

Yat Sun Poon*

Abelian Complex Structures and Generalizations

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Abstract: After a review on the development of deformation theory of abelian complex structures from both the classical and generalized sense, we propose the concept of semi-abelian generalized complex structure. We present some observations on such structure and illustrate this new concept with a variety of examples.

Keywords: abelian complex structures, deformation, nilmanifolds

MSC: Primary 53D18; Secondary 32G07, 14D15, 17B30, 17B62

Dedicated to: In memory of my father Yan Ding Poon

1 Introduction

From the onset generalized geometry is conceived to encompass complex and symplectic geometry [40] [43]. Its local model theory was settled a few years ago [1] [5]. It adds interests to holomorphic Poisson geometry [36] [41] [44] [45] [65].

While complex structures and symplectic structures are often united in the realm of Kähler geometry, generalized geometry enables one to study them with new perspectives, especially from the viewpoint of Kähler geometry with torsion [15] [31]. There is a very rich collection of examples of generalized complex structures beyond classical symplectic structures, complex structures, and holomorphic Poisson structures. A very prominent class of non-Kählerian manifolds on which complex and symplectic structures coexist is nilmanifolds [12] [16] [32] [71].

This author takes the perspective that generalized geometry, especially its deformation theory, is a degree-2 realization of the extended deformation theory developed by Kontsevich et al. [7] [18] [19] [47] [58] [61]. In particular, inspired by the work of Manin and Merkulov [42] [56] [57] [59], this author computed the Frobenius structure on the extended moduli of Kodaira surfaces [61]. One could restrict an analysis of the Frobenius structure from the extended moduli to a generalized moduli in the sense that only degree-2 deformations are allowed. For example, it is proved that the Frobenius structure on the generalized moduli of Kodaira manifolds in all dimensions is trivial [62, Theorem 3].

Lots of geometric consideration and much of the author's work on Frobenius structure and holomorphic Poisson deformation rely on the underlying complex structures being abelian [3] [4] [6] [10] [30] [63] [64]. This class of invariant complex structures was first seen in [9], and is often studied along the line of nilpotent complex structures [28] [29] [69].

Some of the reasons for focusing on abelian complex structures on nilmanifolds M are due to its accessibility in terms of cohomology theory and deformation theory. Deformation theory of generalized complex structures is based on the concept of Lie bialgebroids [40] [49] [50] [51]. Deformation of a generalized complex structure \mathcal{J} is controlled by its differential Gerstenhaber algebra $DGA(M, \mathcal{J})$. Gerstenhaber algebra was invented to study deformation of rings after Kodaira's deformation theory of complex manifolds [35]. Part of

*Corresponding Author: Yat Sun Poon: Department of Mathematics, University of California at Riverside, Riverside, CA 92521, U.S.A., E-mail: ypoon@ucr.edu.

this structure is a Schouten bracket on sections of the exterior algebra of a vector bundle [50]. When \mathcal{J} is a left-invariant abelian complex structure on a nilmanifold $M = \Delta \backslash G$ modeled on a Lie group G , $DGA(M, \mathcal{J})$ is quasi-isomorphic to its invariant counterpart $DGA(\mathfrak{g}, \mathcal{J})$ [17]. The invariant $DGA(\mathfrak{g}, \mathcal{J})$ enjoys additional features as stated in Proposition 1 of the next section.

We review the construction of differential Gerstenhaber algebra $DGA(M, \mathcal{J})$ in Section 3. To motivate our work in Section 4, we present some basic properties when \mathcal{J} is a classical object such as complex structure, holomorphic Poisson structure, and symplectic structure.

The goal of this paper is to present the concept of "semi-abelian" generalized complex structures in terms of $DGA(\mathfrak{g}, \mathcal{J})$. It is formally stated in Definitions 2 and 3. Definition 2 is made on algebra level. It is designed to capture the features of the invariant part of $DGA(M, \mathcal{J})$ as if the generalized complex structure \mathcal{J} is an abelian complex structure. It could be extended to analyze generalized complex structures on unimodular algebras [2] [3]. The definition is made so that all abelian complex structures are semi-abelian (generalized) complex structures.

In Section 5, we provide a collection of examples to illustrate the proposed concept. This collection includes symplectic structure, non-abelian complex structure, and generalized complex structures of various types. In particular we find the existence of semi-abelian (generalized) complex structures that fail to be abelian. There are also nilmanifolds on which there are generalized complex structures but none of them could be semi-abelian.

2 Abelian Complex Structures

2.1 Kodaira surfaces

A collection of examples of compact complex surfaces have been inspiring objects in complex manifold theory. To name a few, we have complex projective plane, cubic surfaces, K3-surfaces, Hopf surfaces, and Kodaira surfaces. Kodaira surfaces were discovered as a collection of elliptic surfaces in classification of compact complex surfaces [46]. They have trivial canonical bundle. It is also known that the manifold M in question is the co-compact quotient of the complex two-dimensional vector space \mathbb{C}^2 and the co-compact lattice transformation leave invariant a complex $(2, 0)$ -form η , subjected to the conditions

$$d\eta = 0, \quad \eta \wedge \eta = 0, \quad \eta \wedge \bar{\eta} > 0 \quad (1)$$

everywhere on the manifold M . As such Kodaira surface is a *nilmanifold* with an invariant complex structure.

In 1976, Thurston published his well-known paper presenting Kodaira surface as a non-Kählerian symplectic manifold [71] and remarked that there are an abundance of similar examples in higher dimensions.

Regarding the complex analytic aspect of Kodaira surfaces, Borcea examined the moduli of complex structures on Kodaira surfaces by varying the co-compact lattices of transformations [13]. The starting point of his computation is to recognize that the algebra \mathfrak{g} on the underlying real vector space for \mathbb{C}^2 is solely given by

$$[X_1, X_2] = 2X_4. \quad (2)$$

All other Lie brackets are equal to zero. Therefore, as a Lie algebra it is the direct sum of the trivial one-dimensional algebra and the real three-dimensional Heisenberg algebra \mathfrak{h}_3 . Borcea's computation of the moduli relied on parametrization of the complex 2-form η in (1) with respect to the basis of the algebra $\{X_1, X_2, X_3, X_4\}$ and its dual.

2.2 Nilpotent complex structures

On any compact complex surface, the Frölicher spectral sequence associated to the Dolbeault bicomplex degenerates at first step [11] [46]. In a cluster of papers [23] [24] [25] [26] [28] [29], Cordero et al. studied the degeneracy of Frölicher spectral sequence of invariant complex structures on nilmanifolds in all dimensions.

A compact manifold M is a nilmanifold if it is the quotient of connected simply-connected nilpotent Lie group G by a discrete subgroup Δ so that $M = \Delta \backslash G$ [54]. It is a fundamental observation that such a Lie group gives rise to a nilmanifold if and only if its Lie algebra \mathfrak{g} admits a basis with respect to which the structure constants are rational [54]. An invariant complex structure on a nilmanifold M as a right quotient space is given by a left-invariant complex structure on the Lie group G . Equivalently, it is a real linear map $J : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $J \circ J = -1$ and its Nijenhuis tensor vanishes. i.e., for all X, Y in \mathfrak{g} ,

$$[[JX, JY]] - [[X, Y]] - J[[X, Y]] - J[[X, JY]] = 0. \tag{3}$$

One of the discoveries by Cordero et al. was the concept of nilpotent complex structures, see [26] as a preprint reference for [23]. This subject was further developed by Salamon [69], which has become a key reference for this subject for many publications including this one.

Given an invariant complex structure J on $M = \Delta \backslash G$, the space $\mathfrak{g}^{1,0}$ of $(+i)$ -eigenvectors for J in complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ forms a complex Lie algebra. Denote the $(-i)$ -eigenspace by $\mathfrak{g}^{0,1}$. The dual spaces are respectively $\mathfrak{g}^{*(1,0)}$ and $\mathfrak{g}^{*(0,1)}$. The complex structure J is nilpotent if there exists an ordered basis $\{\omega^1, \dots, \omega^m\}$ for $\mathfrak{g}^{*(1,0)}$ and constants A_{kl}^j and B_{kl}^j such that for all $1 \leq j \leq m$,

$$d\omega^j = \sum_{k,l < j} A_{kl}^j \omega^k \wedge \bar{\omega}^l + \sum_{k < l < j} B_{kl}^j \omega^k \wedge \omega^l. \tag{4}$$

In the meantime, inspired by a search for geometry with finite holonomy, a class of complex structures on nilmanifolds emerged in [9]. It satisfies the condition that for all $X, Y \in \mathfrak{g}$,

$$[[X, JY]] = [[X, Y]]. \tag{5}$$

This condition is equivalent to require the $(+i)$ -eigenspace $\mathfrak{g}^{1,0}$ to form a complex abelian algebra although \mathfrak{g} and $\mathfrak{g}_{\mathbb{C}}$ are not necessarily abelian. Such complex structures are called *abelian* complex structures. In [69] it is proved that if a nilpotent algebra \mathfrak{g} admits an abelian complex structure then there exists an ordered basis $\{\omega^1, \dots, \omega^m\}$ for $\mathfrak{g}^{*(1,0)}$ and constants A_{kl}^j such that for all $1 \leq j \leq m$,

$$d\omega^j = \sum_{k,l < j} A_{kl}^j \omega^k \wedge \bar{\omega}^l. \tag{6}$$

We address this basis for $\mathfrak{g}^{*(0,1)}$ an ascending basis. The observations above lead to the next proposition.

Proposition 1. *Let J be a left-invariant complex structure on a nilmanifold with Lie algebra \mathfrak{g} . Let $\mathfrak{g}^{1,0}$ be the space of invariant $(1, 0)$ -vectors and $\mathfrak{g}^{*(0,1)}$ the space of invariant $(0, 1)$ -forms. The following conditions are equivalent.*

- $\mathfrak{g}^{1,0}$ is an abelian complex algebra.
- $\bar{\partial}\bar{\omega} = 0$ for all $\bar{\omega} \in \mathfrak{g}^{*(0,1)}$.

J is abelian if and only if one of these conditions is satisfied.

Low-dimension nilpotent algebras are classified in [37]. Given the choice of basis and constraints in (4) for nilpotent complex structures and those in (6) for abelian complex structures, there is a classification of the underlying nilpotent algebras to admit such complex structures when the real dimension of the algebra is at most six [26] [27] [69]. A coarse dimension count on family of invariant complex structures on each admissible algebra was also done in [69]. Classification of abelian complex structures in low dimension is also extended to algebras other than the nilpotent ones [2].

