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The classification of left-invariant para-Kähler structures on simply connected four-dimensional Lie groups

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Abstract: We give a complete classification of left invariant para-Kähler structures on four-dimensional simply connected Lie groups up to an automorphism. As an application we discuss some curvatures properties of the canonical connection associated to these structures as flat, Ricci flat and existence of Ricci solitons.

Keywords: Symplectic Lie algebras, para-Kähler structures, Ricci soliton

MSC: 53D05; 53C30; 53C25

1 Introduction and main results

An almost para-complex structure on a $2n$ -dimensional manifold M is a field of endomorphisms $K : TM \rightarrow TM$ such that $K^2 = Id_{TM}$ and the two eigendistributions $T^\pm M := \ker(Id \pm K)$ have the same rank. An almost para-complex structure K is said to be integrable if the distributions $T^\pm M$ are involutive. This is equivalent to the vanishing of the Nijenhuis tensor N_K defined by

$$N_K(X, Y) = [X, Y] + [KX, KY] - K[KX, Y] - K[X, KY],$$

for vector fields X, Y on M . In such a case K is called a para-complex structure. A para-Kähler structure on a manifold M is a pair $(\langle \cdot, \cdot \rangle, K)$ where $\langle \cdot, \cdot \rangle$ is a pseudo-Riemannian metric and K is a parallel skew-symmetric para-complex structure. If $(\langle \cdot, \cdot \rangle, K)$ is a para-Kähler structure on M , then $\omega = \langle \cdot, \cdot \rangle \circ K$ is a symplectic structure and the ± 1 -eigendistributions $T^\pm M$ of K are two integrable ω -Lagrangian distributions. Due to this, a para-Kähler structure can be identified with a bi-Lagrangian structure $(\omega, T^\pm M)$ where ω is a symplectic structure and $T^\pm M$ are two integrable Lagrangian distributions. Moreover the Levi-Civita connection associated to the neutral metric $\langle \cdot, \cdot \rangle$ coincides with the canonical connection associate to the bi-Lagrangian structure (the unique symplectic connection with parallelizes both foliations [9]). For a survey on paracomplex geometry see [6] and for background on bi-Lagrangian structures and their associated connections, the survey [7] is a good starting point and contains further references (See as well [1] and [5]).

Suppose now that M is a Lie group G and $\omega, \langle \cdot, \cdot \rangle$ and K are left invariant. If we denote by $\mathfrak{g}$ the Lie algebra of G , then $(\langle \cdot, \cdot \rangle, K)$ is determined by their restrictions to the Lie algebra $\mathfrak{g}$. In this situation, $(\mathfrak{g}, \langle \cdot, \cdot \rangle_e, K_e)$ or $(\mathfrak{g}, \omega_e, K_e)$ is called a para-Kähler Lie algebra (e is unit of G). In the rest of this paper a para-Kähler Lie algebra will be noted $(\mathfrak{g}, \omega, K)$. Recall that two para-Kähler Lie algebras $(\mathfrak{g}_1, \omega_1, K_1)$ and $(\mathfrak{g}_2, \omega_2, K_2)$ are said to be equivalent if there exists an isomorphism of Lie algebras $T : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ such as $T^* \omega_2 = \omega_1$ and $T^* K_2 = K_1$. Para-Kähler (bi-Lagrangian) structures on Lie algebras in general have been studied, for example, in [2], [3] and [4]. In [8], there is a study of the existence of bi-Lagrangian structures on symplectic nilpotent Lie algebras.

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bras of dimension 2, 4 and 6. A first classification of para-Kähler structures on four-dimensional Lie algebras was obtained by Calvaruso in [6]. Another description based on the classification of symplectic Lie algebras is proposed by Smolentsev and Shagabudinova in [12]. Benayadi and Boucetta provide in [3] a new characterization of para-Kähler Lie algebras using left symmetric bialgebras introduced by Bai in [2]. Based on this characterization, we propose in this paper the classification of para-Kähler structures on four-dimensional Lie algebras. Notice that our classification is more complete and precise than the other classifications existing in the literature.

Notations: For $\{e_1, e_2, e_3, e_4\}$ a basis of $\mathfrak{g}$, we denote by $\{e^1, e^2, e^3, e^4\}$ the dual basis on $\mathfrak{g}^*$ and e^{ij} the two-form $e^i \wedge e^j$, $\dot{e}^{ij}$ is the symmetric two-form $e^i \odot e^j$ and E_{ij} is the endomorphism which sends e_j to e_i and vanishes on e_k for $k \neq j$.

A para-Kähler Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle, K)$ is necessarily a symplectic Lie algebra $(\mathfrak{g}, \omega)$. It is well known that a symplectic four-dimensional Lie algebra is necessarily solvable. The classification of symplectic four-dimensional Lie algebras $(\mathfrak{g}, \omega)$ is given by the following Table (see [10]).

Table 1: Symplectic four-dimensional Lie algebras

$(\mu \geq 0, -1 \leq \beta < 1, -1 < \alpha < 0, \delta > 0 \text{ and } \lambda \geq \frac{1}{2}, \lambda' = 1, 2).$

Case	No vanishing brackets	ω
$\mathfrak{rh}_3$	$[e_1, e_2] = e_3$	$e^{14} + e^{23}$
$\mathfrak{rr}_{3,0}$	$[e_1, e_2] = e_2$	$e^{12} + e^{34}$
$\mathfrak{rr}_{3,-1}$	$[e_1, e_2] = e_2, [e_1, e_3] = -e_3$	$e^{14} + e^{23}$
$\mathfrak{rr}'_{3,0}$	$[e_1, e_2] = -e_3, [e_1, e_3] = e_2$	$e^{14} + e^{23}$
$\mathfrak{r}_2\mathfrak{r}_2$	$[e_1, e_2] = e_2, [e_3, e_4] = e_4$	$e^{12} + \mu e^{13} + e^{34}$
$\mathfrak{r}'_2$	$[e_1, e_3] = e_3, [e_1, e_4] = e_4,$ $[e_2, e_3] = e_4, [e_2, e_4] = -e_3$	$e^{14} + e^{23}$
$\mathfrak{n}_4$	$[e_4, e_1] = e_2, [e_4, e_2] = e_3$	$e^{12} + e^{34}$
$\mathfrak{r}_{4,0}$	$[e_4, e_1] = e_1, [e_4, e_3] = e_2$	$e^{14} \mp e^{23}$
$\mathfrak{r}_{4,-1}$	$[e_4, e_1] = e_1, [e_4, e_2] = -e_2, [e_4, e_3] = e_2 - e_3$	$e^{13} + e^{24}$
$\mathfrak{r}_{4,-1,\beta}$	$[e_4, e_1] = e_1, [e_4, e_2] = -e_2, [e_4, e_3] = \beta e_3$	$e^{12} + e^{34}$
$\mathfrak{r}_{4,\alpha,-\alpha}$	$[e_4, e_1] = e_1, [e_4, e_2] = \alpha e_2, [e_4, e_3] = -\alpha e_3$	$e^{14} + e^{23}$
$\mathfrak{r}'_{4,0,\delta}$	$[e_4, e_1] = e_1, [e_4, e_2] = -\delta e_3, [e_4, e_3] = \delta e_2$	$e^{14} \mp e^{23}$
$\mathfrak{d}_{4,1}$	$[e_1, e_2] = e_3, [e_4, e_3] = e_3,$ $[e_4, e_1] = e_1$	$e^{12} - e^{34}$ $e^{12} - e^{34} + e^{24}$
$\mathfrak{d}_{4,2}$	$[e_1, e_2] = e_3, [e_4, e_3] = e_3,$ $[e_4, e_1] = 2e_1, [e_4, e_2] = -e_2$	$e^{12} - e^{34}$ $e^{14} \mp e^{23}$
$\mathfrak{d}_{4,\lambda}$	$[e_1, e_2] = e_3, [e_4, e_3] = e_3,$ $[e_4, e_1] = \lambda e_1, [e_4, e_2] = (1 - \lambda)e_2$	$e^{12} - e^{34}$
$\mathfrak{d}'_{4,\delta}$	$[e_1, e_2] = e_3, [e_4, e_1] = \frac{\delta}{2}e_1 - e_2,$ $[e_4, e_3] = \delta e_3, [e_4, e_2] = e_1 + \frac{\delta}{2}e_2$	$\mp(e^{12} - \delta e^{34})$
$\mathfrak{h}_4$	$[e_1, e_2] = e_3, [e_4, e_3] = e_3,$ $[e_4, e_1] = \frac{1}{2}e_1, [e_4, e_2] = e_1 + \frac{1}{2}e_2$	$\mp(e^{12} - e^{34})$

Our main result is the following.

Theorem 1.1. *Let $(\mathfrak{g}, \omega, K)$ be a four-dimensional para-Kähler Lie algebra and $x \in \mathbb{R}$. Then $(\mathfrak{g}, \omega, K)$ is isomorphic to one of the following Lie algebras with the given para-Kähler structures:*

Lie algebra $\mathfrak{rh}_3$

For $\omega = e^{14} + e^{23}$

$$K_1 = \pm(E_{11} - E_{22} + E_{33} - E_{44})$$

$$K_2 = \pm(-E_{11} + E_{21} + E_{22} - E_{33} - E_{43} + E_{44})$$

Lie algebra $\mathfrak{rr}_{3,0}$

$$\text{For } \omega = e^{12} + e^{34}$$

$$K_1 = \pm(-E_{11} + E_{22} - E_{33} + E_{44})$$

$$K_2 = E_{11} + xE_{12} - E_{22} + E_{33} - E_{44}, \quad x \neq 0$$

Lie algebra $\mathfrak{rr}_{3,-1}$

$$\text{For } \omega = e^{14} + e^{23}$$

$$K_1 = \pm(-E_{11} + E_{22} - E_{33} + E_{44})$$

$$K_2 = \pm(E_{11} + E_{22} \pm E_{23} - E_{33} - E_{44})$$

$$K_3 = E_{11} + E_{22} + xE_{14} - E_{33} - E_{44}, \quad x \neq 0$$

$$K_3 = -E_{11} + E_{22} \pm E_{32} - E_{33} + E_{44}$$

Lie algebra $\mathfrak{r}_2\mathfrak{r}_2$

$$\text{For } \omega = e^{12} + \mu e^{13} + e^{34}, \quad (\mu > 0)$$

$$K_1 = \pm(-E_{11} + E_{22} + E_{33} - E_{44})$$

$$K_2 = \pm(E_{11} - 2E_{13} - E_{22} - E_{33} + 2E_{42} + E_{44})$$

$$\text{For } \omega = e^{12} + e^{34}$$

$$K_1 = \pm(E_{11} - E_{22} - 2E_{24} + 2E_{31} - E_{33} + E_{44})$$

$$K_2 = \pm(-E_{11} + xE_{12} + E_{13} + xE_{14} + E_{22} + xE_{32} + xE_{34} - E_{42} + \frac{1}{x}E_{43}), \quad x \neq 0$$

$$K_3 = \pm(-E_{11} + E_{22} + E_{33} + xE_{34} - E_{44})$$

$$K_4 = -E_{11} + E_{22} - E_{33} + E_{44}$$

$$K_5 = E_{11} + xE_{12} - E_{22} + E_{33} + yE_{34} - E_{44}, \quad xy \neq 0$$

$$K_6 = E_{11} - 2E_{13} - E_{22} - E_{33} + 2E_{42} + E_{44}$$

$$K_7 = E_{11} + xE_{12} + xE_{14} - E_{22} + xE_{32} + E_{33} + xE_{34} - E_{44}$$

