \mathcal{L}_{\Phi} f(X)$$

and the Corollary is proved. ∎

Definition 10. *Let $\Phi \in \Lambda^1 M \otimes TM$ be such that $Id_{T(M)} + \Phi$ is invertible. e_{Φ} is called the canonical solution of Maurer-Cartan equation associated to Φ.*

4 Canonical solutions of finite type of Maurer-Cartan equation

In this section we define a stratification of the canonical solutions e_{Φ} of the Maurer-Cartan equation in the DGLA $\left(\mathcal{D}^{\star}(M), [\cdot,\cdot], \urcorner\right)$ which depends on the existence of $r \in \mathbb{N}$ such that $\Phi^r \circ [\Phi, \Phi]_{\mathcal{FN}} = 0$. This stratification implies several properties related to the integrability of distributions in $T(M)$.

Remark 5. *Let $\Phi \in \Lambda^1 M \otimes TM$ be such that $Id_{T(M)} + \Phi$ is invertible. We consider a norm on the algebra $\Lambda^1 M \otimes TM$ and we suppose that $\|\Phi\| < 1$. Then*

$$b(\Phi) = -\frac{1}{2}(Id_{TM} + \Phi)^{-1} \circ [\Phi, \Phi]_{\mathcal{FN}} = -\frac{1}{2}\sum_{j=0}^{\infty} (-1)^j \Phi^j \circ [\Phi, \Phi]_{\mathcal{FN}}. \tag{4.1}$$

So

$$e_{\Phi} = \mathcal{L}_{\Phi} + \mathcal{I}_{b(\Phi)} = \mathcal{L}_{\Phi} + \sum_{j=0}^{\infty} \mathcal{I}_{-\frac{1}{2}(-1)^j \Phi^j \circ [\Phi,\Phi]_{\mathcal{FN}}}. \tag{4.2}$$

Notation 5. *Let $\Phi \in \Lambda^1 M \otimes TM$. Denote*

$$\gamma_0^{\Phi} = \mathcal{L}_{\Phi},\ \gamma_k^{\Phi} = \mathcal{I}_{-\frac{1}{2}(-1)^k \Phi^{k-1} \circ [\Phi,\Phi]_{\mathcal{FN}}},\ k \geqslant 1.$$

By using the previous Notation, under the hypothesis of Remark 5, (4.2) may be written as

$$e_{\Phi} = \sum_{k=0}^{\infty} \gamma_k^{\Phi}. \tag{4.3}$$

The following Theorem shows that, under the hypothesis that there exists $r \in \mathbb{N}$ such that $\Phi^r \circ [\Phi, \Phi]_{\mathcal{FN}} = 0$, (4.1) and (4.3) are still valid (as finite sums) for every $\Phi \in \Lambda^1 M \otimes TM$ such that $Id_{T(M)} + \Phi$ is invertible.

Theorem 2. *Let $\Phi \in \Lambda^1 M \otimes TM$ be such that $Id_{T(M)} + \Phi$ is invertible and suppose that there exists $r \in \mathbb{N}$ such that $\Phi^r \circ [\Phi, \Phi]_{\mathcal{FN}} = 0$. Then*

$$e_{\Phi} = \sum_{k=0}^{r-1} \gamma_k^{\Phi}.$$

For the proof of Theorem 2 we use the following:

Lemma 6. *Let A be a unitary ring with unit e and B an A-module. Let $a \in A$, $h \in B$ and $r \in \mathbb{N}^{\star}$ be such that $e - a$ is invertible and $a^r h = 0$. Then*

$$(e-a)^{-1} h = h + ah + a^2 h + \cdots + a^{r-1} h.$$

Proof. We have

$$\begin{aligned}&(e-a)\left(h+ah+a^2h+\cdots+a^rh\right)\\ =\;& h+ah+a^2h+\cdots+a^rh-ah-a^2h-\cdots-a^rh-a^{r+1}h=h,\end{aligned}$$

so

$$(e-a)^{-1}h=h+ah+a^2h+\cdots+a^{r-1}h.$$

Proof of Theorem 2. Suppose that $r=0$. Then $b(\Phi)=0$ and by (3.2) $e_\Phi=\mathcal{L}_\Phi=\gamma_0^\Phi$.