Notations. To further present our work, we adopt a convention popularized by [69]. Suppose that e^j, e^k, e^l are 1-forms we use e^{jkl} to represent their exterior product $e^j \wedge e^k \wedge e^l$. Likewise when e_j, e_k are vectors, the bivector $e_j \wedge e_k$ is represented by e_{jk} . When $\{e^1, \dots, e^n\}$ is an ordered basis for \mathfrak{g}^* , the structure equations

for the algebra \mathfrak{g} in terms of the Chevalley-Eilenberg differential is represented by the n -tuple (de^1, \dots, de^n) . For example, when $de^n = e^{ab} + e^{kl}$ the last entry in (de^1, \dots, de^n) will be represented by $ab + kl$.

With the above notations, the algebra of the real 3-dimensional Heisenberg algebra \mathfrak{h}_3 is represented by $(0, 0, 12)$ and the non-trivial algebra with invariant complex structure given in (2) is isomorphic to $(0, 0, 0, 12) = \mathbb{R} \oplus \mathfrak{h}_3$.

2.3 Examples of abelian complex structures

There are only five non-trivial six-dimensional 2-step nilpotent algebras admitting abelian complex structures [27] [69]. Each has a high-dimension generalization.

Example 1. *The algebras $(0, 0, 0, 0, 0, 12)$ and $(0, 0, 0, 0, 0, 12 + 34)$.*

Consider $(0, 0, 0, 0, 0, 12)$ as the direct sum $\mathbb{R}^3 \oplus \mathfrak{h}_3$. The resulting nilmanifold is the product of Kodaira surface with a real two-dimensional torus.

The four-dimensional algebra $\mathbb{R} \oplus \mathfrak{h}_3 = (0, 0, 0, 12)$ has non-trivial six-dimensional generalization, namely $\mathbb{R} \oplus \mathfrak{h}_5 = (0, 0, 0, 0, 0, 12 + 34)$. In general, for any natural numbers m and n , we have $\mathbb{R}^{2m+1} \oplus \mathfrak{h}_{2n+1}$ where \mathfrak{h}_{2n+1} is the $(2n+1)$ -dimensional Heisenberg algebra with structure equations

$$(0, \dots, 0, 12 + \dots + (2k-1)(2k) + \dots + (2n-1)(2n))$$

where the first $2n$ -entries are all equal to zero. An invariant abelian complex structure is given by $Je_{2k-1} = e_{2k}$ for all $1 \leq k \leq n$ and $Je_{2n+2m+1} = e_{2n+2m+2}$. The resulting complex manifold corresponding to $\mathbb{R} \oplus \mathfrak{h}_{2n+1}$ is addressed as Kodaira manifolds [61] [62]. We will, however, see the algebra $(0, 0, 0, 0, 0, 12 + 34)$ in a different light in Example 7.

Example 2. *Three other 2-step six-dimensional nilpotent algebras.*

The direct sum of two Heisenberg algebras $(0, 0, 0, 0, 12, 34) = \mathfrak{h}_3 \oplus \mathfrak{h}_3$ admits an abelian complex structure J with $Je_1 = e_2$, $Je_3 = e_4$, and $Je_5 = e_6$. Its high-dimension counterparts $\mathfrak{h}_{2m+1} \oplus \mathfrak{h}_{2n+1}$ also have abelian complex structures.

The algebra $(0, 0, 0, 0, 13+42, 14+23)$ and $(0, 0, 0, 0, 12, 14+23)$ also admit abelian complex structures and high-dimension generalization [52].

Example 3. *A 3-step nilpotent algebra $(0, 0, 0, 12, 14 + 23, 13 + 42)$.*

After a change of bases e^4 to $-e^4$, e^5 to $-e^6$ and e^6 to e^5 , we present the same algebra with structure equations below.

$$de^4 = -e^{12}, \quad de^5 = e^{31} + e^{42}, \quad de^6 = e^{41} - e^{32}. \quad (7)$$

Equivalently,

$$[[e_1, e_2]] = e_4, \quad [[e_3, e_1]] = [[e_4, e_2]] = -e_5, \quad [[e_4, e_1]] = -[[e_3, e_2]] = -e_6.$$

Define a complex structure J by $Je_1 = e_2$, $Je_3 = e_4$, $Je_5 = e_6$ so that $\mathfrak{g}^{*(1,0)}$ is spanned by $\omega^1 = e^1 + ie^2$, $\omega^2 = e^3 + ie^4$, and $\omega^3 = e^5 + ie^6$. As a result of (7),

$$d\omega^1 = 0, \quad d\omega^2 = \frac{1}{2}\omega^1 \wedge \bar{\omega}^1, \quad d\omega^3 = \omega^2 \wedge \bar{\omega}^1.$$

So, the complex structure J is abelian.

2.4 Cohomology and deformation

Given a nilmanifold $M = \Delta \backslash G$, there is an inclusion of left-invariant differential forms in the space of smooth differential forms.

$$\iota : \wedge^k \mathfrak{g}^* \longrightarrow C^\infty(M, \wedge^k T^*M). \quad (8)$$

Nomizu proved that this inclusion is a quasi-isomorphism in the sense that the inclusion map induces an isomorphism at cohomology level [60],

$$\iota : H^*(\mathfrak{g}) \cong H_{DR}^*(M, \mathbb{R})$$

where $H^k(\mathfrak{g})$ is the k -th cohomology with respect to the complex of the Chevalley-Eilenberg differential of \mathfrak{g} . In other words, for any $\omega \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$, $d\omega(X, Y) = -\omega([X, Y])$.

Since the attempt by Sakane on similar quasi-isomorphism for Dolbeault cohomology [68], it has been proved that when M is a nilmanifold with a nilpotent complex structure one obtains a natural quasi-isomorphism.

Theorem 1. [26] *Suppose that $M = \Delta \backslash G$ is a nilmanifold with a nilpotent complex structure, the inclusion of invariant (p, q) -forms in the space of sections of (p, q) -forms is a quasi-isomorphism. i.e., the inclusion map*

$$\iota : \mathfrak{g}_{\mathbb{C}}^{*(p,q)} \longrightarrow C^\infty(M, T^{*(p,q)}M) \quad (9)$$

induces an isomorphism at cohomology level: $H_{\mathfrak{g}}^{p,q}(\mathfrak{g}_{\mathbb{C}}) \cong H_{\mathfrak{g}}^{p,q}(M)$.

In addition, Console and Fino initiated a study on the same issue for all invariant complex structures from the perspective of stability of the desired quasi-isomorphisms [20] [21]. It is remarkable that the above statement remains an open conjecture when the complex structure is merely invariant. For advancement in this direction, please see [66] [67] and the latest development in [33] and references therein.

This author and collaborators took on the issue of deformation of invariant complex structures on nilmanifolds in [52] with a focus on 2-step nilmanifolds with abelian complex structures. The key to enable this investigation was a result similar to Theorem 1. When Θ is the sheaf of germs of holomorphic tangents for the complex nilmanifold M , the result states as below.

Theorem 2. *Suppose that $M = \Delta \backslash G$ is a nilmanifold with an abelian complex structure. The inclusion map*

$$\iota : \mathfrak{g}^{1,0} \otimes \mathfrak{g}^{*(0,k)} \longrightarrow C^\infty(M, T^{1,0} \otimes T^{*(0,k)}) \quad (10)$$

induces an isomorphism at cohomology level: $H_{\mathfrak{g}}^k(\mathfrak{g}^{1,0}) \cong H^k(M, \Theta)$.

The above theorem was initially proved on 2-step nilmanifolds [52, Theorem 1]. It is subsequently expanded to include nilmanifolds with arbitrary number of steps [22, Theorem 3.6]. Both [22] and [52] rely on the fact that the center of the algebra \mathfrak{g} is invariant of the complex structure J when it is abelian. It recreates a (series of) holomorphic fibrations with complex torus as fibers. The proof of Theorem 2 becomes an application of Leray spectral sequence formalism.

The quasi-isomorphism in Theorem 2 enables a construction of moduli of complex structures on Kodaira manifolds in [39] along the line of Borcea's work in [13]. It also enables an analysis on the stability of abelian complex structures under deformation [22]. More broadly, in [52] when an abelian complex structure J is given on a 2-step nilmanifold, the authors considered the Kuranishi space $\text{Kur}(J)$ of the given complex structure J and the subspace $\text{Abel}(J)$ consisting of local deformation parameter space of abelian complex structures with J in the center. Among other results, they found the following.

Proposition 2. [52, Table 1] *Consider $M = \Delta \backslash G$ a nilmanifold with 2-step nilpotent algebra \mathfrak{g} .*

- *If \mathfrak{g} is one of $(0, 0, 0, 0, 0, 0)$, $(0, 0, 0, 0, 0, 12)$, and $(0, 0, 0, 0, 0, 12 + 34)$, there exists an abelian complex structure J on M such that $\dim \text{Abel}(J) = \dim \text{Kur}(J)$.*

- If g is one of $(0, 0, 0, 0, 12, 34)$, $(0, 0, 0, 0, 13 + 42, 14 + 23)$, and $(0, 0, 0, 0, 12, 14 + 23)$, there exists an abelian complex structure J on the manifold M such that $\dim \text{Abel}(J) = \dim \text{Kur}(J) - 1 > 0$.

3 Generalized Complex Structures

A generalized complex structure could be conceived as both a tensorial object and a spinorial object subjected to multiple conditions. Its investigation was initiated by Hitchin [43] and developed by Gualtieri [40]. Given any point x on the manifold M , when X, Y are in $T_x M$ and α, β in $T_x^* M$, one considers the pairing between two elements of $T_x M \oplus T_x^* M$

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\beta(X) + \alpha(Y)). \quad (11)$$

An almost generalized complex structure is a bundle map

$$\mathcal{J} : TM \oplus T^*M \rightarrow TM \oplus T^*M \quad (12)$$

such that $\mathcal{J} \circ \mathcal{J} = -1$ and $\langle \mathcal{J}(X + \alpha), \mathcal{J}(Y + \beta) \rangle = \langle X + \alpha, Y + \beta \rangle$ for all $X + \alpha, Y + \beta$ in $TM \oplus T^*M$ for each point x in the manifold M . It is known that it could be expressed as a map of bundle of direct sum $TM \oplus T^*M$,

$$\mathcal{J} = \begin{pmatrix} J & \Pi \\ B & -J^* \end{pmatrix} \quad (13)$$

where B is a two-form and Π is a bivector [40]. Equivalently, the bundle of $(+i)$ -eigenspace L is maximally isotropic subbundle of the complexification of $TM \oplus T^*M$. Let \bar{L} represent the complex conjugate bundle of L , it is the bundle of $(-i)$ -eigenspace. By virtue of L being maximally isotropic, \bar{L} is also maximally isotropic and $L \oplus \bar{L} = (TM \oplus T^*M)_{\mathbb{C}}$. Since L and \bar{L} are maximally isotropic and the bilinear form $\langle -, - \rangle$ is non-degenerate, it yields naturally isomorphisms by the pairing $\langle -, - \rangle$ as given in (11):

$$L \cong \bar{L}^*, \quad \bar{L} \cong L^*. \quad (14)$$

In subsequent discussion, we often utilize these isomorphisms.