Lie algebra $\mathfrak{r}_2'$

$$\text{For } \omega = e^{14} + e^{23}$$

$$K_1 = \pm(E_{11} + xE_{14} - E_{22} - \frac{4}{x}E_{32} + E_{33} - E_{44}), \quad x \neq 0$$

$$K_2 = xE_{11} + 2yE_{12} + (1-x)E_{13} - 2yE_{14} - 2yE_{21} + xE_{22} + 2yE_{23} + (1-x)E_{24} + (1+x)E_{31} + 2yE_{32} - xE_{33} - 2yE_{34} - 2yE_{41} + (1+x)E_{42} + 2yE_{43} - xE_{44}$$

$$K_3 = -E_{11} - E_{22} + E_{33} + E_{44}$$

Lie algebra $\mathfrak{r}_{4,0}$

$$\text{For } \omega = e^{14} \pm e^{23}$$

$$K = \pm(E_{11} - E_{22} + E_{33} - E_{44})$$

Lie algebra $\mathfrak{r}_{4,-1}$

$$\text{For } \omega = e^{13} + e^{24}$$

$$K_1 = \pm(E_{11} + xE_{13} - E_{22} - E_{33} + E_{44})$$

Lie algebra $\mathfrak{r}_{4,-1,\beta}$ ($-1 < \beta < 1$)

$$\text{For } \omega = e^{12} + e^{34}$$

$$K_1 = \pm(E_{11} \pm E_{12} - E_{22} - E_{33} + E_{44})$$

$$K_2 = \pm(E_{11} - E_{22} - E_{33} + E_{44})$$

$$K_3 = \pm(E_{11} - E_{22} + xE_{34} + \frac{1}{x}E_{43}), \quad x \neq 0$$

$$K_4 = \pm(E_{11} - E_{22} + E_{33} - E_{44})$$

Lie algebra $\mathfrak{v}_{4,-1,-1}$

For $\omega = e^{12} + e^{34}$

$$K_1 = \pm(-E_{11} \pm E_{21} + E_{22} - E_{33} + E_{44})$$

$$K_2 = \pm(-E_{11} + E_{22} - E_{33} + xE_{43} + E_{44})$$

$$K_3 = \pm(E_{11} \pm E_{12} - E_{22} - E_{33} + E_{44})$$

$$K_4 = \pm(E_{11} - E_{22} - E_{33} + xE_{43} + E_{44})$$

$$K_5 = -E_{11} + xE_{21} + E_{22} + E_{23} - E_{33} + E_{41} + E_{44}$$

Lie algebra $\mathfrak{v}_{4,\alpha,-\alpha}$ ($-1 < \alpha < 0$)

For $\omega = e^{14} + e^{23}$

$$K_1 = \pm(-E_{11} + E_{22} \pm E_{23} - E_{33} + E_{44})$$

$$K_2 = \pm(-E_{11} + E_{22} - E_{33} + xE_{41} + E_{44})$$

$$K_3 = \pm(E_{11} + E_{22} - E_{33} - E_{44})$$

$$K_4 = \pm(E_{11} + E_{22} \pm E_{32} - E_{33} - E_{44})$$

Lie algebra $\mathfrak{d}_{4,1}$

For $\omega = e^{12} - e^{34}$

$$K_1 = \pm(E_{11} \pm E_{12} - E_{22} - E_{33} + E_{44})$$

$$K_2 = \pm(E_{11} - E_{22} - E_{33} + xE_{43} + E_{44})$$

$$K_3 = \pm(-E_{11} + E_{22} - E_{33} + xE_{43} + E_{44})$$

$$K_4 = \pm(E_{11} + E_{21} - E_{22} + xE_{23} + E_{33} - xE_{41} - E_{44})$$

$$K_5 = \pm(-E_{11} \pm E_{21} + E_{22} - E_{33} + E_{44})$$

For $\omega = e^{12} - e^{34} + e^{24}$

$$K_1 = \pm(E_{11} + xE_{12} - E_{22} - E_{33} + E_{44})$$

Lie algebra $\mathfrak{d}_{4,2}$

For $\omega = e^{12} - e^{34}$

$$K_1 = \pm(E_{11} + xE_{12} - E_{22} - E_{33} + E_{44})$$

$$K_2 = \pm(E_{11} - E_{22} + E_{33} + xE_{43} - E_{44})$$

For $\omega = e^{14} - e^{23}$

$$K_1 = \pm(E_{11} - E_{22} + E_{33} + xE_{41} - E_{44})$$

$$K_2 = \pm(-E_{11} - E_{22} + 2E_{31} + xE_{32} + E_{33} - 2E_{42} + E_{44})$$

$$K_3 = \pm(-E_{11} + E_{22} + xE_{21} + xE_{23} - E_{33} + xE_{41} + xE_{43} + E_{44})$$

$$K_4 = \pm(E_{11} - E_{22} + xE_{21} + xE_{23} + E_{33} + xE_{41} + xE_{43} - E_{44})$$

$$K_5 = E_{11} - E_{22} - xE_{23} + E_{33} + \frac{x}{2}E_{41} - E_{44}, \quad x \neq 0$$

For $\omega = e^{14} + e^{23}$

$$K_1 = \pm(E_{11} - E_{22} + E_{33} + xE_{41} - E_{44})$$

$$K_2 = E_{11} - E_{22} + xE_{23} + E_{33} - \frac{x}{2}E_{41} - E_{44}, \quad x \neq 0$$

Lie algebra $\mathfrak{d}_{4,\frac{1}{2}}$

For $\omega = e^{12} - e^{34}$

$$K_1 = E_{11} - E_{22} - E_{33} + xE_{43} + E_{44}, \quad x \neq 0$$

$$K_2 = \pm(E_{11} - E_{22} + E_{33} - E_{44})$$

Lie algebra $\mathfrak{d}_{4,\lambda}$ ($\lambda > \frac{1}{2}$, $\lambda = 1, 2$)

For $\omega = e^{12} - e^{34}$

$$\begin{aligned} K_1 &= \pm(E_{11} \pm E_{12} - E_{22} - E_{33} + E_{44}) \\ K_2 &= \pm(-E_{11} \pm E_{21} + E_{22} - E_{33} + E_{44}) \\ K_3 &= \pm(E_{11} - E_{22} - E_{33} + xE_{43} + E_{44}) \\ K_4 &= -E_{11} + E_{22} - E_{33} + xE_{43} + E_{44} \\ K_5 &= E_{11} + xE_{21} - E_{22} + E_{33} - E_{44} \end{aligned}$$

Lie algebra $\mathfrak{h}_4$

For $\omega = \pm(e^{12} - e^{34})$

$$K_1 = \pm(E_{11} - E_{22} - E_{33} + E_{44})$$

Corollary 1.1. The symplectic Lie algebras $\mathfrak{r}'_{3,0}$, $\mathfrak{n}_4$, $\mathfrak{r}'_{4,0,\delta}$ and $\mathfrak{d}'_{4,\delta}$ do not admit a para-Kähler structure.

The paper is organized as follows. Section 2 contains the basic results which are essential to the classification of four-dimensional para-Kähler Lie algebras (proof of the Theorem 1.1). Theorem 2.1 and Theorem 2.2 are the key steps in this proof. Section 3 is devoted to some curvature properties of four-dimensional para-Kähler metrics. Section 4 contains the tables of Theorem 2.2 and the isomorphisms tables used in the proof of Theorem 1.1.

The software Maple 18[®] has been used to check all needed calculations.

2 Proof of Theorem 1.1

In this section we begin with a reminder of the new approach introduced by Benayadi and Boucetta in [3], which characterizes the para-Kähler Lie algebras.

Recall that, a para-Kähler Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle, K)$ carries a Levi-Civita product which is characterized by Koszul's formula:

$$2\langle u, v, w \rangle = \langle [u, v], w \rangle + \langle [w, u], v \rangle + \langle [w, v], u \rangle.$$

The subalgebras $\mathfrak{g}^1 = \ker(K - Id_{\mathfrak{g}})$ and $\mathfrak{g}^{-1} = \ker(K + Id_{\mathfrak{g}})$ have the following properties, $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are isotropic with respect to $\langle \cdot, \cdot \rangle$, Lagrangian with respect to ω and checking that $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$, moreover the restriction of the Levi-Civita product on $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ induces a left symmetric structure. i.e. for any $u, v, w \in \mathfrak{g}^1$ (resp. $\mathfrak{g}^{-1}$),

$$ass(u, v, w) = ass(v, u, w)$$

where $ass(u, v, w) = (u \cdot v) \cdot w - u \cdot (v \cdot w)$. In particular, $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$ are left symmetric algebras.

For any $u \in \mathfrak{g}^{-1}$, let u^* denote the element of $(\mathfrak{g}^1)^*$ given by $u^*(v) = \langle u, v \rangle$. The map $u \mapsto u^*$ realizes an isomorphism between $\mathfrak{g}^{-1}$ and $(\mathfrak{g}^1)^*$. Thus, we can identify $(\mathfrak{g}, \langle \cdot, \cdot \rangle, K)$ relative to the phase space $(\mathfrak{g}^1 \oplus (\mathfrak{g}^1)^*, \langle \cdot, \cdot \rangle_0, K_0)$, where $\langle \cdot, \cdot \rangle_0$ and K_0 are given by:

$$\langle u + \alpha, v + \beta \rangle_0 = \alpha(v) + \beta(u) \quad \text{and} \quad K_0(u + \alpha) = u - \alpha.$$

Both $\mathfrak{g}^1$ and $(\mathfrak{g}^1)^*$ carry a left symmetric algebra structure. For any $u \in \mathfrak{g}^1$ and for any $\alpha \in (\mathfrak{g}^1)^*$, we denote $L_u : \mathfrak{g}^1 \rightarrow \mathfrak{g}^1$ and $L_\alpha : (\mathfrak{g}^1)^* \rightarrow (\mathfrak{g}^1)^*$ as the left multiplication by u and α , respectively, i.e., for any $v \in \mathfrak{g}^1$ and any $\beta \in (\mathfrak{g}^1)^*$,

$$L_u v = u \cdot v \quad \text{and} \quad L_\alpha \beta = \alpha \cdot \beta.$$

The Levi-Civita product and the Lie bracket on $\mathfrak{g}$ are determined entirely by their restrictions to $(\mathfrak{g}^1)^*$ and $\mathfrak{g}^1$: For any $u \in \mathfrak{g}^1$ and for any $\alpha \in (\mathfrak{g}^1)^*$,

$$u \cdot \alpha = L_u^t \alpha \quad \text{and} \quad \alpha \cdot u = -L_\alpha^t u.$$

Conversely, let U be a finite dimensional vector space and U^* is its dual space. We suppose that both U and U^* have the structure of a left symmetric algebra. We extend the products on U and U^* to $U \oplus U^*$ for any $X, Y \in U$ and for any $\alpha, \beta \in U^*$, by putting

$$(X + \alpha).(Y + \beta) = X.Y - L_\alpha^t Y - L_X^t \beta + \alpha.\beta. \quad (1)$$

We say that two left symmetric products on U and U^* are Lie-extendible if the commutator of the product on $U \oplus U^*$ given by (1) is a Lie bracket. In this case we have the following theorem:

Theorem 2.1. [3] Let $(U, .)$ and $(U^*, .)$ be two Lie-extendible left symmetric products. Then, $(U \oplus U^*, \langle ., . \rangle_0, K_0)$, endowed with the Lie algebra bracket associated with the product given by (1) is a para-Kähler Lie algebra. Where ω_0 , $\langle ., . \rangle_0$ and K_0 are given by:

$$\omega_0(u + \alpha, v + \beta) = \beta(u) - \alpha(v), \quad \langle u + \alpha, v + \beta \rangle_0 = \alpha(v) + \beta(u) \quad \text{and} \quad K_0(u + \alpha) = u - \alpha.$$

Moreover, all para-Kähler Lie algebras are obtained in this manner.