Suppose now $r>0$. By using Lemma 6 with $a=\Phi\in A=\Lambda^1M\otimes TM$, $h=[\Phi,\Phi]_{\mathcal{FN}}\in B=\Lambda^2M\otimes TM$, B being considered as the A-module defined in Remark 3 *i*), we have

$$b(\Phi)=-\frac{1}{2}\left(Id_{T(M)}+\Phi\right)^{-1}\circ[\Phi,\Phi]_{\mathcal{FN}}=-\frac{1}{2}\sum_{k=0}^{r-1}(-1)^k\Phi^k\circ[\Phi,\Phi]_{\mathcal{FN}}.$$

By (3.2)

$$e_\Phi=\mathcal{L}_\Phi+\mathcal{I}_{b(\Phi)}=\mathcal{L}_\Phi+\mathcal{I}_{-\frac{1}{2}\sum_{k=0}^{r-1}(-1)^k\Phi^k\circ[\Phi,\Phi]_{\mathcal{FN}}}=\mathcal{L}_\Phi+\sum_{k=1}^{r-1}\gamma_k^\Phi$$

and the Theorem is proved. ∎

Definition 11. *Let $\Phi\in\Lambda^1M\otimes TM$ be such that $Id_{T(M)}+\Phi$ is invertible and e_Φ the canonical solution of Maurer-Cartan equation associated to Φ. e_Φ is called of finite type if there exists $r\in\mathbb{N}$ such that $\Phi^r\circ[\Phi,\Phi]_{\mathcal{FN}}=0$ and of finite type r if $r=\min\left\{s\in\mathbb{N}:\ \Phi^s\circ[\Phi,\Phi]_{\mathcal{FN}}=0\right\}$.*

Proposition 3. *Let $\Phi\in\Lambda^1M\otimes TM$ be such that $Id_{T(M)}+\Phi$ is invertible. The following are equivalent:*

i) The canonical solution e_Φ of Maurer-Cartan equation coresponding to Φ is of finite type 0.

ii) e_Φ is $\urcorner$-closed.

iii) d_Φ is $\urcorner$-closed.

iv) $N_\Phi=0$.

Proof. *i*) $\Longleftrightarrow$ *ii*) Suppose that the canonical solution e_Φ of Maurer-Cartan equation coresponding to Φ is of finite type 0. Then by Theorem 2 it follows that

$$e_\Phi=\mathcal{L}_\Phi$$

and by Lemma 1 (3) it follows that e_Φ is $\urcorner$-closed.

Conversely, suppose $\urcorner e_\Phi=0$. By using again Lemma 1 (3) it follows that $e_\Phi\in\mathcal{L}(M)$. In particular $\mathcal{I}_{b(\Phi)}=0$, so $[\Phi,\Phi]_{\mathcal{FN}}=0$.

ii) $\Longleftrightarrow$ *iii*) We have $d=\mathcal{L}_{Id_{T(M)}}$, so $\urcorner d=0$. Since $e_\Phi=d_\Phi-d$ the assertion follows.

i) $\Longleftrightarrow$ *iv*) Since $[\Phi,\Phi]_{\mathcal{FN}}=2N_\Phi$, $N_\Phi=0$ if and only if $[\Phi,\Phi]_{\mathcal{FN}}=0$. ∎

Remark 6. *For an almost complex structure J on M, we obtain from Proposition 3 that J is integrable if and only if the canonical solution associated to J is of finite type* 0.

Theorem 3. *Let $\Phi\in End(TM)$ be a projection of constant rank. Then $\xi=\ker\Phi$ and $\zeta=\operatorname{Im}\Phi$ are subbundles of TM such that $TM=\xi\oplus\zeta$, $Id_{TM}+\Phi$ is invertible and:*

1. *The canonical solution e_Φ associated to Φ is of finite type* 0 *if and only if ξ and ζ are integrable.*
2. *e_Φ is of finite type* 1 *if and only if ξ is integrable and ζ is not integrable.*
3. *If ξ is not integrable then e_Φ is not of finite type.*

Proof. Let $x\in M$ be such that $x+\Phi(x)=0$. Then $\Phi(x+\Phi(x))=2\Phi(x)=0$. So $x=0$ and it follows that $Id_{TM}+\Phi$ is invertible.