The Courant bracket defined on the space of sections $C^\infty(M, TM \oplus T^*M)$ is

$$\llbracket X + \alpha, Y + \beta \rrbracket = \llbracket X, Y \rrbracket + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2}d(\beta(X) - \alpha(Y)), \quad (15)$$

where $\mathcal{L}_X \beta$ is the Lie derivative of the one-form β along the vector field X and $\llbracket X, Y \rrbracket$ is the Lie bracket of vector fields.

An almost generalized complex structure is integrable if the space of smooth sections of the bundle L , denoted by $C^\infty(M, L)$, is closed with respect to the Courant bracket. Although the Courant bracket on $C^\infty(M, TM \oplus T^*M)$ does not satisfy the Jacobi identity, it is observed in [40, Proposition 3.27] that its restriction on $C^\infty(M, L)$ satisfies the Jacobi identity if and only if $C^\infty(M, L)$ is closed under the Courant bracket. The fact that L is maximally isotropic plays a key role in this observation.

3.1 DGA and cohomology

Under the assumption that a generalized complex structure \mathcal{J} is integrable, one treats L as a Lie algebroid [40] [49] and extends the restriction of the Courant bracket to the space of section of exterior algebras $C^\infty(M, \wedge^\bullet L)$ to obtain what is known as a Gerstenhaber algebra, also as a Schouten algebra [35] [50, Definition 7.5.1]. We will address the restriction of the Courant bracket on $C^\infty(M, \wedge^\bullet L)$ as the Schouten bracket.

Taking complex conjugations, the space $C^\infty(M, \wedge^\bullet \bar{L})$ also inherits a Schouten bracket. Since \bar{L} is a Lie algebroid, it has an associated differential on its dual $\bar{L}^* \cong L$:

$$\bar{\delta} : C^\infty(M, L) \longrightarrow C^\infty(M, \wedge^2 L). \quad (16)$$

It is observed in [40] that the pair (L, \bar{L}) forms a Lie bialgebroid in the sense of [51]. The operator $\bar{\delta}$ extends to an even exterior differential operator and an odd differential Lie superalgebra operator [55] [61]. It means that if $a \in C^\infty(M, \wedge^{|a|}L)$ and $b \in C^\infty(M, \wedge^{|b|}L)$, then

$$\bar{\delta}(a \wedge b) = (\bar{\delta}a) \wedge b + (-1)^{|a|} a \wedge (\bar{\delta}b), \quad \bar{\delta}[[a, b]] = [[\bar{\delta}a, b]] - (-1)^{|a|} [[a, \bar{\delta}b]]. \quad (17)$$

In summary, each generalized complex structure is associated to

$$DGA(M, \mathcal{J}) = \left(C^\infty(M, \wedge^* \bar{L}), \wedge, [[\bullet, \bullet]], \bar{\delta} \right), \quad (18)$$

a differential Gerstenhaber algebra [35] [61].

It is further observed in [40] that the operator $\bar{\delta}$ is elliptic and the integrability of \mathcal{J} is equivalent to $\bar{\delta} \circ \bar{\delta} = 0$. Therefore $DGA(M, \mathcal{J})$ has an associated cohomology theory

$$H^k(M, \mathcal{J}) = \frac{\ker \bar{\delta} : C^\infty(M, \wedge^k L) \longrightarrow C^\infty(M, \wedge^{k+1} L)}{\text{image } \bar{\delta} : C^\infty(M, \wedge^{k-1} L) \longrightarrow C^\infty(M, \wedge^k L)}.$$

As a consequence of (17), the exterior product and the Schouten bracket descend to the cohomology space so that we obtain a Gerstenhaber algebra:

$$(H^*(M, \mathcal{J}), \wedge, [[\bullet, \bullet]]).$$

From the view point of extended deformation [47] [58] [61], one treats an element Γ in $C^\infty(M, \wedge^* L)$ as an integrable *extended* deformation if it satisfies the Maurer-Cartan equation

$$\bar{\delta}\Gamma + \frac{1}{2} [[\Gamma, \Gamma]] = 0. \quad (19)$$

We obtain a generalized deformation when Γ is a degree-2 section, i.e., an element in $C^\infty(M, \wedge^2 L)$.

3.2 Symplectic manifolds

Suppose that M is a smooth manifold and Ω is a symplectic form, we treat a contraction of Ω with a tangent vector as a bundle map $\Omega : TM \rightarrow T^*M$. Since Ω is non-degenerate, its inverse Ω^{-1} is a well-defined bivector. In the matrix representation of a generalized complex structure \mathcal{J} , by choosing $J = 0$, $B = \Omega$, and $\Pi = \Omega^{-1}$, one obtains a generalized complex structure. Pointwisely,

$$L = \{X - i\Omega(X) : X \in TM\}, \quad \bar{L} = \{X + i\Omega(X) : X \in TM\}. \quad (20)$$

The integrability of \mathcal{J} as a generalized complex structure amounts to the identity

$$[[X - i\Omega(X), Y - i\Omega(Y)]] = [[X, Y]] - i\Omega([[X, Y]]) \quad (21)$$

for all X, Y in $C^\infty(M, TM)$. It is equivalent to $d\Omega = 0$. Let X, Y, Z be any smooth vector fields on M .

$$\begin{aligned} & \bar{\delta}(Z - i\Omega(Z))(X + i\Omega(X), Y + i\Omega(Y)) \\ &= -\langle Z - i\Omega(Z), [[X + i\Omega(X), Y + i\Omega(Y)]] \rangle = -\langle Z - i\Omega(Z), [[X, Y]] + i\Omega([[X, Y]]) \rangle \\ &= i(\Omega(Z))[[X, Y]] = -i(d(\Omega(Z)))(X, Y). \end{aligned} \quad (22)$$

Consider a bundle map

$$\varphi : L \hookrightarrow (TM \oplus T^*M)_{\mathbb{C}} \longrightarrow T^*M_{\mathbb{C}} \quad (23)$$

obtained by a composition of inclusion and projection. φ is a bundle isomorphism because Ω is non-degenerate. It extends naturally to a bundle map of exterior powers so that for all $a, b \in C^\infty(M, L)$, $\varphi(a) \wedge \varphi(b) = \varphi(a \wedge b)$. By (22) $\varphi(\bar{\delta}a) = d(\varphi(a))$. Moreover, for any μ, ν in $C^\infty(M, T^*M_{\mathbb{C}})$ define

$$[[\mu, \nu]]_{\Omega} := [[\Omega^{-1}\mu, \Omega^{-1}\nu]] \quad (24)$$

where the bracket on the right hand side is the usual Lie bracket of vector fields. By (21), $\llbracket a, b \rrbracket = \llbracket \varphi(a), \varphi(b) \rrbracket_\Omega$. In particular, the differential Gerstenhaber algebra $DGA(M, \Omega)$ defined on $C^\infty(M, \wedge^\bullet L)$ with Courant bracket $\llbracket \bullet, \bullet \rrbracket$ is isomorphic to the differential Gerstenhaber algebra defined on the space of differential forms

$$\left(C^\infty(M, \wedge^\bullet T^*M_{\mathbb{C}}), \wedge, \llbracket \bullet, \bullet \rrbracket_\Omega, d \right) \quad (25)$$

where the differential d is the deRham differential. The resulting cohomology is the complexified deRham cohomology $(H_{DR}^\bullet(M, \mathbb{C}), \wedge, \llbracket \bullet, \bullet \rrbracket_\Omega)$.

Proposition 3. *When Ω is a symplectic form, the bundle map (23) defines an isomorphism of differential Gerstenhaber algebra.*

$$\varphi : DGA(M, \Omega) \cong \left(C^\infty(M, \wedge^\bullet T^*M_{\mathbb{C}}), \wedge, \llbracket \bullet, \bullet \rrbracket_\Omega, d \right) \quad (26)$$

The isomorphism above demonstrates that the differential Gerstenhaber algebra of a symplectic structure as a generalized complex structure is consistent with the ones in extended deformation theory in [47] [57] [58] [61] and weak mirror symmetry [18] [19] [58].

3.3 Classical complex structures

The matrix representation of \mathcal{J} in (13) for a classical complex structure is given by $B = 0$ and $\Pi = 0$. The bundles of $(+i)$ and $(-i)$ eigenvectors with respect to \mathcal{J} are respectively

$$L = T^{1,0} \oplus T^{*(0,1)}, \quad \bar{L} = T^{0,1} \oplus T^{*(1,0)} \quad (27)$$

where $T^{1,0}$ and $T^{0,1}$ are bundles of $(1,0)$ -vectors and $(0,1)$ -vectors with respect to J . $T^{*(1,0)}$ and $T^{*(0,1)}$ are their respective duals.