Remark 1. In the construction of the paracomplex structure associated to the bi-Lagrangian structure we have the choice between (ω, K_0) and $(\omega, -K_0)$ which are not necessarily isomorphic. We take that into consideration by giving all the para-kähler structures up to isomorphism.

Now let U be a 2-dimensional vector space and U^* its dual space and let $\{e_1, e_2\}$, $\{e_3, e_4\}$ be a basis of U and U^* . We base on the previous theorem and the classification of real left-symmetric algebras in dimension 2 listed below (see Theorem 1.2. of [11])

$b_{1,\alpha} : e_2.e_1 = e_1, e_2.e_2 = \alpha.e_2$ $b_2 : e_2.e_1 = e_1, e_2.e_2 = e_1 + e_2$ $b_{3,\alpha} \alpha \neq 0 : e_1.e_2 = e_1, e_2.e_1 = (1 - \frac{1}{\alpha})e_1, e_2.e_2 = e_2$	$b_4 : e_1.e_2 = e_1, e_2.e_2 = e_1 + e_2$ $b_5^+ : e_1.e_1 = e_2, e_2.e_1 = -e_1, e_2.e_2 = -2e_2$ $b_5^- : e_1.e_1 = -e_2, e_2.e_1 = -e_1, e_2.e_2 = -2e_2$
$c_1 : \text{Trivial left-symmetric algebra}$ $c_5^+ : e_2.e_2 = e_2, e_2.e_1 = e_1, e_1.e_2 = e_1, e_1.e_1 = e_2$ $c_5^- : e_2.e_2 = e_2, e_2.e_1 = e_1, e_1.e_2 = e_1, e_1.e_1 = -e_2$	$c_2 : e_2.e_2 = e_2$ $c_3 : e_2.e_2 = e_1$ $c_4 : e_2.e_2 = e_2, e_2.e_1 = e_1, e_1.e_2 = e_1.$

Remark 2. b stands for algebras with non-commutative associated Lie algebra and c stands for algebras with commutative associated Lie algebra.

Theorem 2.2. Let $(\mathfrak{g}, \langle ., . \rangle, K)$ be a four-dimensional para-Kähler Lie algebra. Then there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that

$$\omega = e^{13} + e^{24}, \quad \langle ., . \rangle = \dot{e}^{13} + \dot{e}^{24} \quad \text{and} \quad K = E_{11} + E_{22} - E_{33} - E_{44}$$

and the non vanishing Lie brackets as listed in the Table 4 and 5.

Proof. We will give the proof in the case $\mathcal{B}_2$ since all cases should be handled in a similar way. In that case the left-symmetric product in U is given by $e_2.e_1 = e_1, e_2.e_2 = e_1 + e_2$ and let

$$\begin{aligned} e_3.e_3 &= a_{33}e_3 + b_{33}e_4 \\ e_3.e_4 &= a_{34}e_3 + b_{34}e_4 \\ e_4.e_3 &= a_{43}e_3 + b_{43}e_4 \\ e_4.e_4 &= a_{44}e_3 + b_{44}e_4 \end{aligned}$$

be an arbitrary product in U^* , let's look for products in U^* which satisfy the Jacobi identity $\oint [[e_i, e_j], e_k] = 0$ with $1 \leq i < j < k \leq 4$, where $\oint$ denotes the cyclic sum.

The identities $\mathcal{F}[[e_1, e_2], e_3] = 0$ and $\mathcal{F}[[e_1, e_2], e_4] = 0$ are equivalent to

$$\begin{cases} b_{34} + a_{33} + a_{43} = 0 \\ a_{44} = 0 \\ b_{44} + a_{43} = 0. \end{cases}$$

Suppose that $a_{44} = 0$, $b_{44} + a_{43} = 0$ and $b_{34} = -a_{33} - a_{43}$, the identities $\mathcal{F}[[e_1, e_3], e_4] = 0$ and $\mathcal{F}[[e_2, e_3], e_4] = 0$ are equivalent to

$$\begin{cases} a_{33}a_{34} - 2a_{33}a_{43} - a_{34}b_{43} - a_{43}^2 - a_{43}b_{43} = 0 \\ a_{34}(a_{34} + a_{43}) = 0 \\ a_{34} = 0 \end{cases} \quad \text{and} \quad \begin{cases} (a_{34} - 3a_{43})b_{33} + b_{43}(a_{33} - b_{43}) = 0 \\ 2a_{43}^2 + (2a_{33} - a_{34} + b_{43})a_{43} - a_{34}(a_{33} - b_{43}) = 0 \\ a_{34} = 0 \\ a_{34} + b_{43} + 2a_{33} = 0. \end{cases}$$

We get $a_{33} = 0$, $a_{34} = 0$, $a_{43} = 0$, $a_{44} = 0$, $b_{34} = 0$, $b_{43} = 0$ and $b_{44} = 0$. Then the product in U^* is given by $e_3 \cdot e_3 = b_{33}e_4$ (who is indeed a left-symmetric product) and the Lie bracket in $U \oplus U^*$ is given by

$$[e_1, e_2] = -e_1, [e_2, e_3] = xe_1 - e_3 - e_4, [e_2, e_4] = -e_4.$$

□

Proof. of Theorem 1.1.

The Theorem 2.2 confirms that for each Lie algebra $\mathfrak{g}$ of the tables 4 and 5 there exists a base $B_0 = (e_1, e_2, e_3, e_4)$ such that the para-kähler structure is given by

$$\omega = e^{13} + e^{24} \quad \text{and} \quad K = E_{11} + E_{22} - E_{33} - E_{44}$$

and the Lie brackets depend on some parameters. In Tables 6 and 7 we build a family of isomorphisms (depending on the values of parameters) from $\mathfrak{g}$ ($\mathfrak{B}_{i,j}$ or $\mathcal{C}_{i,j}$) onto a four-dimensional Lie algebra, (say A) of the Table 1. Each isomorphism is given by the passage matrix P from B_0 to $B = (f_1, f_2, f_3, f_4)$. The image by P of the para-Kähler structure (ω, K) is given by the matrices of its component in the bases B and B^* by

$${}^tP \circ \omega \circ P = \omega_i \quad \text{and} \quad P^{-1} \circ K \circ P = K_i.$$

In this way we collect all the possible para-kähler structures (ω_i, K_i) on A . Thereafter, we proceed to the classification in A (up to automorphism).

We will give the proof in the case $\mathfrak{rr}_{3,0}$ since all cases should be handled in a similar way. We will show that the Lie algebra $\mathfrak{rr}_{3,0}$ admits two non-equivalent para-Kähler structures. Note that in this case the non vanishing Lie bracket is

$$[f_1, f_2] = f_2$$

the symplectic form is $\omega_0 = f^{12} + f^{34}$ and the automorphisms is

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & 0 & a_{3,3} & a_{3,4} \\ a_{4,1} & 0 & a_{4,3} & a_{4,4} \end{pmatrix}.$$

The groups of automorphisms of four-dimensional Lie algebras were given in [10].

From Table 6 and Table 7, $\mathfrak{rr}_{3,0}$ is obtained four times.

1. The transformation: $f_1 = -e_4$, $f_2 = e_2$, $f_3 = e_3$, $f_4 = e_1$ gives an isomorphism from $\mathcal{C}_{1,6}$ to $\mathfrak{rr}_{3,0}$ and the para-Kähler structure obtained on $\mathfrak{rr}_{3,0}$ is

$$\omega_1 = f^{12} - f^{34} \quad \text{and} \quad K_1 = -E_{11} + E_{22} - E_{33} + E_{44}.$$

2. The transformation: $f_1 = -e_2$, $f_2 = -ye_2 + e_4$, $f_3 = e_1$, $f_4 = e_3$ gives an isomorphism from $\mathbb{C}_{2,1}$ with $x = 0$ to $\mathfrak{rr}_{3,0}$ and the para-Kähler structure obtained on $\mathfrak{rr}_{3,0}$ is

$$\omega_2 = -f^{12} + f^{34} \quad \text{and} \quad K_2 = E_{11} + 2yE_{12} - E_{22} + E_{33} - E_{44}.$$

3. The transformation: $f_1 = -e_2$, $f_2 = e_4$, $f_3 = e_1$, $f_4 = e_3$ gives an isomorphism from $\mathbb{C}_{2,2}$ with $x = 0$, $y = 0$ to $\mathfrak{rr}_{3,0}$ and the para-Kähler structure obtained on $\mathfrak{rr}_{3,0}$ is

$$\omega_3 = -f^{12} + f^{34} \quad \text{and} \quad K_3 = E_{11} - E_{22} + E_{33} - E_{44}.$$

4. The transformation: $f_1 = e_1 - e_2$, $f_2 = e_4$, $f_3 = e_1$, $f_4 = e_3$ gives an isomorphism from $\mathbb{C}_{2,3}$ with $x = 0$, $y = 0$ to $\mathfrak{rr}_{3,0}$ and the para-Kähler structure obtained on $\mathfrak{rr}_{3,0}$ is

$$\omega_4 = -f^{12} + f^{14} + f^{34} \quad \text{and} \quad K_4 = E_{11} - E_{22} + E_{33} - E_{44}.$$

the algebra $\mathfrak{rr}_{3,0}$ supports ω_0 as a unique symplectic structure (up to automorphism), therefore there are four families of automorphisms T_i , $i \in \{1, \dots, 4\}$ such that, $T_i^* \omega_i = \omega_0$ for $i \in \{1, \dots, 4\}$, a direct calculation gives us

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ 0 & 0 & \frac{a_{3,4}a_{4,3}-1}{a_{4,4}} & a_{3,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & -1 & 0 & 0 \\ 0 & 0 & \frac{a_{3,4}a_{4,3}+1}{a_{4,4}} & a_{3,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ 0 & 0 & \frac{a_{3,4}a_{4,3}-1}{a_{4,4}} & a_{3,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & -1 & 0 & 0 \\ -1 & 0 & \frac{a_{3,4}a_{4,3}+1}{a_{4,4}} & a_{3,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix}.$$

Thus we obtain four para-Kähler structures on $\mathfrak{rr}_{3,0}$ given by (ω_0, K_{0i}) , $i \in 1, \dots, 4$ with $K_{0i} = T_i^{-1} \circ K_i \circ T_i$ a direct calculation gives us

$$K_{01} = -E_{11} + E_{22} - E_{33} + E_{44}$$

$$K_{02} = E_{11} + 2yE_{12} - E_{22} + E_{33} - E_{44}$$

$$K_{03} = E_{11} - E_{22} + E_{33} - E_{44}$$

$$K_{04} = -E_{11} + E_{22} + E_{33} - E_{44}$$

We note that K_{03} is a sub-case of K_{02} and (ω_0, K_{04}) is isomorphic to (ω_0, K_{01}) . Indeed, we have $L^* \omega_0 = \omega_0$ and $L^{-1} \circ K_{04} \circ L = K_{01}$ with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We complete the proof by showing that (ω_0, K_{01}) is not isomorphic to (ω_0, K_{02}) . In fact, the symplectomorphism group of ω_0 is generated by

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ 0 & 0 & \frac{a_{3,4}a_{4,3}+1}{a_{4,4}} & a_{3,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{2,1} & 1 & 0 & 0 \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & -a_{3,4}^{-1} & 0 \end{pmatrix}.$$

A simple calculation gives us

$$f^2((L_1^{-1} \circ K_{01} \circ L_1 - K_{02})(f_2)) = 2 \quad \text{and} \quad f^1((L_2^{-1} \circ K_{01} \circ L_2 - K_{02})(f_1)) = -2$$

so

$$L_1^{-1} \circ K_{01} \circ L_1 \neq K_{02} \quad \text{and} \quad L_2^{-1} \circ K_{01} \circ L_2 \neq K_{02}.$$

□

3 Application: Curvatures properties of four-dimensional para-Kähler Lie algebras

Now let $(\mathfrak{g}, \omega, K)$ denote a four-dimensional para-Kähler Lie algebra. Let $\nabla : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ be the Levi-Civita product associated to a left-invariant pseudo-Riemannian metric $h(X, Y) = \omega(KX, Y)$. The connection ∇ is also called Hess connection. The curvature tensor is then described in terms of the map

$$\begin{aligned} R : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ (X, Y) &\longmapsto R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y] \end{aligned} \quad (2)$$

The Ricci tensor is the symmetric tensor ric given by $\text{ric}(X, Y) = \text{tr}(Z \mapsto R(X, Z)Y)$ and the Ricci operator $\text{Ric} : \mathfrak{g} \rightarrow \mathfrak{g}$ is given by the relation $h(\text{Ric}(X), Y) = \text{ric}(X, Y)$. The scalar curvature is defined in the standard way by $s = \text{tr}(\text{Ric})$.