Suppose that the canonical solution associated to Φ is of finite type 0. By definition, $[\Phi,\Phi]_{\mathcal{FN}} = R+\overline{R}$ and $\overline{R}(X,Y)=0$ for every $X,Y\in\xi$, where R and $\overline{R}$ are defined in Remark 2. Since $[\Phi,\Phi]_{\mathcal{FN}}=0$ and $Id_{TM}-\Phi$ is a projection on ξ, it follows that for every $X,Y\in\xi$

$$0 = R(X,Y) = \Phi\left([(Id_{TM}-\Phi)X,(Id_{TM}-\Phi)Y]\right) = \Phi[X,Y].$$

So ξ is integrable and analoguously, ζ is integrable.

Conversely, suppose that ξ is integrable. As $Id-\Phi$ is a projection on ξ, for every $X,Y\in\mathfrak{X}(M)$ $[(Id-\Phi)X,(Id-\Phi)Y]\in\xi$ and $(Id-\Phi)[\Phi X,\Phi Y]\in\xi$. It follows that $R=0$ and $\Phi\overline{R}=0$, so $\Phi\circ[\Phi,\Phi]_{\mathcal{FN}} = \Phi\circ\left(2R+2\overline{R}\right) = 2\Phi\circ\overline{R} = 0$ and e_Φ is of finite type $\leqslant 1$. Since ζ is integrable if and only if $\overline{R}=0$ and $[\Phi,\Phi]_{\mathcal{FN}}=2\overline{R}$, the second asertion is proved.

Finally, if ξ is not integrable, there exist $X,Y\in\xi$ such that $\Phi([X,Y])\neq 0$. By Proposition 2, for every $k\in\mathbb{N}^\star$ we have $\Phi^k\circ[\Phi,\Phi]_{\mathcal{FN}}([X,Y]) = \Phi([X,Y])\neq 0$, so e_Φ is not of finite type. ∎

Corollary 3. *Let M be a smooth manifold and $\xi\subset TM$ a co-orientable distribution of codimension 1. There exist $X\in\mathfrak{X}(M)$ and $\gamma\in\Lambda^1(M)$ such $\xi=\ker\gamma$ and $\iota_X\gamma=1$. We have $T(M)=\xi\oplus\mathbb{R}X$ and we consider $\Phi\in End(TM)$ defined by $\Phi=0$ on ξ and $\Phi=Id_{\mathbb{R}X}$ on $\mathbb{R}X$. Then the canonical solution associated to Φ is of finite type 0 if and only if ξ is integrable.*

Proof. We apply Theorem 3 for $\zeta=\mathbb{R}X$ which is obviously integrable. ∎

Theorem 4. *Let M be a smooth manifold and $\xi,\tau\subset TM$ distributions such that $\xi\subsetneq\tau$, $\tau=\xi\oplus\eta$ and let ζ be a distribution such that $TM=\tau\oplus\zeta$. Let $K\in End(\tau)$ be such that $K=0$ on ξ. We suppose that there exists a natural number $m\geqslant 1$ such that $K^m=0$. Let $\Phi\in End(TM)$ be defined by $\Phi=K$ on τ and $\Phi=Id_\zeta$ on ζ. The following are equivalent:*

1. *τ is integrable.*
2. *The canonical solution associated to Φ is of finite type $\leqslant m$.*

Proof. For $X\in\mathfrak{X}(M)$ we denote X_τ (respectively X_ξ) the projections of X on τ (respectively on ξ), where $X_\tau = X_\xi+X_\eta$, with X_ξ (respectivelyX_η) the projections of X on ξ (respectively on η). So $K(X_\tau)=AX_\eta+BX_\eta\in\tau$ with $A:\eta\to\xi$, $B\in End(\eta)$ and $\Phi(X)_\tau=K(X_\tau)$, $\Phi(X)_\zeta=X_\zeta$. Since $K^m=0$, $\Phi^m_{|\tau}=0$, $\Phi^m_{|\zeta}=Id_\zeta$ and $\Phi^{m+j}=\Phi^m$ for every $j\in\mathbb{N}$.

