DE GRUYTER Complex Manifolds 2022; 9:261–284

Research Article Open Access

Maxence Mayrand*

Stratification of singular hyperkähler quotients

https://doi.org/10.1515/coma-2021-0140
Received January 7, 2022; accepted August 17, 2022

Abstract: Hyperkähler quotients by non-free actions are typically singular, but are nevertheless partitioned into smooth hyperkähler manifolds. We show that these partitions are topological stratifications, in a strong sense. We also endow the quotients with global Poisson structures which recover the hyperkähler structures on the strata. Finally, we give a local model which shows that these quotients are locally isomorphic to linear complex-symplectic reductions in the GIT sense. These results can be thought of as the hyperkähler analogues of Sjamaar–Lerman's theorems for singular symplectic reduction. They are based on a local normal form for the underlying complex-Hamiltonian manifold, which may be of independent interest.

Keywords: Hyperkähler quotient; moment map; complex symplectic reduction; geometric invariant theory; stratified space; local normal form

MSC: 53C26, 53D20, 57N80, 58A35, 32M05

1 Introduction

1.1 Overview

Let M be a hyperkähler manifold and K a compact Lie group acting on M by preserving the hyperkähler structure and with a hyperkähler moment map $\mu : M \to \mathfrak{k}^* \otimes \mathbb{R}^3$. If K acts freely, then the hyperkähler quotient

$$M /\!\!/\!\!/ K := \mu^{-1}(0)/K$$

is a smooth manifold endowed with a canonical hyperkähler structure [21].

If the K-action is not necessarily free, then $M /\!\!/\!\!/ K$ is typically singular, but it is still a union of smooth hyperkähler manifolds. This was first observed by Nakajima [34] for quiver varieties and, in general, by Dancer–Swann [5].

This decomposition of $M /\!\!/\!\!/ K$ into hyperkähler manifolds is an adaptation of the work of Sjamaar–Lerman [39] on singular symplectic reduction. On the other hand, singular symplectic reductions have much more structure than a union of symplectic manifolds. Indeed, if X is the symplectic reduction of a symplectic manifold by a non-free action of a compact Lie group, then Sjamaar–Lerman [39] also showed that

1. the symplectic manifolds whose union is X "fit together nicely" in the sense that they form a topological stratification of X (Definition 2.4),
2. there is a subalgebra $C^\infty(X)$ of the algebra of continuous functions on X together with a Poisson bracket which recover the symplectic structures on the strata, and
3. there is a local model for X generalizing the Darboux theorem.

***Corresponding Author: Maxence Mayrand:** Maxence Mayrand: Département de mathématiques, Université de Sherbrooke, E-mail: maxence.mayrand@usherbrooke.ca

The goal of this paper is to prove analogues of (1)–(3) for singular hyperkähler quotients $M/\!\!/\!\!/K$. To get these results, we first prove a local normal form for the underlying complex-Hamiltonian structure of M, analogous to the Guillemin–Sternberg–Marle local normal form for a real moment map [12, 29]. This result may be of independent interest.

1.2 Statements of results

We now give precise statements of our results.

A ***hyperkähler manifold*** is a tuple $(M, g, \mathsf{I}, \mathsf{J}, \mathsf{K})$, where M is a smooth manifold, g is a Riemannian metric on M, and $\mathsf{I}, \mathsf{J}, \mathsf{K}$ are three complex structures which are Kähler with respect to g and satisfy $\mathsf{IJK} = -1$. The corresponding Kähler forms will be denoted $\omega_\mathsf{I}, \omega_\mathsf{J}, \omega_\mathsf{K}$. We say that a ***tri-Hamiltonian hyperkähler manifold*** is a triple (M, K, μ), where M is a hyperkähler manifold, K is a compact Lie group acting on M by preserving the hyperkähler structure, and $\mu : M \to \mathfrak{k}^* \otimes \mathbb{R}^3$ is a hyperkähler moment map, where $\mathfrak{k} := \mathrm{Lie}(K)$.

Let (M, K, μ) be a tri-Hamiltonian hyperkähler manifold. The quotient $M /\!\!/\!\!/ K := \mu^{-1}(0)/K$ has a natural ***orbit-type partition***, whose pieces are the connected components of the subspaces $\mu^{-1}(0)_{(H)}/K$ for all subgroups $H \subseteq K$, where $\mu^{-1}(0)_{(H)}$ is the set of points with stabilizer conjugate to H in K. These pieces are, in fact, smooth hyperkähler manifolds:

Theorem 1.1 (Dancer–Swann [5]). *Let (M, K, μ) be a tri-Hamiltonian hyperkähler manifold, let $\pi : \mu^{-1}(0) \to M /\!\!/\!\!/ K$ be the quotient map, and let $S \subseteq M /\!\!/\!\!/ K$ be an orbit-type piece. Then:*

(i) *S is a topological manifold, $\pi^{-1}(S)$ is a smooth submanifold of M, and there is a unique smooth structure on S such that $\pi^{-1}(S) \to S$ is a smooth submersion.*
(ii) *There is a unique hyperkähler structure $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ on S such that the pullbacks of the Kähler forms $\omega_{\mathsf{I}_S}, \omega_{\mathsf{J}_S}, \omega_{\mathsf{K}_S}$ to $\pi^{-1}(S)$ are the restrictions of the Kähler forms $\omega_\mathsf{I}, \omega_\mathsf{J}, \omega_\mathsf{K}$ of M.*

Our formulation is slightly stronger than the one in Dancer–Swann [5] because of the uniqueness part in (ii). We will need this stronger version, so, for completeness, we provide a full proof of Theorem 1.1 in §2.5.

The question of whether this partition is a topological stratification (Definition 2.4) was left open in Dancer–Swann's work. One issue is that the arguments used by Sjamaar–Lerman [39] in the symplectic case is based on the local normal form for the moment map [12, 29], which has no hyperkähler equivalent.

To prove that it is indeed a topological stratification, we will assume that the K-action extends to a holomorphic $K_\mathbb{C}$-action with respect to one of the complex structures (where $K_\mathbb{C}$ is the complexification of K). In that case, we say that (M, K, μ) is ***integrable*** (or ***I-integrable*** if we need to specify the complex structure). This is a natural assumption in the context of Kähler or hyperkähler quotients and holds in most examples that one encounters (cf. Sjamaar [38], Heinzner–Loose [18], or Kaledin [23]). The terminology comes from the fact that it is equivalent to the completeness of the vector fields $\mathsf{I}x^\#$, for $x \in \mathfrak{k}$. For example, it holds if M is compact [12, Theorem 4.4] or a complex affine variety with a real algebraic action [16, p. 226]. We then obtain:

Theorem 1.2. *Let (M, K, μ) be an integrable tri-Hamiltonian hyperkähler manifold. Then the orbit-type partition of $M /\!\!/\!\!/ K$ is a topological stratification.*

In fact, we will show that it is a Whitney stratification (Definition 2.7) with respect to some complex-analytic structure.

The idea of the proof is to use the close relationship between hyperkähler geometry and complex-symplectic geometry. Namely, $M /\!\!/\!\!/ K$ is isomorphic to a symplectic reduction in the category of complex-analytic spaces, and we can adapt Sjamaar–Lerman's arguments to this setting.

More precisely, let $G := K_\mathbb{C}$ and suppose, without loss of generality, that the action is integrable with respect to the complex structure I. Let $\mu_\mathsf{I}, \mu_\mathsf{J}, \mu_\mathsf{K}$ be the three components of the hyperkähler moment map μ and let $\mu_\mathbb{R} := \mu_\mathsf{I}$ and $\mu_\mathbb{C} := \mu_\mathsf{J} + i\mu_\mathsf{K}$. Then $\mu_\mathbb{C} : M \to \mathfrak{g}^*$, where $\mathfrak{g} := \mathrm{Lie}(G)$, is a holomorphic moment map for the action of G on M with respect to the I-holomorphic complex-symplectic form $\omega_\mathbb{C} := \omega_\mathsf{J} + i\omega_\mathsf{K}$. Moreover, by letting

$$\begin{aligned} M^{\mu_{\mathbb{R}}\text{-ss}} &:= \{p \in M : \overline{G \cdot p} \cap \mu_{\mathbb{R}}^{-1}(0) \neq \emptyset\} \\ \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} &:= \mu_{\mathbb{C}}^{-1}(0) \cap M^{\mu_{\mathbb{R}}\text{-ss}}, \end{aligned}$$

we have $\mu^{-1}(0) \subseteq \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$ and, by a result of Heinzner–Loose [18], this inclusion descends to a homeomorphism

$$M /\!\!/\!\!/ K \cong \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G,$$

where $/\!\!/$ is a categorical quotient in the category of complex-analytic spaces (we will review Heinzner–Loose's work in §2.4). Thus, it suffices to get a local normal form for the complex part $\mu_{\mathbb{C}}$ of the moment map, and this is one of the main results of this paper.

To state this normal form, we first define a ***G*-saturated** subset of a G-space X to be a subset $A \subseteq X$ such that $\overline{G \cdot a} \subseteq A$ for all $a \in A$. Let $p \in \mu^{-1}(0)$ and let

$$V_p := (T_p(G \cdot p))^{\omega_{\mathbb{C}}} / T_p(G \cdot p),$$

where $(\cdot)^{\omega_{\mathbb{C}}}$ is the complex-symplectic complement. Then V_p is a complex-symplectic vector space on which the stabilizer $H := G_p$ acts linearly. The normal form says that the complex-Hamiltonian manifold $(M, \mathsf{I}, \omega_{\mathbb{C}}, G, \mu_{\mathbb{C}})$ is completely determined in a G-saturated neighbourhood of p by the representation of H on V_p. More precisely, the local model is the associated vector bundle $G \times_H (\mathfrak{h}^\circ \times V_p)$ (where $\mathfrak{h}^\circ$ is the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$), which has a canonical structure of a complex-Hamiltonian G-manifold (see §3.2). We then have:

Theorem 1.3. *Let (M, K, μ) be an I-integrable tri-Hamiltonian hyperkähler manifold. For all $p \in \mu^{-1}(0)$, there is a G-saturated neighbourhood of p in $M^{\mu_{\mathbb{R}}\text{-ss}}$ which is isomorphic as a complex-Hamiltonian G-manifold to a G-saturated neighbourhood of* $[1, 0, 0]$ *in $G \times_H (\mathfrak{h}^\circ \times V_p)$, where $H := G_p$ and $\mathfrak{h} := \mathrm{Lie}(H)$.*

See Losev [28] for a similar result in the algebraic setting.

This local form enables us to study the structure of the quotient:

Theorem 1.4. *Let (M, K, μ) be an I-integrable tri-Hamiltonian hyperkähler manifold. For each orbit-type stratum $S \subseteq M /\!\!/\!\!/ K$, let $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ be its hyperkähler structure as in* Theorem 1.1. *Let $\mu_{\mathbb{R}} := \mu_{\mathsf{I}}$ and $\mu_{\mathbb{C}} := \mu_{\mathsf{J}} + i\mu_{\mathsf{K}}$.*

(i) **Complex-analytic structure.** *The inclusion $\mu^{-1}(0) \subseteq \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$ descends to a homeomorphism $M /\!\!/\!\!/ K \cong \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G$ and hence $M /\!\!/\!\!/ K$ inherits the structure $\mathcal{O}_{\mathsf{I}}$ of a complex-analytic space. Moreover, the orbit-type partition is a complex-analytic Whitney stratification with respect to $\mathcal{O}_{\mathsf{I}}$ and is compatible with the complex structures I_S on the strata.*

(ii) **Holomorphic Poisson structure.** *There is a unique holomorphic Poisson bracket on $\mathcal{O}_{\mathsf{I}}$ such that the inclusion of each orbit-type piece $S \hookrightarrow M /\!\!/\!\!/ K$ is holomorphic Poisson with respect to $\omega_{\mathsf{J}_S} + i\omega_{\mathsf{K}_S}$.*

(iii) **Real Poisson structure.** *The first Kähler forms ω_{I_S} are compatible with a stratified symplectic structure in the sense of Sjamaar–Lerman, i.e. there is a subalgebra $C^\infty(M /\!\!/\!\!/ K)$ of the algebra of real-valued continuous functions, together with a Poisson bracket, such that the inclusion of each orbit-type piece $S \hookrightarrow M /\!\!/\!\!/ K$ is smooth and Poisson with respect to ω_{I_S}.*

(iv) **Local model.** *Let $q \in M /\!\!/\!\!/ K$. Take a point $p \in \mu^{-1}(0)$ above q, let $H := G_p$, let $V := (T_p(G \cdot p))^{\omega_{\mathbb{C}}} / T_p(G \cdot p)$, where $\omega_{\mathbb{C}} := \omega_{\mathsf{J}} + i\omega_{\mathsf{K}}$, and let $\Phi_V : V \to \mathfrak{h}^*$ be the moment map $\Phi_V(v)(x) = \frac{1}{2}\omega_{\mathbb{C}}(xv, v)$. Then H is a complex reductive group and q has a neighbourhood biholomorphic with respect to $\mathcal{O}_{\mathsf{I}}$ to a neighbourhood of* 0 *in the GIT quotient $\Phi_V^{-1}(0) /\!\!/ H = \mathrm{Spec}(\mathbb{C}[\Phi_V^{-1}(0)]^H)$. Moreover, this biholomorphism respects the orbit-type stratifications and holomorphic Poisson brackets on both sides.*

Remark 1.5. (i) and (iii) imply that $M /\!\!/\!\!/ K$ is a stratified Kähler space in the sense of Huebschmann [22, Definition 3.1].

Remark 1.6. Using Kempf–Ness type theorems, there are many situations where $M /\!/\!/ K$ is isomorphic to a GIT quotient $\mu_{\mathbb{C}}^{-1}(0)/\!/_{\mathcal{L}} G$ for some linearisation $\mathcal{L}$, i.e. when $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$ coincides with the set of $\mathcal{L}$-semistable points. In that case, the sheaf $\mathcal{O}_{|}$ is simply the underlying complex-analytic structure. For example, this is the case for hyperkähler quotients of T^*G by closed subgroups of $K \times K$ [33], where T^*G has the hyperkähler structure found by Kronheimer [26]. See also [32] for a study of the resulting partition in some family of examples.