Recall that: $(\mathfrak{g}, h)$ is called flat if $R = 0$, Ricci flat if $\text{Ric} = 0$ and Ricci soliton if

$$\mathcal{L}_X h + \text{ric} = \lambda h, \quad (3)$$

where $X = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$ is a vector field and λ is a real constant, in that case if $X = 0$ then h is called Einstein metric and if λ is positive, zero, or negative then h is called a shrinking, steady, or expanding Ricci soliton, respectively. We give in the following theorem some geometrical situations for the left invariant four-dimensional dimensional para-Kähler Lie groups.

Remark 3. *Opposite metrics have the same Levi-Civita product and the same curvature and Ricci tensors. Thus, for a metric h and its opposite $g = -h$, one has that*

$$\mathcal{L}_X g + \text{ric}^g = \lambda g \Leftrightarrow \mathcal{L}_X(-h) + \text{ric}^h = \lambda(-h) \Leftrightarrow \mathcal{L}_{-X} h + \text{ric}^h = -\lambda h,$$

Theorem 3.1. *Let $(\mathfrak{g}, \omega, K)$ be a class of para-Kähler Lie algebras obtained in Theorem 1.1. The associated para-Kähler metric (up to sign) and some of its properties are given in the following table*

Table 3: Curvature properties of four-dimensional para-Kähler Lie algebras

Lie algebra	Para-Kähler metric	$R=0$	$\text{Ric}=0$	Ricci soliton	
				λ	X
$\mathfrak{th}_3$	$\dot{e}^{14} - \dot{e}^{23}$	Yes	Yes	0	$(0, 0, x_3, x_4)$
	$\dot{e}^{13} - \dot{e}^{14} + \dot{e}^{23}$	Yes	Yes	0	$(0, 0, x_3, x_4)$
$\mathfrak{tt}_{3,0}$	$-\dot{e}^{12} - \dot{e}^{34}$	Yes	Yes	0	$(0, 0, x_3, x_4)$
	$\dot{e}^{12} + x\dot{e}^{22} + \dot{e}^{34}, x \neq 0$	No	No		No
$\mathfrak{tt}_{3,-1}$	$-\dot{e}^{14} + \dot{e}^{23}$	Yes	Yes	0	$(x_1, 0, 0, x_4)$
	$\dot{e}^{14} + \dot{e}^{23} \pm \dot{e}^{33}$	No	Yes	0	$(0, 0, 0, x_4)$
	$\dot{e}^{14} + \dot{e}^{23} + x\dot{e}^{44}, x \neq 0$	Yes	Yes	0	$(x_1, 0, 0, x_4)$

	$-\dot{e}^{14} \pm \dot{e}^{22} + \dot{e}^{23}$	No	Yes	0	$(0, 0, 0, x_4)$
$\tau_2 \tau_2$	$-\dot{e}^{12} - \mu \dot{e}^{13} + \dot{e}^{34}$	No	Yes	0	$(0, 0, 0, 0)$
$\mu > 0$	$\dot{e}^{12} + \mu \dot{e}^{13} - 2\dot{e}^{23} - 2\mu \dot{e}^{33} - \dot{e}^{34}$	No	Yes	0	$(0, 0, 0, 0)$
	$\dot{e}^{12} + 2\dot{e}^{14} - \dot{e}^{34}$	Yes	Yes	$-x_3$	$(x_3, 0, x_3, 0)$
	$-\dot{e}^{12} + x\dot{e}^{22} + \dot{e}^{23} + x\dot{e}^{24} - \frac{1}{x}\dot{e}^{33} + x\dot{e}^{44}, x \neq 0$	No	No	$\frac{3}{2}x$	$(0, 0, 0, 0)$
	$\dot{e}^{12} + x\dot{e}^{22} + \dot{e}^{34} + y\dot{e}^{44}, xy \neq 0, x \neq y$	No	No		No
	$\dot{e}^{12} + x\dot{e}^{22} + \dot{e}^{34} + x\dot{e}^{44}, x \neq 0$	No	No	x	$(0, 0, 0, 0)$
$\tau_2 \tau_2$	$-\dot{e}^{12} + \dot{e}^{34} + x\dot{e}^{44}, x \neq 0$	No	No	x	$(-x, 0, 0, 0)$
	$-\dot{e}^{12} + \dot{e}^{34}$	Yes	Yes	$-x_3$	$(x_3, 0, x_3, 0)$
$\mu = 0$	$\dot{e}^{12} - 2\dot{e}^{23} - \dot{e}^{34}$	Yes	Yes	$-x_3$	$(x_3, 0, x_3, 0)$
	$\dot{e}^{12} + x\dot{e}^{22} + x\dot{e}^{24} + \dot{e}^{34} + x\dot{e}^{44}, x \neq 0$	No	No	$\frac{3}{2}x$	$(0, 0, 0, 0)$
	$\dot{e}^{12} + \dot{e}^{34}$	Yes	Yes	$-x_3$	$(x_3, 0, x_3, 0)$
	$\dot{e}^{14} + \frac{4}{x}\dot{e}^{22} - \dot{e}^{23} + x\dot{e}^{44}, x \neq 0$	No	No	$\frac{3}{2}x$	$(0, 0, 0, 0)$
τ_2'	$h_1, \quad y \neq 0$	No	No		No
	$h_2, \quad y \neq 0$	No	No		No
	$h_3, \quad y \neq 0$	No	No	$-4y$	$(0, 0, 0, 0)$
	$-(1+x)\dot{e}^{12} + x(\dot{e}^{14} + \dot{e}^{23}) + (1-x)\dot{e}^{34}, x \neq 0, 1$	No	No		No
	$-2\dot{e}^{12} + \dot{e}^{14} + \dot{e}^{23}$	Yes	Yes	$-x_3$	$(x_3, 0, x_3, 0)$
	$-\dot{e}^{12} + \dot{e}^{34}$	No	No		No
	$-\dot{e}^{14} - \dot{e}^{23}$	Yes	Yes	$-x_1$	$(x_1, 0, 0, 0)$
$\tau_{4,0}$	$\dot{e}^{14} - \dot{e}^{23}$	No	Yes	0	$(0, x_2, 0, 0)$
$\tau_{4,-1}$	$\dot{e}^{13} - \dot{e}^{24} + x\dot{e}^{33}, x \neq 0$	No	Yes	0	$(0, 0, 0, 0)$
	$\dot{e}^{13} - \dot{e}^{24}$	Yes	Yes	0	$(0, 0, 0, 0)$
	$\dot{e}^{12} + \dot{e}^{22} - \dot{e}^{34}, \beta \neq 0$	No	Yes	0	$(0, 0, 0, 0)$
$\tau_{4,-1,\beta}$	$\dot{e}^{12} + \dot{e}^{22} - \dot{e}^{34}, \beta \neq 0$	No	Yes	0	$(0, 0, x_3, 0)$
	$\dot{e}^{12} \pm \dot{e}^{34}, \beta \neq 0$	Yes	Yes	0	$(0, 0, 0, 0)$
	$\dot{e}^{12} \pm \dot{e}^{34}, \beta = 0$	Yes	Yes	0	$(0, 0, x_3, x_4)$
	$\dot{e}^{12} - \frac{1}{x}\dot{e}^{33} + x\dot{e}^{44}, \beta \neq 0$	No	No		No
	$\dot{e}^{12} - \frac{1}{x}\dot{e}^{33} + x\dot{e}^{44}, \beta = 0$	Yes	Yes	0	$(0, 0, x_3, x_4)$
	$\pm \dot{e}^{11} - \dot{e}^{12} - \dot{e}^{34}$	No	Yes	0	$(0, 0, 0, 0)$
$\tau_{4,-1,-1}$	$\pm \dot{e}^{12} - x\dot{e}^{33} - \dot{e}^{34}, x \neq 0$	No	No		No
	$\pm \dot{e}^{12} - \dot{e}^{34}$	Yes	Yes	0	$(0, 0, 0, 0)$
	$\dot{e}^{12} \pm \dot{e}^{22} - \dot{e}^{34}$	No	Yes	0	$(0, 0, 0, 0)$
	$-x\dot{e}^{11} - \dot{e}^{12} - \dot{e}^{13} - \dot{e}^{34}, x \neq 0$	No	Yes	0	$(0, 0, 0, 0)$
	$-\dot{e}^{12} - \dot{e}^{13} - \dot{e}^{34}$	Yes	Yes	0	$(0, 0, 0, 0)$
	$-\dot{e}^{14} + \dot{e}^{23} \pm \dot{e}^{33}$	No	Yes	0	$(0, 0, 0, 0)$
$\tau_{4,\alpha,-\alpha}$	$-x\dot{e}^{11} - \dot{e}^{14} + \dot{e}^{23}, x \neq 0$	No	No		No
	$\pm \dot{e}^{14} + \dot{e}^{23}$	Yes	Yes	0	$(0, 0, 0, 0)$
	$\dot{e}^{14} \pm \dot{e}^{22} + \dot{e}^{23}$	No	Yes	0	$(0, 0, 0, 0)$
	$\dot{e}^{12} \pm \dot{e}^{22} + \dot{e}^{34}$	Yes	Yes	0	$(x_1, 0, 0, 0)$
$\mathfrak{d}_{4,1}$	$\dot{e}^{12} + x\dot{e}^{33} + \dot{e}^{34}, x \neq 0$	No	No	$\frac{3}{2}x$	$(0, 0, 0, 0)$
	$\dot{e}^{12} + \dot{e}^{34}$	Yes	Yes	$-x_4$	$(x_1, 0, 0, x_4)$
	$-\dot{e}^{12} + x\dot{e}^{33} + \dot{e}^{34}, x \neq 0$	No	No		No
	$-\dot{e}^{12} + \dot{e}^{34}$	Yes	Yes	0	$(0, 0, 0, 0)$
	$-\dot{e}^{11} + \dot{e}^{12} - x\dot{e}^{13} - \dot{e}^{34}, x \neq 0$	No	No		No
	$-\dot{e}^{11} + \dot{e}^{12} - \dot{e}^{34}$	No	Yes	0	$(0, 0, 0, 0)$
	$\pm \dot{e}^{11} - \dot{e}^{12} + \dot{e}^{34}$	No	Yes	0	$(0, 0, 0, 0)$
	$\dot{e}^{12} + x\dot{e}^{22} - \dot{e}^{24} + \dot{e}^{34}$	Yes	Yes	0	$(x_1, 0, 0, 0)$
	$\dot{e}^{12} + x\dot{e}^{22} + \dot{e}^{34}, x \neq 0$	No	Yes	0	$(xx_2, x_2, 0, 0)$
	$\dot{e}^{12} + \dot{e}^{34}$	Yes	Yes	$-x_4$	$(0, x_2, 0, x_4)$

