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Almost-complex invariants of families of six-dimensional solvmanifolds

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Abstract: We compute almost-complex invariants $h^{p,0}_{\overline{\partial}}$, $h^{p,0}_{\mathrm{Dol}}$ and almost-Hermitian invariants $h^{p,0}_{\overline{\delta}}$ on families of almost-Kähler and almost-Hermitian 6-dimensional solvmanifolds. Finally, as a consequence of almost-Kähler identities we provide an obstruction to the existence of a compatible symplectic structure on a given compact almost-complex manifold. Notice that, when (X, J, g, ω) is a compact almost Hermitian manifold of real dimension greater than four, not much is known concerning the numbers $h^{p,q}_{\overline{\partial}}$.

Keywords: almost-complex structure; almost-Kähler structure; Hodge number

MSC: 53C15; 58A14; 58J05

1 Introduction

Let (X, J) be a complex manifold, then the Dolbeault cohomology of X

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) := \frac{\mathrm{Ker}\,\overline{\partial}}{\mathrm{Im}\,\overline{\partial}}$$

is well defined and it represents an important holomorphic invariant for the complex manifold. If we drop the integrability assumption on J, then $\overline{\partial}^2 \neq 0$ and such a cohomology is not well defined anymore. However, if we fix a J-Hermitian metric g on an almost-complex manifold (X, J) and with $*$ we denote the associated Hodge-$*$-operator, then

$$\Delta_{\overline{\partial}} := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

is a well-defined second order, elliptic, differential operator. In particular, if X is compact, then $\mathrm{Ker}\Delta_{\overline{\partial}}$ is a finite-dimensional complex vector space and we will denote as usual with $h^{\bullet,\bullet}_{\overline{\partial}}$ its dimension. If J is integrable, then

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \simeq \mathrm{Ker}\Delta_{\overline{\partial}}\,,$$

and in particular the dimension of the space of harmonic forms depends only on the complex structure and not on the choice of the Hermitian metric. In [11, Problem 20] Kodaira and Spencer asked whether this is the case also when J is not integrable. More precisely,

Question I *Let (M, J) be an almost complex manifold. Choose an Hermitian metric on (M, J) and consider the numbers $h^{p,q}_{\overline{\partial}}$. Is $h^{p,q}_{\overline{\partial}}$ independent of the choice of the Hermitian metric?*

In [12] Holt and Zhang answered negatively to this question, showing with an explicit example that there exist almost complex structures on the Kodaira-Thurston manifold with Hodge number $h^{0,1}_{\overline{\partial}}$ varying with different

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choices of Hermitian metrics.
They also proved that if (M, J, g, ω) is a 4-dimensional compact almost-Kähler manifold, then $h^{1,1}_{\bar\partial} = b_- + 1$, where b_- denotes the dimension of the space of anti self-dual harmonic forms, namely in such a case $h^{1,1}_{\bar\partial}$ has a cohomological meaning. In this context, (see [12, Question 6.2]) they asked the following

Question II *Let (M, J) be an almost complex 4-manifold which admits an almost Kähler structure. Does it have a non almost Kähler Hermitian metric such that $h^{1,1}_{\bar\partial} \neq b_- + 1$?.*

About this, in [16, Theorem 3.7] it is proved that if g is a strictly locally conformally Kähler metric on a 4-dimensional compact almost complex manifold (X, J), then $h^{1,1}_{\bar\partial} = b_-$. Therefore, since in the non integrable case almost-Kähler metrics and strictly locally conformally Kähler metrics can coexist, this gives a positive answer to Question II. For other results on the study of the numbers $h^{\bullet,\bullet}_{\bar\partial}$ see [10, 13, 14] and the references therein.
However, when (X, J, g) is a compact almost Hermitian manifold of real dimension greater than four, not much is known concerning the numbers $h^{p,q}_{\bar\partial}$ and this may be due also by the lack of explicit computations of such numbers in the literature.
As a general fact, in special bidegree $(p, 0)$, $h^{p,0}_{\bar\partial}$ is independent of the choice of the Hermitian metric, indeed in this case being $\bar\partial$-harmonic is equivalent to be $\bar\partial$-closed. So, in particular $h^{p,0}_{\bar\partial}$ is a genuine almost-complex invariant.
Notice that $h^{n,0}$ is related to the computation of the *Kodaira dimension* of $2n$-dimensional almost-complex manifolds, recently introduced by H. Chen and W. Zhang in [3] and [4]. For explicit computations of the Kodaira dimension one can refer to [3] for the Kodaira-Thurston manifold and to [1], [2] for several 6-dimensional solvmanifolds and 4-dimensional solvmanifolds with no complex structures.
In this paper we will compute explicitly the numbers $h^{p,0}_{\bar\partial}$, for $p = 1, 2, 3$, on families of six-dimensional manifolds endowed with non-integrable almost-complex structures. More in detail, we will consider a family of completely solvable 6-dimensional solvmanifolds constructed in [9] which is particularly interesting because it admits invariant symplectic structures and invariant almost-complex structures but it does not admit any integrable invariant complex structures. For this reason, in such a case, the computation of these almost-complex invariants is particularly meaningful. We will consider on such manifolds an invariant family of almost-Kähler structures and we will compute $h^{p,0}_{\bar\partial}$, with $p = 1, 2, 3$. Furthermore, we will show that these numbers, differently from the integrable case, can vary when the almost-complex structures are almost-Kähler and vary continuously (cf. [12]).
In fact, we will also construct an almost-complex structure which does not admit any compatible symplectic structure and compute $h^{p,0}_{\bar\partial}$ in this case.
Another example will be provided by the computations of $h^{p,0}_{\bar\partial}$, with $p = 1, 2, 3$ for an almost-Kähler structure on the Iwasawa manifold.

Moreover, denoting with μ the $(2, -1)$-component of the exterior derivative d, in [15] we considered the following differential operator (cf. also [8])

$$\bar\delta := \bar\partial + \mu$$

and studied the corresponding harmonic forms. In particular, we compute on the aforementioned families of almost-Hermitian manifolds the $\bar\delta$-harmonic forms of bidegree $(p, 0)$.
One should notice that the spaces of $\bar\partial$-harmonic and $\bar\delta$-harmonic forms on non-integrable almost-complex manifolds do not have a cohomological counterpart. However, in [6] J. Cirici and S. O. Wilson introduced a generalization of the Dolbeault cohomology on almost-complex manifolds constructing therefore new invariants in this setting. By [5] these cohomology groups on compact almost-complex manifolds are not finite dimensional in general. This means that we have a deep gap between Hodge theory and cohomological theory on almost-complex manifolds. However, as noticed in [6], in special bi-degrees, e.g., $(p, 0)$, the almost-complex Dolbeault cohomology groups have finite dimensions. For this reason, we compute such groups in bi-degree $(p, 0)$, for the families of almost-complex manifolds considered above.
The paper is organized as follows: in Section 2 we start by fixing some notations and recalling the basic facts of almost-complex geometry used in the rest of the paper. In Section 3 we construct families of almost-Kähler

solvmanifolds with no left invariant complex structures and then we compute several numerical almost-complex and almost-Hermitian invariants on them. The basic tools to compute the space of harmonic $(p,0)$-forms are suitable Fourier expansions series adapted to the lattices of the solvmanifolds. In Sections 5 and 6 we perform similar computations respectively on the same differentiable manifold endowed with an almost-complex structure that does not admit any compatible symplectic structures and on the Iwasawa manifold endowed with an almost-Kähler structure. Finally, we apply harmonic theory to give an obstruction to the existence of compatible symplectic structures on almost-complex manifolds.

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2 Preliminaries

In this Section we recall some basic facts about almost-complex manifolds and fix some notations. Let X be a smooth manifold of dimension $2n$ and let J be an almost-complex structure on X, i.e., a $(1,1)$-tensor on X such that $J^2 = -\mathrm{Id}$. Then, J induces a natural bigrading on the space of complex valued differential forms $A^\bullet(X)$, namely

$$A^\bullet(X) = \bigoplus_{p+q=\bullet} A^{p,q}(X)\,.$$

According to this decomposition, the exterior derivative d splits into four operators

$$d : A^{p,q}(X) \to A^{p+2,q-1}(X) \oplus A^{p+1,q}(X) \oplus A^{p,q+1}(X) \oplus A^{p-1,q+2}(X)$$

$$d = \mu + \partial + \overline{\partial} + \bar{\mu}\,,$$

where μ and $\bar{\mu}$ are differential operators that are linear over functions. The almost-complex structure J is integrable, that is J induces a complex structure on X, if and only if $\mu = \bar{\mu} = 0$.
In general, since $d^2 = 0$, one has the following relations

$$\begin{cases} \mu^2 & = & 0 \\ \mu\partial + \partial\mu & = & 0 \\ \partial^2 + \mu\overline{\partial} + \overline{\partial}\mu & = & 0 \\ \partial\overline{\partial} + \overline{\partial}\partial + \mu\bar{\mu} + \bar{\mu}\mu & = & 0 \\ \overline{\partial}^2 + \bar{\mu}\partial + \partial\bar{\mu} & = & 0 \\ \bar{\mu}\overline{\partial} + \overline{\partial}\bar{\mu} & = & 0 \\ \bar{\mu}^2 & = & 0 \end{cases}$$

and so the Dolbeault cohomology of X

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) := \frac{\mathrm{Ker}\,\overline{\partial}}{\mathrm{Im}\,\overline{\partial}}$$

is well defined if and only if J is integrable.
If g is an Hermitian metric on (X,J) with associated fundamental form ω and $*$ denotes the Hodge-$*$-operator, one can consider the following differential operator

$$\Delta_{\overline{\partial}} := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}\,.$$