Remark 1.7. The complex-symplectic GIT quotients $\Phi_V^{-1}(0)/\!/H$ which appear as local models in this theorem have been well studied in the literature. For example, it is known that if H is abelian then $\Phi_V^{-1}(0)/\!/H$ is normal [2]. In particular, Theorem 1.4(iv) implies that hyperkähler quotients by compact tori are normal.

1.3 Organization of the paper

In §2 we give background material on stratified spaces and reduction, in §3 we prove Theorem 1.3, and in §4 we prove Theorem 1.4.

2 Preliminaries

This section gives background material on stratified spaces, symplectic reduction, quotients of complex-analytic spaces, and the links between these notions. We start with a review of the theory of topologically stratified spaces and explain the work of Sjamaar–Lerman [39] on singular symplectic reduction. We then discuss links with complex-analytic geometry, reviewing work of Heinzner–Loose [18] and Sjamaar [38]. We also recall Dancer–Swann's construction [5] of the hyperkähler structures on the orbit-type pieces of a singular hyperkähler quotient and prove Theorem 1.1.

2.1 Stratified spaces

Stratified spaces are topological spaces which can be partitioned into manifolds which "fit together nicely". The underlying object for this theory is thus the following:

Definition 2.1. A ***partitioned space*** is a pair $(X, \mathcal{P})$ where X is a topological space and $\mathcal{P}$ is a partition of X, i.e. a collection of non-empty disjoint subsets of X whose union is X. The elements of $\mathcal{P}$ are called the ***pieces***. An ***isomorphism*** between two partitioned spaces $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ is a homeomorphism $f : X \to Y$ such that for all $S \in \mathcal{P}$ there exists $T \in \mathcal{Q}$ such that $f(S) = T$.

Just like manifolds are topological spaces satisfying additional conditions, stratified spaces are partitioned spaces with additional conditions imposed. The first step is the following notion.

Definition 2.2 ([7, §1.1])**.** A ***decomposed space*** is a partitioned space $(X, \mathcal{P})$ such that X is second countable and Hausdorff, and the following conditions hold:

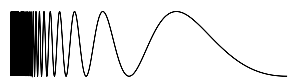
- **Manifold condition.** Each piece is a topological manifold in the subspace topology.
- **Local condition.** $\mathcal{P}$ is locally finite and its pieces are locally closed.
- **Frontier condition.** For all $S, T \in \mathcal{P}$, if $S \cap \overline{T} \neq \emptyset$ then $S \subseteq \overline{T}$.

Remark 2.3. If $(X, \mathcal{P})$ is a decomposed space, then there is a natural relation on $\mathcal{P}$ given by $S \leq T$ if $S \subseteq \overline{T}$. It follows from the local closedness of the strata that this relation is a partial order. Moreover, the frontier condition is equivalent to

$$\overline{S} = \bigcup_{T \leq S} T, \quad \text{for all } S \in \mathcal{P}.$$

This notion is sometimes incorporated in the definition of decomposed space, namely we fix a poset $\mathcal{I}$ and say that an $\mathcal{I}$-decomposed space is a topologically stratified space $(X, \mathcal{P})$ with an isomorphism $\mathcal{P} \cong \mathcal{I}$ of posets.

Decomposed spaces can be rather pathological: for example, the topologist's sine curve



is a perfectly valid one with two strata. Roughly speaking, stratified spaces avoid such pathologies by requiring that every point has a neighbourhood which retracts continuously onto it. We also impose that this neighbourhood is compatible with the partition in some sense. To make this precise, we need a few extra notions. First, the ***dimension*** of a decomposed space $(X, \mathcal{P})$ is

$$\dim(X, \mathcal{P}) := \sup\{\dim S : S \in \mathcal{P}\}.$$

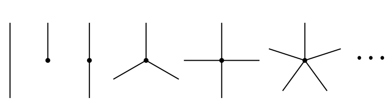
Given two partitioned spaces $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$, their ***cartesian product*** is the partitioned space $(X \times Y, \mathcal{P} \times \mathcal{Q})$ where $\mathcal{P} \times \mathcal{Q} = \{S \times T : S \in \mathcal{P}, T \in \mathcal{Q}\}$. If $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ are decomposed spaces, then so is $(X \times Y, \mathcal{P} \times \mathcal{Q})$, and $\dim(X \times Y, \mathcal{P} \times \mathcal{Q}) = \dim(X, \mathcal{P}) + \dim(Y, \mathcal{Q})$. Next, the ***cone*** over a partitioned space $(X, \mathcal{P})$ is the partitioned space $(CX, C\mathcal{P})$ where CX is the open cone over X, i.e.

$$CX := (X \times [0, \infty))/\{(p, 0) \sim (q, 0), \text{for all } p, q \in X\}$$

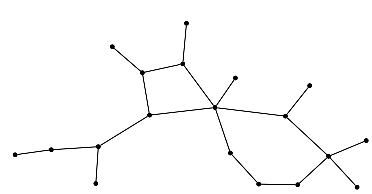
and $C\mathcal{P}$ is the natural partition of CX given by $C\mathcal{P} := \{S \times (0, \infty) : S \in \mathcal{P}\} \cup \{\text{vertex}\}$. The cone over a decomposed space $(X, \mathcal{P})$ is itself a decomposed space and has dimension $\dim(CX, C\mathcal{P}) = \dim(X, \mathcal{P}) + 1$. A topologically stratified space is defined inductively as a decomposed space $(X, \mathcal{P})$ which is locally isomorphic to $\mathbb{R}^n$ times a cone over a lower-dimensional topologically stratified space:

Definition 2.4 ([7, 39])**.** A zero-dimensional topologically stratified space is any countable set of points with the discrete topology and with any partition. A ***topologically stratified space*** is a finite-dimensional decomposed space $(X, \mathcal{P})$ such that every point $p \in X$ has a neighbourhood isomorphic as a partitioned space to $\mathbb{R}^n \times CL$ for some $n \geq 0$ and some compact topologically stratified space L, by a map sending $p \mapsto \{0\} \times \{\text{vertex}\}$.

For example, one-dimensional topologically stratified spaces are locally modelled on cones over finite sets of points, i.e. they are graphs:



one-dimensional local models



a one-dimensional topologically stratified space

Then two-dimensional topologically stratified spaces are locally modelled on cones over graphs, etc. Also, manifolds with corners are special cases.

A typical way of proving that a decomposed space $(X, \mathcal{P})$ is a topologically stratified space is by the Whitney conditions [43].

Definition 2.5. Let S and T be two disjoint smooth submanifolds of $\mathbb{R}^n$. We say that S is ***regular*** over T if the following two conditions hold:

- **Whitney Condition A.** If $x_i \in S$ is a sequence converging to $y \in T$ and the sequence of subspaces $T_{x_i}S \subseteq \mathbb{R}^n$ converges (in the Grassmannian) to $V \subseteq \mathbb{R}^n$, then $T_yT \subseteq V$.
- **Whitney Condition B.** If $x_i \in S$ and $y_i \in T$ are two sequences converging to $y \in T$ in such a way that that the sequence of lines $\mathbb{R}(x_i - y_i) \subseteq \mathbb{R}^n$ converges to $l \in \mathbb{RP}^{n-1}$ and the subspaces $T_{x_i}S$ to $V \subseteq \mathbb{R}^n$, then $l \subseteq V$.

A ***Whitney stratification*** of a subset X of $\mathbb{R}^n$ is a decomposition $\mathcal{P}$ of X into smooth submanifolds of $\mathbb{R}^n$ such that S is regular over T for all $S, T \in \mathcal{P}$.

We have (see e.g. Goresky–MacPherson [7, Ch. 1, §1.4] or Mather [31]):

Proposition 2.6. *A Whitney stratification is a topological stratification in the sense of* Definition 2.4. □

Although Whitney stratifications are initially defined in $\mathbb{R}^n$, the definition is purely local and is invariant under diffeomorphisms [31, §2]. In particular, it makes sense for complex-analytic spaces:

Definition 2.7. A ***complex-analytic Whitney stratified space*** is a complex-analytic space $(X, \mathcal{O}_X)$ together with a decomposition $\mathcal{P}$ of X into complex submanifolds satisfying Whitney conditions A and B.

Finally, we end this subsection by recalling the notion of a smooth structure on partitioned space (see e.g. [39, p. 380]).

Definition 2.8. A ***smooth structure*** on a partitioned space $(X, \mathcal{P})$ consists of a smooth structure on each strata and a subalgebra $C^\infty(X)$ of the algebra $C^0(X)$ of continuous functions such that for each $f \in C^\infty(X)$ and $S \in \mathcal{P}$, the restriction $f|_S : S \to \mathbb{R}$ is smooth. A continuous map $\varphi : X \to Y$ between partitioned space endowed with smooth structures is ***smooth*** if for all $f \in C^\infty(Y)$ we have $f \circ \varphi \in C^\infty(X)$.

2.2 Smooth manifold quotients

Let K be a compact Lie group acting smoothly on a smooth manifold M. Then the quotient M/K is a topologically stratified space with respect to a natural partition by orbit-types. To define this partition, for each subgroup $H \subseteq K$, let (H) be the conjugacy class of H in K. We say that $p \in M$ has ***orbit-type*** (H) if its stabilizer subgroup K_p is in (H). Denote the set of points of orbit-type (H) by

$$M_{(H)} := \{p \in M : K_p \in (H)\}.$$

The ***orbit-type partition*** is the partition whose pieces are the connected components of the sets $M_{(H)}/K$ for $H \subseteq K$. This is a topological stratification as a consequence of the slice theorem for proper group actions (see e.g. [6, Theorem 2.7.4]).

2.3 Stratified symplectic spaces

A ***Hamiltonian manifold*** is a triple (M, K, μ), where M is a symplectic manifold, K a compact Lie group acting on M by symplectomorphisms, and $\mu : M \to \mathfrak{k}^*$ a K-equivariant moment map. Sjamaar–Lerman [39] generalized the Marsden–Weinstein theorem [30] by showing that the orbit-type partition of $M /\!\!/_\mu K := \mu^{-1}(0)/K$ is a topological stratification (even though $\mu^{-1}(0)$ is not a smooth manifold) and that each piece has a canonical symplectic structure.

Moreover, these symplectic structures are compatible with a Poisson bracket on an appropriate substitute for the algebra of smooth functions. More precisely, let $C^\infty(M /\!\!/_\mu K)$ be $\mathbb{R}$-algebra of continuous functions on $M /\!\!/_\mu K$ which descend from smooth K-invariant functions on M. Then there is a unique Poisson bracket on $C^\infty(M /\!\!/_\mu K)$ such that the inclusion of each symplectic stratum $S \hookrightarrow M /\!\!/_\mu K$ is a smooth Poisson map.

Hence, the main result of [39] is that singular symplectic reductions are examples of the following notion.

Definition 2.9 (Sjamaar–Lerman [39]). A ***stratified symplectic space*** is a topologically stratified space $(X, \mathcal{P})$ with a symplectic structure on each stratum, a smooth structure $C^\infty(X)$, and a Poisson bracket on $C^\infty(X)$ such that for each stratum $S \in \mathcal{P}$ the inclusion $S \hookrightarrow X$ is a smooth Poisson map, i.e. for all $f, g \in C^\infty(X)$ the restrictions $f|_S, g|_S$ are smooth and $\{f|_S, g|_S\} = \{f, g\}|_S$.

We recall the construction of the symplectic forms on the orbit-type pieces of $M/\!\!/_\mu K$ [39, Theorem 3.5], as this will be useful for our discussion on hyperkähler quotients. For a closed subgroup $H \subseteq K$, let M_H be the set of points $p \in M$ whose stabilizer is precisely H. Then the connected components of M_H are smooth symplectic submanifolds of M (of possibly different dimensions) and the group $L := N_K(H)/H$ (where $N_K(H)$ is the normalizer of H in K) is compact and acts freely on M_H by preserving the symplectic forms. Now, $\mathfrak{l}^* := \mathrm{Lie}(L)^*$ can be identified with a subspace of $\mathfrak{k}^*$, namely, $\mathfrak{h}^\circ \cap (\mathfrak{k}^*)^H$, where $\mathfrak{h}^\circ$ is the annihilator of $\mathfrak{h} := \mathrm{Lie}(H)$ and $(\mathfrak{k}^*)^H$ is the set of points fixed by H. Moreover, if M_H' denotes the union of the connected components of M_H which intersect $\mu^{-1}(0)$, then μ restricts to a moment map $\mu_H : M_H' \to \mathfrak{l}^*$ for the action of L on M_H'. Since this action is free, each connected component of $M_H /\!\!/_{\mu_H} L = \mu_H^{-1}(0)/L$ is a smooth symplectic manifold by the standard Marsden–Weinstein theorem [30]. Then the inclusion $\mu_H^{-1}(0) \subseteq \mu^{-1}(0)_{(H)}$ descends to a homeomorphism $M_H/\!\!/_{\mu_H} L \cong \mu^{-1}(0)_{(H)}/K$, and this endows each connected component of $\mu^{-1}(0)_{(H)}/K$ with a symplectic structure. Furthermore, the pullback of each symplectic form to the corresponding connected component of $\mu^{-1}(0)_{(H)}$ (which is a smooth submanifold of M) is the restriction of the symplectic form of M.

2.4 Kähler quotients

A ***Hamiltonian Kähler manifold*** is a Hamiltonian manifold (M, K, μ) with a K-invariant Kähler structure compatible with the symplectic form. If K acts freely, then $M /\!\!/_\mu K$ has a Kähler structure compatible with the reduced symplectic form (see e.g. [21, Theorem 3.1]). More generally, each symplectic stratum in Sjamaar–Lerman's stratification is Kähler. To see this, it suffices to note that for each closed subgroup $H \subseteq K$, the space M_H of points with stabiliser H is now a complex submanifold of M and hence is Kähler. Thus, the connected components of $M_H /\!\!/_{\mu_H} L$ (where μ_H and L are as in §2.3) are Kähler manifolds, and the homeomorphism $M_H /\!\!/_{\mu_H} L \cong \mu^{-1}(0)_{(H)}/K$ gives the desired Kähler structures.