$\mathfrak{d}_{4,2}$	$\dot{e}^{12} + x\dot{e}^{33} - \dot{e}^{34}, \chi = 0$	No	No	$\frac{3}{2}x$	$(0, 0, 0, 0)$
	$\dot{e}^{12} - \dot{e}^{34}$	Yes	Yes	$-x_4$	$(0, 0, 0, x_4)$
	$-x\dot{e}^{11} + \dot{e}^{14} + \dot{e}^{23}, \chi = 0$	No	No		No
	$\dot{e}^{14} \pm \dot{e}^{23}$	Yes	Yes	0	$(0, 0, 0, 0)$
	$2\dot{e}^{12} - \dot{e}^{14} + x\dot{e}^{22} + \dot{e}^{23}$	No	No		No
	$-x\dot{e}^{11} - x\dot{e}^{13} - \dot{e}^{14} - \dot{e}^{23} - x\dot{e}^{33}, \chi = 0$	No	No		No
	$-x\dot{e}^{11} - x\dot{e}^{13} + \dot{e}^{14} + \dot{e}^{23} - x\dot{e}^{33}, \chi = 0$	No	No		No
	$2\dot{e}^{12} + \dot{e}^{14} + x\dot{e}^{22} - \dot{e}^{23}$	No	No		No
	$-\frac{x}{2}\dot{e}^{11} + \dot{e}^{14} + \dot{e}^{23} + x\dot{e}^{33}, \chi = 0$	No	No		No
	$-x\dot{e}^{11} + \dot{e}^{14} - \dot{e}^{23}, x \neq 0$	No	No		No
$\mathfrak{d}_{4,\frac{1}{2}}$	$\frac{x}{2}\dot{e}^{11} + \dot{e}^{14} - \dot{e}^{23} + x\dot{e}^{33}, \chi = 0$	No	No		No
	$\dot{e}^{12} - \dot{e}^{34}$	Yes	Yes	$-x_4$	$(0, 0, 0, x_4)$
	$\dot{e}^{12} + x\dot{e}^{33} + \dot{e}^{34}, \chi = 0$	No	No	$\frac{3}{2}x$	$(0, 0, 0, 0)$
$\mathfrak{d}_{4,\lambda}$	$\dot{e}^{12} \pm \dot{e}^{22} + \dot{e}^{34}$	No	Yes	0	$(0, 0, 0, 0)$
	$\pm \dot{e}^{11} - \dot{e}^{12} + \dot{e}^{34}$	No	Yes	0	$(0, 0, 0, 0)$
	$\pm \dot{e}^{12} + x\dot{e}^{33} + \dot{e}^{34}, \chi = 0$	No	No	$\frac{3}{2}x$	$(0, 0, 0, 0)$
	$-x\dot{e}^{11} + \dot{e}^{12} - \dot{e}^{34}, \chi = 0$	No	Yes	0	$(0, 0, 0, 0)$
	$\pm \dot{e}^{12} + \dot{e}^{34}$	Yes	Yes	$-x_4$	$(0, 0, 0, x_4)$
	$\dot{e}^{12} - \dot{e}^{34}$	Yes	Yes	$-x_4$	$(0, 0, 0, x_4)$
$\mathfrak{h}_4$	$\dot{e}^{12} + \dot{e}^{34}$	No	Yes	0	$(0, 0, 0, 0)$

$$h_1 = 2y(\dot{e}^{11} - \dot{e}^{13} - \dot{e}^{22} + \dot{e}^{24} + \dot{e}^{33} - \dot{e}^{44}) - (1+x)\dot{e}^{12} + x(\dot{e}^{14} + \dot{e}^{23}) + (1-x)\dot{e}^{34}$$

$$h_2 = 2y(\dot{e}^{11} - \dot{e}^{13} - \dot{e}^{22} + \dot{e}^{24} + \dot{e}^{33} - \dot{e}^{44}) - \dot{e}^{12} + \dot{e}^{34}$$

$$h_3 = 2y(\dot{e}^{11} - \dot{e}^{13} - \dot{e}^{22} + \dot{e}^{24} + \dot{e}^{33} - \dot{e}^{44}) - 2\dot{e}^{12} + \dot{e}^{14} + \dot{e}^{23}$$

Proof. We report below the details for the case of $\mathfrak{d}_{4,\frac{1}{2}}$ the other cases are treated in the same way. Let $\{e_1, e_2, e_3, e_4\}$ denote the basis used in theorem 1.1 for $\mathfrak{d}_{4,\frac{1}{2}}$. The non isomorphic para-Kähler structures in $\mathfrak{d}_{4,\frac{1}{2}}$ are (ω, K_1) and (ω, K_2) where $\omega = e^{12} - e^{34}$: $K_1 = E_{11} - E_{22} - E_{33} + xE_{43} + E_{44}$, $x \neq 0$ and $K_2 = E_{11} - E_{22} + E_{33} - E_{44}$.

The corresponding compatible metric to (ω, K_i) is uniquely determined by $h_i(X, Y) = \omega(K_i X, Y)$. Hence, para-Kähler metrics in $\mathfrak{d}_{4,\frac{1}{2}}$ are of the form :

$$h_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & x & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad x \in \mathbb{R}^*, \quad h_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

• For h_1 , using the Koszul formula the Levi-Civita connection is described by

$$\begin{aligned} \nabla_{e_1} &= \begin{pmatrix} 0 & 0 & -\frac{1}{2}x & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2}x & 0 & 0 \end{pmatrix}, & \nabla_{e_2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}x & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}x & 0 & 0 & 0 \end{pmatrix}, \\ \nabla_{e_3} &= \begin{pmatrix} -\frac{1}{2}x & 0 & 0 & 0 \\ 0 & \frac{1}{2}x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & -x^2 & -x \end{pmatrix}, & \nabla_{e_4} &= \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & -1 \end{pmatrix}. \end{aligned}$$

Then, we calculate the curvature matrices $R(e_i, e_j)$ (for $1 \leq i < j \leq 4$) and we find

$$\begin{aligned} R(e_1, e_2) &= \begin{pmatrix} -x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & \frac{x}{2} & 0 \\ 0 & 0 & -\frac{1}{2}x^2 & -\frac{1}{2}x \end{pmatrix}, & R(e_1, e_3) &= \begin{pmatrix} 0 & 0 & -\frac{1}{4}x^2 & -\frac{x}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{x}{2} & 0 & 0 \\ 0 & -\frac{1}{4}x^2 & 0 & 0 \end{pmatrix} \\ R(e_2, e_3) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4}x^2 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4}x^2 & 0 & 0 & 0 \end{pmatrix}, & R(e_2, e_4) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{x}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{x}{2} & 0 & 0 & 0 \end{pmatrix} \\ R(e_3, e_4) &= \begin{pmatrix} \frac{x}{2} & 0 & 0 & 0 \\ 0 & -\frac{x}{2} & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & x^2 & x \end{pmatrix} & \text{and } R(e_1, e_4) &= 0. \end{aligned}$$

The Ricci tensor ric and the Ricci operator Ric are given by

$$ric = \begin{pmatrix} 0 & \frac{3}{2}x & 0 & 0 \\ \frac{3}{2}x & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2}x^2 & \frac{3}{2}x \\ 0 & 0 & \frac{3}{2}x & 0 \end{pmatrix} \quad \text{and} \quad Ric = \begin{pmatrix} \frac{3}{2}x & 0 & 0 & 0 \\ 0 & \frac{3}{2}x & 0 & 0 \\ 0 & 0 & \frac{3}{2}x & 0 \\ 0 & 0 & 0 & \frac{3}{2}x \end{pmatrix}.$$

The Lie derivative $\mathcal{L}_X h_1$ of the metric h_1 with respect to an arbitrary vector field $X = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 \in \mathfrak{g}$ is given by

$$\mathcal{L}_X h_1 = \begin{pmatrix} 0 & -x_4 & x_2 x & \frac{3}{2}x_2 \\ -x_4 & 0 & -x_1 x & -\frac{1}{2}x_1 \\ x_2 x & -x_1 x & -2x_4 x & x x_3 - x_4 \\ \frac{3}{2}x_2 & -\frac{1}{2}x_1 & x x_3 - x_4 & 2x_3 \end{pmatrix}.$$

Then, solving equation $\mathcal{L}_X h_1 + ric = \lambda h_1$, $x \neq 0$ we obtain

$$\lambda = \frac{3}{2}x \quad \text{and} \quad X = 0.$$

Notice that, in this case the para-Kähler metric is an Einstein metric not Ricci flat.

• For h_2

$$\nabla_{e_2} e_1 = -e_3, \quad \nabla_{e_2} e_4 = -e_2, \quad \nabla_{e_4} e_1 = \frac{1}{2}e_1, \quad \nabla_{e_4} e_2 = -\frac{1}{2}e_2, \quad \nabla_{e_4} e_3 = e_3, \quad \nabla_{e_4} e_4 = -e_4.$$

This para-Kähler structure is flat ($R(e_i, e_j) = 0$ for $1 \leq i < j \leq 4$) and also Ricci flat. The Lie derivative $\mathcal{L}_X h_2$ of the metric h_2 is given by

$$\mathcal{L}_X h_2 = \begin{pmatrix} 0 & -x_4 & 0 & -\frac{1}{2}x_2 \\ -x_4 & 0 & 0 & \frac{3}{2}x_1 \\ 0 & 0 & 0 & x_4 \\ -\frac{1}{2}x_2 & \frac{3}{2}x_1 & x_4 & -2x_3 \end{pmatrix}.$$