This is a second order, elliptic, differential operator and we will denote its kernel by

$$\mathcal{H}^{p,q}_{\overline{\partial}}(X) := \mathrm{Ker}\,\Delta_{\overline{\partial}_{|A^{p,q}(X)}}\,.$$

If X is compact this space is finite-dimensional and its dimension will be denoted by $h^{p,q}_{\overline{\partial}}(X)$. By [12] we know that these Hodge numbers are not almost-complex invariants, more precisely they depend on the choice of

the Hermitian metric.
In [15] we considered the following differential operator (cf. also [8])

$$\bar{\delta} := \bar{\partial} + \mu$$

and we set

$$\Delta_{\bar{\delta}} := \bar{\delta}\bar{\delta}^* + \bar{\delta}^*\bar{\delta}\,.$$

This is a second order, elliptic, differential operator and we denote with

$$\mathcal{H}^k_{\bar{\delta}}(X) := \operatorname{Ker} \Delta_{\bar{\delta}_{|A^k(X)}}$$

the space of $\bar{\delta}$-harmonic k-forms and with

$$\mathcal{H}^{p,q}_{\bar{\delta}}(X) := \operatorname{Ker} \Delta_{\bar{\delta}_{|A^{p,q}(X)}}$$

the space of $\bar{\delta}$-harmonic (p,q)-forms. If X is compact these spaces are finite dimensional, and we will set $h^k_{\bar{\delta}}(X)$ and $h^{p,q}_{\bar{\delta}}(X)$ for their dimensions respectively.
Moreover, if we set

$$\Delta_\mu := \mu\mu^* + \mu^*\mu\,,$$

we have that the associated spaces of harmonic forms $\mathcal{H}^{\bullet,\bullet}_{\mu}(X)$ and $\mathcal{H}^{\bullet}_{\mu}(X)$ are infinite-dimensional in general. Indeed, μ is linear over functions.
In [15, Proposition 5.5] we showed that on a compact almost-Hermitian manifold (X, J, g) we have

$$\mathcal{H}^{\bullet}_{\bar{\partial}}(X) \cap \mathcal{H}^{\bullet}_{\mu}(X) \subseteq \mathcal{H}^{\bullet}_{\bar{\delta}}(X)$$

and on bi-graded forms we have the equality (cf. [15, Remark 5.6])

$$\mathcal{H}^{\bullet,\bullet}_{\bar{\partial}}(X) \cap \mathcal{H}^{\bullet,\bullet}_{\mu}(X) = \mathcal{H}^{\bullet,\bullet}_{\bar{\delta}}(X)\,.$$

3 Families of Almost-Kähler solvmanifolds with no left-invariant complex structures

We recall the following construction from [9]. Let G be the following connected 2-step solvable 6-dimensional Lie group

$$G := \left\{ \begin{bmatrix} e^t & 0 & xe^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & z_1 \\ 0 & 0 & 0 & e^{-t} & 0 & z_2 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; y_1, y_2, z_1, z_2, t, x \in \mathbb{R} \right\}$$

and set

$$\begin{cases} e^1 &= dt \\ e^2 &= dx \\ e^3 &= e^{-t}dy_1 - xe^{-t}dz_1 \\ e^4 &= e^t dy_2 - xe^t dz_2 \\ e^5 &= e^{-t}dz_1 \\ e^6 &= e^t dz_2 \end{cases},$$

for a basis of left-invariant 1-forms on G, and the dual basis is given by

$$\begin{cases} e_1 &= \frac{\partial}{\partial t} \\ e_2 &= \frac{\partial}{\partial x} \\ e_3 &= e^t \frac{\partial}{\partial y_1} \\ e_4 &= e^{-t} \frac{\partial}{\partial y_2} \\ e_5 &= e^t \frac{\partial}{\partial z_1} + x e^t \frac{\partial}{\partial y_1} \\ e_6 &= e^{-t} \frac{\partial}{\partial z_2} + x e^{-t} \frac{\partial}{\partial y_2} \end{cases} .$$

In particular, the following structure equations hold

$$\begin{cases} de^1 &= 0 \\ de^2 &= 0 \\ de^3 &= -e^{13} - e^{25} \\ de^4 &= e^{14} - e^{26} \\ de^5 &= -e^{15} \\ de^6 &= e^{16} \end{cases} ,$$

where, as usual, we set $e^{ij} := e^i \wedge e^j$, and

$$[e_1, e_3] = [e_2, e_5] = e_3, \quad [e_1, e_4] = -[e_2, e_6] = -e_4, \quad [e_1, e_5] = e_5, \quad [e_1, e_6] = -e_6 .$$

Let $\mathfrak{g}$ be the Lie algebra of G, then $\mathfrak{g}$ is completely solvable. In fact, G can be seen as a semidirect product $G = \mathbb{R}^2 \ltimes_{\Phi} \mathbb{R}^4$, where for every $(t, x) \in \mathbb{R}^2$,

$$\Phi(t,x) : \mathbb{R}^4 \to \mathbb{R}^4, \quad \Phi(t,x) = \begin{bmatrix} e^t & 0 & xe^t & 0 \\ 0 & e^{-t} & 0 & xe^{-t} \\ 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

and the group operation on G is given by

$$(t, x, y_1, y_2, z_1, z_2) \star \left(t', x', y_1', y_2', z_1', z_2'\right) =$$

$$\left(t + t', x + x', y_1' e^t + x z_1' e^t + y_1, y_2' e^{-t} + x z_2' e^{-t} + y_2, z_1' e^t + z_1, z_2' e^{-t} + z_2\right) .$$

A lattice Γ for G can be constructed as follows. Let $B \in SL(2, \mathbb{Z})$ be a unimodular matrix with integer entries and distinct eigenvalues e^{a_0}, e^{-a_0}. Then there exists a real invertible matrix P such that

$$PBP^{-1} = \begin{bmatrix} e^{a_0} & 0 \\ 0 & e^{-a_0} \end{bmatrix} .$$

Let $\tilde{\Gamma} := a_0\mathbb{Z} \times \mathbb{Z}$ and $L := \left((m_1, m_2)P^t, (n_1, n_2)P^t\right)$ with $m_1, m_2, n_1, n_2 \in \mathbb{Z}$. Then, $\Gamma := \tilde{\Gamma} \ltimes_{\Phi} L$ is a lattice in G and we set $X := \Gamma \backslash G$ for the associated solvmanifold. In fact, X has the structure of a $\mathbb{T}^4$-bundle over $\mathbb{T}^2$. As proven in [9], X is a completely solvable solvmanifold which admits symplectic structures but none of them satisfies the Hard Lefschetz condition. Moreover, X is not formal but all the triple Massey products vanish. Finally, X does not admit any invariant integrable almost complex structure.

Now we construct a family of left-invariant almost-complex structures on X. As noticed in [9] the arbitrary left-invariant symplectic structure on X is given by

$$\omega_{a,b,c} = ae^{12} + be^{56} + c(e^{36} + e^{45}) \tag{3.1}$$

with $a, b, c \in \mathbb{R}$ and $a, c \neq 0$. We define the following compatible almost-complex structure $J_{a,b,c}$,

$$\begin{cases} J_{a,b,c}e_1 &= ae_2 \\ J_{a,b,c}e_2 &= -\frac{1}{a}e_1 \\ J_{a,b,c}e_3 &= ce_6 \\ J_{a,b,c}e_4 &= ce_5 - be_3 \\ J_{a,b,c}e_5 &= -\frac{1}{c}e_4 + be_6 \\ J_{a,b,c}e_6 &= -\frac{1}{c}e_3 \end{cases},$$

and it acts on forms by

$$\begin{cases} J_{a,b,c}e^1 &= -\frac{1}{a}e^2 \\ J_{a,b,c}e^2 &= ae^1 \\ J_{a,b,c}e^3 &= -be^4 - \frac{1}{c}e^6 \\ J_{a,b,c}e^4 &= -\frac{1}{c}e^5 \\ J_{a,b,c}e^5 &= ce^4 \\ J_{a,b,c}e^6 &= be^5 + ce^3 \end{cases}.$$

Hence, $(J_{a,b,c}, \omega_{a,b,c})$ is a family of left-invariant almost-Kähler structures on X.
A global co-frame of $(1,0)$-forms is provided by

$$\varphi^1 := ae^1 + ie^2\,, \qquad \varphi^2 := be^5 + ce^3 + ie^6\,, \qquad \varphi^3 := ce^4 + ie^5\,,$$

and the dual frame of $(1,0)$-vectors is given by

$$V_1 := \frac{1}{2}\left(\frac{1}{a}e_1 - ie_2\right)\,, \qquad V_2 := \frac{1}{2}\left(\frac{1}{c}e_3 - ie_6\right)\,, \qquad V_3 := \frac{1}{2}\left(\frac{1}{c}e_4 - ie_5 + i\frac{b}{c}e_3\right)\,.$$

In particular, the complex structure equations become

$$\begin{cases} d\varphi^1 &= 0 \\ d\varphi^2 &= \frac{c}{4}\varphi^{13} - \frac{1}{2a}\varphi^{1\bar{2}} - \frac{c}{4}\varphi^{1\bar{3}} + \frac{c}{4}\varphi^{3\bar{1}} - \frac{1}{2a}\varphi^{\bar{1}\bar{2}} + \frac{c}{4}\varphi^{\bar{1}\bar{3}} \\ d\varphi^3 &= \frac{c}{4}\varphi^{12} - \frac{c}{4}\varphi^{1\bar{2}} + \frac{1}{2a}\varphi^{1\bar{3}} + \frac{c}{4}\varphi^{2\bar{1}} + \frac{c}{4}\varphi^{\bar{1}\bar{2}} + \frac{1}{2a}\varphi^{\bar{1}\bar{3}} \end{cases}.$$

4 Numerical almost-complex and almost-Hermitian invariants on $(X, J_{a,b,c}, \omega_{a,b,c})$

In this section we compute several almost-complex invariants on $(X, J_{a,b,c}, \omega_{a,b,c})$ where $\omega_{a,b,c}$ was defined in (3.1). In particular, we start with the Hodge numbers $h^{p,0}_{\bar\partial}$, with $p = 1, 2, 3$.