Proposition 4. *Let \mathcal{J} be a classical complex structure and $\bar{\delta}$ the Lie algebroid differential on $C^\infty(M, T^{1,0} \oplus T^{*(0,1)})$. For any $(1,0)$ -vector field U , $\bar{\delta}U = \frac{1}{2}\bar{\partial}U$. For any $(0,1)$ -form $\bar{\omega}$, $\bar{\delta}\bar{\omega} = \frac{1}{2}\bar{\partial}\bar{\omega}$.*

Proof: Given $U, V, W \in C^\infty(M, T^{1,0})$ and $\omega, \mu, \nu \in C^\infty(M, T^{*(0,1)})$, we test $\bar{\delta}U$ and $\bar{\delta}\bar{\omega}$ against the pairs $\{\bar{V}, \bar{W}\}$, $\{\bar{V}, \nu\}$, and $\{\mu, \nu\}$ respectively. As $\llbracket \mu, \nu \rrbracket = 0$, we have only two pairs to work on.

$$\begin{aligned} \bar{\delta}U(\bar{V}, \bar{W}) &= -\langle U, \llbracket \bar{V}, \bar{W} \rrbracket \rangle = 0, \\ \bar{\delta}U(\bar{V}, \nu) &= -\langle U, \llbracket \bar{V}, \nu \rrbracket \rangle = -\frac{1}{2}\llbracket \bar{V}, \nu \rrbracket(U) = -\frac{1}{2}d\nu(\bar{V}, U) = \frac{1}{2}\nu(\llbracket \bar{V}, U \rrbracket). \end{aligned}$$

Since ν is a $(1,0)$ -form, $\nu(\llbracket \bar{V}, U \rrbracket) = \nu(\llbracket \bar{V}, U \rrbracket^{1,0}) = \nu(\bar{\partial}_{\bar{V}}U)$ [34] [61]. We obtain $\bar{\delta}U = \frac{1}{2}\bar{\partial}U$. Similarly $\bar{\delta}\bar{\omega}(\bar{V}, \nu) = 0$ and

$$\bar{\delta}\bar{\omega}(\bar{V}, \bar{W}) = -\langle \bar{\omega}, \llbracket \bar{V}, \bar{W} \rrbracket \rangle = \frac{1}{2}d\bar{\omega}(\bar{V}, \bar{W}) = \frac{1}{2}\bar{\partial}\bar{\omega}(\bar{V}, \bar{W}).$$

It follows that $\bar{\delta}\bar{\omega} = \frac{1}{2}\bar{\partial}\bar{\omega}$. ■

The observation above implies that a $(1,0)$ -vector field U is holomorphic if and only if $\bar{\delta}U = 0$. Equivalently, the $(1,0)$ -component of $\llbracket \bar{V}, U \rrbracket$ vanishes for any $(0,1)$ -vector field \bar{V} . For instance, in terms of the ascending basis of a nilpotent complex structure as given in (6) and its dual basis $\{T_1, \dots, T_m\}$, $\bar{\partial}T_m = 0$. For further reference, we have

Corollary 1. *Let Λ be a $(2,0)$ -vector field. It is holomorphic $\bar{\delta}\Lambda = 0$ if and only if for any $(0,1)$ -vector field \bar{V} , the $(2,0)$ -component of $\llbracket \bar{V}, \Lambda \rrbracket$ vanishes.*

As a result, it is now also obvious that in terms of the ascending basis and its dual $\{T_1, \dots, T_m\}$, $\bar{\partial}(T_{m-1} \wedge T_m) = 0$ and $\llbracket T_{m-1} \wedge T_m, T_{m-1} \wedge T_m \rrbracket = 0$, a fact observed in [16, Theorem 5.1].

3.4 Holomorphic Poisson structures

Holomorphic Poisson structure is built upon a complex structure J . In a matrix representation for \mathcal{J} , J is a complex structure, $B = 0$, and $\Pi \neq 0$. In the presence of J , it is convenient to put Π in complex terms. A contraction with Π defines a linear map $\Pi : T^*M \rightarrow TM$. The complex structure J maps from TM to TM while J^* maps from T^*M to T^*M . Define $Y : T^*M \rightarrow TM$ by $Y = -\Pi \circ J^*$. It follows that $\Pi = Y \circ J^*$ because $J \circ J = -1$. $\Pi + iY$ is a type-(2,0) bivector because its contraction with any (0,1)-form is equal to zero. Define

$$\Lambda = \frac{i}{2}(\Pi + iY). \quad (28)$$

The (+i)-eigenbundle L with respect to \mathcal{J} has a real representation in terms of Π and a complex representation in terms of Λ .

$$\begin{aligned} L &= \{X - iJX, \alpha + iJ^*\alpha - i\Pi(\alpha) : X \in TM, \alpha \in T^*M\}, \\ L &= \{U, \bar{\omega} + \Lambda\bar{\omega} : U \in T^{1,0}, \bar{\omega} \in T^{*(0,1)}\}. \end{aligned}$$

It is most convenient to express the integrability of L in terms of the second representation because its integrability is equivalent to J being an integrable classical complex structure and Λ is a holomorphic Poisson structure, i.e., $\bar{\partial}\Lambda = 0$ and $[\Lambda, \Lambda] = 0$.

Lemma 1. *For any (1,0)-vector field U on a holomorphic Poisson manifold, $\bar{\delta}U = 0$ if and only if $\bar{\partial}U = 0$ and $\mathcal{L}_U\Lambda = 0$, i.e., U is an infinitesimal holomorphic Poisson transformation.*

Proof: Note that for any (0,1)-vector fields \bar{V} and \bar{W} , $(\bar{\delta}U)(\bar{V}, \bar{W}) = 0$ because $TM_{\mathbb{C}}$ is isotropic. When ω is a (1,0)-form,

$$\begin{aligned} (\bar{\delta}U)(\bar{V}, \omega + \Lambda\omega) &= -\langle U, [\bar{V}, \omega + \Lambda\omega] \rangle = -\langle U, [\bar{V}, \Lambda\omega] + [\bar{V}, \omega] \rangle \\ &= -\langle U, [\bar{V}, \omega] \rangle = -\frac{1}{2}(t_{\bar{V}}d\omega)(U) = -\frac{1}{2}d\omega(\bar{V}, U) = \frac{1}{2}\omega([\bar{V}, U]). \end{aligned}$$

The last is equal to zero for all ω and all \bar{V} if and only if $\bar{\partial}U = 0$ [34]. Next for any (1,0)-forms μ, ν ,

$$\begin{aligned} (\bar{\delta}U)(\mu + \Lambda\mu, \nu + \Lambda\nu) &= -\langle U, [\mu + \Lambda\mu, \nu + \Lambda\nu] \rangle \\ &= \langle U, [\Lambda\mu, \nu] - [\Lambda\nu, \mu] \rangle = -\frac{1}{2}(d\nu(\Lambda\mu, U) - d\mu(\Lambda\nu, U)) \\ &= \frac{1}{2}(\nu([\Lambda\mu, U]) - \mu([\Lambda\nu, U])) = -\frac{1}{2}(\mathcal{L}_U\Lambda)(\mu, \nu) = \frac{1}{2}[\Lambda, U](\mu, \nu). \end{aligned}$$

It is equal to zero for all μ, ν if and only if $\mathcal{L}_U\Lambda = 0$. ■

Lemma 2. *For any (0,1)-form $\bar{\omega}$, $\bar{\delta}(\bar{\omega} + \Lambda\bar{\omega}) = 0$ if and only if $[\Lambda, \bar{\omega}] = 0$ and $\bar{\partial}\bar{\omega} = 0$.*

Proof: For any (0,1)-vector fields \bar{V} and \bar{W} ,

$$\begin{aligned} \bar{\delta}(\bar{\omega} + \Lambda\bar{\omega})(\bar{V}, \bar{W}) &= -\langle \bar{\omega} + \Lambda\bar{\omega}, [\bar{V}, \bar{W}] \rangle \\ &= -\frac{1}{2}\bar{\omega}([\bar{V}, \bar{W}]) = \frac{1}{2}(d\bar{\omega})(\bar{V}, \bar{W}) = \frac{1}{2}(\bar{\partial}\bar{\omega})(\bar{V}, \bar{W}). \end{aligned}$$

Let ν be any (1,0)-form,

$$\begin{aligned} \bar{\delta}(\bar{\omega} + \Lambda\bar{\omega})(\bar{V}, \nu + \Lambda\nu) &= -\langle \bar{\omega} + \Lambda\bar{\omega}, [\bar{V}, \nu + \Lambda\nu] \rangle = -\langle \bar{\omega} + \Lambda\bar{\omega}, [\bar{V}, \Lambda\nu] + [\bar{V}, \nu] \rangle \\ &= -\frac{1}{2}(\bar{\omega}([\bar{V}, \Lambda\nu]) + [\bar{V}, \nu](\Lambda\bar{\omega})) = -\frac{1}{2}(\bar{\omega}([\bar{V}, \Lambda\nu]) + (t_{\bar{V}}d\nu)(\Lambda\bar{\omega})). \end{aligned}$$

As the complex structure is integrable, the type-(0,2) component of $d\nu$ vanishes. Therefore,

$$\bar{\delta}(\bar{\omega} + \Lambda\bar{\omega})(\bar{V}, \nu + \Lambda\nu) = -\frac{1}{2}\bar{\omega}([\bar{V}, \Lambda\nu]) = \frac{1}{2}[\Lambda, \bar{\omega}](\nu, \bar{V}). \quad (29)$$

Next, for any (1,0)-forms μ, ν ,

$$\begin{aligned}
& \bar{\delta}(\bar{\omega} + \bar{\Lambda}\bar{\omega})(\mu + \Lambda\mu, \nu + \Lambda\nu) = -\langle \bar{\omega} + \bar{\Lambda}\bar{\omega}, \llbracket \mu + \Lambda\mu, \nu + \Lambda\nu \rrbracket \rangle \\
& = -\frac{1}{2} (\bar{\omega}(\llbracket \Lambda\mu, \Lambda\nu \rrbracket) + \llbracket \Lambda\mu, \nu \rrbracket(\bar{\Lambda}\bar{\omega}) - \llbracket \Lambda\nu, \mu \rrbracket(\bar{\Lambda}\bar{\omega})) \\
& = -\frac{1}{2} (\llbracket \Lambda\mu, \nu \rrbracket(\bar{\Lambda}\bar{\omega}) - \llbracket \Lambda\nu, \mu \rrbracket(\bar{\Lambda}\bar{\omega})) = -\frac{1}{2} (\bar{\Lambda}(\bar{\omega}, \llbracket \Lambda\mu, \nu \rrbracket) - \bar{\Lambda}(\bar{\omega}, \llbracket \Lambda\nu, \mu \rrbracket)) \\
& = \frac{1}{2} \llbracket \Lambda, \bar{\Lambda}\bar{\omega} \rrbracket(\mu, \nu).
\end{aligned}$$

By Corollary 1, when Λ is holomorphic the (2, 0)-component of $\llbracket \Lambda, \bar{\Lambda}\bar{\omega} \rrbracket$ vanishes for any (0,1)-form $\bar{\omega}$. Therefore, the only obstructions for $\bar{\omega} + \bar{\Lambda}\bar{\omega}$ to be $\bar{\delta}$ -closed is when $\bar{\delta}\bar{\omega} = 0$ and $\llbracket \Lambda, \bar{\omega} \rrbracket = 0$ as stated. \blacksquare

3.5 Different types of generalized complex structures

There are many examples of generalized complex structures different from the symplectic or complex types. For example, one could have a symplectic bundle over complex manifold [4]. On the total space of such a fiber bundle, there exists a closed 2-form such that its restriction to each fiber is a symplectic form. The base manifold is a complex manifold. Below we apply a well-known method to construct a generalized complex structure on a manifold that is neither symplectic type or complex type. Let $U(1)$ represent the one-dimensional unitary group.