Then, solving equation $\mathcal{L}_X h_2 + ric = \lambda h_2$, we obtain

$$\lambda = -x_4 \quad \text{and} \quad X = x_4 e_4.$$

□

4 Tables

Table 4: Four-dimensional Para-Kähler Lie algebras coming from b

Lie algebra	No zero brackets
$\mathcal{B}_{1,\alpha}^1$ $\alpha \in \{-2, -1, 1\}$	$[e_1, e_2] = -e_1$, $[e_2, e_3] = xe_1 - e_3$, $[e_2, e_4] = -\alpha e_4$
$\mathcal{B}_{1,\alpha}^2$ $\alpha \in \{-2, -1, 0, 1\}$	$[e_1, e_2] = -e_1$, $[e_1, e_4] = -\frac{x}{\alpha}e_1$, $[e_2, e_3] = -e_3$, $[e_2, e_4] = xe_2 - \alpha e_4$, $[e_3, e_4] = \frac{x}{\alpha}e_3$
$\mathcal{B}_{1,-2}^1$	$[e_1, e_2] = -e_1$, $[e_1, e_4] = xe_1$, $[e_2, e_3] = ye_1 - e_3$, $[e_2, e_4] = 2xe_2 + 2e_4$, $[e_3, e_4] = -xe_3$
$\mathcal{B}_{1,-2}^2$	$[e_1, e_2] = -e_1$, $[e_1, e_3] = xe_1$, $[e_2, e_3] = ye_1 - xe_2 - e_3$, $[e_2, e_4] = xe_1 + 2e_4$, $[e_3, e_4] = -2xe_4$
$\mathcal{B}_{1,-1}^1$	$[e_1, e_2] = -e_1$, $[e_1, e_3] = -xe_1$, $[e_2, e_3] = ye_1 + xe_2 - e_3$, $[e_2, e_4] = e_4$, $[e_3, e_4] = xe_4$
$\mathcal{B}_{1,-1}^2$	$[e_1, e_2] = -e_1$, $[e_1, e_4] = xe_1$, $[e_2, e_3] = -e_3$, $[e_2, e_4] = xe_2 + e_4$, $[e_3, e_4] = -xe_3$
$\mathcal{B}_{1,0}$	$[e_1, e_2] = -e_1$, $[e_1, e_4] = xe_1$, $[e_2, e_3] = -e_3$, $[e_3, e_4] = -xe_3$
$\mathcal{B}_{1,1}^1$ $x \neq 0$	$[e_1, e_2] = -e_1$, $[e_1, e_3] = -\frac{y}{2}e_1$, $[e_1, e_4] = -xe_1$, $[e_2, e_3] = \frac{y^2}{2x}e_1 + \frac{y}{2}e_2 - e_3$ $[e_2, e_4] = ye_1 + xe_2 - e_4$, $[e_3, e_4] = xe_3 - \frac{y}{2}e_4$
$\mathcal{B}_{1,1}^2$	$[e_1, e_2] = -e_1$, $[e_2, e_3] = xe_1 - e_3$, $[e_2, e_4] = -e_4$
$\mathcal{B}_2$	$[e_1, e_2] = -e_1$, $[e_2, e_3] = xe_1 - e_3 - e_4$, $[e_2, e_4] = -e_4$
$\mathcal{B}_{3,\alpha}^1$ $\alpha \neq 0$	$[e_1, e_2] = \frac{1}{\alpha}e_1$, $[e_1, e_3] = [e_2, e_4] = -e_4$, $[e_2, e_3] = xe_1 + \frac{1-\alpha}{\alpha}e_3$
$\mathcal{B}_{3,\alpha}^2$ $\alpha \neq 0$	$[e_1, e_2] = \frac{1}{\alpha}e_1$, $[e_1, e_3] = [e_2, e_4] = x\alpha e_2 - e_4$, $[e_1, e_4] = xe_1$, $[e_2, e_3] = \frac{1-\alpha}{\alpha}e_3$, $[e_3, e_4] = x(\alpha - 1)e_3$
$\mathcal{B}_{3,\frac{1}{2}}^1$	$[e_1, e_2] = 2e_1$, $[e_1, e_3] = xe_1 - e_4$, $[e_2, e_3] = ye_1 - \frac{x}{2}e_2 + e_3$, $[e_2, e_4] = -e_4$, $[e_3, e_4] = -\frac{x}{2}e_4$
$\mathcal{B}_{3,\frac{1}{2}}^2$ $y \neq 0$	$[e_1, e_2] = 2e_1$, $[e_1, e_3] = -2xe_1 + \frac{y}{2}e_2 - e_4$, $[e_1, e_4] = ye_1$, $[e_2, e_3] = -\frac{3x^2}{y}e_1 + \frac{x}{2}e_2 + e_3$ $[e_2, e_4] = xe_1 + \frac{y}{2}e_2 - e_4$, $[e_3, e_4] = -\frac{y}{2}e_3 - \frac{x}{2}e_4$
$\mathcal{B}_{3,\frac{1}{2}}^3$	$[e_1, e_2] = 2e_1$, $[e_1, e_3] = [e_2, e_4] = xe_1 - e_4$, $[e_2, e_3] = ye_1 - xe_2 + e_3$, $[e_3, e_4] = -2xe_4$
$\mathcal{B}_{3,1}$	$[e_1, e_2] = e_1$, $[e_1, e_3] = xe_1 + ye_2 - e_4$, $[e_1, e_4] = ye_1$, $[e_2, e_3] = ze_1$, $[e_2, e_4] = ye_2 - e_4$
$\mathcal{B}_4$	$[e_1, e_2] = e_1$, $[e_1, e_3] = xe_1 - e_4$, $[e_2, e_3] = ye_1 - e_4$, $[e_2, e_4] = -e_4$
$\mathcal{B}_{5,1}^+$	$[e_1, e_2] = e_1$, $[e_1, e_4] = xe_1 - e_3$, $[e_2, e_3] = e_3$, $[e_2, e_4] = -2xe_2 + 2e_4$, $[e_3, e_4] = -xe_3$
$\mathcal{B}_{5,2}^+$	$[e_1, e_2] = e_1$, $[e_1, e_4] = -\frac{x}{4}e_1 - e_3$, $[e_2, e_3] = xe_1 + e_3$, $[e_2, e_4] = -\frac{x}{2}e_2 + 2e_4$, $[e_3, e_4] = \frac{x}{4}e_3$
$\mathcal{B}_{5,3}^+$	$[e_1, e_2] = e_1$, $[e_1, e_3] = -xe_1$, $[e_1, e_4] = -2xe_1 - xe_2 - e_3$, $[e_2, e_3] = 2xe_1 + xe_2 + e_3$ $[e_2, e_4] = 3xe_1 + 2xe_2 + 2e_4$, $[e_3, e_4] = 2xe_3 - 2xe_4$
$\mathcal{B}_{5,4}^+$	$[e_1, e_2] = e_1$, $[e_1, e_3] = -xe_1$, $[e_1, e_4] = 2xe_1 - xe_2 - e_3$ $[e_2, e_3] = -2xe_1 + xe_2 + e_3$, $[e_2, e_4] = 3xe_1 - 2xe_2 + 2e_4$, $[e_3, e_4] = -2xe_3 - 2xe_4$
$\mathcal{B}_{5,1}^-$	$[e_1, e_2] = e_1$, $[e_1, e_4] = xe_1 + e_3$, $[e_2, e_3] = e_3$, $[e_2, e_4] = -2xe_2 + 2e_4$, $[e_3, e_4] = -xe_3$
$\mathcal{B}_{5,2}^-$	$[e_1, e_2] = e_1$, $[e_1, e_4] = \frac{x}{4}e_1 + e_3$, $[e_2, e_3] = xe_1 + e_3$, $[e_2, e_4] = \frac{x}{2}e_2 + 2e_4$, $[e_3, e_4] = -\frac{x}{4}e_3$

Table 5: Four-dimensional Para-Kähler Lie algebras coming from c

Lie algebra	No zero brackets
$\mathcal{C}_{1,1}$	$[e_1, e_4] = e_1, [e_2, e_4] = \alpha e_2, [e_3, e_4] = -e_3$
$\mathcal{C}_{1,2}$	$[e_1, e_4] = e_1 + e_2, [e_2, e_4] = e_2, [e_3, e_4] = -e_3$
$\mathcal{C}_{1,3}$	$[e_1, e_3] = [e_2, e_4] = e_2, [e_1, e_4] = (1 - \frac{1}{\alpha})e_1, [e_3, e_4] = \frac{1}{\alpha}e_3$
$\mathcal{C}_{1,4}$	$[e_1, e_3] = [e_1, e_4] = [e_2, e_4] = e_2, [e_3, e_4] = e_3$
$\mathcal{C}_{1,5}^+$	$[e_1, e_4] = -e_1, [e_2, e_3] = e_1, [e_2, e_4] = -2e_2, [e_3, e_4] = e_3$
$\mathcal{C}_{1,5}^-$	$[e_1, e_4] = [e_2, e_3] = -e_1, [e_2, e_4] = -2e_2, [e_3, e_4] = e_3$
$\mathcal{C}_{1,6}$	$[e_2, e_4] = e_2$
$\mathcal{C}_{1,7}$	$[e_1, e_4] = e_2$
$\mathcal{C}_{1,8}$	$[e_1, e_3] = [e_2, e_4] = e_2, [e_1, e_4] = e_1$
$\mathcal{C}_{1,9}$	$[e_1, e_3] = [e_2, e_4] = e_2, [e_1, e_4] = [e_2, e_3] = e_1$
$\mathcal{C}_{1,10}$	$[e_1, e_3] = [e_2, e_4] = e_2, [e_1, e_4] = e_1, [e_2, e_3] = -e_1$
$\mathcal{C}_{2,1}$	$[e_1, e_3] = xe_1, [e_2, e_4] = ye_2 - e_4$
$\mathcal{C}_{2,2}$	$[e_1, e_3] = xe_1, [e_2, e_3] = ye_1, [e_2, e_4] = -e_4$
$\mathcal{C}_{2,3}$	$[e_1, e_3] = xe_1, [e_2, e_3] = ye_1, [e_2, e_4] = xe_1 - e_4, [e_3, e_4] = -xe_4$
$\mathcal{C}_{3,1}$	$[e_1, e_3] = xe_1, [e_2, e_3] = ye_1 + ze_2 - e_4, [e_3, e_4] = ze_4$
$\mathcal{C}_{3,2}$	$[e_1, e_3] = [e_2, e_4] = xe_1, [e_2, e_3] = ye_1 + ze_2 - e_4, [e_3, e_4] = (z - x)e_4$
$\mathcal{C}_{4,1}$	$[e_1, e_3] = [e_2, e_4] = xe_1 - e_4, [e_2, e_3] = ye_1 + xe_2 - e_3$
$\mathcal{C}_{4,2}$	$[e_1, e_3] = [e_2, e_4] = xe_2 - e_4, [e_2, e_3] = -e_3, [e_3, e_4] = xe_3$
$\mathcal{C}_{5,1}^+$	$[e_1, e_3] = [e_2, e_4] = xe_1 + ye_2 - e_4, [e_1, e_4] = [e_2, e_3] = ye_1 + xe_2 - e_3$
$\mathcal{C}_{5,2}^+$	$[e_1, e_3] = [e_2, e_4] = xe_2 - e_4, [e_1, e_4] = [e_2, e_3] = -e_3, [e_3, e_4] = xe_3$
$\mathcal{C}_{5,1}^-$	$[e_1, e_3] = [e_2, e_4] = xe_1 + ye_2 - e_4, [e_1, e_4] = ye_1 - xe_2 + e_3, [e_2, e_3] = -ye_1 + xe_2 - e_3$
$\mathcal{C}_{5,2}^-$	$[e_1, e_3] = [e_2, e_4] = xe_2 - e_4, [e_1, e_4] = e_3, [e_2, e_3] = -e_3, [e_3, e_4] = xe_3$

Table 6: Isomorphisms from the Lie algebras obtained in Table 4 onto the Lie algebras in Table 1