4.1 Computations for $\mathcal{H}^{3,0}_{\bar\partial}$

We compute now $\mathcal{H}^{3,0}_{\bar\partial}$ for $X := (X, J_{a,b,c}, \omega_{a,b,c})$. Let

$$\psi = A\varphi^{123}$$

with A smooth function on X, be an arbitrary $(3,0)$-form on X. By degree reasons, ψ is $\bar\partial$-harmonic if and only if $\bar\partial\psi = 0$. Since φ^{123} is $\bar\partial$-closed we have

$$\bar\partial\psi = -\bar{V}_1(A)\varphi^{123\bar{1}} - \bar{V}_2(A)\varphi^{123\bar{2}} - \bar{V}_3(A)\varphi^{123\bar{3}},$$

hence $\bar\partial\psi = 0$ if and only if

$$\bar{V}_1(A) = \bar{V}_2(A) = \bar{V}_3(A) = 0$$

hence $(V_1\bar{V}_1 + V_2\bar{V}_2 + V_3\bar{V}_3)(A) = 0$. A direct computation shows that

$$4(V_1\bar{V}_1 + V_2\bar{V}_2 + V_3\bar{V}_3)(A) = \frac{1}{a^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \left(\frac{1+b^2}{c^2} + x^2\right) e^{2t}\frac{\partial^2}{\partial y_1^2} + e^{2t}\frac{\partial^2}{\partial z_1^2} + \left(\frac{1}{c^2} + x^2\right) e^{-2t}\frac{\partial^2}{\partial y_2^2} + e^{-2t}\frac{\partial^2}{\partial z_2^2}.$$

Hence, $V_1\bar{V}_1 + V_2\bar{V}_2 + V_3\bar{V}_3$ is an elliptic differential operator and consequently we have that A is constant. Therefore,

$$\mathcal{H}^{3,0}_{\bar\partial}(X) = \left\langle \varphi^{123} \right\rangle$$

and $h^{3,0}_{\bar\partial} = 1$.

4.2 Computations for $\mathcal{H}^{1,0}_{\bar\partial}$

Let

$$\psi = A\varphi^1 + B\varphi^2 + D\varphi^3$$

with A, B, D smooth functions on X, be an arbitrary $(1,0)$-form on X. By degree reasons, ψ is $\bar\partial$-harmonic if and only if $\bar\partial\psi = 0$. Using the structure equations we have

$$\bar\partial\psi = -\bar{V}_1(A)\varphi^{1\bar1} - \bar{V}_2(A)\varphi^{1\bar2} - \bar{V}_3(A)\varphi^{1\bar3} - \bar{V}_1(B)\varphi^{2\bar1} - \bar{V}_2(B)\varphi^{2\bar2} - \bar{V}_3(B)\varphi^{2\bar3}$$
$$-\bar{V}_1(D)\varphi^{3\bar1} - \bar{V}_2(D)\varphi^{3\bar2} - \bar{V}_3(D)\varphi^{3\bar3} - \frac{B}{2a}\varphi^{1\bar2} - \frac{1}{4}B\varphi^{1\bar3} + B\frac{c}{4}\varphi^{3\bar1} - \frac{c}{4}D\varphi^{1\bar2} + \frac{1}{2a}D\varphi^{1\bar3} + \frac{c}{4}D\varphi^{2\bar1},$$

hence $\bar\partial\psi = 0$ if and only if

$$\begin{cases} \bar{V}_1(A) &= 0\\ \bar{V}_2(A) + \frac{1}{2a}B + \frac{c}{4}D &= 0\\ \bar{V}_3(A) + \frac{1}{4}B - \frac{1}{2a}D &= 0\\ \bar{V}_1(B) - \frac{c}{4}D &= 0\\ \bar{V}_2(B) &= 0\\ \bar{V}_3(B) &= 0\\ \bar{V}_1(D) - \frac{c}{4}B &= 0\\ \bar{V}_2(D) &= 0\\ \bar{V}_3(D) &= 0 \end{cases}.$$

In particular, by $\bar{V}_2(B) = \bar{V}_3(B) = 0$ we have that $V_2\bar{V}_2(B) = V_3\bar{V}_3(B) = 0$ and $V_2\bar{V}_2 + V_3\bar{V}_3$ is a strictly elliptic operator without zero order terms when B is viewed as function of y_1, y_2, z_1, z_2. Since the fiber is compact by the maximum principle B is constant on the fibers, then B is a function on the base with (t, x) as coordinates. Namely, $B = B(t, x)$ and similarly by the previous system, $D = D(t, x)$.
As a consequence, from the first three equations

$$(V_1\bar{V}_1 + V_2\bar{V}_2 + V_3\bar{V}_3)(A) = 0$$

then A is constant.
The previous system reduces to

$$\begin{cases} \frac{1}{2a}B + \frac{c}{4}D &= 0\\ \frac{1}{4}B - \frac{1}{2a}D &= 0\\ \bar{V}_1(B) - \frac{c}{4}D &= 0\\ \bar{V}_1(D) - \frac{c}{4}B &= 0 \end{cases}.$$

In particular,

$$B = -\frac{ac}{2}D, \qquad \text{and} \qquad \frac{a^2c+4}{4a}D = 0.$$

Therefore we have two cases to consider. First, if $a^2c + 4 \neq 0$ then

$$D = 0, \quad B = 0, \quad A = \text{const}$$

hence

$$\mathcal{H}^{1,0}_{\bar\partial} = \left\langle \varphi^1 \right\rangle$$

and $h^{1,0}_{\bar\partial} = 1$.

If $a^2c + 4 = 0$, since $B = -\frac{ac}{2}D$, the system reduces to

$$\begin{cases} \frac{ac}{2}\bar V_1(D) + \frac{c}{4}D &= 0 \\ \bar V_1(D) + \frac{ac^2}{8}D &= 0 \end{cases} .$$

that is

$$\begin{cases} \bar V_1(D) + \frac{1}{2a}D &= 0 \\ \left(-\frac{ac^2}{8} + \frac{1}{2a}\right)D &= 0 \end{cases} .$$

By the first equation we have $(-a^2c^2 + 4)D = 0$, and recalling that $a^2c + 4 = 0$, we have two cases. If $a \neq \pm 2$ then

$$D = 0, \quad B = 0, \quad A = \text{const}$$

hence

$$\mathcal{H}^{1,0}_{\bar\partial} = \left\langle \varphi^1 \right\rangle$$

and $h^{1,0}_{\bar\partial} = 1$.

If $a = \pm 2$, we are left with

$$\bar V_1(D) \pm \frac{1}{4}D = 0, \quad B = \pm D, \quad A = \text{const} .$$

Since $D = D(t, x)$, we can expand in Fourier series and get

$$D = \sum_{\lambda,\mu\in\mathbb{Z}} D_{\lambda\mu} e^{2\pi i(\lambda x + \frac{\mu}{a_0}t)}$$

with $D_{\lambda\mu}$ constants for every $\lambda, \mu \in \mathbb{Z}$. The equation $\bar V_1(D) \pm \frac{1}{4}D = 0$ becomes

$$(\frac{1}{a}2\pi i\frac{\mu}{a_0} - 2\pi\lambda)D_{\lambda\mu} \pm \frac{1}{2}D_{\lambda\mu} = 0$$

namely,

$$\left((-4\pi\lambda \pm 1) + i(4\pi\frac{\mu}{a_0}\frac{1}{a})\right) D_{\lambda\mu} = 0$$

and since $-4\pi\lambda \pm 1 \neq 0$ for every $\lambda \in \mathbb{Z}$ we have that $D_{\lambda\mu} = 0$ for every $\lambda, \mu \in \mathbb{Z}$. Therefore,

$$D = 0, \quad B = 0, \quad A = \text{const}$$

hence

$$\mathcal{H}^{1,0}_{\bar\partial} = \left\langle \varphi^1 \right\rangle$$

and $h^{1,0}_{\bar\partial} = 1$.