So

$$\begin{aligned}\left(\Phi^m\circ[\Phi,\Phi]_{\mathcal{FN}}\right)[Y,Z] &= \Phi^m\left([\Phi Y,\Phi Z]+\Phi^2[Y,Z]-\Phi[\Phi Y,Z]-\Phi[Y,\Phi Z]\right)\\ &= \Phi^m[\Phi Y,\Phi Z]+\Phi^m[Y,Z]-\Phi^m[\Phi Y,Z]-\Phi^m[Y,\Phi Z].\end{aligned} \tag{4.4}$$

Suppose that τ is integrable. Since $\Phi=0$ on ξ,

$$\begin{aligned}\Phi Y &= \Phi Y_\eta+\Phi Y_\zeta = AY_\eta+BY_\eta+Y_\zeta = C_\tau+Y_\zeta,\\ \Phi Z &= \Phi Z_\eta+\Phi Z_\zeta = AZ_\eta+BZ_\eta+Z_\zeta = D_\tau+Z_\zeta,\end{aligned}$$

where $C_\tau=AY_\eta+BY_\eta\in\tau$ and $D_\tau=AZ_\eta+BZ_\eta\in\tau$. Since τ is integrable, $[C_\tau,D_\tau]\in\tau=Ker\,\Phi^m$, so $\Phi^m([C_\tau,D_\tau])=0$ and

$$\Phi^m\circ[\Phi Y,\Phi Z] = \Phi^m\left(\left[C_\tau,Z_\zeta\right]\right)+\Phi^m\left(\left[Y_\zeta,D_\tau\right]\right)+\Phi^m\left(\left[Y_\zeta,Z_\zeta\right]\right). \tag{4.5}$$

Similarly, since $[Y_\tau,Z_\tau]$, $[C_\tau,Z_\tau]$, $[Y_\tau,D_\tau]\in\tau=Ker\,\Phi^m$

$$\Phi^m[Y,Z] = \Phi^m\left(\left[Y_\tau+Y_\zeta,Z_\tau+Z_\zeta\right]\right) = \Phi^m\left(\left[Y_\zeta,Z_\tau\right]\right)+\Phi^m\left(\left[Y_\tau,Z_\zeta\right]\right)+\Phi^m\left(\left[Y_\zeta,Z_\zeta\right]\right), \tag{4.6}$$

$$\Phi^m[\Phi Y,Z] = \Phi^m\left[C_\tau+Y_\zeta,Z_\tau+Z_\zeta\right] = \Phi^m\left[C_\tau,Z_\zeta\right]+\Phi^m\left[Y_\zeta,Z_\tau\right]+\Phi^m\left[Y_\zeta,Z_\zeta\right], \tag{4.7}$$

$$\Phi^m [Y, \Phi Z] = \Phi^m \left[Y_\tau + Y_\zeta, D_\tau + Z_\zeta\right] = \Phi^m \left[Y_\zeta, D_\tau\right] + \Phi^m \left[Y_\tau, Z_\zeta\right] + \Phi^m \left[Y_\zeta, Z_\zeta\right]. \quad (4.8)$$

Replacing (4.5), (4.6), (4.7), (4.8) in (4.4) we obtain

$$\begin{aligned}\left(\Phi^m \circ [\Phi, \Phi]_{\mathcal{FN}}\right)[Y, Z] &= \Phi^m \left(\left[C_\tau, Z_\zeta\right]\right) + \Phi^m \left(\left[Y_\zeta, D_\tau\right]\right) + \Phi^m \left(\left[Y_\zeta, Z_\zeta\right]\right) \\ &\quad + \Phi^m \left(\left[Y_\zeta, Z_\tau\right]\right) + \Phi^m \left(\left[Y_\tau, Z_\zeta\right]\right) + \Phi^m \left(\left[Y_\zeta, Z_\zeta\right]\right) \\ &\quad - \left(\Phi^m \left[C_\tau, Z_\zeta\right] + \Phi^m \left[Y_\zeta, Z_\tau\right] + \Phi^m \left[Y_\zeta, Z_\zeta\right]\right) \\ &\quad - \left(\Phi^m \left[Y_\zeta, D_\tau\right] + \Phi^m \left[Y_\tau, Z_\zeta\right] + \Phi^m \left[Y_\zeta, Z_\zeta\right]\right) \\ &= 0,\end{aligned}$$

and it follows that the canonical solution associated to Φ is of finite type $\leqslant m$.