But we can say much more about the holomorphic aspect of $M /\!\!/_\mu K$ when the action is integrable, i.e. extends to a holomorphic action of $G := K_\mathbb{C}$, as shown by Sjamaar [38] and Heinzner–Loose [18]. Indeed, $M/\!\!/_\mu K$ is homeomorphic to a categorical quotient $M^{\mu\text{-ss}}/\!\!/G$ in the category of complex-analytic spaces, or more precisely, an analytic Hilbert quotient: the complex-analytic analogue of GIT quotients. Good expositions can be found in Heinzner–Huckleberry [15, 16] or Greb [9, §2–3]; we summarise the main points in this section. See also [41, §2.4] [8, §2] [10, §2] [9, §1] [14, §0] [19] [17] [18].

2.4.1 Analytic Hilbert quotients

Definition 2.10. Let $(X, \mathcal{O}_X)$ be a complex-analytic space and G a complex reductive group acting holomorphically on X. An ***analytic Hilbert quotient*** of X by G is a complex-analytic space $(Y, \mathcal{O}_Y)$ together with a G-invariant surjective holomorphic map $\pi : X \to Y$ such that:

(i) the map $\pi : X \to Y$ is ***locally Stein***, i.e. Y has a cover by Stein open sets whose preimages are Stein;
(ii) $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$.

An important consequence of this definition is that, if it exists, an analytic Hilbert quotient is a categorial quotient for complex-analytic spaces. In particular, it is unique up to biholomorphisms. We denote it

$$X/\!\!/G := \text{the analytic Hilbert quotient of } X \text{ by } G \text{ (if it exists).}$$

Topologically, $X/\!\!/G$ is the quotient of X by the equivalence relation $x \sim y$ if $\overline{G\cdot x} \cap \overline{G\cdot y} \neq \emptyset$ and $\pi : X \to X/\!\!/G$ is the corresponding quotient map. The space $X/\!\!/G$ can also be viewed as the set of closed G-orbits, i.e. by defining the set of ***polystable*** points

$$X^{\mathrm{ps}} := \{x \in X : \text{the orbit } G \cdot x \text{ is closed in } X\},$$

the inclusion $X^{\mathrm{ps}} \subseteq X$ descends to a bijection $X^{\mathrm{ps}}/G \to X/\!\!/G$. In particular, for every $p \in X/\!\!/G$, there is a unique closed G-orbit in the fibre $\pi^{-1}(p) \subseteq X$.

Example 2.11 (GIT quotients)**.** Let X be a complex affine variety, G a complex reductive group acting algebraically on X, and consider the affine GIT quotient $X/\!\!/G := \mathrm{Spec}(\mathbb{C}[X]^G)$ together with the morphism $X \to X/\!\!/G$ induced by the inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$. Then the analytification of $X \to X/\!\!/G$ is an analytic Hilbert quotient [13, §6.4]. More generally, since complex affine varieties are Stein, the analytification of any GIT quotient is an analytic Hilbert quotient.

We will later need the following properties of analytic Hilbert quotients:

Proposition 2.12. *Let $\pi : X \to X/\!\!/G$ be an analytic Hilbert quotient.*

(i) *An open set $U \subseteq X$ is G-saturated if and only if it is saturated with respect to π. In that case, $U/\!\!/G := \pi(U)$ is open in $X/\!\!/G$ and the restriction $U \to U/\!\!/G$ is an analytic Hilbert quotient.*
(ii) *If $Y \subseteq X$ is a G-invariant closed complex-analytic subspace, then $Y/\!\!/G := \pi(Y)$ is a closed complex-analytic subspace of $X/\!\!/G$ and the restriction $Y \to Y/\!\!/G$ is an analytic Hilbert quotient.* □

For (i) see [19, §2 Remark and §1 Corollary] and for (ii) see [19, §1(ii)].

2.4.2 Analytic quotient theorem due to Heinzner and Loose

Just as for GIT quotients, the question of existence of analytic Hilbert quotients is a subtle one. In complete analogy, for an action of a complex reductive group G on a complex-analytic space X, there does not always exist an analytic Hilbert quotient, but in good cases, one can find a large open subset of X on which the quotient exists. For GIT, this set depends on a choice of a linearisation, and for analytic Hilbert quotients, it depends on a choice of a moment map for the action of a maximal compact subgroup $K \subseteq G$, as we now explain.

Let (M, K, μ) be an integrable Hamiltonian Kähler manifold and let $G := K_{\mathbb{C}}$. Define the set of ***μ-semistable*** points by

$$M^{\mu\text{-ss}} := \{p \in M : \overline{G\cdot p} \cap \mu^{-1}(0) \neq \emptyset\}$$

and the set of ***μ-polystable*** points by

$$M^{\mu\text{-ps}} := \{p \in M : G \cdot p \text{ is closed in } M^{\mu\text{-ss}}\}.$$

Theorem 2.13 (Heinzner–Loose [18])**.** *The set $M^{\mu\text{-ss}}$ is a G-invariant open subset of M and the analytic Hilbert quotient $M^{\mu\text{-ss}}/\!\!/G$ exists. For all $p \in M$, we have*

$$p \in M^{\mu\text{-ps}} \iff G\cdot p \cap \mu^{-1}(0) \neq \emptyset. \tag{2.1}$$

Moreover, the inclusion $\mu^{-1}(0) \hookrightarrow M^{\mu\text{-ss}}$ descends to a homeomorphism $M/\!\!/_{\mu} K \to M^{\mu\text{-ss}}/\!\!/G$. Also, for every $p \in M^{\mu\text{-ps}}$ we have $G_p = (K_p)_{\mathbb{C}}$, so G_p is a complex reductive group. □

Remark 2.14. Special cases of Theorem 2.13 were known long before [18]. See, for example, Guillemin–Sternberg [11, §4] and Kirwan [25, §7.5]. It was also obtained independently by Sjamaar [38] under an additional assumption on the moment map. This result can be thought of as an “analytic” version of the Kempf–Ness theorem.

Remark 2.15. Heinzner–Loose [18] do not mention analytic Hilbert quotients directly, but the above theorem can be deduced from their proofs. The reformulation which we gave can be found in Heinzner–Huckleberry [14, §0].

The main ingredient in the proof of Heinzner–Loose's theorem is the Holomorphic Slice Theorem. We briefly review it here, since we will use it later. If H is a complex Lie subgroup of a complex Lie group G and S is a complex H-manifold, we denote by $G \times_H S$ the quotient of $G \times S$ by the H-action $h \cdot (g, x) = (gh^{-1}, h \cdot x)$. Since the H-action is free and proper, there is a unique complex manifold structure on $G \times_H S$ such that $G \times S \to G \times_H S$ is a holomorphic submersion.

Definition 2.16. Let G be a complex reductive group acting holomorphically on a complex manifold M. A ***holomorphic slice*** at a point p in M is a G_p-invariant complex submanifold $S \subseteq M$ containing p such that $G \cdot S$ is open in M and the map

$$G \times_{G_p} S \longrightarrow G \cdot S, \quad [g, x] \longmapsto g \cdot x$$

is a G-equivariant biholomorphism.

Theorem 2.17 (Holomorphic Slice Theorem [18, §2.7] [38, Theorem 1.12]). *Let (M, K, μ) be an integrable Hamiltonian Kähler manifold. Then there exists a holomorphic slice at every point $p \in M^{\mu\text{-ps}}$.* □

Remark 2.18. In [18], this is stated only for points $p \in M$ such that $\mu(p)$ is fixed by the coadjoint action, but since $M^{\mu\text{-ps}} = G \cdot \mu^{-1}(0)$ we deduce the above version.

This theorem enables us to study G-equivariant local properties of the complex manifold M near a closed orbit of $M^{\mu\text{-ss}}$ by the local model $G \times_{G_p} S$. This was used by Heinzner–Loose to prove the existence of the analytic Hilbert quotient.

2.4.3 Stratification of analytic Hilbert quotients

Let $\pi : X \to X /\!\!/ G$ be an analytic Hilbert quotient (e.g. a GIT quotient). Then as in §2.2, the orbit space X^{ps}/G has a partition by G-orbit-types, i.e. the pieces are the connected components of the sets $(X^{\text{ps}})_{(H)}/G$ for $H \subseteq G$. Then the bijection $X^{\text{ps}}/G \to X /\!\!/ G$ defines a natural partition on $X /\!\!/ G$ which we call the ***G-orbit-type partition***. Equivalently, the orbit-type of a point $p \in X /\!\!/ G$ is defined to be the orbit-type of the unique closed orbit in $\pi^{-1}(p)$.

If (M, K, μ) is a Hamiltonian Kähler manifold, then $M /\!\!/_\mu K \cong M^{\mu\text{-ss}} /\!\!/ G$ is an analytic Hilbert quotient and hence has a G-orbit-type partition. But it also has the K-orbit-type partition of Sjamaar–Lerman. Moreover, each stratum in the K-orbit-type partition is a Kähler manifold, and hence has a complex structure. The next result shows that these partitions and complex structures are the same.

Theorem 2.19 (Sjamaar [38, Theorem 2.10]).

(i) *The homeomorphism $M /\!\!/_\mu K \to M^{\mu\text{-ss}} /\!\!/ G$ is an isomorphism of partitioned spaces.*
(ii) *The G-orbit-type strata of $M^{\mu\text{-ss}} /\!\!/ G$ are complex submanifolds.*
(iii) *Let S be a K-orbit-type stratum in $M /\!\!/_\mu K$ and S' the corresponding G-orbit-type stratum in $M^{\mu\text{-ss}} /\!\!/ G$. Then the restriction $S \to S'$ is a biholomorphism with respect to Kähler structure on S and the complex structure on S' obtained from* (ii). □

Remark 2.20. (iii) is not stated in this way in [38], but is part of the proof.

Remark 2.21. As explained earlier, Sjamaar [38] obtained Heinzner–Loose's theorem (Theorem 2.13) independently, but under an additional assumption on the moment map which he called *admissibility*. He then stated Theorem 2.19 under the same assumption, but his proof relies only on the validity of Theorem 2.13 but not on the admissibility of the moment map.

2.5 Hyperkähler quotients

The goal of this section is to prove Theorem 1.1. We follow Dancer–Swann [5], refining slightly their arguments to get the uniqueness part of the theorem. The proof is similar to the construction of the Kähler structures on the orbit-type strata of a Kähler quotient explained in §2.4.

We use the notation and terminology of the introduction (§1.2). Let (M, K, μ) be a tri-Hamiltonian hyperkähler manifold. Then the orbit-type partition of the hyperkähler quotient $M/\!/\!/K$ is the partition whose pieces are the connected components of the subspaces $\mu^{-1}(0)_{(H)}/K$ for subgroups $H \subseteq K$. Let $S \subseteq \mu^{-1}(0)_{(H)}/K$ be such a piece. The set M_H of points with stabiliser H is now a hyperkähler submanifold of M and μ restricts to a hyperkähler moment map $\mu_H : M_H' \to \mathfrak{l}^* \otimes \mathbb{R}^3 \subseteq \mathfrak{k}^* \otimes \mathbb{R}^3$ for the free action of $L := N_K(H)/H$ on the union M_H' of the connected components of M_H intersecting $\mu^{-1}(0)$. Hence, the connected components of $M_H/\!/\!/L = \mu_H^{-1}(0)/L$ are smooth hyperkähler manifolds by the usual hyperkähler quotient construction [21, Theorem 3.2]. Moreover, the inclusion $\mu_H^{-1}(0) \subseteq \mu^{-1}(0)_{(H)}$ descends to a homeomorphism $M_H /\!/\!/ L \to \mu^{-1}(0)_{(H)}/K$ and hence endows each connected component of $\mu^{-1}(0)_{(H)}/K$ with a hyperkähler structure. To show that this map is indeed a homeomorphism and also to characterise the hyperkähler structures, we will need the following lemma. This result is implicit in Sjamaar–Lerman [39] and Dancer–Swann [5], but we give a short proof for completeness.

Lemma 2.22. *Let K be a compact Lie group acting smoothly on a smooth manifold M, let H be a closed subgroup of K, and let $L = N_K(H)/H$. Then M_H and $M_{(H)}$ are smooth submanifolds of M, and the quotients M_H/L and $M_{(H)}/K$ are topological manifolds with unique smooth structures such that the quotients maps $M_H \to M_H/L$ and $M_{(H)} \to M_{(H)}/K$ are smooth submersions. Moreover, the inclusion $M_H \hookrightarrow M_{(H)}$ descends to a diffeomorphism $M_H/L \to M_{(H)}/K$.*

Proof. This follows easily from the slice theorem for proper group actions. The map $M_H/L \to M_{(H)}/K$ is bijective, so everything reduces to local statements. Hence we may assume (by the slice theorem) that $M = K \times_H W$ for some representation W of H. Then $M_H = L \times W_H$, $M_{(H)} = K/H \times W_H$, and W_H is a linear subspace of W (the set of fixed points of H), so M_H and $M_{(H)}$ are smooth submanifolds of M. Moreover, $M_H/L = W_H$ and the quotient map $M_H \to M_H/L$ is the projection $L \times W_H \to W_H$ and hence is a smooth submersion. Similarly, the quotient map $M_{(H)} \to M_{(H)}/K$ is the projection $K/H \times W_H \to W_H$. Under these identifications, the map $M_H/L \to M_{(H)}/K$ is the identity map $W_H \to W_H$. □

Proof of Theorem 1.1. Let $S \subseteq M/\!/\!/K$ be an orbit-type piece. Let $Z := \mu^{-1}(0)$ so that S is a connected component of $Z_{(H)}/K$ for some $H \subseteq K$. As explained in the paragraph above Lemma 2.22, Z_H is a smooth submanifold of M_H and Z_H/L is a hyperkähler manifold, where $L := N_K(H)/K$. Now, Z_H/L is a smooth submanifold of M_H/L and its image under the diffeomorphism $M_H/L \to M_{(H)}/K$ is $Z_{(H)}/K$, so the latter is also smooth. Hence, by the transversality theorem applied to $M_{(H)} \to M_{(H)}/K$, $Z_{(H)}$ is a smooth submanifold of $M_{(H)}$ and $Z_{(H)} \to Z_{(H)}/K$ is a smooth submersion. Note that $\pi^{-1}(S)$ is open in $Z_{(H)}$, so it is also a smooth submanifold and the restriction $\pi^{-1}(S) \to S$ is a smooth submersion. Moreover, $\pi^{-1}(S)$ has pure dimension since S is connected and all fibres are diffeomorphic to K/H.