Source	Isomorphism	Target
$\mathcal{B}_{1,\alpha}^1 \quad \alpha < 1$	$f_1 = e_1, f_2 = -\frac{x}{2}e_1 + e_3, f_3 = e_4, f_4 = e_2$	$\mathfrak{r}_{4,-1,-\alpha}$
$\mathcal{B}_{1,\alpha}^1 \quad \alpha > 1, \alpha \neq -2$	$f_1 = e_4, f_2 = e_1, f_3 = -\frac{x}{2}e_1 + e_3, f_4 = -\frac{1}{\alpha}e_2$	$\mathfrak{r}_{4,-\frac{1}{ \alpha }, \frac{1}{ \alpha }}$
$\mathcal{B}_{1,\alpha}^2 \quad \alpha < 1, \alpha \neq 0$	$f_1 = e_1, f_2 = e_3, f_3 = -\frac{x}{\alpha}e_2 + e_4, f_4 = e_2$	$\mathfrak{r}_{4,-1,-\alpha}$
$\mathcal{B}_{1,\alpha}^2 \quad \alpha > 1, \alpha \neq -2$	$f_1 = -\frac{x}{\alpha}e_2 + e_4, f_2 = e_1, f_3 = e_3, f_4 = -\frac{1}{\alpha}e_2$	$\mathfrak{r}_{4,-\frac{1}{ \alpha }, \frac{1}{ \alpha }}$
$\mathcal{B}_{1,-2}^1 \quad xy \neq 0$	$f_1 = e_2 + \frac{1}{x}e_4, f_2 = -\frac{y}{2}e_1 + e_3, f_3 = ye_1, f_4 = e_2$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{1,-2}^1 \quad y = 0$	$f_1 = xe_2 + e_4, f_2 = e_3, f_3 = e_1, f_4 = \frac{1}{2}e_2$	$\mathfrak{r}_{4,-\frac{1}{2}, \frac{1}{2}}$
$\mathcal{B}_{1,-2}^1 \quad x = 0$	$f_1 = e_4, f_2 = -\frac{y}{2}e_1 + e_3, f_3 = e_1, f_4 = \frac{1}{2}e_2$	$\mathfrak{r}_{4,-\frac{1}{2}, \frac{1}{2}}$
$\mathcal{B}_{1,-2}^2 \quad x \neq 0$	$f_1 = e_4, f_2 = \frac{1}{2x}e_1 + e_2 + \frac{1}{x}e_3, f_3 = -xe_1, f_4 = -\frac{1}{x}e_3 + (\frac{y+1}{x^2})e_4$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{1,-2}^2 \quad x = 0$	$f_1 = e_4, f_2 = -\frac{y}{2}e_1 + e_3, f_3 = e_1, f_4 = \frac{1}{2}e_2$	$\mathfrak{r}_{4,-\frac{1}{2}, \frac{1}{2}}$
$\mathcal{B}_{1,-1}^1$	$f_1 = -\frac{1}{2}ye_1 - xe_2 + e_3, f_2 = e_1, f_3 = e_4, f_4 = -e_2$	$\mathfrak{r}_{4,-1,-1}$
$\mathcal{B}_{1,-1}^2$	$f_1 = e_3, f_2 = e_1, f_3 = xe_2 + e_4, f_4 = -e_2$	$\mathfrak{r}_{4,-1,-1}$
$\mathcal{B}_{1,0}^1$	$f_1 = e_2, f_2 = e_1, f_3 = e_3, f_4 = xe_2 + e_4$	$\mathfrak{rr}_{3,-1}$
$\mathcal{B}_{1,0}^2$	$f_1 = -e_2, f_2 = -\frac{x}{2}e_1 + e_3, f_3 = e_1, f_4 = e_4$	$\mathfrak{rr}_{3,-1}$
$\mathcal{B}_{1,1}^1 \quad x \neq 0$	$f_1 = e_1, f_2 = xe_3 - \frac{y}{2}e_4, f_3 = \frac{y}{2}e_1 + xe_2 - e_4, f_4 = e_2$	$\mathfrak{r}_{4,-1,-1}$
$\mathcal{B}_{1,1}^2$	$f_1 = e_1, f_2 = -\frac{x}{2}e_1 + e_3, f_3 = e_4, f_4 = e_2$	$\mathfrak{r}_{4,-1,-1}$
$\mathcal{B}_2$	$f_1 = e_1, f_2 = -e_4, f_3 = -\frac{x}{2}e_1 + e_3, f_4 = e_2$	$\mathfrak{r}_{4,-1}$
$\mathcal{B}_{3,\alpha}^1 \quad \frac{1}{\alpha} > \frac{1}{2}$	$f_1 = e_1, f_2 = -xe_1 + \frac{\alpha-2}{\alpha}e_3, f_3 = -\frac{\alpha-2}{\alpha}e_4, f_4 = -e_2$	$\mathfrak{d}_{4, \frac{1}{\alpha}}$
$\mathcal{B}_{3,\alpha}^1 \quad \frac{1}{\alpha} < \frac{1}{2}$	$f_1 = -xe_1 + \frac{\alpha-2}{\alpha}e_3, f_2 = e_1, f_3 = \frac{\alpha-2}{\alpha}e_4, f_4 = -e_2$	$\mathfrak{d}_{4, \frac{\alpha-1}{\alpha}}$

$\mathcal{B}_{3,2}^1$	$x \neq 0$	$f_1 = -xe_1, f_2 = e_3, f_3 = xe_4, f_4 = -e_2$	$\mathfrak{h}_4$
$\mathcal{B}_{3,2}^1$	$x = 0$	$f_1 = e_3, f_2 = e_1, f_3 = e_4, f_4 = -e_2$	$\mathfrak{d}_{4,\frac{1}{2}}$
$\mathcal{B}_{3,\alpha}^2$	$\frac{1}{\alpha} > \frac{1}{2}$	$f_1 = e_1 + xae_2 - e_4, f_2 = e_3, f_3 = xae_2 - e_4, f_4 = -e_2 + \frac{\alpha-1}{\alpha}e_3$	$\mathfrak{d}_{4,\frac{1}{\alpha}}$
$\mathcal{B}_{3,\alpha}^2$	$\frac{1}{\alpha} < \frac{1}{2}$	$f_1 = -xae_2 + e_3 + e_4, f_2 = e_1, f_3 = -xae_2 + e_4, f_4 = \frac{1}{\alpha}e_1 - e_2$	$\mathfrak{d}_{4,\frac{\alpha-1}{\alpha}}$
$\mathcal{B}_{3,2}^2$		$f_1 = e_3, f_2 = e_1, f_3 = -2xe_2 + e_4, f_4 = -e_2$	$\mathfrak{d}_{4,\frac{1}{2}}$
$\mathcal{B}_{3,\frac{1}{2}}^1$		$f_1 = e_1, f_2 = \frac{1}{3}ye_1 - \frac{1}{2}xe_2 + e_3, f_3 = -e_4, f_4 = -e_2$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{3,\frac{1}{2}}^2$	$y \neq 0$	$f_1 = e_1, f_2 = -e_3 - \frac{x}{y}e_4, f_3 = xe_1 - \frac{1}{2}ye_2 + e_4, f_4 = -\frac{2}{y}e_4$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{3,1}$	$x = 0, y = 0$	$f_1 = e_1, f_2 = ze_1 + e_3, f_3 = -e_4, f_4 = -e_2$	$\mathfrak{d}_{4,1}$
$\mathcal{B}_{3,1}$	$x \neq 0, y = 0$	$f_1 = \frac{z}{x}e_1 - e_2 + \frac{1}{x}e_3, f_2 = e_4, f_3 = -\frac{z}{x}e_1 - \frac{1}{x}e_3, f_4 = e_1 - \frac{1}{x}e_4$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{B}_{3,1}$	$x = 0, yz > 0$	$f_1 = -\frac{\sqrt{yz}}{2y}e_1 - \frac{1}{2}e_2 - \frac{1}{2\sqrt{yz}}e_3, f_2 = -\sqrt{yz}e_1 - ye_2 + e_4, f_3 = \frac{\sqrt{yz}}{2y}e_1 - \frac{1}{2}e_2 + \frac{1}{2\sqrt{yz}}e_3, f_4 = \sqrt{yz}e_1 - ye_2 + e_4$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{B}_{3,1}$	$x = 0, yz < 0$	$f_1 = -\frac{1}{y}e_4, f_2 = \frac{1}{\sqrt{-yz}}e_3, f_3 = \sqrt{-yz}e_1, f_4 = -ye_2 + e_4$	$\mathfrak{r}_2'$
$\mathcal{B}_{3,1}$	$x = 0, y \neq 0, z = 0$	$f_1 = e_1, f_2 = e_3, f_3 = ye_2 - e_4, f_4 = -e_2$	$\mathfrak{d}_{4,1}$
$\mathcal{B}_{3,1}$	$xy \neq 0, x^2 + 4yz = 0$	$f_1 = -ye_2 + e_4, f_2 = -\frac{x^2}{4y}e_1 - \frac{x}{2}e_2 + e_3, f_3 = \frac{x^2}{4}e_1 + \frac{xy}{2}e_2 - \frac{x}{2}e_4, f_4 = -e_2$	$\mathfrak{d}_{4,1}$
$\mathcal{B}_{3,1}$	$xy \neq 0, x^2 + 4yz > 0$	$f_1 = \frac{z}{\sqrt{x^2+4yz}}e_1 - \frac{x+\sqrt{x^2+4yz}}{2\sqrt{x^2+4yz}}e_2 + \frac{1}{\sqrt{x^2+4yz}}e_3, f_2 = (-x+\sqrt{x^2+4yz})e_1 - 2ye_2 + 2e_4, f_3 = -\frac{z}{\sqrt{x^2+4yz}}e_1 + \frac{x-\sqrt{x^2+4yz}}{2\sqrt{x^2+4yz}}e_2 - \frac{1}{\sqrt{x^2+4yz}}e_3, f_4 = -\frac{x+\sqrt{x^2+4yz}}{2}e_1 - ye_2 + e_4$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{B}_{3,1}$	$xy \neq 0, x^2 + 4yz < 0$	$f_1 = \frac{\sqrt{-x^2-4yz}}{2}e_1 - e_2, f_2 = -\frac{x\sqrt{-x^2-4yz}-4z}{2\sqrt{-x^2-4yz}}e_1 - \frac{y\sqrt{-x^2-4yz}+x}{\sqrt{-x^2-4yz}}e_2 + \frac{2}{\sqrt{-x^2-4yz}}e_3 + e_4, f_3 = \frac{\sqrt{-x^2-4yz}}{2}e_1, f_4 = -\frac{x}{2}e_1 - ye_2 + e_4$	$\mathfrak{r}_2'$
$\mathcal{B}_4$	$x = 0$	$f_1 = -e_1, f_2 = ye_1 + e_3 - e_4, f_3 = e_4, f_4 = -e_2$	$\mathfrak{d}_{4,1}$
$\mathcal{B}_4$	$x \neq 0$	$f_1 = \frac{x-y}{x}e_1 - \frac{1}{x}e_3, f_2 = -xe_1 + e_4, f_3 = \frac{y}{x}e_1 - e_2 + \frac{1}{x}e_3, f_4 = e_4$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{B}_{5,1}^+$		$f_1 = -xe_2 - e_3 + e_4, f_2 = e_1, f_3 = e_3, f_4 = e_1 + e_2$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{5,2}^+$	$x \neq 0$	$f_1 = \frac{1}{2}e_2 - \frac{2}{x}e_4, f_2 = e_3, f_3 = \frac{x}{2}e_1 + e_3, f_4 = \frac{4}{x}e_4$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{5,2}^+$	$x = 0$	$f_1 = e_3 + e_4, f_2 = e_1, f_3 = e_3, f_4 = -e_1 + e_2$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{5,3}^+$	$x \neq 0$	$f_1 = -e_3 - e_4, f_2 = e_1, f_3 = xe_1 + xe_2 + e_3, f_4 = -\frac{1}{x}e_3$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{5,3}^+$	$x = 0$	$f_1 = e_4, f_2 = e_1, f_3 = e_3, f_4 = e_2$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{5,4}^+$	$x \neq 0$	$f_1 = e_3 + e_4, f_2 = e_1, f_3 = -xe_1 + xe_2 + e_3, f_4 = -e_1 + e_2$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{5,4}^+$	$x = 0$	$f_1 = -e_3 + e_4, f_2 = e_1, f_3 = e_3, f_4 = e_1 + e_2$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{5,1}^-$		$f_1 = -xe_2 + e_4, f_2 = e_1, f_3 = -e_3, f_4 = e_2$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{5,2}^-$	$x \neq 0$	$f_1 = e_2 + \frac{4}{x}e_4, f_2 = e_3, f_3 = xe_1 + 2e_3, f_4 = -\frac{4}{x}e_4$	$\mathfrak{d}_{4,2}$
$\mathcal{B}_{5,2}^-$	$x = 0$	$f_1 = e_3 + e_4, f_2 = e_1, f_3 = -e_3, f_4 = e_1 + e_2$	$\mathfrak{d}_{4,2}$

Table 7: Isomorphisms from the Lie algebras obtained in Table 5 onto the Lie algebras in Table 1