4.3 Computations for $\mathcal{H}^{2,0}_{\overline{\partial}}$

Let

$$\psi = A\varphi^{12} + B\varphi^{13} + D\varphi^{23}$$

with A, B, D smooth functions on X, be an arbitrary $(2,0)$-form on X. By degree reasons, ψ is $\overline{\partial}$-harmonic if and only if $\overline{\partial}\psi = 0$. Using the structure equations we have

$$\begin{aligned}\overline{\partial}\psi \;=\;& \bar{V}_1(A)\varphi^{12\bar{1}} + \bar{V}_2(A)\varphi^{12\bar{2}} + \bar{V}_3(A)\varphi^{12\bar{3}} + \bar{V}_1(B)\varphi^{13\bar{1}} + \bar{V}_2(B)\varphi^{13\bar{2}} + \\ & + \bar{V}_3(B)\varphi^{13\bar{3}} + \bar{V}_1(D)\varphi^{23\bar{1}} + \bar{V}_2(D)\varphi^{23\bar{2}} + \bar{V}_3(D)\varphi^{23\bar{3}} - \tfrac{c}{4}A\varphi^{13\bar{1}} + \\ & - \tfrac{c}{4}B\varphi^{12\bar{1}} + D\tfrac{1}{2a}\varphi^{13\bar{2}} + \tfrac{c}{4}D\varphi^{13\bar{3}} - \tfrac{c}{4}D\varphi^{12\bar{2}} + \tfrac{1}{2a}D\varphi^{12\bar{3}},\end{aligned}$$

hence $\overline{\partial}\psi = 0$ if and only if

$$\begin{cases} \bar{V}_1(A) - \frac{c}{4}B &= 0 \\ \bar{V}_2(A) - \frac{c}{4}D &= 0 \\ \bar{V}_3(A) + \frac{1}{2a}D &= 0 \\ \bar{V}_1(B) - \frac{c}{4}A &= 0 \\ \bar{V}_2(B) + \frac{1}{2a}D &= 0 \\ \bar{V}_3(B) + \frac{c}{4}D &= 0 \\ \bar{V}_1(D) &= 0 \\ \bar{V}_2(D) &= 0 \\ \bar{V}_3(D) &= 0 \end{cases}.$$

From the last three equations we obtain immediately that $D = \text{const}$. Hence, from the system we have that

$$V_2\bar{V}_2(A) = V_3\bar{V}_3(A) = V_2\bar{V}_2(B) = V_3\bar{V}_3(B) = 0$$

hence, with a similar argument used before we have that

$$A = A(t,x), \quad B = B(t,x).$$

In particular, this implies that

$$D = 0.$$

We can expand in Fourier series and get

$$A = \sum_{\lambda,\mu\in\mathbb{Z}} A_{\lambda\mu}e^{2\pi i(\lambda x + \frac{\mu}{a_0}t)}, \quad B = \sum_{\lambda,\mu\in\mathbb{Z}} B_{\lambda\mu}e^{2\pi i(\lambda x + \frac{\mu}{a_0}t)}$$

with $A_{\lambda\mu}, B_{\lambda\mu}$ constants for every $\lambda, \mu \in \mathbb{Z}$. The first and fourth equations become respectively

$$\left(\frac{1}{a}2\pi i\frac{\mu}{a_0} - 2\pi\lambda\right)A_{\lambda\mu} - \frac{c}{2}B_{\lambda\mu} = 0$$

$$\left(\frac{1}{a}2\pi i\frac{\mu}{a_0} - 2\pi\lambda\right)B_{\lambda\mu} - \frac{c}{2}A_{\lambda\mu} = 0.$$

Summing the two equations we get

$$\left((-2\pi\lambda - \frac{c}{2}) + i(\frac{1}{a}2\pi\frac{\mu}{a_0})\right)(A_{\lambda\mu} + B_{\lambda\mu}) = 0.$$

Now we consider two cases: $c \notin 4\pi\mathbb{Z}$ and $c \in 4\pi\mathbb{Z}$.
If $c \notin 4\pi\mathbb{Z}$, then $A_{\lambda\mu} + B_{\lambda\mu} = 0$ for every $\lambda\mu \in \mathbb{Z}$, implying that $A = -B$. In this case, we obtain the following equation

$$\bar{V}_1(A) + \frac{c}{4}A = 0$$

and so

$$\left((-2\pi\lambda + \frac{c}{2}) + i(\frac{1}{a}2\pi\frac{\mu}{a_0})\right) A_{\lambda\mu} = 0.$$

Therefore, under our assumption $A_{\lambda\mu} = 0$ for every $\lambda, \mu \in \mathbb{Z}$ and therefore $B_{\lambda\mu} = 0$ for every $\lambda, \mu \in \mathbb{Z}$. As a consequence we have that if $c \notin 4\pi\mathbb{Z}$,

$$A = 0, \quad B = 0, \quad D = 0$$

hence

$$\mathcal{H}^{2,0}_{\bar\partial} = 0$$

and $h^{2,0}_{\bar\partial} = 0$.
If $c \in 4\pi\mathbb{Z}$, we set $c = 4\pi k$ with $k \in \mathbb{Z} \setminus \{0\}$, since by construction $c \neq 0$. The equation becomes

$$\left((-2\pi\lambda - 2\pi k) + i(\frac{1}{a}2\pi\frac{\mu}{a_0})\right) (A_{\lambda\mu} + B_{\lambda\mu}) = 0.$$

If $(\lambda, \mu) \neq (-k, 0)$ then $A_{\lambda\mu} + B_{\lambda\mu} = 0$, otherwise the equation is trivially satisfied.
Suppose that $(\lambda, \mu) \neq (-k, 0)$, then $A_{\lambda\mu} = -B_{\lambda\mu}$ and the first equation becomes

$$\left((-2\pi\lambda + 2\pi k) + i(\frac{1}{a}2\pi\frac{\mu}{a_0})\right) A_{\lambda\mu} = 0.$$

Hence, if, moreover $(\lambda, \mu) \neq (k, 0)$ then $A_{\lambda\mu} = -B_{\lambda\mu} = 0$. Namely, resuming we have that

- $A_{\lambda\mu} = B_{\lambda\mu} = 0$ if $(\lambda, \mu) \neq (\pm k, 0)$
- $A_{k0} = -B_{k0} = 0$
- we have no informations on A_{-k0}, B_{-k0}.

The Fourier expansions reduces to

$$A = A_{k0}e^{2\pi ikx} + A_{-k0}e^{-2\pi ikx}$$

and

$$B = -A_{k0}e^{2\pi ikx} + B_{-k0}e^{-2\pi ikx}.$$

In particular, the equation $\bar{V}_1(A) - \frac{c}{4}B = 0$ becomes

$$2\pi k(A_{-k0} - B_{-k0})e^{-2\pi ikx} = 0$$

giving $A_{-k0} = B_{-k0}$, and also the other equations are now satisfied. Therefore,

$$A = A_{k0}e^{2\pi ikx} + A_{-k0}e^{-2\pi ikx}, \quad B = -A_{k0}e^{2\pi ikx} + A_{-k0}e^{-2\pi ikx}, \quad D = 0$$

satisfy the system of equations for $\mathcal{H}^{2,0}_{\bar\partial}$ hence, if $c \in 4\pi\mathbb{Z}$, $c \neq 0$, $h^{2,0}_{\bar\partial} = 2$.

Therefore, we just proved the following

Theorem 4.1. *Let $(X, J_{a,b,c}, \omega_{a,b,c})$ be the family of almost-Kähler manifolds previously constructed. Then,*

- $h^{1,0}_{\bar\partial} = 1$,
- $h^{2,0}_{\bar\partial} = \begin{cases} 0 & \textit{if} \quad c \notin 4\pi\mathbb{Z} \\ 2 & \textit{if} \quad c \in 4\pi\mathbb{Z} \end{cases}$,
- $h^{3,0}_{\bar\partial} = 1$.

An immediate consequence is the following result that marks a difference with the integrable case (cf. also [12]).

Corollary 4.2. *The Hodge numbers can vary when the almost-complex structures are almost-Kähler and vary continuously.*

We compute now the almost-Hermitian invariants $h^{p,0}_{\bar\delta}$, with $p = 1, 2, 3$.
First of all we recall that on bi-graded forms $\mathcal{H}^{\bullet,\bullet}_{\bar\delta} = \mathcal{H}^{\bullet,\bullet}_{\bar\partial} \cap \mathcal{H}^{\bullet,\bullet}_{\mu}$, in particular for bidegree reasons

$$\mathcal{H}^{1,0}_{\bar\delta} = \mathcal{H}^{1,0}_{\bar\partial} ,$$

hence we are left to compute $\mathcal{H}^{2,0}_{\bar\delta}$ and $\mathcal{H}^{3,0}_{\bar\delta}$.

4.4 Computations for $\mathcal{H}^{3,0}_{\bar\delta}$

Let $g_{a,b,c}$ be the Hermitian metric associated to $(J_{a,b,c}, \omega_{a,b,c})$ where $\omega_{a,b,c}$ is defined in (3.1).

It is immediate to see that

$$\mathcal{H}^{3,0}_{\bar\delta} = \mathcal{H}^{3,0}_{\bar\partial} \cap \mathrm{Ker}\,(\mu^*) .$$

Since $\mathcal{H}^{3,0}_{\bar\partial} = \langle \varphi^{123} \rangle$ we set $\psi = A\varphi^{123}$ with $A \in \mathbb{C}$. Then, $\psi \in \mathrm{Ker}\,(\mu^*)$ if and only if $\bar\mu * \psi = 0$. Since $*\psi = A \cdot \mathrm{const} \cdot \varphi^{123}$ and, by the structure equation

$$\bar\mu\varphi^{123} = \frac{1}{2a}\varphi^{13\bar1\bar2} - \frac{c}{4}\varphi^{13\bar1\bar3} + \frac{c}{4}\varphi^{12\bar1\bar2} + \frac{1}{2a}\varphi^{12\bar1\bar3},$$

we have that $\bar\mu * \psi = 0$ if and only if $A = 0$. Therefore,

$$\mathcal{H}^{3,0}_{\bar\delta} = \{0\}$$

and $h^{3,0}_{\bar\delta} = 0$.