To prove the claim about the hyperkähler structure, let $\eta_{\mathsf{I}}, \eta_{\mathsf{J}}, \eta_{\mathsf{K}}$ be the Kähler forms on $Z_{(H)}/K$ induced by the diffeomorphism $Z_H/L \to Z_{(H)}/K$ and consider the commutative diagram

$$\begin{array}{ccccc} Z_H & \overset{i}{\hookrightarrow} & Z_{(H)} & \overset{j}{\hookrightarrow} & M \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} & & \\ Z_H/L & \xrightarrow{\varphi} & Z_{(H)}/K. & & \end{array}$$

We want to show that $\pi^\star\eta_\mathsf{I} = j^\star\omega_\mathsf{I}$ and similarly for J and K. By construction, we have $i^\star\pi^\star\eta_\mathsf{I} = \rho^\star\varphi^\star\eta_\mathsf{I} = i^\star j^\star\omega_\mathsf{I}$ and hence $\pi^\star\eta_\mathsf{I}$ and $j^\star\omega_\mathsf{I}$ agree on T_pZ_H for all $p \in Z_H$. Note that since $d\varphi_p$ and $d\rho_p$ are surjective we have $T_pZ_{(H)} = T_pZ_H + \ker d\pi_p$. Thus, to prove that $\pi^\star\eta_\mathsf{I}$ and $j^\star\omega_\mathsf{I}$ agree on $T_pZ_{(H)}$ it suffices to show that if $u \in \ker d\pi_p$ and $v \in T_pZ_{(H)}$ then $\pi^\star\eta_\mathsf{I}(u,v) = j^\star\omega_\mathsf{I}(u,v)$. Clearly, $\pi^\star\eta_\mathsf{I}(u,v) = 0$ since $d\pi_p(u) = 0$. To show that also $j^\star\omega_\mathsf{I}(u,v) = 0$, note that $\ker d\pi_p = T_p(K\cdot p)$ so $u = x_p^\#$ for some $x \in \mathfrak{k}$ and hence $\omega_\mathsf{I}(u,v) = i_{x^\#}\omega_\mathsf{I}(v) = d\langle\mu_\mathsf{I}, x\rangle(v) = 0$ since $v \in T_pZ_{(H)} \subseteq \ker(d\mu_\mathsf{I})_p$. Hence, $\pi^\star\eta_\mathsf{I}$ and $j^\star\omega_\mathsf{I}$ agree on $T_pZ_{(H)}$ for all $p \in Z_H$ and since they are K-invariant and $K\cdot Z_H = Z_{(H)}$ we conclude that $\pi^\star\eta_\mathsf{I} = j^\star\omega_\mathsf{I}$. The same argument also shows that $\pi^\star\eta_\mathsf{J} = j^\star\omega_\mathsf{J}$ and $\pi^\star\eta_\mathsf{K} = j^\star\omega_\mathsf{K}$. Since a hyperkähler structure is completely determined by its three symplectic forms (e.g. $\mathsf{I} = \omega_\mathsf{K}^{-1}\omega_\mathsf{J}$), this proves the proposition. □

3 A local normal form for the underlying complex-Hamiltonian manifold

3.1 Overview

The goal of this section is to prove Theorem 1.3, which establishes a local normal form for the underlying complex-Hamiltonian manifold of a tri-Hamiltonian hyperkähler manifold analogous to the local normal form of Guillemin–Sternberg [12] and Marle [29].

Throughout this section, (M, K, μ) is an I-integrable tri-Hamiltonian hyperkähler manifold and $G := K_\mathbb{C}$. Then $\omega_\mathbb{C} := \omega_\mathsf{J} + i\omega_\mathsf{K}$ is a complex-symplectic form on (M, I) and $\mu_\mathbb{C} := \mu_\mathsf{J} + i\mu_\mathsf{K} : M \to \mathfrak{g}^\star$ is a holomorphic moment map for the action of G on $(M, \mathsf{I}, \omega_\mathbb{C})$ (see [21, §3(D)]). We call $(M, \mathsf{I}, \omega_\mathbb{C}, G, \mu_\mathbb{C})$ the ***underlying complex-Hamiltonian manifold*** of (M, K, μ).

Let $\mu_\mathbb{R} := \mu_\mathsf{I}$ so that $(M, K, \mu_\mathbb{R})$ is an integrable Hamiltonian Kähler manifold as in §2.4. In particular, we have the sets $M^{\mu_\mathbb{R}\text{-ss}}$ and $M^{\mu_\mathbb{R}\text{-ps}}$ as in §2.4.2, and we will use the notations

$$\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} := \mu_\mathbb{C}^{-1}(0) \cap M^{\mu_\mathbb{R}\text{-ss}} \quad \text{and} \quad \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ps}} := \mu_\mathbb{C}^{-1}(0) \cap M^{\mu_\mathbb{R}\text{-ps}}.$$

The idea of the local normal form for the moment map is to show that in a neighbourhood of a point $p \in \mu^{-1}(0)$, the underlying complex-Hamiltonian manifold $(M, \mathsf{I}, \omega_\mathbb{C}, G, \mu_\mathbb{C})$ is completely determined by the representation of $H := G_p$ on the ***complex-symplectic slice***

$$V := (T_p(G\cdot p))^{\omega_\mathbb{C}}/T_p(G\cdot p).$$

By the Holomorphic Slice Theorem 2.17, the orbit $G\cdot p$ is an immersed submanifold of M, so the tangent space $T_p(G\cdot p)$ is well-defined. We have $T_p(G\cdot p) \subseteq \ker(d\mu_\mathbb{C})_p = (T_p(G\cdot p))^{\omega_\mathbb{C}}$ (see e.g. [37, p. 168]) and hence V is a well-defined complex-symplectic vector space. Moreover, $H := G_p = (K_p)_\mathbb{C}$ (by Theorem 2.13), so H is a complex reductive group acting linearly on V and preserving its complex-symplectic form. In other words, p determines a complex-symplectic representation $\rho : H \to \mathrm{Sp}(V, \omega_\mathbb{C})$. The goal of Theorem 1.3 is to construct a complex-Hamiltonian manifold E from G and ρ which is isomorphic to a neighbourhood of p in $(M, \mathsf{I}, \omega_\mathbb{C}, G, \mu_\mathbb{C})$. The construction of E is the same as the one used by Guillemin–Sternberg [12], but in a complex-symplectic setting; see also [39, §2] and [28].

In §3.2 we recall the construction of the local model E. In §3.3, we prove a holomorphic version of the Darboux–Weinstein theorem which we will need for the proof of Theorem 1.3. In §3.4, we reformulate the Holomorphic Slice Theorem in a way that is more suitable for our purpose. Finally, we prove Theorem 1.3 in §3.5.

3.2 The local model

Let G be a complex reductive group, H a reductive subgroup of G, and $(V, \omega_\mathbb{C})$ a complex-symplectic representation of H. Then $T^\star G$ has the canonical complex-symplectic form $-d\theta$, where θ is the tautological holomorphic 1-form. We identify T^*G with $G \times \mathfrak{g}^*$ via left translation, i.e. via the biholomorphism

$$G \times \mathfrak{g}^* \longrightarrow T^*G, \quad (g, \xi) \longmapsto (dL_{g^{-1}})^*(\xi), \tag{3.1}$$

where $L_{g^{-1}} : G \to G$ is left multiplication by g^{-1}. Recall that a Lie group action on any manifold lifts to a Hamiltonian action on the cotangent bundle. By considering the action of $G \times G$ on G by left and right multiplications (i.e. $(a, b) \cdot g := agb^{-1}$) its lift to $T^*G = G \times \mathfrak{g}^*$ is $(a, b) \cdot (g, \xi) = (agb^{-1}, \mathrm{Ad}_b^* \xi)$, and the moment map is

$$T^*G \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*, \quad (g, \xi) \longmapsto (\mathrm{Ad}_g^* \xi, -\xi) \tag{3.2}$$

(see e.g. [1, §4.4]). The representation $H \to \mathrm{Sp}(V, \omega_{\mathbb{C}})$ can also be viewed as a complex-Hamiltonian H-manifold with moment map $\Phi_V : V \to \mathfrak{h}^*$, $\Phi_V(v)(x) = \frac{1}{2}\omega_{\mathbb{C}}(xv, v)$. Thus, there is a Hamiltonian action of H on $T^*G \times V$, where H acts on T^*G as the subgroup $1 \times H \subseteq G \times G$ and on V by the given representation. Let E be the complex-symplectic reduction of $T^*G \times V$ by H. Since the action of H on $T^*G \times V$ is free and proper, E is a complex-symplectic manifold. Moreover, the Hamiltonian action of $G \times 1$ on T^*G descends to a Hamiltonian action of G on E, making E into a complex-Hamiltonian G-manifold.

We can also rewrite E in a more convenient form where the complex moment map for the G-action is explicit. First, note that the complex moment map for the action of H on $T^*G \times V$ is

$$\lambda : T^*G \times V \longrightarrow \mathfrak{h}^*, \quad \lambda(g, \xi, v) = \Phi_V(v) - \xi|_{\mathfrak{h}}.$$

Take a Hermitian inner-product on $\mathfrak{g}$ invariant under the maximal compact subgroup $K \subseteq G$ and let $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. This defines an H-equivariant isomorphism $\mathfrak{h}^* \cong \mathfrak{m}^\circ \subseteq \mathfrak{g}^*$ so we can view Φ_V as taking values in $\mathfrak{g}^*$. Then the map

$$G \times \mathfrak{h}^\circ \times V \longrightarrow \lambda^{-1}(0), \quad (g, \xi, v) \longmapsto (g, \xi + \Phi_V(v), v)$$

is a biholomorphism since it has an inverse $(g, \xi, v) \mapsto (g, \xi - \Phi_V(v), v)$. The H-action on $\lambda^{-1}(0)$ corresponds to the H-action on $G \times \mathfrak{h}^\circ \times V$ given by $h \cdot (g, \xi, v) = (gh^{-1}, \mathrm{Ad}_h^* \xi, h \cdot v)$, so E is the holomorphic vector bundle

$$E = G \times_H (\mathfrak{h}^\circ \times V) \tag{3.3}$$

over G/H. In this setup, the Hamiltonian G-action is

$$G \times E \longrightarrow E, \quad a \cdot [g, \xi, v] = [ag, \xi, v] \tag{3.4}$$

and the complex moment map is

$$\kappa : G \times_H (\mathfrak{h}^\circ \times V) \longrightarrow \mathfrak{g}^*, \quad [g, \xi, v] \longmapsto \mathrm{Ad}_g^*(\xi + \Phi_V(v)). \tag{3.5}$$

We summarise this discussion in the following proposition.

Proposition 3.1. *Let G be a complex reductive group, H a reductive subgroup of G, and V a complex-symplectic representation of H. Then the complex-symplectic manifold* (3.3) *with the action* (3.4) *and moment map* (3.5) *is a complex-Hamiltonian manifold.* □

Remark 3.2. Dancer–Swann [4] showed that E is a tri-Hamiltonian hyperkähler manifold whose underlying complex-Hamiltonian manifold is the one described above.

3.3 Holomorphic Darboux–Weinstein Theorem

The Darboux–Weinstein theorem [42] is a standard result in symplectic geometry which says that if two symplectic forms ω_0 and ω_1 on a manifold M agree on a submanifold $N \subseteq M$, then we can find a diffeomorphism f on a neighbourhood of N such that $f^*\omega_1 = \omega_0$. There is also an equivariant version of the theorem, where if ω_0, ω_1 and N are invariant under the action of a compact Lie group, then f can be taken to be equivariant. By the tubular neighbourhood theorem, it suffices to prove the result when M is a vector bundle and N the zero-section, and this is indeed how Weinstein's original proof goes [42]. In the holomorphic setting, there is no tubular neighbourhood theorem, but we can still adapt Weinstein's proof to formulate a similar statement on holomorphic vector bundles:

Theorem 3.3. *Let G be a group acting on a holomorphic vector bundle E by bundle automorphisms (not necessarily fixing the base). Let ω_0 and ω_1 be two G-invariant holomorphic symplectic forms on a G-invariant neighbourhood U of the zero-section $Z \subseteq E$ such that $\omega_0|_Z = \omega_1|_Z$. Then there are G-invariant neighbourhoods U_0 and U_1 of Z in U and a G-equivariant biholomorphism $f : U_0 \to U_1$ such that $f^\star\omega_1 = \omega_0$ and $f|_Z = \mathrm{Id}_Z$.*

Remark 3.4. Here $\omega_i|_Z$ is the restriction of ω_i to $(\Lambda^2 T^\star E)|_Z$ (this is not the same as the pullback to Z).

The rest of this subsection is devoted to the proof of this theorem. Let us first briefly sketch how we will proceed. The first step is to get a "Poincaré lemma" for the retraction of U onto Z, i.e. to construct an explicit homotopy operator $I : \Omega^k(U) \to \Omega^{k-1}(U)$ between the identity map and $\pi^\star$, where $\pi : U \to U, v \mapsto 0 \cdot v$. Then $\alpha = I(\omega_0 - \omega_1)$ is a 1-form on U and, for t small enough, $\omega_t := \omega_0 + t(\omega_1 - \omega_0)$ is non-degenerate, so we get a time-dependent holomorphic vector field $X_t = \omega_t^{-1}(\alpha)$ on a neighbourhood of Z. The proof concludes by showing that the time-dependent flow of X gives a biholomorphism with the desired properties.