Proposition 5. *Suppose that M is the total space of a principal $U(1) \times U(1)$ -bundle over a complex n -dimensional manifold X . If the curvature of a connection on this bundle is represented by type-(1,1) forms on X , then M admits a generalized complex structure of type- n on the manifold M .*

Proof: We prove this theorem by mimicking a well-known construction of integrable complex structure on toric bundles. See e.g., [38].

Denote the principal bundle projection by $\pi : M \rightarrow X$. Let $\theta = (\theta^1, \theta^2) : M \rightarrow \mathbb{R} \oplus \mathbb{R}$ be the connection 1-form. The curvature form of this connection is $(d\theta^1, d\theta^2)$. By assumption, there are type-(1,1) forms ϑ^1, ϑ^2 on the manifold X such that $d\theta^1 = \pi^* \vartheta^1$ and $d\theta^2 = \pi^* \vartheta^2$. Let \mathcal{V} be the bundle of vertical vector fields generated by the principal action of $U(1) \times U(1)$. It yields an exact sequence of vector bundles on the manifold M :

$$0 \rightarrow \mathcal{V} \hookrightarrow TM \xrightarrow{d\pi} \pi^* TX \rightarrow 0.$$

Let $\mathcal{H} = \ker \theta = \ker \theta^1 \cap \ker \theta^2$ be the horizontal space of the connection θ . It yields a splitting: $TM = \mathcal{H} \oplus \mathcal{V}$. The projection $d\pi : TM \rightarrow \pi^* TX$ yields an isomorphism $d\pi : \mathcal{H} \rightarrow \pi^* TX$ via horizontal lift. For any tangent vector e on the manifold X , denote its horizontal lift to M by e^h so that $d\pi(e^h) = e$.

Consider $B = \theta^1 \wedge \theta^2$ as a 2-form on the manifold M . Let (v_1, v_2) be the fundamental vector field on M generated by the principal $U(1) \times U(1)$ -action. In particular, they are vertical vector fields trivializing the vertical bundle \mathcal{V} over the manifold M . Fiberwise, they form a dual to the vertical 1-forms (θ_1, θ_2) . Let $\Pi = v_1 \wedge v_2$ be a bivector field on the manifold M . We now obtain two of the three components of the matrix representation (13) of an (almost) generalized complex structure \mathcal{J} on M . The last component J acts on \mathcal{H} and it is equal to zero on \mathcal{V} .

Define J on \mathcal{H} by the horizontal lift of the action of J on the base manifold X . i.e., for each e^h local section of \mathcal{H} ,

$$d\pi(Je^h) = Jd\pi(e^h). \quad (30)$$

Therefore, we obtain an *almost* generalized complex structure \mathcal{J} with (+i)-eigenbundles given by

$$L = \mathcal{H}^{1,0} \oplus \pi^* T^{*(0,1)} X \oplus \{v - iB(v), \rho - i\Pi(\rho) : v \in \mathcal{V}, \rho \in \mathcal{V}^*\}.$$

It is apparent that L is isotropic with respect to the non-degenerate pairing (11). The space $\{v - iB(v), \rho - i\Pi(\rho) : v \in \mathcal{V}, \rho \in \mathcal{V}^*\}$ is trivialized by

$$v_1 - iB(v_1) = v_1 - i\theta^2, \quad v_2 - iB(v_2) = v_2 + i\theta^1. \quad (31)$$

If u, u_1, u_2 are (1,0)-vector fields on the base manifold X , u_1^h and u_2^h are (+i)-eigenvectors with respect to \mathcal{J} on the manifold M . On M ,

$$\llbracket u_1^h, u_2^h \rrbracket = (d\theta^1, d\theta^2)(u_1^h, u_2^h) = (\pi^* \vartheta^1, \pi^* \vartheta^2)(u_1^h, u_2^h) = (\vartheta^1(u_1, u_2), \vartheta^2(u_1, u_2)) = 0$$

because ϑ^1 and ϑ^2 are both type-(1,1) forms.

On the other hand, a vertical (+i)-eigenvector is given by $v_1 - iB(v_1) = v_1 - i\theta^2$. Since all horizontal distributions are invariant of the principal action and u^h is in $\ker \theta^1 \cap \ker \theta^2$,

$$\begin{aligned} \llbracket u^h, v_1 - iB(v_1) \rrbracket &= \llbracket u^h, v_1 - i\theta^2 \rrbracket = -i \llbracket u^h, \theta^2 \rrbracket \\ &= -i(\mathcal{L}_{u^h} \theta^2 - \frac{1}{2} d(\theta^2(u^h))) = -i \iota_{u^h} d\theta^2 = -i \iota_{u^h} \pi^* \vartheta^2 = -i \pi^* (\iota_u \vartheta^2). \end{aligned}$$

Since ϑ^2 is type-(1,1) and u is a type-(1,0), $\iota_u \vartheta^2$ is a type-(0,1) form on the manifold X . It follows that last term is a section of $\pi^* T^{*(0,1)} X \subset L$. If \bar{w} is a (0,1)-form on the base complex manifold X ,

$$\llbracket \pi^* \bar{w}, v_1 - iB(v_1) \rrbracket = \llbracket \pi^* \bar{w}, v_1 - i\theta^2 \rrbracket = \llbracket \pi^* \bar{w}, v_1 \rrbracket = -\mathcal{L}_{v_1}(\pi^* \bar{w}) + \frac{1}{2} d((\pi^* \bar{w})(v_1)).$$

Since the pull-back form $\pi^* \bar{w}$ is invariant of vertical vector field, the above is equal to zero.

A similar computation works for $\llbracket u^h, v_2 - iB(v_2) \rrbracket = \llbracket u^h, v_2 + i\theta^1 \rrbracket$. Therefore, the space of sections for L is closed with respect to the Courant bracket. \blacksquare

There are lots examples of non-trivial toric bundles as described by Proposition 5 with very interesting complex geometry; see e.g., [38]. In this note, we illustrate the above construction with Example 7.

3.6 Deformation

Recall that the construction for the differential Gerstenhaber algebra $DGA(M, \mathcal{J})$ for a generalized complex structure \mathcal{J} on a smooth manifold M is grounded on the Lie bialgebroid (L, \bar{L}) of (+i) and (-i)-eigenbundles with respect to bundle map \mathcal{J} . The differential $\bar{\delta}$ on L is a reflection of the structure of the Schouten bracket on sections of $\wedge^* \bar{L}$ and L is treated as the dual bundle of \bar{L} via the natural non-degenerate pairing. Given $\Gamma \in C^\infty(M, \wedge^2 L)$, we treat it as a map $\Gamma : \bar{L} = L^* \rightarrow L$ and its conjugation as another map $\bar{\Gamma} : L = \bar{L}^* \rightarrow \bar{L}$. Define

$$L_{\bar{\Gamma}} = \{\sigma + \bar{\Gamma}(\sigma) : \sigma \in L\}, \quad \bar{L}_{\Gamma} = \{\bar{\sigma} + \Gamma(\bar{\sigma}) : \bar{\sigma} \in \bar{L}\}. \quad (32)$$

The pair forms a new Lie bialgebroid, or equivalently new generalized complex structure in our context if and only if Γ satisfies the Maurer-Cartan equation (19) and the pair stays maximally isotropic [40] [49]. Furthermore, the direct sum $L_{\bar{\Gamma}} \oplus \bar{L}_{\Gamma}$ has alternative representation when Γ is sufficiently close to zero.

$$L_{\bar{\Gamma}} \oplus \bar{L}_{\Gamma} = (TM \oplus T^*M)_{\mathbb{C}} = L \oplus \bar{L}_{\Gamma}. \quad (33)$$

From the right hand side, the space of sections for $\wedge^* L$ continues to inherit the Courant bracket on $(TM \oplus T^*M)_{\mathbb{C}}$. However, the natural non-degenerate pairing identifies L to the dual of \bar{L}_{Γ} . Therefore $\wedge^* L = \wedge^*(\bar{L}_{\Gamma})^*$, and it inherits a differential $\bar{\delta}_{\Gamma}$ determined by the restriction of the Courant bracket on $\wedge^* \bar{L}_{\Gamma}$.

Theorem 3. [49] *The differential $\bar{\delta}$ on L as a dual to \bar{L} and the differential $\bar{\delta}_{\Gamma}$ on L as a dual to \bar{L}_{Γ} are related by*

$$\bar{\delta}_{\Gamma} a = \bar{\delta} a + \llbracket \Gamma, a \rrbracket \quad (34)$$

for all section a in $C^\infty(M, \wedge^* L)$. In particular, $DGA(M, \mathcal{J}_{\Gamma})$ after deformation by Γ is isomorphic to $(C^\infty(M, \wedge^* L), \wedge, \llbracket -, - \rrbracket, \bar{\delta}_{\Gamma})$.

From now on, we denote $\llbracket \Gamma, a \rrbracket$ by $ad_\Gamma(a)$. In the notations above, the Maurer-Cartan equation (19) is translated to $\bar{\delta}_\Gamma \circ \bar{\delta}_\Gamma = 0$ [49]. The next corollary is trivial.

Corollary 2. *Let \mathcal{J} be a generalized complex structure on a manifold M with differential Gerstenhaber algebra $DGA(M, \mathcal{J})$. If Γ is an element in $C^\infty(M, \wedge^2 L)$ such that $\bar{\delta}\Gamma = 0$ and $ad_\Gamma = 0$. Then $DGA(M, \mathcal{J}) = DGA(M, \mathcal{J}_\Gamma)$.*

When a generalized complex structure \mathcal{J} is given by a holomorphic Poisson structure (J, Λ) , from the viewpoint of Theorem 3 we treat Λ as a deformation of the complex structure J . In terms of a matrix representation with components J and Π , we have

$$\mathcal{J}_t = \begin{pmatrix} J & t\Pi \\ 0 & -J^* \end{pmatrix} \quad (35)$$

where t is the deformation parameter. When $t = 0$, we reach the underlying complex structure. When $t = 1$, we have our holomorphic Poisson structure. In this perspective, the algebroid differential $\bar{\delta}$ with respect to the generalized complex structure associated to (J, Λ) is identified to $\bar{\delta} + ad_\Lambda$ acting on the Lie algebroid of the complex structure J , and Lemma 1 and Lemma 2 follow easily.