4.5 Computations for $\mathcal{H}^{2,0}_{\bar\delta}$

It is immediate to see that

$$\mathcal{H}^{2,0}_{\bar\delta} = \mathcal{H}^{2,0}_{\bar\partial} \cap \mathrm{Ker}\,(\mu^*) .$$

If $c \notin 4\pi\mathbb{Z}$ then $\mathcal{H}^{2,0}_{\bar\partial} = \{0\}$, hence $\mathcal{H}^{2,0}_{\bar\delta} = \{0\}$.
Let us assume that $c \in 4\pi\mathbb{Z}$, namely $c = 4\pi k$, with $k \in \mathbb{Z} \setminus \{0\}$.
Since

$$\mathcal{H}^{2,0}_{\bar\partial} = \left\langle e^{2\pi ikx}\varphi^{12} - e^{2\pi ikx}\varphi^{13}, e^{-2\pi ikx}\varphi^{12} + e^{-2\pi ikx}\varphi^{13} \right\rangle$$

We set

$$\psi = A(e^{2\pi ikx}\varphi^{12} - e^{2\pi ikx}\varphi^{13}) + B(e^{-2\pi ikx}\varphi^{12} + e^{-2\pi ikx}\varphi^{13})$$

with $A, B \in \mathbb{C}$. Then, $\psi \in \mathrm{Ker}\,(\mu^*)$ if and only if $\bar\mu * \psi = 0$.
We get

$$*\varphi^{12} = \frac{i}{2}\varphi^{123\bar3} , \qquad *\varphi^{13} = -\frac{i}{2}\varphi^{123\bar2} .$$

For instance, by the definition of the $\mathbb{C}$-linear Hodge $*$ operator we have that

$$\varphi^{\bar1\bar2} \wedge *\varphi^{12} = |\varphi^{12}|^2 \frac{\omega^3_{a,b,c}}{6} = -\frac{i}{8}|\varphi^{12}|^2\varphi^{1\bar12\bar23\bar3} = -\frac{i}{8}2^2\varphi^{1\bar12\bar23\bar3} = -\frac{i}{2}\varphi^{1\bar12\bar23\bar3} = \frac{i}{2}\varphi^{\bar1\bar2123\bar3}$$

and $\varphi^{\bar i\bar j} \wedge *\varphi^{12} = 0$ for $(i, j) \neq (1, 2)$. This shows that $*\varphi^{12} = \frac{i}{2}\varphi^{123\bar3}$.
Hence, we have that

$$*\psi = A\frac{i}{2}(e^{2\pi ikx}\varphi^{123\bar3} + e^{2\pi ikx}\varphi^{123\bar2}) + B\frac{i}{2}(e^{-2\pi ikx}\varphi^{123\bar3} - e^{-2\pi ikx}\varphi^{123\bar2}) .$$

By the structure equations

$$\bar\mu\varphi^{123\bar 2} = \frac{c}{4}\varphi^{13\bar 1\bar 2\bar 3} - \frac{1}{2a}\varphi^{12\bar 1\bar 2\bar 3}\,, \quad \bar\mu\varphi^{123\bar 3} = \frac{1}{2a}\varphi^{13\bar 1\bar 2\bar 3} + \frac{c}{4}\varphi^{12\bar 1\bar 2\bar 3}\,.$$

Hence, we obtain

$$\bar\mu \star \psi = \varphi^{12\bar 1\bar 2\bar 3}\left[A\frac{i}{2}(\frac{c}{4} - \frac{1}{2a})e^{2\pi ikx} + B\frac{i}{2}(\frac{c}{4} + \frac{1}{2a})e^{-2\pi ikx}\right] +$$

$$\varphi^{13\bar 1\bar 2\bar 3}\left[A\frac{i}{2}(\frac{c}{4} + \frac{1}{2a})e^{2\pi ikx} + B\frac{i}{2}(\frac{1}{2a} - \frac{c}{4})e^{-2\pi ikx}\right]\,.$$

Therefore, $\bar\mu \star \psi = 0$ if and only if

$$A(\frac{c}{4} - \frac{1}{2a})e^{4\pi ikx} + B(\frac{c}{4} + \frac{1}{2a}) = 0\,,$$

and

$$A(\frac{c}{4} + \frac{1}{2a})e^{4\pi ikx} + B(\frac{1}{2a} - \frac{c}{4}) = 0.$$

This implies that $A = B = 0$, namely $\psi = 0$.
Therefore,

$$\mathcal{H}^{2,0}_{\bar\delta} = \{0\}$$

and $h^{2,0}_{\bar\delta} = 0$.

Therefore, we just proved the following

Theorem 4.3. *Let $(X, J_{a,b,c}, \omega_{a,b,c})$ be the family of almost-Kähler manifolds previously constructed. Then,*

- $h^{1,0}_{\bar\delta} = 1$,
- $h^{2,0}_{\bar\delta} = 0$,
- $h^{3,0}_{\bar\delta} = 0$.

Now we compute the dimension of the almost-complex Dolbeault cohomology groups $H^{p,0}_{\mathrm{Dol}}$.

First of all, notice that by [6, Proposition 4.10],

$$H^{p,0}_{\mathrm{Dol}} \simeq \mathcal{H}^{p,0}_{\bar\partial} \cap \operatorname{Ker}\bar\mu$$

4.6 Computation of $H^{1,0}_{\mathrm{Dol}}$ and $H^{3,0}_{\mathrm{Dol}}$

Clearly, by the structure equations and by the previous computations

$$H^{1,0}_{\mathrm{Dol}} \simeq \mathcal{H}^{1,0}_{\bar\partial} \cap \operatorname{Ker}\bar\mu = \left\langle \varphi^1 \right\rangle\,.$$

Now, since $\mathcal{H}^{3,0}_{\bar\partial} = \langle\varphi^{123}\rangle$ and by a direct computation $\bar\mu\varphi^{123} \neq 0$, one has that

$$H^{3,0}_{\mathrm{Dol}} = \{0\}\,.$$

4.7 Computation of $H^{2,0}_{\mathrm{Dol}}$

Notice that, if $c \notin 4\pi\mathbb{Z}$, then $\mathcal{H}^{2,0}_{\bar\partial} = \{0\}$ and so

$$H^{2,0}_{\mathrm{Dol}} = \{0\}\,.$$

Let now $c \in 4\pi\mathbb{Z}$, then

$$\mathcal{H}^{2,0}_{\bar\partial} = \left\langle e^{2\pi ikx}\varphi^{12} - e^{2\pi ikx}\varphi^{13}, e^{-2\pi ikx}\varphi^{12} + e^{-2\pi ikx}\varphi^{13} \right\rangle$$

We set

$$\psi = A(e^{2\pi ikx}\varphi^{12} - e^{2\pi ikx}\varphi^{13}) + B(e^{-2\pi ikx}\varphi^{12} + e^{-2\pi ikx}\varphi^{13})$$

with $A, B \in \mathbb{C}$. Since

$$\bar\mu\varphi^{12} = \frac{1}{2a}\varphi^{1\bar1\bar2} - \frac{c}{4}\varphi^{1\bar1\bar3}, \quad \bar\mu\varphi^{13} = -\frac{c}{4}\varphi^{1\bar1\bar2} - \frac{1}{2a}\varphi^{1\bar1\bar3},$$

then, $\bar\mu\psi = 0$ if and only if

$$A(\frac{c}{4} + \frac{1}{2a})e^{4\pi ikx} + B(\frac{1}{2a} - \frac{c}{4}) = 0.$$

and

$$A(-\frac{c}{4} + \frac{1}{2a})e^{4\pi ikx} + B(-\frac{c}{4} - \frac{1}{2a}) = 0.$$

This implies that $A = B = 0$, and so

$$H^{2,0}_{\mathrm{Dol}} = \{0\}.$$

Therefore we proved the following

Theorem 4.4. *Let $(X, J_{a,b,c}, \omega_{a,b,c})$ be the family of almost-Kähler manifolds previously constructed. Then,*

- $h^{1,0}_{Dol} = 1$,
- $h^{2,0}_{Dol} = 0$,
- $h^{3,0}_{Dol} = 0$.

5 An almost-complex structure with no compatible symplectic structures

We will construct now an almost-complex structure J on X which does not admit any compatible symplectic structures. We set as a global co-frame of $(1,0)$-forms

$$\Phi^1 := e^1 + ie^2, \qquad \Phi^2 := e^3 + ie^4, \qquad \Phi^3 := e^5 + ie^6,$$

and the dual frame of $(1,0)$-vectors is given by

$$W_1 := \frac{1}{2}(e_1 - ie_2), \qquad W_2 := \frac{1}{2}(e_3 - ie_4), \qquad W_3 := \frac{1}{2}(e_5 - ie_6).$$

The complex structure equations become

$$\begin{cases} d\Phi^1 &= 0 \\ d\Phi^2 &= \frac{i}{2}\Phi^{13} - \frac{1}{2}\Phi^{1\bar2} + \frac{i}{2}\Phi^{3\bar1} - \frac{1}{2}\Phi^{\bar1\bar2} \\ d\Phi^3 &= -\frac{1}{2}\Phi^{1\bar3} - \frac{1}{2}\Phi^{\bar1\bar3} \end{cases}.$$

Notice that the almost-complex manifold just constructed does not admit any compatible symplectic structures. Indeed, by contradiction, if (X, J) admits a compatible symplectic structure then, by a symmetrization process it also admits a compatible left-invariant symplectic structure. As noticed before, every left-invariant symplectic structure on X is given by

$$\omega_{a,b,c} = ae^{12} + be^{56} + c(e^{36} + e^{45})$$

with $a, b, c \in \mathbb{R}$ and $a, c \neq 0$. Hence, by construction J cannot be compatible with any of these symplectic structures.

We compute now the Hodge numbers $h^{p,0}_{\bar\partial}$, for $p = 1, 2, 3$.