Let us now construct the homotopy operator. Let $\overline{\mathbb{D}}$ be the closed unit disc centred at 0 in $\mathbb{C}$ and let $U \subseteq E$ be as in Theorem 3.3. By shrinking U if necessary, we may assume that it is preserved by $\overline{\mathbb{D}}$, i.e. $zu \in U$ for all $z \in \overline{\mathbb{D}}$ and $u \in U$. Let

$$W := \{(z, u) \in \mathbb{C} \times U : zu \in U\}.$$

Then W is open in $\mathbb{C} \times U$ and $\overline{\mathbb{D}} \times U \subseteq W$. Let

$$\lambda : W \longrightarrow U, \quad (z, u) \longmapsto zu,$$

and for each $z \in \overline{\mathbb{D}}$ let

$$\iota_z : U \longrightarrow W, \quad u \longmapsto (z, u).$$

Let $\Omega^k(U)$ be the space of holomorphic k-forms on U. Then, for all $\omega \in \Omega^k(U)$, we have a family of holomorphic $(k-1)$-forms $\iota_z^\star i_{\partial_z}\lambda^\star\omega \in \Omega^{k-1}(U)$ depending holomorphically on $z \in \bar{\mathbb{D}}$, where i_{∂_z} is the interior product with the vector field $\partial_z := \frac{\partial}{\partial z}$ on $W \subseteq \mathbb{C} \times E$. Hence, we have a linear operator

$$I : \Omega^k(U) \longrightarrow \Omega^{k-1}(U), \quad I\omega := \int_0^1 (\iota_z^\star i_{\partial_z}\lambda^\star\omega)dz.$$

Let $\pi : U \to U$ be the projection onto the zero-section.

Proposition 3.5. *We have $d(I\omega) + I(d\omega) = \omega - \pi^\star\omega$ for all $\omega \in \Omega^k(U)$, i.e. I is a homotopy operator between the identity map and $\pi^\star$.*

Proof. We have

$$d(I\omega) + I(d\omega) = \int_0^1 \iota_z^\star(di_{\partial_z}\lambda^\star\omega + i_{\partial_z}d\lambda^\star\omega)dz = \int_0^1 (\iota_z^\star\mathcal{L}_{\partial_z}\lambda^\star\omega)dz.$$

Now, the flow θ_t of ∂_z is $\theta_t(z, u) = (z + t, u) = \iota_{z+t}(u)$, so $\theta_t \circ \iota_0 = \iota_t$. Hence $\iota_t^\star\mathcal{L}_{\partial_z}\lambda^\star\omega = \iota_0^\star\theta_t^\star\mathcal{L}_{\partial_z}\lambda^\star\omega = \iota_0^\star\frac{d}{dt}\theta_t^\star\lambda^\star\omega = \frac{d}{dt}\iota_0^\star\theta_t^\star\lambda^\star\omega$, so we get

$$d(I\omega) + I(d\omega) = \int_0^1 \frac{d}{dt}\iota_0^\star\theta_t^\star\lambda^\star\omega\, dt = \iota_0^\star\theta_1^\star\lambda^\star\omega - \iota_0^\star\theta_0^\star\lambda^\star\omega = \omega - \pi^\star\omega. \qquad \square$$

The following observation will be useful.

Lemma 3.6. *Let $\omega \in \Omega^k(U)$ and let $p \in Z$. If $\omega_p = 0$ then $(I\omega)_p = 0$.*

Proof. For all $v \in (T_pU)^{k-1}$, we have $(\iota_z^\star i_{\partial_z}\lambda^\star\omega)_p(v) = \omega_{zp}(d\lambda(\partial_z), d\lambda(d\iota_z(v))) = 0$, where the last equality follows from the fact that $zp = p$ (since $p \in Z$) and the assumption that $\omega_p = 0$. Thus, $(\iota_z^\star i_{\partial_z}\lambda^\star\omega)_p = 0$ for all z and hence $(I\omega)_p = 0$. □

Proof of Theorem 3.3. Let $\eta = \omega_1 - \omega_0$ and let $\alpha = -I\eta \in \Omega^1(U)$. Then $\eta = -d\alpha$ by Proposition 3.5. Since η is G-invariant, it follows from the definition of I that α is also G-invariant. Moreover, since $\eta|_Z = 0$ we have $\alpha|_Z = 0$ by Lemma 3.6.

For each $z \in \mathbb{C}$, define a G-invariant holomorphic 2-form on U by $\omega_z = \omega_0 + z\eta$. We have $\omega_z|_Z = \omega_0|_Z$, so in particular, $\omega_z|_p$ is non-degenerate for all $(z,p) \in \mathbb{C} \times Z$. Let $\mathbb{D}_r$ be the open disc of radius r centred at 0 in $\mathbb{C}$. By compactness of $\overline{\mathbb{D}}_2$, we can find a neighbourhood $U' \subseteq U$ of Z such that $\omega_z|_p$ is non-degenerate for all $(z,p) \in \overline{\mathbb{D}}_2 \times U'$. Moreover, by G-invariance of ω_z, we can take U' to be G-invariant. Thus, we may assume that $\omega_z|_p$ is non-degenerate for all $(z,p) \in \overline{\mathbb{D}}_2 \times U$. In particular, the maps

$$\hat{\omega}_z : TU \longrightarrow T^\star U, \quad v \longmapsto \omega_z(v, \cdot)$$

are vector bundle isomorphisms for all $z \in \overline{\mathbb{D}}_2$. Define a holomorphic family of vectors fields on U by

$$X : \mathbb{D}_2 \times U \longrightarrow TU, \quad (z,p) \longmapsto (\hat{\omega}_z)^{-1}(\alpha_p).$$

Let $J = \mathbb{D}_2 \cap \mathbb{R} = (-2, 2)$ and let $\psi : \mathcal{E} \to U$ be the smooth time-dependent flow of the restriction $X|_{J\times U}$. That is, $\mathcal{E}$ is the open subset of $J \times J \times M$ such that for all $(t_0, p) \in J \times M$, the map $\psi^{(t_0,p)}(t) := \psi(t, t_0, p)$ is the maximally extended integral curve of $X|_{J\times U}$ starting at (t_0, p). From the general theory of smooth time-dependent flows (see e.g. [27, Theorem 9.48]), for all $(t_1, t_0) \in J \times J$ the set

$$U_{(t_1,t_0)} := \{p \in U : (t_1, t_0, p) \in \mathcal{E}\}$$

is open, and the map

$$\psi_{(t_1,t_0)} : U_{(t_1,t_0)} \longrightarrow U_{(t_0,t_1)}, \quad p \longmapsto \psi(t_1, t_0, p)$$

is a diffeomorphism. Moreover, since X is holomorphic, $\psi_{(t_1,t_0)}$ is a biholomorphism (this follows from the holomorphic dependence of solutions to linear systems of ODEs on the initial conditions; see e.g. [3, Ch. 1, §8]). Since $\alpha|_Z = 0$ we have $X_{(t_0,p)} = 0$ for all $(t_0, p) \in J \times Z$, and hence $\psi(t_1, t_0, p) = p$ for all $(t_1, t_0, p) \in J \times J \times Z$. In particular, $J \times J \times Z \subseteq \mathcal{E}$, so $U_{(1,0)}$ and $U_{(0,1)}$ contain Z. We claim that the biholomorphism $\psi_{1,0} : U_{1,0} \to U_{0,1}$ is the one we need. First, since α and ω_z are G-invariant, so is X. Hence, $U_{1,0}$ and $U_{0,1}$ are G-invariant, and $\psi_{1,0}$ is G-equivariant. Moreover, from [27, Proposition 22.15] we have for all $t_1 \in J$,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_1} \psi_{t,0}^\star \omega_t = \psi_{t_1,0}^\star \left(\mathcal{L}_{X_{t_1}} \omega_{t_1} + \frac{d}{dt}\Big|_{t=t_1} \omega_t \right) &= \psi_{t_1,0}^\star \left(i_{X_{t_1}} d\omega_t + d i_{X_{t_1}} \omega_{t_1} + \eta \right) \\ &= \psi_{t_1,0}^\star (d\alpha + \eta) = 0.\end{aligned}$$

Thus, $\psi_{1,0}^\star \omega_1 = \psi_{0,0}^\star \omega_0 = \omega_0$. □

3.4 Linearisation of the Holomorphic Slice Theorem

In this subsection, we explain how to put the Holomorphic Slice Theorem 2.17 in a form which will be more convenient for our purpose. First, we want to linearise the slice and realise neighbourhoods of orbits in M as neighbourhoods of zero-sections of vector bundles.

Let (M, K, μ) be an integrable Hamiltonian Kähler manifold, let $p \in M^{\mu\text{-ps}}$, let $G := K_{\mathbb{C}}$, let $H := G_p$, and let

$$W := T_pM / T_p(G \cdot p).$$

Proposition 3.7. *There is an open ball B centred at 0 in W, a G-invariant neighbourhood U of p in M, and a G-equivariant biholomorphism $G \times_H (H \cdot B) \longrightarrow U$ mapping $[1,0]$ to p.*

Proof. This is an intermediate step in Sjamaar's proof of the Holomorphic Slice Theorem: see the top of p. 101 in [38]. It can also be proved by linearising the action of G_p on the slice S at p [38, Theorem 1.21]. □

It will be important to know that the open set U of the preceding proposition can be taken to be G-saturated. First, we have:

Proposition 3.8. *Every G-invariant neighbourhood of p contains a neighbourhood of p which is G-saturated in $M^{\mu\text{-ss}}$.*

Proof. Our argument is similar to [20, Remark 14.24]. As observed in [19, Remark 1.1], the quotient map $\pi : M^{\mu\text{-ss}} \to M^{\mu\text{-ss}} /\!\!/ G$ sends G-invariant closed subsets to closed subsets. Let U be a G-invariant neighbourhood of p in $M^{\mu\text{-ss}}$. Then $C := M^{\mu\text{-ss}} \setminus U$ is a G-invariant closed subset of $M^{\mu\text{-ss}}$, so $\pi(C)$ is closed in $M^{\mu\text{-ss}} /\!\!/ G$. Moreover, since $G \cdot p$ is closed in $M^{\mu\text{-ss}}$, we have $\pi(p) \notin \pi(C)$. Hence, $\pi^{-1}(M^{\mu\text{-ss}} /\!\!/ G \setminus \pi(C))$ is a G-saturated neighbourhood of p contained in U. □

The set $H \cdot B$ in Proposition 3.7 is also H-saturated [40, Corollary 4.9] and it follows that $G \times_H (H \cdot B)$ is G-saturated in $G \times_H W$. We can then restate the Holomorphic Slice Theorem in the following form:

Theorem 3.9. *There is a G-saturated neighbourhood U of p in $M^{\mu\text{-ss}}$, a G-saturated neighbourhood U' of the zero-section of the vector bundle $G \times_H W$, and a G-equivariant biholomorphism $U' \to U$ mapping $[1,0]$ to p.*

Proof. Let $\varphi : G \times_H (H \cdot B) \to U$ be the biholomorphism of Proposition 3.7. By Proposition 3.8, there is a G-saturated neighbourhood U' of p contained in U. Let $B' \subseteq B$ be an open ball sufficiently small so that $U'' := \varphi(G \times_H (H \cdot B')) \subseteq U'$. Then U'' is G-saturated. □

3.5 Proof of the Complex-Hamiltonian Local Normal Form

We now complete the proof of Theorem 1.3. The first step is to have an explicit expression for the complex-symplectic form $\eta_{\mathbb{C}}$ of the local model $E = G \times_H (\mathfrak{h}^\circ \times V)$ at the point $q = [1,0,0]$. Note that $G_q = H$, so H acts linearly on $T_q E$. Since the G-action is Hamiltonian, this is a complex-symplectic representation of H on $T_q E$. Recall that $\mathfrak{m} \subseteq \mathfrak{g}$ is the orthogonal complement to $\mathfrak{h}$.

Proposition 3.10. *We have $T_q E \cong \mathfrak{m} \times \mathfrak{m}^* \times V$ as complex-symplectic H-representations, where $\mathfrak{m} \times \mathfrak{m}^*$ has the canonical symplectic form $((x,\varphi),(y,\psi)) \mapsto \psi(x) - \varphi(y)$. Moreover, $T_q(G \cdot q) \cong \mathfrak{m} \times 0 \times 0$ under the isomorphism $T_q E \cong \mathfrak{m} \times \mathfrak{m}^* \times V$.*

Proof. The canonical symplectic form on $T^* G = G \times \mathfrak{g}^*$ at $T_{(g,\xi)}(T^* G) = \mathfrak{g} \times \mathfrak{g}^*$ is

$$((x,\varphi),(y,\psi)) \longmapsto \psi(x) - \varphi(y) + \xi([x,y]) \tag{3.6}$$

(see e.g. [1, Proposition 4.4.1]). In particular, if $\hat{q} := (1,0,0) \in T^* G \times V$, the symplectic form on $T^* G \times V$ at $T_{\hat{q}}(T^* G \times V) = \mathfrak{g} \times \mathfrak{g}^* \times V$ is

$$((x,\varphi,u),(y,\psi,v)) \mapsto \psi(x) - \varphi(y) + \omega_{\mathbb{C}}(u,v).$$

Now, we have $d\lambda_{\hat{q}}(x,\xi,v) = -\xi|_{\mathfrak{h}}$, so the tangent space to $\lambda^{-1}(0)$ at $\hat{q}$ is $\mathfrak{g} \times \mathfrak{h}^\circ \times V$. Moreover, $T_{\hat{q}}(H \cdot \hat{q}) = \mathfrak{h} \times 0 \times 0$, so

$$T_q(\lambda^{-1}(0)/H) = T_{\hat{q}} \lambda^{-1}(0) / T_{\hat{q}}(G \cdot \hat{q}) = \mathfrak{g}/\mathfrak{h} \times \mathfrak{h}^\circ \times V.$$

Identifying $\mathfrak{g}/\mathfrak{h}$ with $\mathfrak{m}$ and $\mathfrak{h}^\circ$ with $\mathfrak{m}^*$ gives the result. □

Lemma 3.11. *Let $H \to \mathrm{Sp}(R,\omega)$ be a complex-symplectic representation and $S \subseteq R$ an H-invariant isotropic subspace. Then $R/S \cong S^* \times S^\omega / S$ as H-modules.*

Proof. Let $R \to S^\star$ be the composition of the isomorphism $R \to R^\star$ induced by ω with the restriction map $R^\star \to S^\star$. Let $R \to S^\omega$ be the projection along the H-invariant complement of S^ω in R (by complete reducibility). These maps give an H-equivariant surjective map $R \to S^\star \times S^\omega/S$ with kernel $S^\omega \cap S$. Since S is isotropic, we have $S^\omega \cap S = S$. ☐

Proof of Theorem 1.3. Since $T_p(G{\cdot}p) \cong \mathfrak{g}/\mathfrak{h}$ is isotropic in T_pM, Lemma 3.11 implies that $T_pM/T_p(G{\cdot}p) \cong \mathfrak{h}^\circ \times V$, where V is the complex-symplectic slice at p. Thus, by Theorem 3.9, there is a G-saturated neighbourhood of p in $M^{\mu_\mathbb{R}\text{-ss}}$ which is G-equivariantly biholomorphic to a G-saturated neighbourhood of $q = [1, 0, 0]$ in $E = G \times_H (\mathfrak{h}^\circ \times V)$. Note that by Proposition 3.10, $T_q(G{\cdot}q)$ is also isotropic with respect to the canonical complex-symplectic form on E. Note also that any G-invariant neighbourhood of the zero-section $0_E = G{\cdot}q$ of E contains a G-saturated neighbourhood, namely $G \times_H (H \cdot B)$ for a sufficiently small open ball B. Hence, it suffices to show that, for any two G-invariant complex-symplectic forms $\omega_\mathbb{C}$ and $\eta_\mathbb{C}$ on a G-invariant neighbourhood of the zero-section $0_E = G \cdot q$ in E such that T_q0_E is isotropic with respect to both, there is a G-equivariant biholomorphism on a possibly smaller neighbourhood of 0_E which pulls back $\eta_\mathbb{C}$ to $\omega_\mathbb{C}$. By the holomorphic version of the Darboux–Weinstein theorem given in Theorem 3.3, it suffices to find such a biholomorphism that makes them match on 0_E. This can be reduced to a linear algebraic problem, as we now explain. The proof is inspired from [28, Proposition 2].