Example 4. *A holomorphic Poisson structure associated to an abelian complex structure on the algebra $(0, 0, 0, 12, 14 + 23, 13 + 42)$.*

We revisit Example 3. Recall that $Je_1 = e_2, Je_3 = e_4, Je_5 = e_6$. Let $T_1 = \frac{1}{2}(e_1 - ie_2), T_2 = \frac{1}{2}(e_3 - ie_4)$, and $T_3 = \frac{1}{2}(e_5 - ie_6)$. Then $\{T_1, T_2, T_3\}$ spans $\mathfrak{g}^{1,0}$. Now we could work out the structure equations for $DGA(\mathfrak{g}, J)$.

$$d\bar{\omega}^1 = 0, \quad d\bar{\omega}^2 = -\frac{1}{2}\omega^1 \wedge \bar{\omega}^1, \quad d\bar{\omega}^3 = -\omega^1 \wedge \bar{\omega}^2. \quad (36)$$

In particular, the non-zero Courant brackets on $\mathfrak{g}_\mathbb{C}$ are

$$\llbracket T_1, \bar{T}_1 \rrbracket = -\frac{1}{2}T_2 + \frac{1}{2}\bar{T}_2, \quad \llbracket T_1, \bar{T}_2 \rrbracket = \bar{T}_3, \quad \llbracket T_2, \bar{T}_1 \rrbracket = -T_3. \quad (37)$$

Since $\llbracket T_j, \bar{\omega}^k \rrbracket = \iota_{T_j} d\bar{\omega}^k$, the non-zero brackets between elements in $\mathfrak{g}^{1,0}$ and elements in $\mathfrak{g}^{*(0,1)}$ are

$$\llbracket T_1, \bar{\omega}^2 \rrbracket = -\frac{1}{2}\bar{\omega}^1, \quad \llbracket T_1, \bar{\omega}^3 \rrbracket = -\bar{\omega}^2. \quad (38)$$

Finally, as for any $T \in \mathfrak{g}^{1,0}$, $\bar{\delta}T = \sum_{k=1}^3 \llbracket \bar{T}_k, T \rrbracket^{1,0} \wedge \bar{\omega}^k$, $\bar{\delta}T_3 = 0$ and

$$\begin{aligned} \bar{\delta}T_1 &= \llbracket \bar{T}_1, T_1 \rrbracket^{1,0} \wedge \bar{\omega}^1 + \llbracket \bar{T}_2, T_1 \rrbracket^{1,0} \wedge \bar{\omega}^2 = \frac{1}{2}T_2 \wedge \bar{\omega}^1, \\ \bar{\delta}T_2 &= \llbracket \bar{T}_1, T_2 \rrbracket^{1,0} \wedge \bar{\omega}^1 + \llbracket \bar{T}_2, T_2 \rrbracket^{1,0} \wedge \bar{\omega}^2 = T_3 \wedge \bar{\omega}^1. \end{aligned}$$

Therefore, $\Lambda = T_2 \wedge T_3$ is a holomorphic Poisson structure. By structure equations (36), $T_2 \wedge T_3$ commutes with every element in $\mathfrak{g}^{*(0,1)}$. By Corollary 2, $DGA(M, J) = DGA(M, \mathcal{J}_\Lambda)$.

3.7 Nilmanifolds

Suppose that M is a nilmanifold $M = \Delta \backslash G$ and the generalized complex structure \mathcal{J} is invariant, there is an inclusion of left-invariant sections in the space of smooth sections for various bundles.

Since the evaluation of invariant forms on invariant vectors are constants, the restriction of the Courant bracket to $\mathfrak{g} \oplus \mathfrak{g}^*$ is reduced to

$$\llbracket X + \alpha, Y + \beta \rrbracket = \llbracket X, Y \rrbracket + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha = \llbracket X, Y \rrbracket + \iota_X d\beta - \iota_Y d\alpha.$$

If ℓ and $\bar{\ell}$ represent the space of invariant sections for the Lie bialgebroid L and \bar{L} respectively, we have $\ell \oplus \bar{\ell} = (\mathfrak{g} \oplus \mathfrak{g}^*)_\mathbb{C}$ and the inclusions

$$\iota : \wedge^k \ell \longrightarrow C^\infty(M, \wedge^k L) \quad \text{and} \quad \iota : \wedge^k \bar{\ell} \longrightarrow C^\infty(M, \wedge^k \bar{L}). \quad (39)$$

Since the differential $\bar{\delta}$ on $C^\infty(M, \wedge^k L)$ is due to the invariant Lie algebroid structure on $C^\infty(M, \bar{L})$, the inclusion map ι of ℓ intertwines with the differential $\bar{\delta}$ on ℓ so that we have an inclusion of invariant differential Gerstenhaber algebra:

$$\iota : DGA(\ell) \hookrightarrow DGA(M, \mathcal{J}). \quad (40)$$

Problem 1. *When will the inclusion map be a quasi-isomorphism?*

In view of Proposition 3 and Nomizu's Theorem [60], when the generalized complex structure is a symplectic structure, we have an isomorphism. As noted in previous sections, the work regarding generalizing Nomizu's work to Dolbeault cohomology on complex structure on nilmanifolds remains an on-going effort in the past twenty years.

Theorem 4. [17, Theorem 1] *Suppose that $M = \Delta \backslash G$ is a nilmanifold with an abelian complex structure. The inclusion map ι in (40) is a quasi-isomorphism.*

The above theorem enables a collection of work on generalized deformation on holomorphic Poisson cohomology on nilmanifolds [63] [64]. In [4] Angella et al. study symplectic bundles over complex manifolds as generalized complex structures. They are fiber bundles over generalized complex manifolds such that a global 2-form on the total space of the bundle is restricted to a symplectic form on each fiber. Taking advantage of Nomizu's work and the quasi-isomorphism when a complex structure J is abelian, we paraphrase a result of [4].

Theorem 5. [4, Theorem 5.3] *When a generalized complex structure on a nilmanifold $M = \Delta \backslash G$ is realized as the symplectic bundle over an abelian complex nilmanifold, then the inclusion map ι in (40) is a quasi-isomorphism.*

4 Semi-Abelian Generalized Complex Structure

We are interested in exploring ways to extend the concept of abelian complex structure to generalized complex manifolds. Consider the definition of abelian complex structure as given in (5). Let \mathcal{J} be a generalized complex structure. One may attempt to expand it naively so that for all $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$,

$$\llbracket \mathcal{J}(X + \alpha), \mathcal{J}(Y + \beta) \rrbracket = \llbracket X + \alpha, Y + \beta \rrbracket. \quad (41)$$

However, when \mathcal{J} is a symplectic structure Ω it forces the algebra \mathfrak{g} to be abelian because we would have

$$\llbracket X, Y \rrbracket = \llbracket \Omega(X), \Omega(Y) \rrbracket.$$

Yet the Courant bracket between a pair of 1-forms is equal to zero.

As our goal is to create a theory to adopt to cohomological computation to facilitate investigation deformation of generalized complex structures, we focus on the structure of the differential Gerstenhaber algebra for a generalized complex structure as given in (18). Let us once again recall the key features of a classical complex structure J being abelian is presented in the two equivalent conditions in Proposition 1. This proposition and our need in computing cohomology effectively at least in some situation drives our proposed concept below.

Let \mathcal{G} be the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ equipped with the Courant bracket induced by the Lie bracket on \mathfrak{g} . It is a Lie algebra with the properties that \mathfrak{g} is a subalgebra and \mathfrak{g}^* is an abelian ideal. Therefore, \mathcal{G} is a semi-direct product $\mathcal{G} = \mathfrak{g} \rtimes \mathfrak{g}^*$.

Definition 1. *Suppose that \mathcal{A} is a subalgebra of $\mathcal{G} = \mathfrak{g} \rtimes \mathfrak{g}^*$, \mathcal{K} is an abelian ideal of \mathcal{G} , and both \mathcal{A} and \mathcal{K} are maximally isotropic with respect to the natural pairing (11), the semi-direct product presentation $\mathcal{G} = \mathcal{A} \rtimes \mathcal{K}$ is called admissible. We will also call $(\mathcal{A}, \mathcal{K})$ an admissible pair associated to the Lie algebra \mathfrak{g} .*

By virtual of being maximally isotropic, the non-degenerate pairing defines an isomorphism $\mathcal{K}^* \cong \mathcal{A}$.

Definition 2. Let \mathfrak{g} be a Lie algebra with an integrable generalized complex structure \mathcal{J} . Suppose that there exists an admissible pair $(\mathcal{A}, \mathcal{K})$ associated to \mathfrak{g} such that (a) both \mathcal{A} and \mathcal{K} are invariant of the map \mathcal{J} and (b) the restriction of \mathcal{J} on \mathcal{A} satisfies Identity (5), then the generalized complex structure \mathcal{J} is said to be semi-abelian and the pair $(\mathcal{A}, \mathcal{K})$ is said to be \mathcal{J} -admissible.

Definition 3. Let $M = \Delta \backslash G$ be a nilmanifold with an invariant generalized complex structure \mathcal{J} . It is semi-abelian if \mathcal{J} is semi-abelian on the Lie algebra \mathfrak{g} of the group G .

Since \mathcal{A} and \mathcal{K} are \mathcal{J} -invariant, their complexification have respective decomposition by eigenspaces. Let \mathfrak{a} and \mathfrak{k} be their respective $(+i)$ -eigenspaces. Their conjugations $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{k}}$ are the $(-i)$ -eigenspace so that

$$\ell = \mathfrak{a} \oplus \mathfrak{k}, \quad \bar{\ell} = \bar{\mathfrak{a}} \oplus \bar{\mathfrak{k}}, \quad \ell \oplus \bar{\ell} = (\mathfrak{g} \oplus \mathfrak{g}^*)_{\mathbb{C}}. \quad (42)$$

Since the restriction of \mathcal{J} on \mathcal{A} satisfies Identity (5), \mathfrak{a} is an abelian complex algebra. Therefore, ℓ is a semi-direct product $\mathfrak{a} \rtimes \mathfrak{k}$ of two abelian subalgebras. As $\mathcal{A} \cong \mathcal{K}^*$ by the natural non-degenerate pairing,

$$\mathfrak{a} = \bar{\mathfrak{k}}^*, \quad \mathfrak{k} = \bar{\mathfrak{a}}^*. \quad (43)$$

Since for any element $\ell_1 \in \ell$, $\bar{\delta}\ell_1$ is obtained by the evaluation on every pair of elements $\bar{\ell}_2, \bar{\ell}_3$ in $\bar{\ell}$:

$$(\bar{\delta}\ell_1)(\bar{\ell}_2, \bar{\ell}_3) = -\langle \ell_1, \llbracket \bar{\ell}_2, \bar{\ell}_3 \rrbracket \rangle,$$

Given the structure of $\bar{\ell} = \bar{\mathfrak{a}} \rtimes \bar{\mathfrak{k}}$ and \mathfrak{a} and \mathfrak{k} are abelian, $\llbracket \bar{\ell}, \bar{\ell} \rrbracket \subseteq \bar{\mathfrak{k}}$. By (43), we find that \mathfrak{k} is in the kernel of $\bar{\delta}$ and the image of \mathfrak{a} via $\bar{\delta}$ is contained in $\mathfrak{a} \otimes \mathfrak{k}$.