5.1 Computations for $\mathcal{H}^{1,0}_{\overline{\partial}}$

Let

$$\psi = A\Phi^1 + B\Phi^2 + C\Phi^3$$

with A, B, C smooth functions on X, be an arbitrary $(1,0)$-form on X. By degree reasons, ψ is $\overline{\partial}$-harmonic if and only if $\overline{\partial}\psi = 0$. Using the structure equations we have that $\overline{\partial}\psi = 0$ if and only if

$$\begin{cases} \bar{W}_1(A) & = & 0 \\ \bar{W}_2(A) + \frac{1}{2}B & = & 0 \\ \bar{W}_3(A) + \frac{1}{2}C & = & 0 \\ \bar{W}_1(B) & = & 0 \\ \bar{W}_2(B) & = & 0 \\ \bar{W}_3(B) & = & 0 \\ \bar{W}_1(C) - \frac{i}{2}B & = & 0 \\ \bar{W}_2(C) & = & 0 \\ \bar{W}_3(C) & = & 0 \end{cases}.$$

Then from $\bar{W}_1(B) = \bar{W}_2(B) = \bar{W}_3(B) = 0$ we get with similar arguments used before that B is constant. Hence

$$(W_1\bar{W}_1 + W_2\bar{W}_2 + W_3\bar{W}_3)(C) = 0$$

and so C is also constant. As a consequence, the same holds for A. Therefore, having A constant, this implies that $B = C = 0$. Therefore,

$$B = 0, \quad C = 0, \quad A = \mathrm{const}$$

hence

$$\mathcal{H}^{1,0}_{\overline{\partial}} = \left\langle \Phi^1 \right\rangle$$

and $h^{1,0}_{\overline{\partial}} = 1$.

5.2 Computations for $\mathcal{H}^{2,0}_{\overline{\partial}}$

Let

$$\psi = A\Phi^{12} + B\Phi^{13} + C\Phi^{23}$$

with A, B, C smooth functions on X, be an arbitrary $(2,0)$-form on X. By degree reasons, ψ is $\overline{\partial}$-harmonic if and only if $\overline{\partial}\psi = 0$. Using the structure equations we have that $\overline{\partial}\psi = 0$ if and only if

$$\begin{cases} \bar{W}_1(A) & = & 0 \\ \bar{W}_2(A) & = & 0 \\ \bar{W}_3(A) - \frac{1}{2}C & = & 0 \\ \bar{W}_1(B) - \frac{i}{2}A & = & 0 \\ \bar{W}_2(B) + \frac{1}{2}C & = & 0 \\ \bar{W}_3(B) & = & 0 \\ \bar{W}_1(C) & = & 0 \\ \bar{W}_2(C) & = & 0 \\ \bar{W}_3(C) & = & 0 \end{cases}.$$

Then from $\bar{W}_1(C) = \bar{W}_2(C) = \bar{W}_3(C) = 0$ we get with similar arguments used before that C is constant. Hence $(W_1\bar{W}_1 + W_2\bar{W}_2 + W_3\bar{W}_3)(A) = 0$ and so A is also constant. This implies that $C = 0$ and therefore B is constant leading to A being zero. Namely

$$A = 0, \quad C = 0, \quad B = \mathrm{const}$$

hence

$$\mathcal{H}^{2,0}_{\bar\partial} = \left\langle \Phi^{13} \right\rangle$$

and $h^{2,0}_{\bar\partial} = 1$.

5.3 Computations for $\mathcal{H}^{3,0}_{\bar\partial}$

Let

$$\psi = A\Phi^{123}$$

with A smooth function on X, be an arbitrary $(3,0)$-form on X. By degree reasons, ψ is $\bar\partial$-harmonic if and only if $\bar\partial\psi = 0$. Since Φ^{123} is $\bar\partial$-closed we have that $\bar\partial\psi = 0$ if and only if

$$\bar W_1(A) = \bar W_2(A) = \bar W_3(A) = 0$$

hence $(W_1\bar W_1 + W_2\bar W_2 + W_3\bar W_3)(A) = 0$ and so we have that A is constant. Therefore,

$$\mathcal{H}^{3,0}_{\bar\partial}(X) = \left\langle \Phi^{123} \right\rangle$$

and $h^{3,0}_{\bar\partial} = 1$.

Therefore, we just proved the following

Theorem 5.1. *Let (X, J) be the almost complex manifold previously constructed. Then,*

- $h^{1,0}_{\bar\partial} = 1$,
- $h^{2,0}_{\bar\partial} = 1$,
- $h^{3,0}_{\bar\partial} = 1$.

Let now ω be the following Hermitian metric

$$\omega = \frac{i}{2}\left(\Phi^{1\bar 1} + \Phi^{2\bar 2} + \Phi^{3\bar 3}\right).$$

We compute now the numbers $h^{p,0}_{\bar\delta}$, for $p = 1, 2, 3$.

First of all, as noticed before, for bidegree reasons

$$\mathcal{H}^{1,0}_{\bar\delta} = \mathcal{H}^{1,0}_{\bar\partial},$$

hence we are left to compute $\mathcal{H}^{2,0}_{\bar\delta}$ and $\mathcal{H}^{3,0}_{\bar\delta}$.

5.4 Computations for $\mathcal{H}^{2,0}_{\bar\delta}$

It is immediate to see that

$$\mathcal{H}^{2,0}_{\bar\delta} = \mathcal{H}^{2,0}_{\bar\partial} \cap \operatorname{Ker}(\mu^*).$$

Since $\mathcal{H}^{2,0}_{\bar\partial} = \langle \Phi^{13} \rangle$ we set $\psi = A\Phi^{13}$ with $A \in \mathbb{C}$. Then, $\psi \in \operatorname{Ker}(\mu^*)$ if and only if $\bar\mu \star \psi = 0$. Since $\star\psi = -A\frac{i}{2}\Phi^{123\bar 2}$ and, by the structure equations

$$\bar\mu\Phi^{23} = -\frac{1}{2}\Phi^{3\bar 1\bar 2} + \frac{1}{2}\Phi^{2\bar 1\bar 3}$$

we have that

$$\bar{\mu} \star \psi = A\frac{i}{2}\Phi^1 \wedge \bar{\mu}(\Phi^{23}) \wedge \Phi^{\bar{2}} = -A\frac{i}{4}\Phi^{12\bar{1}\bar{2}\bar{3}}.$$

Then, $\bar{\mu} \star \psi = 0$ if and only if $A = 0$. Therefore,

$$\mathcal{H}^{2,0}_{\bar{\delta}} = \{0\}$$

and $h^{2,0}_{\bar{\delta}} = 0$.

5.5 Computations for $\mathcal{H}^{3,0}_{\bar{\delta}}$

Clearly, as before

$$\mathcal{H}^{3,0}_{\bar{\delta}} = \mathcal{H}^{3,0}_{\bar{\partial}} \cap \operatorname{Ker}(\mu^\star).$$

Since $\mathcal{H}^{3,0}_{\bar{\partial}} = \langle \Phi^{123} \rangle$ we set $\psi = A\Phi^{123}$ with $A \in \mathbb{C}$. Then, $\psi \in \operatorname{Ker}(\mu^\star)$ if and only if $\bar{\mu} \star \psi = 0$. Since $\star\psi = A\Phi^{123}$ and, by the structure equations

$$\bar{\mu} \star \psi = A\left(\frac{1}{2}\Phi^{13\bar{1}\bar{2}} - \frac{1}{2}\Phi^{12\bar{1}\bar{3}}\right).$$

Then, $\bar{\mu} \star \psi = 0$ if and only if $A = 0$. Therefore,

$$\mathcal{H}^{3,0}_{\bar{\delta}} = \{0\}$$

and $h^{3,0}_{\bar{\delta}} = 0$.

Therefore, we just proved the following

Theorem 5.2. *Let (X, J, ω) be the almost-Hermitian manifold previously constructed. Then,*

- $h^{1,0}_{\bar{\delta}} = 1$,
- $h^{2,0}_{\bar{\delta}} = 0$,
- $h^{3,0}_{\bar{\delta}} = 0$.

We compute now the dimensions of the almost-complex Dolbeault cohomology groups $H^{p,0}_{\mathrm{Dol}}$, for $p = 1, 2, 3$.

As done above, notice that by [6, Proposition 4.10],

$$H^{p,0}_{\mathrm{Dol}} \simeq \mathcal{H}^{p,0}_{\bar{\partial}} \cap \operatorname{Ker}\bar{\mu}.$$

5.6 Computations for $\mathcal{H}^{1,0}_{\mathrm{Dol}}$, $\mathcal{H}^{2,0}_{\mathrm{Dol}}$ and $\mathcal{H}^{3,0}_{\mathrm{Dol}}$

Clearly, by the structure equations and by the previous computations

$$H^{1,0}_{\mathrm{Dol}} \simeq \mathcal{H}^{1,0}_{\bar{\partial}} \cap \operatorname{Ker}\bar{\mu} = \left\langle \Phi^1 \right\rangle.$$

Now, since $\mathcal{H}^{2,0}_{\bar{\partial}} = \left\langle \Phi^{13} \right\rangle$ and by a direct computation $\bar{\mu}\Phi^{13} = \frac{1}{2}\Phi^{1\bar{1}\bar{3}} \neq 0$, one has that

$$H^{2,0}_{\mathrm{Dol}} = \{0\}.$$

Similarly, since $\mathcal{H}^{3,0}_{\bar{\partial}} = \left\langle \Phi^{123} \right\rangle$ and by a direct computation $\bar{\mu}\Phi^{123} \neq 0$, one has that

$$H^{3,0}_{\mathrm{Dol}} = \{0\}.$$

Therefore, we just proved the following

Theorem 5.3. *Let (X, J) be the almost complex manifold previously constructed. Then,*

- $h^{1,0}_{Dol} = 1$,
- $h^{2,0}_{Dol} = 0$,
- $h^{3,0}_{Dol} = 0$.