Source	Isomorphism	Target
$\mathcal{C}_{1,1}$ $-1 \leq \alpha < 1$	$f_1 = e_1, f_2 = e_3, f_3 = e_2, f_4 = -e_4$	$\mathfrak{r}_{4,-1,\alpha}$
$\mathcal{C}_{1,1}$ $\alpha < -1$	$f_1 = e_2, f_2 = e_1, f_3 = e_3, f_4 = -\frac{1}{\alpha}e_4$	$\mathfrak{r}_{4,\frac{1}{\alpha},-\frac{1}{\alpha}}$
$\mathcal{C}_{1,1}$ $\alpha > 1$	$f_1 = e_2, f_2 = e_3, f_3 = e_1, f_4 = -\frac{1}{\alpha}e_4$	$\mathfrak{r}_{4,-\frac{1}{\alpha},\frac{1}{\alpha}}$
$\mathcal{C}_{1,1}$ $\alpha = 1$	$f_1 = e_3, f_2 = e_1, f_3 = e_2, f_4 = e_4$	$\mathfrak{r}_{4,-1,-1}$
$\mathcal{C}_{1,2}$	$f_1 = e_3, f_2 = -e_2, f_3 = e_1, f_4 = e_4$	$\mathfrak{r}_{4,-1}$
$\mathcal{C}_{1,3}$ $0 < \alpha \leq 2$	$f_1 = e_2 - e_3, f_2 = e_1, f_3 = e_2, f_4 = \frac{\alpha-1}{\alpha}e_1 - e_4$	$\mathfrak{d}_{4,\frac{1}{\alpha}}$
$\mathcal{C}_{1,3}$ $\alpha < 0$ or $\alpha > 2$	$f_1 = e_1 + ae_2, f_2 = e_3, f_3 = e_2, f_4 = e_3 - e_4$	$\mathfrak{d}_{4,\frac{\alpha-1}{\alpha}}$

$\mathcal{C}_{1,4}$	$f_1 = e_3, f_2 = e_1 - e_2, f_3 = -e_2, f_4 = -e_4$	$\mathfrak{d}_{4,1}$
$\mathcal{C}_{1,5}^+$	$f_1 = e_2, f_2 = e_3, f_3 = e_1, f_4 = e_4$	$\mathfrak{d}_{4,2}$
$\mathcal{C}_{1,5}^-$	$f_1 = e_2, f_2 = -e_3, f_3 = e_1, f_4 = e_4$	$\mathfrak{d}_{4,2}$
$\mathcal{C}_{1,6}$	$f_1 = -e_4, f_2 = e_2, f_3 = e_3, f_4 = e_1$	$\mathfrak{rr}_{3,0}$
$\mathcal{C}_{1,7}$	$f_1 = e_1, f_2 = e_4, f_3 = e_2, f_4 = e_3$	$\mathfrak{rh}_3$
$\mathcal{C}_{1,8}$	$f_1 = e_1, f_2 = e_3, f_3 = e_2, f_4 = -e_4$	$\mathfrak{d}_{4,1}$
$\mathcal{C}_{1,9}$	$f_1 = \frac{1}{2}e_3 - \frac{1}{2}e_4, f_2 = e_1 - e_2, f_3 = -\frac{1}{2}e_3 - \frac{1}{2}e_4, f_4 = e_1 + e_2$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{C}_{1,10}$	$f_1 = -e_4, f_2 = e_3, f_3 = e_1, f_4 = -e_2$	$\mathfrak{r}_2'$
$\mathcal{C}_{2,1} \quad x = 0$	$f_1 = -e_2, f_2 = -ye_2 + e_4, f_3 = e_1, f_4 = e_3$	$\mathfrak{rr}_{3,0}$
$\mathcal{C}_{2,1} \quad x \neq 0$	$f_1 = -\frac{1}{x}e_3, f_2 = e_1, f_3 = -e_2, f_4 = -xye_2 + xe_4$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{C}_{2,2} \quad x \neq 0$	$f_1 = -\frac{1}{x}e_3, f_2 = e_1, f_3 = \frac{y}{x}e_1 - e_2, f_4 = e_4$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{C}_{2,2} \quad x = 0, y \neq 0$	$f_1 = e_4, f_2 = -ye_1, f_3 = e_3, f_4 = -e_2$	$\mathfrak{r}_{4,0}$
$\mathcal{C}_{2,2} \quad x = 0, y = 0$	$f_1 = -e_2, f_2 = e_4, f_3 = e_1, f_4 = e_3$	$\mathfrak{rr}_{3,0}$
$\mathcal{C}_{2,3} \quad x \neq 0$	$f_1 = \frac{y}{x}e_1 - e_2, f_2 = -xe_1 + e_4, f_3 = e_2 - \frac{1}{x}e_3, f_4 = e_1$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{C}_{2,3} \quad x = 0, y \neq 0$	$f_1 = e_4, f_2 = -ye_1, f_3 = e_3, f_4 = -e_2$	$\mathfrak{r}_{4,0}$
$\mathcal{C}_{2,3} \quad x = 0, y = 0$	$f_1 = e_1 - e_2, f_2 = e_4, f_3 = e_1, f_4 = e_3$	$\mathfrak{rr}_{3,0}$
$\mathcal{C}_{3,1} \quad z = 0, x \neq 0$	$f_1 = e_1, f_2 = -\frac{1}{x}e_4, f_3 = -\frac{y}{x}e_1 + e_2, f_4 = -\frac{1}{x}e_3$	$\mathfrak{r}_{4,0}$
$\mathcal{C}_{3,1} \quad z = 0, x = 0$	$f_1 = -e_2, f_2 = e_2 - e_3, f_3 = ye_1 - e_4, f_4 = e_1$	$\mathfrak{rh}_3$
$\mathcal{C}_{3,1} \quad z \neq 0, x = 0$	$f_1 = \frac{1}{z}e_3, f_2 = e_4, f_3 = -2ye_1 - 2ze_2 + e_4, f_4 = e_1$	$\mathfrak{rr}_{3,-1}$
$\mathcal{C}_{3,1} \quad z \neq 0, -1 < \frac{-x}{z} < 1$	$f_1 = e_4, f_2 = \frac{2yz}{x-z}e_1 - 2ze_2 + e_4, f_3 = e_1, f_4 = \frac{1}{z}e_3$	$\mathfrak{r}_{4,-1,-\frac{x}{z}}$
$\mathcal{C}_{3,1} \quad z \neq 0, x \neq 0, \frac{-x}{z} > 1$	$f_1 = e_1, f_2 = \frac{2yz}{x-z}e_1 - 2ze_2 + e_4, f_3 = e_4, f_4 = -\frac{1}{x}e_3$	$\mathfrak{r}_{4,\frac{z}{x},-\frac{z}{x}}$
$\mathcal{C}_{3,1} \quad z \neq 0, x \neq 0, \frac{-x}{z} < -1$	$f_1 = e_1, f_2 = e_4, f_3 = \frac{2yz}{x-z}e_1 - 2ze_2 + e_4, f_4 = -\frac{1}{x}e_3$	$\mathfrak{r}_{4,-\frac{z}{x},\frac{z}{x}}$
$\mathcal{C}_{3,1} \quad z \neq 0, \frac{-x}{z} = 1$	$f_1 = -\frac{y}{2x}e_1 + e_2 + \frac{1}{2x}e_4, f_2 = e_4, f_3 = e_1, f_4 = \frac{1}{x}e_3$	$\mathfrak{r}_{4,-1,-1}$
$\mathcal{C}_{3,1} \quad z \neq 0, y \neq 0, x = z$	$f_1 = e_4, f_2 = -\frac{y}{z}e_1, f_3 = e_2 - \frac{1}{2z}e_4, f_4 = \frac{1}{z}e_3$	$\mathfrak{r}_{4,-1}$
$\mathcal{C}_{3,1} \quad z \neq 0, y = 0, x = z$	$f_1 = e_4, f_2 = -2ze_2 + e_4, f_3 = e_1, f_4 = \frac{1}{z}e_3$	$\mathfrak{r}_{4,-1,-1}$
$\mathcal{C}_{3,2} \quad z = 0, x = 0$	$f_1 = -e_2, f_2 = e_2 - e_3, f_3 = ye_1 - e_4, f_4 = e_1$	$\mathfrak{rh}_3$
$\mathcal{C}_{3,2} \quad z = 0, x \neq 0$	$f_1 = e_4, f_2 = -\frac{y}{x}e_1 + e_2 + \frac{1}{x}e_4, f_3 = -xe_1, f_4 = -\frac{1}{x}e_3$	$\mathfrak{d}_{4,1}$
$\mathcal{C}_{3,2} \quad z \neq 0, x = 0$	$f_1 = \frac{1}{z}e_3, f_2 = e_4, f_3 = -2ye_1 - 2ze_2 + e_4, f_4 = e_1$	$\mathfrak{rr}_{3,-1}$
$\mathcal{C}_{3,2} \quad z \neq 0, x \neq 0, \frac{z}{x} = \frac{1}{2}$	$f_1 = -\frac{1}{2z}e_4, f_2 = -\frac{y}{z}e_1 + e_2, f_3 = e_1, f_4 = -\frac{1}{2z}e_3$	$\mathfrak{h}_4$
$\mathcal{C}_{3,2} \quad z \neq 0, x \neq 0, \frac{z}{x} < \frac{1}{2}$	$f_1 = \frac{y(x-2z)}{z}e_1 + e_4, f_2 = (x-2z)e_2 + e_4, f_3 = -x(x-2z)e_1, f_4 = -\frac{y(x-2z)}{x^2}e_2 - \frac{1}{x}e_3$	$\mathfrak{d}_{4,\frac{x-z}{x}}$
$\mathcal{C}_{3,2} \quad z \neq 0, x \neq 0, \frac{z}{x} > \frac{1}{2}$	$f_1 = (x-2z)e_2 + e_4, f_2 = \frac{y(x-2z)}{z}e_1 + e_4, f_3 = x(x-2z)e_1, f_4 = -\frac{y(x-2z)}{x^2}e_2 - \frac{1}{x}e_3$	$\mathfrak{d}_{4,\frac{z}{x}}$
$\mathcal{C}_{4,1}$	$f_1 = -ye_1 - xe_2 + e_3, f_2 = e_1, f_3 = -xe_1 + e_4, f_4 = -xe_1 - e_2 + e_4$	$\mathfrak{d}_{4,1}$
$\mathcal{C}_{4,2}$	$f_1 = e_3, f_2 = e_1, f_3 = -xe_2 + e_4, f_4 = -e_2$	$\mathfrak{d}_{4,1}$
$\mathcal{C}_{5,1}^+$	$f_1 = \frac{1}{2}e_1 - \frac{1}{2}e_2, f_2 = (-x+y)e_1 + (x-y)e_2 - e_3 + e_4, f_3 = -(\frac{1}{2}+x+y)e_1 - (\frac{1}{2}+x+y)e_2 + e_3 + e_4, f_4 = -(x+y)e_1 - (x+y)e_2 + e_3 + e_4$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{C}_{5,2}^+$	$f_1 = \frac{1}{2}e_1 - \frac{1}{2}e_2, f_2 = -xe_2 - e_3 + e_4, f_3 = -\frac{1}{2}e_1 - \frac{1}{2}e_2, f_4 = -xe_2 + e_3 + e_4$	$\mathfrak{r}_2\mathfrak{r}_2$
$\mathcal{C}_{5,1}^-$	$f_1 = ye_1 - (x+1)e_2 + e_3, f_2 = -(x+1)e_1 - ye_2 + e_4, f_3 = ye_1 - xe_2 + e_3, f_4 = -xe_1 - ye_2 + e_4$	$\mathfrak{r}_2'$
$\mathcal{C}_{5,2}^-$	$f_1 = -e_2 - e_3, f_2 = e_1 - xe_2 + e_4, f_3 = -e_3, f_4 = -xe_2 + e_4$	$\mathfrak{r}_2'$

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