Since $T_q0_E \subseteq T_qE$ is isotropic with respect to both $\omega_\mathbb{C}$ and $\eta_\mathbb{C}$, [28, Lemma 6] says that there exists an H-equivariant linear isomorphism $\varphi : T_qE \to T_qE$ which restricts to the identity on T_q0_E and such that $\varphi^\star\eta_\mathbb{C} = \omega_\mathbb{C}$. We have $T_qE = \mathfrak{m} \times \mathfrak{m}^\star \times V$ and $T_q0_E = \mathfrak{m} \times 0 \times 0$, so φ is of the form

$$\varphi : \mathfrak{m} \times \mathfrak{m}^\star \times V \longrightarrow \mathfrak{m} \times \mathfrak{m}^\star \times V, \quad \varphi(x, \xi, v) = (x + A(\xi, v), B(\xi, v)),$$

where $A : \mathfrak{m}^\star \times V \to \mathfrak{m}$ and $B : \mathfrak{m}^\star \times V \to \mathfrak{m}^\star \times V$ are some linear maps, with B invertible. Then

$$\psi : E \longrightarrow E, \quad \psi([g, \xi, v]) = [ge^{A(\xi,v)}, B(\xi, v)]$$

is a G-equivariant biholomorphism with $d\psi_q = \varphi$. In particular, $\psi^\star\eta_\mathbb{C}|_q = \omega_\mathbb{C}|_q$ and, since $\omega_\mathbb{C}$ and $\eta_\mathbb{C}$ are G-invariant and ψ is G-equivariant, this implies that $\psi^\star\eta_\mathbb{C}|_{g\cdot q} = \omega_\mathbb{C}|_{g\cdot q}$ for all $g \in G$, i.e. $\psi^\star\eta_\mathbb{C}|_{0_E} = \omega_\mathbb{C}|_{0_E}$.

We can now apply Theorem 3.3, which shows the existence of a G-equivariant complex-symplectic isomorphism $f : U \to U'$ such that $f(p) = q$, where U is a G-saturated neighbourhood of p in $M^{\mu_\mathbb{R}\text{-ss}}$ and U' a G-saturated neighbourhood of q in E. It remains to show that $\kappa \circ f = \mu_\mathbb{C}$. Since $(\kappa \circ f)(p) = 0 = \mu_\mathbb{C}(p)$ and since moment maps are unique up to a constant (see e.g. [37, Ch. 26]) it suffices to show that $\kappa \circ f$ is a moment map for the G-action on $M^{\mu_\mathbb{R}\text{-ss}}$. This follows from the fact that f is a G-equivariant complex-symplectic isomorphism. ☐

4 Stratification of singular hyperkähler quotients

The goal of this section is to prove Theorem 1.4, describing the structure of singular hyperkähler quotients. Throughout this section, (M, K, μ) will be a fixed I-integrable tri-Hamiltonian hyperkähler manifold.

4.1 Complex-analytic structure

Let us first explain how the results on analytic Hilbert quotients of §2.4 help us define a complex-analytic structure on $M /\!/\!/ K$. We use the notation of §3. First, note that $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} := \mu_\mathbb{C}^{-1}(0) \cap M^{\mu_\mathbb{R}\text{-ss}}$ is a G-invariant closed complex-analytic subspace of $M^{\mu_\mathbb{R}\text{-ss}}$. Hence, by Proposition 2.12(ii), its image $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} /\!/ G$ in $M^{\mu_\mathbb{R}\text{-ss}} /\!/ G$ is a closed complex-analytic subspace, and the restriction

$$\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} \longrightarrow \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} /\!/ G$$

is an analytic Hilbert quotient. Note that the space $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} /\!/ G$ has a G-orbit-type partition as in §2.4.3 and $M /\!/\!/ K$ has a K-orbit-type partition into hyperkähler manifolds by Theorem 1.1. Moreover, by Theorem 2.13 and

Theorem 2.19(i), we have $\mu^{-1}(0) \subseteq \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$, and this inclusion descends to an isomorphism

$$M /\!\!/\!\!/ K \longrightarrow \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} /\!\!/ G$$

of partitioned spaces. In particular, $M /\!\!/\!\!/ K$ has the structure of a complex-analytic space. We denote the structure sheaf by $\mathcal{O}_{\mathsf{I}}$.

We have shown the first part Theorem 1.4(i); it remains to show that the orbit-type partition of $M /\!\!/\!\!/ K$ is a complex-analytic Whitney stratification with respect to this sheaf $\mathcal{O}_{\mathsf{I}}$ and is compatible with the complex structures I_S on the pieces. This will be shown later as a consequence of the local model Theorem 1.4(iv), which we show next.

4.2 Linear hyperkähler quotients

Let us first consider the case of a linear hyperkähler quotient. Let V be a quaternionic vector space, i.e. a real vector space endowed with three endomorphisms $\mathsf{I}, \mathsf{J}, \mathsf{K}$ such that $\mathsf{I}^2 = \mathsf{J}^2 = \mathsf{K}^2 = \mathsf{IJK} = -1$. Then $V \cong \mathbb{H}^n$ for some n, so we may endow V with a real inner-product $\langle \cdot, \cdot \rangle$ such that $\mathsf{I}, \mathsf{J}, \mathsf{K}$ are skew-symmetric. This makes V into a hyperkähler manifold, with Kähler forms $\omega_{\mathsf{I}}(u, v) = \langle \mathsf{I}u, v \rangle$, etc. Let L be a compact Lie group acting linearly on V by preserving $\langle \cdot, \cdot \rangle$ and $\mathsf{I}, \mathsf{J}, \mathsf{K}$. Then there is a hyperkähler moment map, namely,

$$\phi : V \longrightarrow \mathfrak{l}^* \otimes \mathbb{R}^3, \quad \phi(v)(x_1, x_2, x_3) = \frac{1}{2}(\omega_{\mathsf{I}}(x_1 v, v), \omega_{\mathsf{J}}(x_2 v, v), \omega_{\mathsf{K}}(x_3 v, v)).$$

Moreover, the L-action extends to an I-complex-linear action of $H := L_{\mathbb{C}}$, and the underlying complex-Hamiltonian manifold is $(V, \mathsf{I}, \omega_{\mathbb{C}}, H, \Phi_V)$, where $\omega_{\mathbb{C}} := \omega_{\mathsf{J}} + i\omega_{\mathsf{K}}$ and Φ_V is the canonical moment map $\Phi_V(v)(x) = \frac{1}{2}\omega_{\mathbb{C}}(xv, v)$. By the Kempf–Ness theorem [24], every point in V is ϕ_{I}-semistable (see e.g. [35, Proposition 3.9]), so the complex-analytic space $(V /\!\!/\!\!/_{\phi_V} L, \mathcal{O}_{\mathsf{I}})$ is simply the analytification of the affine GIT quotient $\Phi_V^{-1}(0) /\!\!/ H = \operatorname{Spec}(\mathbb{C}[\Phi_V^{-1}(0)]^H)$.

Conversely, if H is any complex reductive group and $H \to \operatorname{Sp}(V, \omega_{\mathbb{C}})$ is a complex-symplectic representation (e.g. a complex-symplectic slice) then $V \cong \mathbb{C}^{2n} \cong \mathbb{H}^n$ for some n, so we may endow V with the structure of a quaternionic vector space invariant under the action of a maximal compact subgroup L of H (by averaging). Hence, the GIT quotient $\Phi_V^{-1}(0) /\!\!/ H$ can always be viewed as a hyperkähler quotient.

4.3 Local model

We now prove the first part of Theorem 1.4(iv) which gives a local model for the complex-analytic structure $\mathcal{O}_{\mathsf{I}}$. It will be shown later that there is a holomorphic Poisson bracket on $\mathcal{O}_{\mathsf{I}}$ (Theorem 1.4(ii)) compatible with this model.

Let $q \in M /\!\!/\!\!/ K$ and let $p \in \mu^{-1}(0)$ be a point above q. Let $H := G_p$ and let $V := (T_p(G \cdot p))^{\omega_{\mathbb{C}}} / T_p(G \cdot p)$ be the complex-symplectic slice at p. Let $\Phi_V : V \to \mathfrak{h}^*$ be the canonical moment map. We want to show that q has a neighbourhood U which is isomorphic as a complex-analytic and partitioned space to a neighbourhood U' of 0 in the GIT quotient $\Phi_V^{-1}(0) /\!\!/ H$. However, note that the natural partition of $\Phi_V^{-1}(0) /\!\!/ H$ is by H-orbit-types rather than G-orbit-types. More precisely, if X is a space acted on by a group H and G is a group containing H but not necessarily acting on X, we say that the ***G*-orbit type partition** of X is the partition whose pieces are the connected components of the sets induced by the relation $x \sim y$ if H_x and H_y are conjugate by an element of G. To show that the biholomorphism $U \to U'$ is an isomorphism of partitioned spaces, we need to show that the G-orbit-type partition of $\Phi_V^{-1}(0) /\!\!/ H$ is identical to the standard H-orbit-type partition.

Lemma 4.1. *Let (M, K, μ) be a Hamiltonian Kähler manifold and let $\tilde{K}$ be a compact Lie group containing K as a Lie subgroup (but not necessarily acting on M). Then the K- and $\tilde{K}$-orbit-type partitions of $M /\!\!/_\mu K$ coincide. Moreover, if (M, K, μ) is integrable, then the $\tilde{K}_{\mathbb{C}}$- and $K_{\mathbb{C}}$-orbit-type partitions of $M^{\mu\text{-ss}} /\!\!/ K_{\mathbb{C}}$ also coincide.*

Proof. Let $X = \mu^{-1}(0)$ and let $\pi : X \to X/K$ be the quotient map. Let $S \subseteq X/K$ be a $\tilde{K}$-orbit-type piece, i.e. a connected component of a set of the form $X_{(H)_{\tilde{K}}}/K$ for some closed subgroup $H \subseteq K$, where $(H)_{\tilde{K}}$ is the conjugacy class of H in $\tilde{K}$. We have $S = \pi(T)$ for some connected component T of $X_{(H)_{\tilde{K}}}$. Fix $x \in T$. We want to show that if $y \in T$ then K_x and K_y (which are conjugate in $\tilde{K}$) are in fact conjugate in K. Let

$$A := \{y \in T : K_y \text{ is conjugate to } K_x \text{ in } K\}.$$

It suffices to show that A is both open and closed in T. To show that A is closed in T, let $y \in \overline{A} \cap T$ and write $y = \lim_{n\to\infty} y_n$ with $y_n \in A$. Then there exist $k_n \in K$ such that $k_n K_x k_n^{-1} = K_{y_n}$ for all n. Since K is compact, we may assume that $\lim_{n\to\infty} k_n = k$ for some $k \in K$. Then $kK_x k^{-1} \subseteq K_y$ by continuity of the action. Moreover, $kK_x k^{-1}$ and K_y are isomorphic since they are conjugate in $\tilde{K}$ and since they have finitely many connected components, the inclusion $kK_x k^{-1} \subseteq K_y$ implies that $kK_x k^{-1} = K_y$. Thus, A is closed. To show that A is open in T, let $y \in A$. By Palais [36, Corollary 2 on p. 313] there is a neighbourhood V of y in X such that if $z \in V$ then K_z is conjugate (in K) to a subgroup of K_y. Then $V \cap T$ is a neighbourhood of y in T and $V \cap T \subseteq A$, so A is open in T.

The second statement amounts to show that if H and L are two closed subgroups of a compact Lie group R, then H and L are conjugate in R if and only if $H_{\mathbb{C}}$ and $L_{\mathbb{C}}$ are conjugate in $R_{\mathbb{C}}$. This follows from Mostow's decomposition, as explained by Sjamaar [38, Proof of Theorem 2.10, first paragraph]. □

Now, by picking a quaternionic structure on the complex-symplectic slice V as explained in §4.2, we can apply this result to $(V, K_p, \phi_{\mathrm{I}})$ and infer that the G- and H-orbit-type partitions of $\Phi_V^{-1}(0)/\!/H$ coincide. This will be used for the last part of the following result.