Proposition 6. Let \mathcal{J} be a semi-abelian complex structure on a Lie algebra \mathfrak{g} with admissible pair $(\mathcal{A}, \mathcal{K})$, the following holds.

- \mathfrak{a} is abelian subalgebra and \mathfrak{k} is an abelian ideal such that $\ell = \mathfrak{a} \rtimes \mathfrak{k}$.
- $\mathfrak{a} = \bar{\mathfrak{k}}^*$ and $\mathfrak{k} = \bar{\mathfrak{a}}^*$.
- $\bar{\delta}\mathfrak{k} = \{0\}$ and $\bar{\delta}\mathfrak{a} \subseteq \mathfrak{a} \otimes \mathfrak{k}$.

The last point makes obvious restriction on the DGA($\mathfrak{g}, \mathcal{J}$) in terms of a lower bound on the dimension of its first cohomology $H^1(\mathfrak{g}, \mathcal{J})$. This proposition mirrors the observation on abelian complex structures as noted in Proposition 1. Next, we find constraints on symplectic structure being semi-abelian.

Proposition 7. Suppose that Ω is a symplectic structure on a nilpotent algebra \mathfrak{g} . If $(\mathcal{A}, \mathcal{K})$ is Ω -admissible, there exist an abelian subalgebra \mathfrak{b} in \mathfrak{g} and an abelian ideal \mathfrak{h} in \mathfrak{g} such that

$$\mathcal{A} = \{X, \Omega(X) : X \in \mathfrak{b}\}, \quad \mathcal{K} = \{Y, \Omega(Y) : Y \in \mathfrak{h}\}$$

with $d\Omega(X) = 0$ for all $X \in \mathfrak{h}$. In particular, \mathfrak{g} is a semi-direct product $\mathfrak{g} = \mathfrak{b} \rtimes \mathfrak{h}$.

Proof: When Ω is a semi-abelian symplectic form, suppose that $\ell = \mathfrak{a} \rtimes \mathfrak{k}$. Then

$$\mathfrak{a} = \{X - i\Omega(X) : X \in \mathfrak{b}\}, \quad \mathfrak{k} = \{Y - i\Omega(Y) : Y \in \mathfrak{h}\}$$

for some vector subspaces \mathfrak{b} and \mathfrak{h} in \mathfrak{g} . For any $X, Y \in \mathfrak{g}$,

$$\llbracket X - i\Omega(X), Y - i\Omega(Y) \rrbracket = \llbracket X, Y \rrbracket - i\Omega(\llbracket X, Y \rrbracket).$$

Therefore, it is equal to zero if and only if $\llbracket X, Y \rrbracket = 0$. Therefore \mathfrak{b} is an abelian subalgebra and \mathfrak{h} is an abelian ideal. Furthermore, $\bar{\delta}(X - i\Omega(X)) = 0$ if and only if $d\Omega(X) = 0$. The dimension restriction forces \mathfrak{g} to be a direct sum of \mathfrak{b} and \mathfrak{h} . The Lie algebra structure follows. \blacksquare

We may now restrict the scope of Problem 1 to our current perspective.

Problem 2. *Suppose that $M = \Delta \setminus G$ is a nilmanifold with a semi-abelian generalized complex structure. Is the inclusion map (40) necessarily a quasi-isomorphism?*

Corollary 2 indicates that the concept of *semi-abelian* is invariant of deformations generated by center of the underlying Schouten algebra of $DGA(M, \mathcal{J})$. In view of the work in [22] [52] and the results as noted in Proposition 2, there are non-trivial deformation keeping the property of *abelian stable*.

Problem 3. *Suppose that \mathcal{J} is a semi-abelian generalized complex structure. Find all Γ in $\wedge^2 \ell$ such that $(L_{\bar{F}}, \bar{L}_{\Gamma})$ is a semi-abelian generalized complex structure.*

Remark 1. *The concept of semi-abelian generalized complex structure as given in Definition 2 on algebra level potentially could be extended in a context similar to those in [2] [8] [48].*

5 Examples

By construction, all abelian complex structures are semi-abelian (generalized) complex structures. In this section, we present a collection of examples of semi-abelian generalized complex structures, including symplectic structures, non-abelian nilpotent complex structures, together with non-complex and non-symplectic type structures.

If a symplectic structure is semi-abelian as a generalized complex structure, we will address it as a semi-abelian symplectic structure. In Example 10, we find a non-abelian complex structure such that as a generalized complex structure, it is semi-abelian. In short, there are semi-abelian complex structures that fail to be abelian. We start on four-dimension cases.

Example 5. *The algebra $(0, 0, 0, 12)$, revisit.*

Since $de^4 = e^{12}$, the structure equations of Courant bracket on $\mathcal{G} = \mathfrak{g} \rtimes \mathfrak{g}^*$ are

$$[[e_1, e_2]] = -e_4, \quad [[e_1, e^4]] = e^2, \quad [[e_2, e^4]] = -e^1. \tag{44}$$

When we choose $Je_1 = e_2$ and $Je_3 = e_4$, we have the well-known complex structure of Kodaira surface as noted in Section 2.1. Its corresponding choice of admissible pair is $\mathcal{A} = \mathfrak{g}$ and $\mathcal{K} = \mathfrak{g}^*$.

On the other hand, we may choose $\mathcal{A} = \{e_1, e_2, e_4, e^3\}$ and $\mathcal{K} = \{e^1, e^2, e^4, e_3\}$. Consider a type-1 generalized complex structure \mathcal{J} defined by its components.

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad B = e^{34}, \quad \Pi = e_{34}.$$

In particular, $\mathcal{J}(e_3) = e^4$, $\mathcal{J}(e_4) = -e^3$. The pair $(\mathcal{A}, \mathcal{K})$ becomes \mathcal{J} -admissible. Moreover, the $(+i)$ -eigenspace is spanned by the following elements.

$$\ell_1 = e_1 - ie_2, \quad \ell_2 = e_4 + ie^3, \quad \ell_3 = e^1 - ie^2, \quad \ell_4 = e_3 - ie^4.$$

Since e_4, e^1, e^2, e^3 are in the center of the algebra \mathcal{G} , the integrability is due to

$$[[\ell_1, \ell_4]] = [[e_1 - ie_2, e_3 - ie^4]] = -i[[e_1 - ie_2, e^4]] = -i(e^2 + ie^1) = \ell_3.$$

Therefore, \mathcal{J} is integrable and semi-abelian.

Example 6. *An example admitting no semi-abelian generalized complex structures.*

Consider the algebra $(0, 0, 12, 13)$. Since $de^3 = e^{12}$, $de^4 = e^{13}$, it admits symplectic structures such as $e^{23} + e^{14}$. However, this algebra does allow a semi-direct product as prescribed by Proposition 7, there is no semi-abelian symplectic structure on this algebra.

The structure of Courant bracket on $\mathcal{G} = \mathfrak{g} \ltimes \mathfrak{g}^*$ are

$$\begin{aligned} \llbracket e_1, e_2 \rrbracket &= -e_3, & \llbracket e_1, e_3 \rrbracket &= -e_4 \\ \llbracket e_1, e^3 \rrbracket &= e^2, & \llbracket e_1, e^4 \rrbracket &= e^3, & \llbracket e_2, e^3 \rrbracket &= -e^1, & \llbracket e_3, e^4 \rrbracket &= -e^1. \end{aligned}$$

The center of \mathcal{G} is spanned by e_4, e^1, e^2 .

Utilizing ascending basis similar to (4) as in [54] [69], Cavalcanti et al. provided a characterization of generalized complex structures in terms of spinor formalism on nilmanifolds [16, Corollary 1]. In particular, when \mathcal{J} is type-1, there exists a spinor representation $\rho = e^{B+i\omega}\theta$ with real 2-forms B and ω , and complex 1-form θ such that $\omega \wedge \theta \wedge \bar{\theta} \neq 0$ and $d\theta = 0$. Given the algebra at hand, the only choice for θ , up to linear combination, is $\theta = e^1 + ie^2$. It follows that up to a constant

$$\omega = e^{34} + e^3 \wedge (a_{31}e^1 + a_{32}e^2) + e^4 \wedge (a_{41}e^1 + a_{42}e^2)$$

for some real numbers $a_{31}, a_{32}, a_{41}, a_{42}$. Therefore the generalized complex structure \mathcal{J} is given by

$$\begin{aligned} \mathcal{J}e_1 &= e_2, & \mathcal{J}e^1 &= e^2, \\ \mathcal{J}e_3 &= e^4 + a_{31}e^1 + a_{32}e^2, & \mathcal{J}e_4 &= -e^3 + a_{41}e^1 + a_{42}e^2. \end{aligned}$$

It follows that the (+i)-eigenspace is spanned as

$$\ell = \{e_1 - ie_2, e^1 - ie^2, e_3 - i(e^4 + a_{31}e^1 + a_{32}e^2), e_4 - i(-e^3 + a_{41}e^1 + a_{42}e^2)\}.$$

As ℓ is isotropic, $a_{31} - ia_{32} = 0$, $a_{41} - ia_{42} = 0$. Therefore, all such constants are equal to zero so that

$$\ell = \{e_1 - ie_2, e^1 - ie^2, e_3 - ie^4, e_4 + ie^3\}. \quad (45)$$

As e^1 and e^2 are in the center of the algebra \mathcal{G} , to verify integrability of \mathcal{J} it suffices to see that

$$\begin{aligned} \llbracket e_1 - ie_2, e_3 - ie^4 \rrbracket &= \llbracket e_1, e_3 - ie^4 \rrbracket - i\llbracket e_2, e_3 - ie^4 \rrbracket = -e_4 - ie^3 = -(e_4 + ie^3), \\ \llbracket e_1 - ie_2, e_4 + ie^3 \rrbracket &= i\llbracket e_1, e^3 \rrbracket + \llbracket e_2, e^3 \rrbracket = ie^2 - e^1 = -(e^1 - ie^2). \end{aligned}$$

In terms of $\bar{\ell}$, $\llbracket e_1 + ie_2, e_3 + ie^4 \rrbracket = -(e_4 - ie^3)$, $\llbracket e_1 + ie_2, e_4 - ie^3 \rrbracket = -(e^1 + ie^2)$. It follows that no combination of $e_3 - ie^4$ and $e_1 - ie_2$ is in the kernel of the operator $\bar{\delta}$. Moreover,

$$\ker \bar{\delta} = \{e^1 - ie^2, e_4 + ie^3\}.$$

Now suppose that $(\mathcal{A}, \mathcal{G})$ is a \mathcal{J} -admissible pair. By Proposition 6, \mathfrak{k} is in the kernel of $\bar{\delta}$. A dimension restriction shows that $\mathcal{K} = \{e_4, e^1, e^2, e^3\}$. While \mathcal{K} is indeed an abelian ideal in \mathcal{G} , the dual space is $\mathcal{K}^* = \mathcal{A} = \{e^4, e_1, e_2, e_3\}$, and it fails to be a Lie subalgebra. Therefore, the generalized complex structure \mathcal{J} is not semi-abelian.