Conversely, suppose that the canonical solution of the Maurer-Cartan equation associated to Φ is of finite type $\leqslant m$, i.e. $\Phi^m \circ [\Phi, \Phi]_{\mathcal{FN}} = 0$. We will prove that $[Y, Z] \in \tau = Ker\, \Phi^m$ for every $Y, Z \in \tau$ by taking in account several cases.

a) Let $Y, Z \in \xi$. Then $\Phi Y = \Phi Z = 0$ and by using (4.4) we obtain

$$\Phi^m [\Phi, \Phi]_{\mathcal{FN}} ([Y, Z]) = \Phi^{m+2} [Y, Z] = \Phi^m [Y, Z] = 0.$$

So $[Y, Z] \in \tau$.

b) Let $Y \in \xi$, $Z \in \tau$, $Z = Z_\xi + Z_\eta$. Then

$$[Y, Z] = \left[Y, Z_\xi\right] + [Y, Z_\eta]. \quad (4.9)$$

By a) $\left[Y, Z_\xi\right] \in \tau$ and from (4.9) it follows that $[Y, Z] \in \tau$ if and only if $[Y, Z_\eta] \in \tau$.

Since $\Phi Y = 0$, by using (4.4) we have

$$\begin{aligned}[\Phi, \Phi]_{\mathcal{FN}} ([Y, Z_\eta]) &= [\Phi Y, \Phi Z_\eta] + \Phi^2 [Y, Z_\eta] - \Phi [\Phi Y, Z_\eta] - \Phi [Y, \Phi Z_\eta] \\ &= \Phi^2 [Y, Z_\eta] - \Phi [Y, \Phi Z_\eta]\end{aligned}$$

and

$$\begin{aligned}\Phi^m \circ [\Phi, \Phi]_{\mathcal{FN}} ([Y, Z_\eta]) &= \Phi^m [Y, Z_\eta] - \Phi^m [Y, \Phi Z_\eta] \\ &= \Phi^m ([Y, (Id - \Phi) Z_\eta]) = 0.\end{aligned}$$

In particular $[Y, (Id - \Phi) Z_\eta] \in \tau$ for every $Z_\eta \in \eta$.

But

$$\det\left((Id - \Phi)_{|\eta}\right) = \det (Id_\eta - B) = 1, \quad (4.10)$$

so $(Id - \Phi)_{|\eta}$ is a bijection. It follows that $[Y, Z] \in \tau$ for every $Y \in \xi$ and $Z \in \tau$.

c) Let $Y, Z \in \tau$. Since

$$[Y, Z] = \left[Y_\xi + Y_\eta, Z_\xi + Z_\eta\right] = \left[Y_\xi, Z_\xi\right] + \left[Y_\eta, Z_\xi\right] + \left[Y_\xi, Z_\eta\right] + [Y_\eta, Z_\eta]$$

and $\left[Y_\xi, Z_\xi\right], \left[Y_\eta, Z_\xi\right], \left[Y_\xi, Z_\eta\right] \in \tau$ it follows that $[Y, Z] \in \tau$ if and only if $[Y_\eta, Z_\eta] \in \tau$.