Proposition 4.2. *Let $q \in M/\!/\!/K$. Take a point $p \in \mu^{-1}(0)$ above q, let $H := G_p = (K_p)_{\mathbb{C}}$, and let $V := (T_p(G\cdot p))^{\omega_{\mathbb{C}}}/T_p(G\cdot p)$. Then there is a neighbourhood U of q in $M/\!/\!/K$, an open ball $B \subseteq V$ around 0, and a biholomorphism (with respect to $\mathcal{O}_{\mathrm{I}}$) from U to the image of $(H\cdot B)\cap\Phi_V^{-1}(0)$ in the GIT quotient $\Phi_V^{-1}(0)/\!/H = \operatorname{Spec}(\mathbb{C}[\Phi_V^{-1}(0)]^H)$ which maps q to the image of $0 \in \Phi_V^{-1}(0)$. Moreover, this biholomorphism is an isomorphism of partitioned spaces.*

Proof. Let $E = G\times_H(\mathfrak{h}^\circ\times V)$. Since H is reductive and acts freely on $G\times(\mathfrak{h}^\circ\times V)$, E is an affine variety. Moreover, the moment map $\kappa : E \to \mathfrak{g}^*$ is algebraic, so $\kappa^{-1}(0)$ is an affine variety in E and we can consider the GIT quotient $\kappa^{-1}(0)/\!/G = \operatorname{Spec}(\mathbb{C}[\kappa^{-1}(0)]^G)$. We claim that $\kappa^{-1}(0)/\!/G \cong \Phi_V^{-1}(0)/\!/H$ as affine varieties. Indeed, we have $\kappa^{-1}(0) = G\times_H\Phi_V^{-1}(0)$, so the inclusion $\Phi_V^{-1}(0)\to\kappa^{-1}(0) : v\mapsto[1,v]$ descends to a morphism $\psi : \Phi_V^{-1}(0)/\!/H \to \kappa^{-1}(0)/\!/G$. Also, the projection $\kappa^{-1}(0) = G\times_H\Phi_V^{-1}(0)\to\Phi_V^{-1}(0)/\!/H$ onto the second factor descends to a morphism $\kappa^{-1}(0)/\!/G\to\Phi_V^{-1}(0)/\!/H$ which is an inverse of ψ.

Now, for an element $[g,v]\in G\times_H\Phi_V^{-1}(0) = \kappa^{-1}(0)$ we have $G_{[g,v]} = gH_vg^{-1}$, so ψ is an isomorphism of partitioned spaces with the G-orbit-type partitions on both sides. As explained above, Lemma 4.1 implies that the G-orbit-type partition on $\Phi_V^{-1}(0)/\!/H$ coincides with the H-orbit-type partition.

By the local normal form (Theorem 1.3), there are G-saturated neighborhoods $U \subseteq M^{\mu_{\mathbb{R}}\text{-ss}}$ and $U' \subseteq E$ of p and $[1,0,0]$, and an isomorphism $f : U \to U'$ of complex-Hamiltonian G-manifolds. Note that U' can be taken to be of the form $U' = G\times_H(H\cdot B)$ for some open ball B around zero in $\mathfrak{m}^*\times V$ (this is how U' was constructed in the proof). Then $W := U\cap\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$ is a G-saturated open subset of $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}$, and so is $W' := U'\cap\kappa^{-1}(0)$ in $\kappa^{-1}(0)$. Moreover, by Proposition 2.12(i), the image $W/\!/G$ of W in $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}/\!/G$ is open and $W\to W/\!/G$ is an analytic Hilbert quotient. Similarly, $W'\to W'/\!/G\subseteq\kappa^{-1}(0)/\!/G$ is an analytic Hilbert quotient. Since $f : U\to U'$ is a G-equivariant biholomorphism with $\kappa\circ f = \mu_{\mathbb{C}}$, it restricts to a G-equivariant biholomorphism $W\to W'$ and hence to a biholomorphism $W/\!/G\to W'/\!/G$ which respects the G-orbit-type partitions. Moreover, under the isomorphism $\kappa^{-1}(0)/\!/G\cong\Phi_V^{-1}(0)/\!/H$ above we have an isomorphism $W'/\!/G\cong(H\cdot B\cap\Phi_V^{-1}(0))/\!/H$ of complex-analytic and partitioned spaces. □

4.4 The orbit-type pieces are complex submanifolds

As a first application of Proposition 4.2, we will show that the pieces in the orbit-type partition of $M/\!/\!/K$ are complex submanifolds with respect to $\mathcal{O}_{\mathrm{I}}$; this is one of the requirements in the definition of complex-analytic Whitney stratifications.

We shall achieve this by describing the orbit-type partition of $\Phi_V^{-1}(0)/\!/H$, where H is a complex reductive group, $H \to \mathrm{Sp}(V, \omega_{\mathbb{C}})$ a complex-symplectic representation, and $\Phi_V : V \to \mathfrak{h}^*$ the canonical moment map. The set V^H of fixed points of H is a complex-symplectic subspace, so $V = W \oplus V^H$, where W is the symplectic complement. Then W is complex-symplectic and H-invariant, so it provides a complex-symplectic representation of H. The moment map $\Phi_W : W \to \mathfrak{h}^*$ associated with this representation is simply the restriction of Φ_V to W, so we have the decomposition

$$\Phi_V^{-1}(0)/\!/H = (\Phi_W^{-1}(0)/\!/H) \times V^H.$$

For each $L \subseteq H$, let $(\Phi_W^{-1}(0)/\!/H)_{(L)}$ be the image of $\Phi_W^{-1}(0)^{\mathrm{ps}}_{(L)}/H$ under the bijection $\Phi_W^{-1}(0)^{\mathrm{ps}}/H \to \Phi_W^{-1}(0)/\!/H$. Then the pieces of the orbit-type partition of $\Phi_V^{-1}(0)/\!/H$ are the connected components of the sets of the form $(\Phi_W^{-1}(0)/\!/H)_{(L)} \times V^H$.

Lemma 4.3. *The orbit-type piece of $\Phi_V^{-1}(0)/\!/H$ containing 0 is $\{0\} \times V^H$.*

Proof. Note that $V_{(H)} = V^H$ since if $v \in V$ and $H_v = gHg^{-1}$ for some $g \in H$, then $gHg^{-1} \subseteq H$, and since gHg^{-1} and H are isomorphic Lie groups with finitely many connected components this implies $gHg^{-1} = H$ and hence $H_v = H$. In particular, $W_{(H)} = W \cap V^H = 0$, so the piece containing 0 is $(\Phi_W^{-1}(0)/\!/H)_{(H)} \times V^H = \{0\} \times V^H$. □

Proposition 4.4. *The orbit-type pieces of $M/\!/\!/K$ are non-singular complex-analytic subspaces with respect to $\mathcal{O}_{\mathsf{I}}$.*

Proof. By Lemma 4.3 and Proposition 4.2, the embedding of a K-orbit-type piece in $M/\!/\!/K$ is locally biholomorphic to the embedding of $\{0\} \times V^H$ in $(\Phi_W^{-1}(0)/\!/H) \times V^H$. □

4.5 Compatibility with the hyperkähler structures

Let $S \subseteq M/\!/\!/K$ be an orbit-type piece. Then by Proposition 4.4, S is a complex manifold. But also, S has a complex structure I_S as part of its hyperkähler structure $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ of Theorem 1.1. We want to show that those are equal, or in other words:

Proposition 4.5. *The inclusion $S \hookrightarrow M/\!/\!/K$ is holomorphic with respect to I_S and $\mathcal{O}_{\mathsf{I}}$.*

Proof. We want to show that the composition $S \hookrightarrow M/\!/\!/K \to \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}/\!/G$ is holomorphic, where S has the complex structure I_S. Since $\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}/\!/G$ is a closed complex-analytic subspace of $M^{\mu_{\mathbb{R}}\text{-ss}}/\!/G$, it suffices to show that the composition $S \to \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}}/\!/G \hookrightarrow M^{\mu_{\mathbb{R}}\text{-ss}}/\!/G$ is holomorphic, which is the same as the composition $S \hookrightarrow M/\!/\!/K \hookrightarrow M/\!/_{\mu_{\mathbb{R}}} K \to M^{\mu_{\mathbb{R}}\text{-ss}}/\!/G$. The set S is a connected component of $\mu^{-1}(0)_{(H)}/K$ for some $H \subseteq K$. Hence, S is a subset of a connected component T of $\mu_{\mathbb{R}}^{-1}(0)_{(H)}/K$. Moreover, T is a stratum in the Kähler quotient $M/\!/_{\mu_{\mathbb{R}}} K$ and, from the definition of the Kähler structure on T given in §2.4 and the definition of I_S given in the paragraph above this proposition, the inclusion $S \hookrightarrow T$ is holomorphic. Hence, it suffices to show that the composition $T \hookrightarrow M/\!/_{\mu_{\mathbb{R}}} K \to M^{\mu_{\mathbb{R}}\text{-ss}}/\!/G$ is holomorphic, and this follows from Theorem 2.19(iii). □

4.6 The frontier condition

At this point we have shown Theorem 1.4(i) except for the fact that the orbit-type partition is a Whitney stratification. In this section we prove the first step, which is that this partition is a decomposition in the sense of Definition 2.2 (this is a requirement in the definition of Whitney stratified spaces). Since K is compact, $\mu^{-1}(0)/K$ satisfies the local condition, so it only remains to show the frontier condition. This will be achieved by the local model of Proposition 4.2, so we first need to discuss how the frontier condition can be inferred locally.

Given a partitioned space $(X, \mathcal{P})$ we will denote by $\mathcal{P}^\circ$ the refinement of $\mathcal{P}$ obtained by separating every piece of $\mathcal{P}$ into its connected components. In particular, the orbit-type partition of $M /\!\!/\!\!/ K$ which we are considering is the refinement $\mathcal{P}^\circ$ of $\mathcal{P} := \{\mu^{-1}(0)_{(K_p)}/K : p \in \mu^{-1}(0)\}$. Also, we will say that a partitioned space $(X, \mathcal{P})$ is ***conical*** at a stratum $S \in \mathcal{P}$ if $S \subseteq \overline{T}$ for all $T \in \mathcal{P}$.

The following lemma provides a local criterion for partitioned spaces to satisfy the frontier condition.

Lemma 4.6. *Let $(X, \mathcal{P})$ be a partitioned space. Suppose that every point $x \in X$ has a neighbourhood U such that if S is the stratum containing x, then $S \cap U$ is connected and $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$. Then $\mathcal{P}^\circ$ satisfies the frontier condition.*

Proof. Let $S, T \in \mathcal{P}$ and let $S = \bigsqcup_i S_i$, $T = \bigsqcup_j T_j$ be their connected components. Suppose that $S_{i_0} \cap \overline{T_{j_0}} \neq \emptyset$ for some i_0, j_0. We want to show that $S_{i_0} \subseteq \overline{T_{j_0}}$. The set $R := S_{i_0} \cap \overline{T_{j_0}}$ is closed in S_{i_0}, so it suffices to show that R is also open in S_{i_0}. Let $x \in R$. Take a neighbourhood U of x in X such that $S \cap U$ is connected and $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$. We claim that $S_{i_0} \cap U \subseteq R$, or equivalently, $S_{i_0} \cap U \subseteq \overline{T_{j_0}}$. If $T \cap U = \bigsqcup_k C_k$ are the connected components of $T \cap U$, then, since $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$, we have $S \cap U \subseteq \overline{C_k}$ for all k. But the set of connected components of $T \cap U$ is the union of the set of connected components of $T_j \cap U$ for all j, so there exists k_0 such that $C_{k_0} \subseteq T_{j_0} \cap U$ and hence $S_{i_0} \cap U \subseteq S \cap U \subseteq \overline{C_{k_0}} \subseteq \overline{T_{j_0}}$. □

Proposition 4.7. *The orbit-type partition of $M /\!\!/\!\!/ K$ satisfies the frontier condition and hence is a decomposition.*

Proof. Let $q \in M /\!\!/\!\!/ K$, let V, H and $B \subseteq V$ be as in Proposition 4.2, and let $U = (H \cdot B) \cap \Phi_V^{-1}(0) /\!\!/ H$. We denote by $[v]$ the image of a point $v \in \Phi_V^{-1}(0)$ in the GIT quotient $\Phi_V^{-1}(0) /\!\!/ H$. Then q has a neighbourhood isomorphic to U as partitioned spaces, with an isomorphism sending q to $[0]$. Let $\mathcal{P}$ be the orbit-type partition of $\Phi_V^{-1}(0) /\!\!/ H$ and let $S \in \mathcal{P}$ be the piece containing $[0]$. By Lemma 4.6, it suffices to show that $S \cap U$ is connected and $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$. By Lemma 4.3, $S = \{[0]\} \times V^H$ so $S \cap U = \{[0]\} \times (V^H \cap B)$ is connected. To show that $(\mathcal{P}|_U)^\circ$ is conical at $S \cap U$, let $T' \in (\mathcal{P}|_U)^\circ$. Then T' is a connected component of $T \cap U$, where $T := (\Phi_W^{-1}(0) /\!\!/ H)_{(L)} \times V^H$ for some $L \subseteq H$. We need to show that $S \cap U \subseteq \overline{T'}$. Let $([0], v) \in S \cap U$, where $v \in V^H \cap B$. Take any point $([w], u)$ of T', where $w \in (\Phi_W^{-1}(0)^{\mathrm{ps}})_{(L)}$, $u \in V^H$, and $w + u \in H \cdot B$. It suffices to find a continuous path $\gamma : (0, 1] \to T \cap U$ such that $\gamma(1) = ([w], u)$ and $\lim_{t \to 0} \gamma(t) = ([0], v)$. Let $h \in H$ be such that $w + u \in h^{-1} \cdot B$. Then $hw + u \in B$. We also have $v \in B$, so there exists $t_0 > 0$ small enough so that $t_0 hw + v \in B$ and hence $t_0 w + v \in H \cdot B$. Now, $\Phi_W(tw) = t^2 \Phi_W(w) = 0$ and hence $([tw], v) \in T \cap U$ for all $t > 0$ and $([tw], v) \to ([0], v)$ as $t \to 0$. Moreover, since B is convex, the straight line from $t_0 w + v$ to $w + u$ will stay in $(H \cdot B) \cap ((\Phi_W^{-1}(0)^{\mathrm{ps}})_{(L)} \times V^H)$ and hence $([t_0 w], v)$ and $([w], u)$ are in the same path component T' of $T \cap U$. □

4.7 Whitney conditions

We show that the orbit-type partition of $M /\!\!/\!\!/ K$ is a complex-analytic Whitney stratification with respect to $\mathcal{O}_\mathsf{I}$ and hence a stratification in the sense of Definition 2.4. In particular, this completes the proof of Theorem 1.4(i). Our proof is similar to that of Sjamaar–Lerman [39, §6]. Let us first recall the following result of Whitney.