Finally by [16, Theorem 3.2] as well as [70, Proposition 2.1], the algebra $(0, 0, 12, 13)$ does not admit any invariant complex structure, let alone a semi-abelian one.

It is an elementary fact that the only non-abelian 4-dimensional nilpotent algebras are $(0, 0, 0, 12)$ and $(0, 0, 12, 13)$, see e.g., [53]. The observation above shows that the latter admits type-1 as well as type-0 generalized complex structure, but none of them are semi-abelian.

Proposition 8. *The only four-dimensional nilpotent algebras admitting semi-abelian generalized complex structures are $(0, 0, 0, 0)$ and $(0, 0, 0, 12)$.*

Example 7. *A type-2 example on $(0, 0, 0, 0, 0, 12 + 34)$.*

As the structure for Lie algebra \mathfrak{g} is $de^6 = e^{12} + e^{34}$, the structural equations associated to $\mathcal{G} = \mathfrak{g} \ltimes \mathfrak{g}^*$ are

$$\begin{aligned} \llbracket e_1, e_2 \rrbracket &= -e_6, & \llbracket e_3, e_4 \rrbracket &= -e_6, \\ \llbracket e_1, e^6 \rrbracket &= e^2, & \llbracket e_2, e^6 \rrbracket &= -e^1, & \llbracket e_3, e^6 \rrbracket &= e^4, & \llbracket e_4, e^6 \rrbracket &= e^3. \end{aligned}$$

As seen in Example 1, $(\mathfrak{g}, \mathfrak{g}^*)$ forms an admissible pair for an abelian complex structure. On the other hand, the pair below is also admissible.

$$\mathcal{A} = \{e_1, e_2, e_3, e_4, e_6, e^5\}, \quad \mathcal{K} = \{e^1, e^2, e^3, e^4, e^6, e_5\}.$$

Consider a generalized complex structure \mathcal{J} given below.

$$\begin{aligned} \mathcal{J}e_1 &= e_2, \mathcal{J}e_2 = -e_1, \mathcal{J}e_3 = e_4, \mathcal{J}e_4 = -e_3, \mathcal{J}e_6 = -e^5, \mathcal{J}e^5 = e_6, \\ \mathcal{J}e^1 &= e^2, \mathcal{J}e^2 = -e^1, \mathcal{J}e^3 = e^4, \mathcal{J}e^4 = -e^3, \mathcal{J}e_5 = e^6, \mathcal{J}e^6 = -e_5. \end{aligned}$$

It leaves \mathcal{A} and \mathcal{K} invariant. In terms of components in the matrix representation for \mathcal{J} ,

$$J = e^1 \otimes e_2 - e^2 \otimes e_1 + e^3 \otimes e_4 - e^4 \otimes e_3, \quad B = e^5 \wedge e^6, \quad \Pi = e_5 \wedge e_6. \quad (46)$$

The $(+i)$ -eigenspace for \mathcal{J} is

$$\ell = \{e_1 - ie_2, e_3 - ie_4, e^1 - ie^2, e^3 - ie^4, e_5 - ie^6, e_6 + ie^5\}.$$

Since $de^1 = de^2 = de^3 = de^4 = 0$, $e^1 - ie^2, e^3 - ie^4$ are in the center of the Schouten bracket on ℓ . Similarly, e_6 is in the center of \mathfrak{g} and $de^5 = 0$, $e_6 + ie^5$ is also in the center of the Schouten bracket on ℓ . Therefore, a computation of Schouten bracket on ℓ is reduced to

$$\begin{aligned} \llbracket e_1 - ie_2, e_3 - ie_4 \rrbracket &= 0, \\ \llbracket e_1 - ie_2, e_5 - ie^6 \rrbracket &= -i\llbracket e_1 - ie_2, e^6 \rrbracket = -i(e^2 + ie^1) = e^1 - ie^2, \\ \llbracket e_3 - ie_4, e_5 - ie^6 \rrbracket &= i\llbracket e_3 - ie_4, e^6 \rrbracket = -i(e^4 + ie^3) = e^3 - ie^4. \end{aligned}$$

Since ℓ is closed with respect to the Schouten bracket, \mathcal{J} is an integrable generalized complex structure. Moreover, $\ell = \alpha \times \mathfrak{k}$ where

$$\alpha = \{e_1 - ie_2, e_3 - ie_4, e_6 + ie^5\}, \quad \mathfrak{k} = \{e^1 - ie^2, e^3 - ie^4, e_5 - ie^6\}. \quad (47)$$

It follows that \mathcal{J} is a type-2 semi-abelian complex structure. Since $de^5 = 0$ and $de^6 = e^{12} + e^{34}$, they are type-(1,1) when treated as 2-form on the quotient space spanned by $\{e_1, e_2, e_3, e_4\}$, we obtain an example for Proposition 5 when we consider taking the quotient of the algebra $(0, 0, 0, 0, 0, 12 + 34)$ by its center as a principal bundle projection on the nilmanifold level.

Example 8. A semi-abelian symplectic structure on $(0, 0, 0, 0, 12, 14 + 25)$.

Let $\mathfrak{b} = \{e_2, e_3, e_4\}$ and $\mathfrak{h} = \{e_1, e_5, e_6\}$, then $\mathfrak{g} = \mathfrak{b} \rtimes \mathfrak{h}$. This algebra has a symplectic form, $\Omega = e^{13} + e^{26} + e^{45}$. Since $\Omega(e_1) = e^3$, $\Omega(e_5) = -e^4$, and $\Omega(e_6) = -e^2$, the pair $(\mathfrak{b}, \mathfrak{h})$ satisfies the conditions in Proposition 7 with respect to Ω . Therefore Ω is an abelian symplectic structure with $\ell = \alpha \times \mathfrak{k}$ where

$$\alpha = \{e_2 - ie^6, e_3 + ie^1, e_4 - ie^5\}, \quad \mathfrak{k} = \{e_1 - ie^3, e_5 + ie^4, e_6 + ie^2\}.$$

Example 9. A semi-abelian, non-abelian complex structure on $(0, 0, 0, 0, 12, 13)$.

The structure equations for this $\mathcal{G} = \mathfrak{g} \rtimes \mathfrak{g}^*$ are given below.

$$\begin{aligned} \llbracket e_1, e_2 \rrbracket &= -e_5, & \llbracket e_1, e_3 \rrbracket &= -e_6, \\ \llbracket e_1, e^5 \rrbracket &= e^2, & \llbracket e_2, e^5 \rrbracket &= -e^1, & \llbracket e_1, e^6 \rrbracket &= e^3, & \llbracket e_3, e^6 \rrbracket &= -e^1. \end{aligned}$$

Although the algebra \mathfrak{g} does not admit any abelian complex structure [27, Proposition 3.3], it does admit nilpotent complex structures [69]. We adopt the one given in [16, Table 1]. Namely, $\mathcal{J} = J$, $B = 0$, $\Pi = 0$, with

$$Je_1 = e_4, \quad Je_2 = e_3, \quad Je_5 = e_6. \quad (48)$$

The pair $\mathcal{A} = \{e_2, e_3, e^1, e^4, e^5, e^6\}$, $\mathcal{K} = \{e_1, e_4, e_5, e_6, e^2, e^3\}$ is \mathcal{J} -admissible. Therefore, the nilpotent complex structure in (48) is semi-abelian although it is not abelian.

Example 10. A semi-abelian holomorphic Poisson structure on $(0, 0, 0, 0, 12, 13)$.

A basis for $\mathfrak{g}^{1,0}$ with respect to the complex structure (48) is

$$T_1 = \frac{1}{2}(e_1 - ie_4), \quad T_2 = \frac{1}{2}(e_2 - ie_3), \quad T_3 = \frac{1}{2}(e_5 - ie_6). \quad (49)$$

It follows that $\bar{\omega}^1 = e^1 - ie^4$, $\bar{\omega}^2 = e^2 - ie^3$, and $\bar{\omega}^3 = e^5 - ie^6$ forms a basis for $\mathfrak{g}^{*(0,1)}$. Moreover, $d\bar{\omega}^1 = 0$, $d\bar{\omega}^2 = 0$, and

$$d\bar{\omega}^3 = \frac{1}{2}\omega^1 \wedge \bar{\omega}^2 + \frac{1}{2}\bar{\omega}^1 \wedge \bar{\omega}^2.$$

Equivalently,

$$\llbracket T_1, T_2 \rrbracket = -\frac{1}{2}T_3, \quad \llbracket \bar{T}_1, T_2 \rrbracket = -\frac{1}{2}T_3. \quad (50)$$

By Proposition 4, $\bar{\partial}T_2 = -\frac{1}{2}T_3 \wedge \bar{\omega}^1$. It follows that $\Lambda = T_2 \wedge T_3$ is a holomorphic Poisson bivector. Moreover, $\llbracket T_2, \bar{\omega}^3 \rrbracket = 0$ and $\llbracket T_3, \bar{\omega}^3 \rrbracket = 0$, $ad_\Lambda = 0$. By Corollary 2, \mathcal{J}_Λ defines a semi-abelian generalized complex structure.

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