We have

$$\begin{aligned}[\Phi, \Phi]_{\mathcal{FN}} ([Y_\eta, Z_\eta]) &= [\Phi Y_\eta, \Phi Z_\eta] + \Phi^2 [Y_\eta, Z_\eta] - \Phi [\Phi Y_\eta, Z_\eta] - \Phi [Y_\eta, \Phi Z_\eta] \\ &= [AY_\eta + BY_\eta, AZ_\eta + BZ_\eta] + \Phi^2 [Y_\eta, Z_\eta] \\ &\quad - \Phi [AY_\eta + BY_\eta, Z_\eta] - \Phi [Y_\eta, AZ_\eta + BZ_\eta].\end{aligned}$$

Since $AY_\eta, AZ_\eta \in \xi$ it follows that $[AY_\eta, AZ_\eta], [AY_\eta, BZ_\eta], [BY_\eta, AZ_\eta], [AY_\eta, Z_\eta], [Y_\eta, AZ_\eta] \in \tau = \ker \Phi^m$ and

$$\begin{aligned}\Phi^m [\Phi, \Phi]_{\mathcal{FN}} ([Y_\eta, Z_\eta]) &= \Phi^m [BY_\eta, BZ_\eta] + \Phi^m [Y_\eta, Z_\eta] - \Phi^m [BY_\eta, Z_\eta] - \Phi^m [Y_\eta, BZ_\eta] \\ &= \Phi^m [Y_\eta, (Id - B) Z_\eta] - \Phi^m [BY_\eta, (Id - B) Z_\eta] \\ &= \Phi^m [(Id - B) Y_\eta, (Id - B) Z_\eta] = 0\end{aligned}$$

for every $Y_\eta, Z_\eta \in \eta$.

As before, by (4.10) it follows that $\Phi^m [Y_\eta, Z_\eta] = 0$ for every $Y_\eta, Z_\eta \in \eta$ and this implies that $[Y, Z] \in \tau$ for every $Y, Z \in \tau$. ∎

Notation 6. *Let $\xi \subset TM$ be a distribution. We denote by $\xi^\star$ the smallest involutive distribution of TM such that $\xi \subset \xi^\star$. If $\mathcal{E} = \{X_1, \cdots, X_s\}$ are generators of ξ on an open subset U of M, then for every $x \in U$, $\xi^\star_x$ is the linear subspace of T_xM generated by $\left[X_{i_1}, \left[X_{i_2}, \left[\cdots, X_{i_k}\right]\right]\right](x)$, $k \geqslant 1$, $1 \leqslant i_k \leqslant s$.*

Remark 7. *If* $\dim \xi^\star_x$ *is independent of x, $\xi^\star$ is a distribution, but in general* $\dim \xi^\star_x$ *depends on x. If $\xi^\star$ is a distribution, then $\xi^\star$ is the smallest integrable distribution containing ξ [11].*

Corollary 4. *Let M be a smooth manifold of dimension n, $\xi \subset TM$ a distribution of dimension s such that $\xi^\star$ is a distribution of dimension $d > s$. Then for every $x \in M$ there exists a neighborhood U of x and $\Phi \in \Lambda^1(U, TU)$ such that* $\ker \Phi = \xi_{|U}$ *and the canonical solution of the Maurer-Cartan equation associated to Φ is of finite type r, where $r = \min\left\{m \in \mathbb{N} : m \geqslant \frac{d}{s}\right\}$.*

Proof. Since $d > s$, ξ is not integrable (the case $d = s$ and $r = 1$ follows by Theorem 3).

For each $x \in M$ there exists a neighborhood U of x and a basis $(X_1, \cdots, X_n)$ of TM on U such that $(X_1, \cdots, X_s)$ is a basis of ξ and $(X_1, \cdots, X_d)$ is a basis of $\xi^\star$ on U.

We define $\Phi \in End(TU)$ as $\Phi X_i = 0, i = 1, \cdots, s$ $\Phi(X_i) = X_{i-s}, i = s+1, \cdots, d$, $\Phi(X_i) = X_i$, $i = d+1, \cdots, n$.

Since $K^j \neq 0$ if $d - js > 0$ and $K^j = 0$ if $d - js \leqslant 0$, $j \in \mathbb{N}$, by Theorem 4, the canonical solution of the Maurer-Cartan equation associated to Φ is of finite type $\leqslant r$. ∎