6 The Iwasawa manifold

We study now another 6-dimensional example. Let $\mathbb{I}$ be the Iwasawa manifold defined as the quotient $\mathbb{I} := \Gamma\backslash\mathbb{H}_3$ where

$$\mathbb{H}_3 := \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}$$

and

$$\Gamma := \left\{ \begin{bmatrix} 1 & \gamma_1 & \gamma_3 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix} \mid \gamma_1, \gamma_2, \gamma_3 \in \mathbb{Z}[i] \right\} .$$

Then, setting $z_j = x_j + iy_j$, there exists a basis of left-invariant 1-forms $\{e_i\}$ on $\mathbb{I}$ given by

$$\begin{cases} e^1 &= dx_1 \\ e^2 &= dy_1 \\ e^3 &= dx_2 \\ e^4 &= dy_2 \\ e^5 &= dx_3 - x_1 dx_2 + y_1 dy_2 \\ e^6 &= dy_3 - x_1 dy_2 - y_1 dx_2 \end{cases} ,$$

and the dual basis is given by

$$\begin{cases} e_1 &= \frac{\partial}{\partial x_1} \\ e_2 &= \frac{\partial}{\partial y_1} \\ e_3 &= \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + y_1 \frac{\partial}{\partial y_3} \\ e_4 &= \frac{\partial}{\partial y_2} - y_1 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial y_3} \\ e_5 &= \frac{\partial}{\partial x_3} \\ e_6 &= \frac{\partial}{\partial y_3} \end{cases} .$$

The following structure equations hold

$$\begin{cases} de^1 &= 0 \\ de^2 &= 0 \\ de^3 &= 0 \\ de^4 &= 0 \\ de^5 &= -e^{13} + e^{24} \\ de^6 &= -e^{14} - e^{23} \end{cases} .$$

We define the almost-complex structure J setting as global co-frame of $(1, 0)$-forms

$$\varphi^1 := e^1 + ie^6 , \qquad \varphi^2 := e^2 + ie^5 , \qquad \varphi^3 := e^3 + ie^4$$

and let

$$V_1 := \frac{1}{2}(e_1 - ie_6) , \qquad V_2 := \frac{1}{2}(e_2 - ie_5) , \qquad V_3 := \frac{1}{2}(e_3 - ie_4)$$

be the dual frame of vectors. In particular, the complex structure equations become

$$\begin{cases} d\varphi^1 &= -\frac{1}{4}\varphi^{13} - \frac{i}{4}\varphi^{23} + \frac{1}{4}\varphi^{1\bar{3}} - \frac{i}{4}\varphi^{2\bar{3}} + \frac{1}{4}\varphi^{3\bar{1}} + \frac{i}{4}\varphi^{3\bar{2}} + \frac{1}{4}\varphi^{\bar{1}\bar{3}} - \frac{i}{4}\varphi^{\bar{2}\bar{3}} \\ d\varphi^2 &= -\frac{i}{4}\varphi^{13} + \frac{1}{4}\varphi^{23} - \frac{i}{4}\varphi^{1\bar{3}} - \frac{1}{4}\varphi^{2\bar{3}} + \frac{i}{4}\varphi^{3\bar{1}} - \frac{1}{4}\varphi^{3\bar{2}} - \frac{i}{4}\varphi^{\bar{1}\bar{3}} - \frac{1}{4}\varphi^{\bar{2}\bar{3}} \\ d\varphi^3 &= 0 \end{cases} .$$

Notice that

$$\omega := \frac{i}{2}\sum_{j=1}^{3}\varphi^{j\bar{j}}$$

is an almost-Kähler metric on $\mathbb{I}$, in particular (J, ω) is an almost-Kähler structure on $\mathbb{I}$.

We compute now the Hodge numbers $h^{p,0}_{\bar{\partial}}$, for $p = 1, 2, 3$.

6.1 Computations for $\mathcal{H}^{1,0}_{\bar{\partial}}$

Let

$$\psi = A\varphi^1 + B\varphi^2 + C\varphi^3$$

with A, B, C smooth functions on $\mathbb{I}$, be an arbitrary $(1,0)$-form on $\mathbb{I}$. By degree reasons, ψ is $\bar{\partial}$-harmonic if and only if $\bar{\partial}\psi = 0$. Using the structure equations we have that $\bar{\partial}\psi = 0$ if and only if

$$\begin{cases} \bar{V}_1(A) &= 0 \\ \bar{V}_2(A) &= 0 \\ -\bar{V}_3(A) + \frac{1}{4}A - \frac{i}{4}B &= 0 \\ \bar{V}_1(B) &= 0 \\ \bar{V}_2(B) &= 0 \\ \bar{V}_3(B) + \frac{i}{4}A + \frac{1}{4}B &= 0 \\ -\bar{V}_1(C) + \frac{1}{4}A + \frac{i}{4}B &= 0 \\ -\bar{V}_2(C) + \frac{i}{4}A - \frac{1}{4}B &= 0 \\ \bar{V}_3(C) &= 0 \end{cases} .$$

From $\bar{V}_1(A) = \bar{V}_2(A) = \bar{V}_1(B) = \bar{V}_2(B) = 0$ we get that

$$(V_1\bar{V}_1 + V_2\bar{V}_2)(A) = 0 \qquad \text{and} \qquad (V_1\bar{V}_1 + V_2\bar{V}_2)(B) = 0$$

and so $A = A(x_2, y_2)$ and $B = B(x_2, y_2)$ depend only on x_2 and y_2.
Hence, from the last three equations we obtain $(V_1\bar{V}_1 + V_2\bar{V}_2 + V_3\bar{V}_3)(C) = 0$ implying that C is constant. Therefore, $A + iB = 0$ giving

$$-\bar{V}_3(A) + \frac{1}{2}A = 0 \qquad \text{and} \qquad -\bar{V}_3(B) - \frac{1}{2}B = 0.$$

We can expand in Fourier series and get

$$A = \sum_{\lambda,\mu\in\mathbb{Z}} A_{\lambda\mu}e^{2\pi i(\lambda x_2 + \mu y_2)}, \quad B = \sum_{\lambda,\mu\in\mathbb{Z}} B_{\lambda\mu}e^{2\pi i(\lambda x_2 + \mu y_2)}$$

with $A_{\lambda\mu}, B_{\lambda\mu}$ constants for every $\lambda, \mu \in \mathbb{Z}$. Therefore, $\bar{V}_3(A) - \frac{1}{2}A = 0$ gives

$$\left(-\pi i\lambda + \pi\mu + \frac{1}{2}\right)A_{\lambda\mu} = 0$$

and since $\mu \in \mathbb{Z}$ we have that $A_{\lambda\mu} = 0$ for every $\lambda, \mu \in \mathbb{Z}$. Hence,

$$A = 0 \qquad \text{and} \qquad B = 0.$$

Therefore,

$$A = 0, \quad B = 0, \quad C = \text{const}$$

hence

$$\mathcal{H}^{1,0}_{\bar\partial} = \left\langle \varphi^3 \right\rangle$$

and $h^{1,0}_{\bar\partial} = 1$.

6.2 Computations for $\mathcal{H}^{2,0}_{\bar\partial}$

Let

$$\psi = A\varphi^{12} + B\varphi^{13} + C\varphi^{23}$$

with A, B, C smooth functions on $\mathbb{I}$, be an arbitrary $(2,0)$-form on $\mathbb{I}$. By degree reasons, ψ is $\bar\partial$-harmonic if and only if $\bar\partial\psi = 0$. Using the structure equations we have that $\bar\partial\psi = 0$ if and only if

$$\begin{cases} \bar V_1(A) & = & 0 \\ \bar V_2(A) & = & 0 \\ \bar V_3(A) & = & 0 \\ \bar V_1(B) - \frac{i}{4}A & = & 0 \\ \bar V_2(B) + \frac{1}{4}A & = & 0 \\ \bar V_3(B) - \frac{1}{4}B + \frac{i}{4}C & = & 0 \\ \bar V_1(C) + \frac{1}{4}A & = & 0 \\ \bar V_2(C) + \frac{i}{4}A & = & 0 \\ \bar V_3(C) + \frac{i}{4}B + \frac{1}{4}C & = & 0 \end{cases} .$$

With similar arguments used above we have that $A = \text{const}$, $B = B(x_2, y_2)$ and $C = C(x_2, y_2)$. In particular, since $\bar V_1(B) = 0$ we get that $A = 0$. Therefore, from

$$\bar V_3(B) - \frac{1}{4}B + \frac{i}{4}C = 0 \qquad \text{and} \qquad \bar V_3(C) + \frac{i}{4}B + \frac{1}{4}C = 0$$

we obtain $\bar V_3(B - iC) = 0$ hence, $B - iC = \text{const} =: k$. In particular,

$$\bar V_3(B) - \frac{1}{4}k = 0$$

and so B is constant implying that also C is constant. Therefore, $k = 0$ giving $B = iC$.
Therefore,

$$A = 0, \quad B = iC = \text{const},$$

hence

$$\mathcal{H}^{2,0}_{\bar\partial} = \left\langle i\varphi^{13} + \varphi^{23} \right\rangle$$

and $h^{2,0}_{\bar\partial} = 1$.

6.3 Computations for $\mathcal{H}^{3,0}_{\bar\partial}$

Let

$$\psi = A\varphi^{123}$$

with A smooth function on $\mathbb{I}$, be an arbitrary $(3,0)$-form on $\mathbb{I}$. By degree reasons, ψ is $\bar\partial$-harmonic if and only if $\bar\partial\psi = 0$. Hence $\bar\partial\psi = 0$ if and only if

$$\bar V_1(A) = \bar V_2(A) = \bar V_3(A) = 0$$

hence $(V_1\bar{V}_1 + V_2\bar{V}_2 + V_3\bar{V}_3)(A) = 0$ and, since $V_1\bar{V}_1 + V_2\bar{V}_2 + V_3\bar{V}_3$ is an elliptic differential operator we have that A is constant. Therefore,

$$\mathcal{H}^{3,0}_{\bar{\partial}}(X) = \left\langle \varphi^{123} \right\rangle$$

and $h^{3,0}_{\bar{\partial}} = 1$.