5 Deformations of foliations of codimension 1

One of the major open problems in Complex Analysis is the nonexistence of Levi-flat hypersurfaces in the complex projective plane. By using deformations in an adapted DGLA, the authors proved in [1] the nonexistence of transversally parallelizable Levi-flat hypersurfaces in the complex projective plane. This was done by using partial differential equations for infinitesimal deformations of Levi-flat hypersurfaces. A possible idea would be to use the solutions of Maurer-Cartan equation in the DGLA $\left(\mathcal{D}^\star(M), \urcorner, [\cdot,\cdot]\right)$, where M is a Levi-flat hypersurface in the complex projective plane to solve the above mentioned conjecture. The Proposition 4 below gives an equation for infinitesimal deformations of Levi-flat hypersurfaces by means of the solutions of Maurer-Cartan equation in the DGLA $\left(\mathcal{D}^\star(M), \urcorner, [\cdot,\cdot]\right)$. But in [6] it was proved that this leads to the same second order elliptic differential equation for the infinitesimal deformations of Levi-flat hypersurfaces as in [1]. We also mention that it was demonstrated in [2] that the deformation theory in the DGLA $\left(\mathcal{D}^\star(M), \urcorner, [\cdot,\cdot]\right)$ is not obstructed but it is level-wise obstructed.

Definition 12. *By a differentiable family of deformations of an integrable distribution ξ we mean a differentiable family $\omega : \mathcal{D} = (\xi_t)_{t\in I} \mapsto t \in I =]-a, a[$, $a > 0$, of integrable distributions such that $\xi_0 = \omega^{-1}(0) = \xi$.*

Remark 8. *An integrable distribution ξ of codimension 1 in a smooth manifold L is called co-orientable if the normal space to the foliation defined by ξ is orientable. We recall that ξ is co-orientable if and only if there exists a 1-form γ on L such that $\xi =$* $\ker$ *γ (see for ex. [5]). A couple (γ, X) where $\gamma \in \wedge^1(L)$ and X is a vector field on L such that* $\ker$ *$\gamma = \xi$ and $\gamma(X) = 1$ is called a DGLA defining couple and verifies $d\gamma = -\iota_X d\gamma \wedge \gamma$ (see [1]).*

If $(\xi_t)_{t\in I}$ is a differentiable family of deformations of an integrable co-orientable distribution ξ, then the distribution ξ_t is co-orientable for t small enough. So, if ξ is an integrable co-orientable distribution of codimension 1 in L and $(\xi_t)_{t\in I}$ is a differentiable family of deformations of ξ we may consider a DGLA defining couple (γ_t, X_t) for every t small enough such that $t \mapsto (\gamma_t, X_t)$ is differentiable on a neighborhood of the origin.

Lemma 7. *Let L be a C^∞ manifold and $\xi \subset T(L)$ a co-orientable distribution of codimension 1. Let (γ, X) be a DGLA defining couple and denote $\Phi \in End(TM)$ the endomorphism corresponding to $\gamma \otimes X \in \Lambda^1 M \otimes TM$. Then Φ is defined on $TM = \xi \oplus \mathbb{R}X$ as $\Phi = 0$ on ξ and $\Phi = Id_{\mathbb{R}X}$ on $\mathbb{R}X$.*

Proof. Let $Y = Y_\xi + \lambda X$ be a vector field on L, $Y_\xi \in \xi$, $\lambda \in \mathbb{R}$. Then

$$(\gamma \otimes X)(Y) = \gamma(Y)X = \gamma\left(Y_\xi + \lambda X\right)X = \lambda X.$$

∎

We recall the following lemma from [1]:

Lemma 8. *Let L be a C^∞ manifold and X a vector field on L. For $\alpha, \beta \in \Lambda^\star(L)$, set*

$$\{\alpha, \beta\} = \mathcal{L}_X\alpha \wedge \beta - \alpha \wedge \mathcal{L}_X\beta \tag{5.1}$$

where $\mathcal{L}_X$ is the Lie derivative. Then $\left(\Lambda^\star(L), d, \{\cdot,\cdot\}\right)$ is a DGLA.

Proposition 4. *Let L be a C^2 manifold and $\xi \subset T(L)$ an integrable co-orientable distribution of codimension 1. Let $(\xi_t)_{t\in I}$ be a differentiable family of deformations of ξ such that ξ_t is co-orientable and integrable for every $t \in I$ and let (γ_t, X_t) be a DGLA defining couple for ξ_t such that $t \mapsto (\gamma_t, X_t)$ is differentiable on I. Denote $\gamma = \gamma_0$, $\alpha = \frac{d\gamma_t}{dt}_{|t=0}$, $X = X_0$, $Y = \frac{dX_t}{dt}_{|t=0}$. Then*

$$\delta\alpha + \mathcal{L}_Y\gamma \wedge \gamma = 0$$

where

$$\delta = d + \{\gamma, \cdot\}$$

and $\{\cdot,\cdot\}$ is defined in (5.1).

In particular $\delta\alpha(V, W) = 0$ for every vector fields V, W tangent to ξ.

Proof. Since

$$\gamma_t(X_t) = (\gamma + t\alpha + o(t))(X + tY + o(t)) = 1 + t(\alpha(X) + \gamma(Y)) + o(t) = 1$$

it follows that

$$\alpha(X) + \gamma(Y) = 0. \tag{5.2}$$

Denote $\sigma(t) = \gamma_t \otimes X_t \in \Lambda^1 M \otimes TM$. By Corollary 3 and Lemma 7 the canonical solution of the Maurer-Cartan equation in $\left(\mathcal{D}^\star(L), [\cdot,\cdot], \urcorner\right)$ associated to $\sigma(t)$ is of finite type 0 for each t, so $[\sigma(t), \sigma(t)]_{\mathcal{FN}} = 0$ for every t. We have

$$\begin{aligned} \sigma(t) &= \gamma_t \otimes X_t = (\gamma + t\alpha + o(t)) \otimes (X + tY + o(t)) \\ &= \gamma \otimes X + t(\alpha \otimes X + \gamma \otimes Y) + o(t). \end{aligned}$$

Since $[\gamma \otimes X, \gamma \otimes X]_{\mathcal{FN}} = [\sigma(0), \sigma(0)]_{\mathcal{FN}} = 0$,

$$[\sigma(t), \sigma(t)]_{\mathcal{FN}} = 2t\left([\gamma \otimes X, \alpha \otimes X + \gamma \otimes Y]_{\mathcal{FN}}\right) + o(t) = 0$$

and it follows that

$$[\gamma \otimes X, \alpha \otimes X + \gamma \otimes Y]_{\mathcal{FN}} = 0. \tag{5.3}$$

But Proposition 1 gives

$$\begin{aligned} [\gamma \otimes X, \alpha \otimes X]_{\mathcal{FN}} &= \gamma \wedge \mathcal{L}_X\alpha \otimes X - \mathcal{L}_X\gamma \wedge \alpha \otimes X - (d\gamma \wedge \iota_X\alpha \otimes X + \iota_X\gamma \wedge d\alpha \otimes X) \\ &= -\{\gamma, \alpha\} \otimes X - \alpha(X)\, d\gamma \otimes X - d\alpha \otimes X \\ &= -\delta\alpha \otimes X - \alpha(X)\, d\gamma \otimes X \end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
[\gamma \otimes X, \gamma \otimes Y]_{\mathcal{FN}} &= \gamma \wedge \mathcal{L}_X \gamma \otimes Y - \mathcal{L}_Y \gamma \wedge \gamma \otimes X \\
&\quad -d\gamma \wedge \iota_X \gamma \otimes Y - \iota_Y \gamma \wedge d\gamma \otimes X \\
&= \gamma \wedge \iota_X d\gamma \otimes Y - \mathcal{L}_Y \gamma \wedge \gamma \otimes X \\
&\quad -d\gamma \otimes Y - \gamma (Y)\, d\gamma \otimes X.
\end{aligned}$$

By using Remark 8 it follows that

$$[\gamma \otimes X, \gamma \otimes Y]_{\mathcal{FN}} = -\mathcal{L}_Y \gamma \wedge \gamma \otimes X - \gamma (Y)\, d\gamma \otimes X \tag{5.5}$$

and by (5.3), (5.4) (5.5) and (5.2) we obtain

$$-\delta\alpha - (\alpha (X) + \gamma (Y))\, d\gamma - \mathcal{L}_Y \gamma \wedge \gamma = -\delta\alpha - \mathcal{L}_Y \gamma \wedge \gamma = 0.$$

∎

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