Therefore, we just proved the following

Theorem 6.1. *Let $(\mathbb{I}, J, \omega)$ be the almost-Kähler Iwasawa manifold constructed above. Then,*

- $h^{1,0}_{\bar{\partial}} = 1$,
- $h^{2,0}_{\bar{\partial}} = 1$
- $h^{3,0}_{\bar{\partial}} = 1$.

We compute now the numbers $h^{p,0}_{\bar{\delta}}$, for $p = 1, 2, 3$.

First of all, as noticed before, for bidegree reasons

$$\mathcal{H}^{1,0}_{\bar{\delta}} = \mathcal{H}^{1,0}_{\bar{\partial}}\,,$$

hence we are left to compute $\mathcal{H}^{2,0}_{\bar{\delta}}$ and $\mathcal{H}^{3,0}_{\bar{\delta}}$.

6.4 Computations for $\mathcal{H}^{2,0}_{\bar{\delta}}$

It is immediate to see that

$$\mathcal{H}^{2,0}_{\bar{\delta}} = \mathcal{H}^{2,0}_{\bar{\partial}} \cap \operatorname{Ker}(\mu^*)\,.$$

Since

$$\mathcal{H}^{2,0}_{\bar{\partial}} = \langle i\varphi^{13} + \varphi^{23}\rangle,$$

we set

$$\psi = A(i\varphi^{13} + \varphi^{23})$$

with $A \in \mathbb{C}$. Then, $\psi \in \operatorname{Ker}(\mu^*)$ if and only if $\bar{\mu} * \psi = 0$. Since $*\psi = A \cdot \mathrm{const} \cdot (-i\varphi^{123\bar{2}} + \varphi^{123\bar{1}})$ and by the structure equations we have that

$$\bar{\mu}\varphi^{123\bar{2}} = -\frac{1}{4}\varphi^{23\bar{1}\bar{2}\bar{3}} - \frac{i}{4}\varphi^{13\bar{1}\bar{2}\bar{3}}$$

and

$$\bar{\mu}\varphi^{123\bar{1}} = -\frac{i}{4}\varphi^{23\bar{1}\bar{2}\bar{3}} + \frac{1}{4}\varphi^{13\bar{1}\bar{2}\bar{3}}$$

we get that

$$\bar{\mu} * \psi = 0$$

Therefore,

$$\mathcal{H}^{2,0}_{\bar{\delta}} = \mathcal{H}^{2,0}_{\bar{\partial}} = \langle i\varphi^{13} + \varphi^{23}\rangle$$

and $h^{2,0}_{\bar{\delta}} = 1$.

6.5 Computations for $\mathcal{H}^{3,0}_{\bar{\delta}}$

Clearly, as before

$$\mathcal{H}^{3,0}_{\bar{\delta}} = \mathcal{H}^{3,0}_{\bar{\partial}} \cap \operatorname{Ker}(\mu^*)\,.$$

Since $\mathcal{H}^{3,0}_{\bar\partial} = \langle\varphi^{123}\rangle$, we set $\psi = A\varphi^{123}$ with $A \in \mathbb{C}$. Then, $\psi \in \operatorname{Ker}(\mu^*)$ if and only if $\bar\mu * \psi = 0$. By the definition of the Hodge operator, we have that

$$*\psi = A \cdot \mathrm{const} \cdot \varphi^{123};$$

in view of the the structure equations, we obtain that

$$\bar\mu * \psi = A \cdot \mathrm{const} \cdot \left(\frac{1}{4}\varphi^{23\bar1\bar3} - \frac{i}{4}\varphi^{23\bar2\bar3} + \frac{i}{4}\varphi^{13\bar1\bar3} + \frac{1}{4}\varphi^{13\bar2\bar3} \right).$$

Hence $\bar\mu * \psi = 0$ if and only if $A = 0$. Therefore,

$$\mathcal{H}^{3,0}_{\bar\delta} = \{0\}$$

and $h^{3,0}_{\bar\delta} = 0$.

Therefore, we just proved the following

Theorem 6.2. *Let $(\mathbb{I}, J, \omega)$ be the almost-Kähler Iwasawa manifold previously constructed. Then,*

- $h^{1,0}_{\bar\delta} = 1$,
- $h^{2,0}_{\bar\delta} = 1$,
- $h^{3,0}_{\bar\delta} = 0$.

We compute now the dimensions of the almost-complex Dolbeault cohomology groups $H^{p,0}_{\mathrm{Dol}}$, for $p = 1, 2, 3$.

As done above, notice that by [6, Proposition 4.10],

$$H^{p,0}_{\mathrm{Dol}} \simeq \mathcal{H}^{p,0}_{\bar\partial} \cap \operatorname{Ker} \bar\mu.$$

6.6 Computations for $\mathcal{H}^{1,0}_{\mathrm{Dol}}$, $\mathcal{H}^{2,0}_{\mathrm{Dol}}$ and $\mathcal{H}^{3,0}_{\mathrm{Dol}}$

Clearly, by the structure equations and by the previous computations

$$H^{1,0}_{\mathrm{Dol}} \simeq \mathcal{H}^{1,0}_{\bar\partial} \cap \operatorname{Ker} \bar\mu = \left\langle \varphi^3 \right\rangle .$$

Now, since $\mathcal{H}^{2,0}_{\bar\partial} = \langle i\varphi^{13} + \varphi^{23}\rangle$ and by a direct computation $\bar\mu(i\varphi^{13} + \varphi^{23}) = 0$, one has that

$$H^{2,0}_{\mathrm{Dol}} = \left\langle i\varphi^{13} + \varphi^{23} \right\rangle .$$

Since $\mathcal{H}^{3,0}_{\bar\partial} = \langle \varphi^{123}\rangle$ and by a direct computation $\bar\mu\varphi^{123} \neq 0$, one has that

$$H^{3,0}_{\mathrm{Dol}} = \{0\} .$$

In particular, we have the following

Theorem 6.3. *Let $(\mathbb{I}, J, \omega)$ be the almost-Kähler Iwasawa manifold previously constructed. Then,*

- $h^{1,0}_{Dol} = 1$,
- $h^{2,0}_{Dol} = 1$,
- $h^{3,0}_{Dol} = 0$.

7 Obstructions to the existence of a compatible symplectic structure on an almost-complex manifold

Let (X, J) be an almost-complex manifold and fix a Hermitian metric g with fundamental form ω. Then, setting $\bar{\delta} := \bar{\partial} + \mu$ and $\delta := \partial + \bar{\mu}$ one can consider the following differential operators

$$\Delta_{\bar{\delta}} := \bar{\delta}\bar{\delta}^* + \bar{\delta}^*\bar{\delta}\,,$$

$$\Delta_{\delta} := \delta\delta^* + \delta^*\delta\,.$$

In [15] we studied Hodge theory for such operators, and even though they do not coincide in general, as a consequence of the almost-Kähler identities, if (X, J, g, ω) is an almost-Kähler manifold, then $\Delta_{\bar{\delta}}$ and Δ_{δ} are related by

$$\Delta_{\bar{\delta}} = \Delta_{\delta}\,.$$

In particular, their spaces of harmonic forms coincide, i.e. $\mathcal{H}^{\bullet}_{\delta}(X) = \mathcal{H}^{\bullet}_{\bar{\delta}}(X)$.
We can use now this result to prove an obstruction to the existence of a compatible symplectic structure on an almost-complex manifold.

Theorem 7.1. *Let (X, J) be a compact almost-complex manifold. Suppose that there exists $\varphi \in A^{1,0}(X)$ such that $\bar{\partial}\varphi = 0$ and $d\varphi \neq 0$. Then, there exists no compatible symplectic structure on (X, J).*

Proof. Since, $\bar{\partial}\varphi = 0$ then, for degree reasons $\varphi \in \operatorname{Ker}\Delta_{\bar{\delta}}$ for any arbitrary Hermitian metric. However, since $d\varphi \neq 0$ then, for any fixed Hermitian metric, $\varphi \notin \operatorname{Ker}\Delta_{\delta}$. Namely, $\Delta_{\bar{\delta}} \neq \Delta_{\delta}$ and the thesis follows, since, by [15] on almost-Kähler manifolds $\Delta_{\bar{\delta}} = \Delta_{\delta}$. ☐

An immediate corollary is the following

Corollary 7.2. *Let (X, J) be a compact almost-complex manifold such that there exists a global co-frame of $(1, 0)$-forms $\left\{\varphi^i\right\}$ such that, there exists an index j with*

$$d\varphi^j \in A^{2,0}(X) \oplus A^{0,2}(X)$$

and $d\varphi^j \neq 0$. Then, there exists no compatible symplectic structure on (X, J).

We apply this result to the following example.

Example 7.3. *Let $\mathbb{I}$ be the Iwasawa manifold defined as the quotient $\mathbb{I} := \Gamma\backslash\mathbb{H}_3$ where*

$$\mathbb{H}_3 := \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}$$

and

$$\Gamma := \left\{ \begin{bmatrix} 1 & \gamma_1 & \gamma_3 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix} \mid \gamma_1, \gamma_2, \gamma_3 \in \mathbb{Z}[\,i\,] \right\}.$$

Set $\psi^1 := d\bar{z}_1$, $\psi^2 := d\bar{z}_2$ $\psi^3 := d\bar{z}_3 - z_1 dz_2$. Hence, the structure equations are

$$d\psi^1 = 0, \quad d\psi^2 = 0, \quad d\psi^3 = -\psi^{\bar{1}\bar{2}},$$

therefore, by Corollary 7.2 the Iwasawa manifold with this almost-complex structure does not admit any compatible symplectic structure.

Clearly, the converse implication does not hold as we have seen in Section 5.

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