Lemma 4.8 (Whitney [43, Lemma 19.3]). *Let S and T be disjoint complex submanifolds of a complex-analytic space X with $S \subseteq \overline{T}$ and* $\dim S < \dim T$. *There is a (possibly empty) complex-analytic subspace A of S with* $\dim A < \dim S$ *such that T is regular over $S \setminus A$.* □

Corollary 4.9. *Let X be a complex-analytic space and $T \subseteq X$ a complex submanifold with* $\dim T > 0$. *Then T is regular over $\{x\}$ for all $x \in \overline{T} \setminus T$.*

Proof. Use Lemma 4.8 with $S = \{x\}$. □

Proposition 4.10. *The orbit-type partition of $M /\!\!/\!\!/ K$ is a complex-analytic Whitney stratification with respect to $\mathcal{O}_\mathsf{I}$. In particular, it is a stratification in the sense of* Definition 2.4.

Proof. By Proposition 4.2, the problem reduces to checking Whitney conditions for the H-orbit-type partition of $\Phi_V^{-1}(0)/\!\!/ H$ at [0]. By §4.4, we have $\Phi_V^{-1}(0)/\!\!/ H = (\Phi_W^{-1}(0)/\!\!/ H) \times V^H$ and hence it suffices to check Whitney condition for $\Phi_W^{-1}(0)/\!\!/H$ at [0]. But the piece containing [0] is the singleton $\{[0]\}$, so this follows from Corollary 4.9. □

4.8 Poisson structure

We now show Theorem 1.4(ii), which says that there is a natural Poisson bracket on $\mathcal{O}_\mathsf{I}$ making $M/\!\!/\!\!/K$ a stratified symplectic space as in Sjamaar–Lerman's work (§2.3) but in a complex-analytic sense.

The definition of the Poisson bracket on $\mathcal{O}_\mathsf{I}$ is as follows. Let $U \subseteq M/\!\!/\!\!/ K$ be open, let $f, g \in \mathcal{O}_\mathsf{I}(U)$ and let $q \in U$. To define $\{f, g\}(q)$, let $S \subseteq M/\!\!/\!\!/ K$ be the orbit-type stratum containing q and let $(g_S, \mathsf{I}_S, \mathsf{J}_S, \mathsf{K}_S)$ be its hyperkähler structure. Then $(\omega_S)_\mathbb{C} := \omega_{\mathsf{J}_S} + i\omega_{\mathsf{K}_S}$ is a complex-symplectic form on (S, I_S). By Proposition 4.5, the restrictions $f|_{S\cap U}, g|_{S\cap U}$ are I_S-holomorphic, and hence we can take their Poisson bracket $\{f|_{S\cap U}, g|_{S\cap U}\} : S\cap U \to \mathbb{C}$ with respect to $(\omega_S)_\mathbb{C}$ and define $\{f, g\}(q) := \{f|_{S\cap U}, g|_{S\cap U}\}(q)$. This defines a function $\{f, g\} : U \to \mathbb{C}$ pointwise, and the goal is to show that it is holomorphic, i.e. $\{f, g\} \in \mathcal{O}_\mathsf{I}(U)$.

In what follows, we identify S with a G-orbit-type stratum in $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G$, i.e. S is a connected component of $(\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G)_{(H)}$ for some reductive subgroup $H \subseteq G$. By the definition of the G-orbit-type partition, the map $(\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ps}})_{(H)} \to (\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G)_{(H)}$ is surjective (note that on the left-hand side we use polystable points), so S is the image under the quotient map $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} \to \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G$ of an open subset Z of $(\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ps}})_{(H)}$.

Lemma 4.11. *The set Z is a complex submanifold of M, the map $\pi : Z \to S$ is a holomorphic submersion, and $\pi^*(\omega_S)_\mathbb{C} = i^*\omega_\mathbb{C}$ where $i : Z \hookrightarrow M$.*

Proof. By the local normal form (Theorem 1.3), the embedding of Z in M is locally biholomorphic to the embedding of $G/H \times V^H$ in $G \times_H (\mathfrak{h}^\circ \times V)$ and π is locally biholomorphic to the projection $G/H \times V^H \to V^H$. This proves the first and second assertions. For the third assertion, we first note that, since the pullbacks of the symplectic forms $\omega_{\mathsf{I}_S}, \omega_{\mathsf{J}_S}, \omega_{\mathsf{K}_S}$ on $\mu^{-1}(0)_{(H)}$ are the restrictions of the symplectic forms $\omega_\mathsf{I}, \omega_\mathsf{J}, \omega_\mathsf{K}$ on M, we have $j^*(\pi^*(\omega_S)_\mathbb{C}) = j^*(i^*\omega_\mathbb{C})$ where $j : \mu^{-1}(0)_{(H)} \hookrightarrow Z$. Since j descends to a diffeomorphism $\mu^{-1}(0)_{(H)}/K \to (\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G)_{(H)}$ we get that for all $p \in \mu^{-1}(0)_{(H)}$, $T_pZ = T_p\mu^{-1}(0)_{(H)} + T_p(G \cdot p)$. Hence, the result follows by the same argument as in the proof of Theorem 1.1 given in §2.5. □

Lemma 4.12. *Let $f : U \to \mathbb{C}$ be a holomorphic G-invariant function on an open set $U \subseteq M$, and let Ξ_f be the holomorphic vector field on U dual to df under $\omega_\mathbb{C}$. Then Ξ_f is tangent to Z, i.e. $\Xi_f(p) \in T_pZ$ for all $p \in Z \cap U$.*

Proof. Let $\mathfrak{m} = \mathfrak{h}^\perp$ as in §3.2. By the local normal form we may assume that $M = G \times_H (\mathfrak{m}^* \times V)$, $p = [1, 0, 0]$ and $Z = G/H \times V^H$. By Lemma 3.10, $T_pM = \mathfrak{m} \times \mathfrak{m}^* \times V$, $Z = \mathfrak{m} \times 0 \times V^H$, and $T_p(G \cdot p) = \mathfrak{m} \times 0 \times 0$. Let $(x, \xi, v) := \Xi_f(p) \in \mathfrak{m} \times \mathfrak{m}^* \times V$. Then

$$df_p(y, \eta, w) = \eta(x) - \xi(y) + \omega_\mathbb{C}(v, w) \tag{4.1}$$

for all $(y, \eta, w) \in \mathfrak{m} \times \mathfrak{m}^* \times V$. Since f is G-invariant, we have $df_p(\mathfrak{m} \times 0 \times 0) = 0$, so $\xi = 0$. Also, G-equivariance implies that for all $w \in V$ and $h \in H$ we have $df_p(0, 0, h \cdot w) = df_p(0, 0, w)$, so (4.1) implies that

$$\omega_\mathbb{C}(v, h \cdot w) = df_p(0, 0, h \cdot w) = df_p(0, 0, w) = \omega_\mathbb{C}(v, w).$$

Since $\omega_\mathbb{C}$ is H-invariant, this implies $\omega_\mathbb{C}(h^{-1}v - v, w) = 0$ for all $w \in V$ and $h \in H$, so $v \in V^H$. Thus, $\Xi_f(p) = (y, 0, v) \in \mathfrak{m} \times 0 \times V^H = T_pZ$. □

Lemma 4.13. *For every open set $U \subseteq M/\!\!/\!\!/ K$ and $f, g \in \mathcal{O}_\mathsf{I}(U)$, we have $\{f, g\} \in \mathcal{O}_\mathsf{I}(U)$.*

Proof. We identify $M/\!\!/\!\!/ K$ with $\mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G$. Let $\Pi : \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}} \to \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}\text{-ss}}/\!\!/ G$ be the quotient map. Then $\{f, g\} \in \mathcal{O}_\mathsf{I}(U)$ if and only if the pullback $\Pi^*\{f, g\} : \Pi^{-1}(U) \to \mathbb{C}$ is holomorphic. This is a local statement, so we may assume that $\Pi^{-1}(U) = \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ss}} \cap U'$ for some G-invariant open set $U' \subseteq M^{\mu_{\mathbb{R}}\text{-ss}}$ such that $\Pi^{\star}f$ and $\Pi^{\star}g$ extend to holomorphic G-invariant functions $\hat{f}, \hat{g} : U' \to \mathbb{C}$ (using Proposition 2.12(ii)). It then suffices to show that $\Pi^{\star}\{f, g\} = \{\hat{f}, \hat{g}\}|_{\Pi^{-1}(U)}$. Since $\hat{f}$, $\hat{g}$ and $\omega_{\mathbb{C}}$ are G-invariant, so is $\{\hat{f}, \hat{g}\}$. Thus, it suffices to show that $\Pi^{\star}\{f, g\}(p) = \{\hat{f}, \hat{g}\}(p)$ for every polystable point $p \in \Pi^{-1}(U) \cap \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}\text{-ps}}$. We have $p \in Z$ for some Z as above, and $\Xi_{\hat{f}}(p) \in T_pZ$ by Lemma 4.12. Let $S = \Pi(Z)$, $\pi = \Pi|_Z : Z \to S$ and $i : Z \to M$, as before. Then we have $d\pi(\Xi_{\hat{f}}(p)) = \Xi_f(\pi(p))$, where Ξ_f is the Hamiltonian vector field of f on $U \cap S$, since for all $v \in T_pZ$,

$$\begin{aligned}(\omega_S)_{\mathbb{C}}(d\pi(\Xi_{\hat{f}}(p)), d\pi(v)) &= \omega_{\mathbb{C}}(\Xi_{\hat{f}}(p), v) = d\hat{f}_p(v) = df_{\pi(p)}(d\pi(v))\\ &= (\omega_S)_{\mathbb{C}}(\Xi_f(\pi(p)), d\pi(v)).\end{aligned}$$

Thus,

$$\begin{aligned}\{f, g\}(\Pi(p)) &:= (\omega_S)_{\mathbb{C}}(\Xi_f(\pi(p)), \Xi_f(\pi(p))) = (\omega_S)_{\mathbb{C}}(d\pi(\Xi_{\hat{f}}(p)), d\pi(\Xi_{\hat{f}}(p)))\\ &= \omega_{\mathbb{C}}(\Xi_{\hat{f}}(p), \Xi_{\hat{g}}(p)) = \{\hat{f}, \hat{g}\}(p).\end{aligned}$$

So $\Pi^{\star}\{f, g\} = \{\hat{f}, \hat{g}\}|_{\Pi^{-1}(U)}$ and hence $\{f, g\} \in \mathcal{O}_{|}(U)$. □

By construction, the Poisson bracket is uniquely determined by the property that the inclusions of the strata are Poisson maps. Thus, we have shown Theorem 1.4(ii).

4.9 Compatibility of the local model

We now show the remaining part of Theorem 1.4(iv), which is that the local model is compatible with the holomorphic Poisson bracket constructed in the previous section.

Let H be a complex reductive group and $H \to \mathrm{Sp}(V, \omega_{\mathbb{C}})$ a complex-symplectic representation. Then as explained in §4.2, we can view the affine GIT quotient $V_0 := \Phi_V^{-1}(0)/\!\!/ H$ as a hyperkähler quotient. Hence, if $\mathcal{O}_{V_0}$ denotes the underlying complex-analytic structure of V_0, then $(V_0, \mathcal{O}_{V_0})$ together with the H-orbit-type partition is a complex-analytic Whitney stratified space with a holomorphic Poisson bracket (which does not depend on the choice of quaternionic structure). Recall from Proposition 4.2 that $\Phi_V^{-1}(0)/\!\!/ H$ provides a local model for the complex-analytic structure of $M/\!\!/\!\!/ K$. Here we show that $\Phi_V^{-1}(0)/\!\!/ H$ is also a local model for the Poisson structure.

Proposition 4.14. *The biholomorphism of* Proposition 4.2 *is compatible with the holomorphic Poisson brackets.*

Proof. Since the local normal form for (M, K, μ) is an isomorphism of complex-symplectic manifolds, we only need to show that the isomorphism $\kappa^{-1}(0)/\!\!/ G = \Phi_V^{-1}(0)/\!\!/ H$ of affine varieties in the proof of Proposition 4.2 respects the Poisson brackets. This follows from the fact that V is a complex-symplectic submanifold of E via the embedding $\iota : V \hookrightarrow G \times_H (\mathfrak{h}^{\circ} \times V)$, $v \mapsto [1, 0, v]$ and that the isomorphism descends from this map. □

4.10 Real Poisson structure

We now prove Theorem 1.4(iii), i.e. we show that $M/\!\!/\!\!/ K$ has the structure of a stratified symplectic space (Definition 2.9) compatible with the first Kähler forms.

Let $C^{\infty}(M/\!\!/\!\!/ K)$ be the subalgebra of the $\mathbb{R}$-algebra of continuous functions on $M/\!\!/\!\!/ K$ consisting functions descending from smooth K-invariant functions on M. In other words, $f \in C^{\infty}(M/\!\!/\!\!/ K)$ if and only if there exists $F \in C^{\infty}(M)^K$ such that $\pi^{\star}f = F|_{\mu^{-1}(0)}$, where $\pi : \mu^{-1}(0) \to M/\!\!/\!\!/ K$ is the quotient map.

Lemma 4.15. *The inclusion $S \hookrightarrow M/\!\!/\!\!/ K$ of an orbit-type stratum is smooth.*

Proof. Let $F \in C^\infty(M)^K$ be such that $f \circ \pi = F|_{\mu^{-1}(0)}$. Recall that, by Theorem 1.1, $\pi^{-1}(S)$ is a smooth submanifold of M and the restriction $\pi^{-1}(S) \to S$ is a surjective submersion. Hence, $F|_{\pi^{-1}(S)}$ descends to a unique smooth function $S \to \mathbb{R}$, which is just $f|_S$. □

To define a Poisson bracket on $C^\infty(M /\!\!/\!\!/ K)$, we first define it pointwise as in [39] by letting

$$\{f, g\}(x) := \{f|_S, g|_S\}_{\omega_{I_S}}(x),$$

where S is the unique orbit-type stratum containing x and $\{\cdot,\cdot\}_{\omega_{I_S}}$ is the real Poisson bracket on S induced by ω_{I_S}. It only remains to show that $\{f, g\} \in C^\infty(M /\!\!/\!\!/ K)$ (the Leibniz rule and Jacobi identity follow from that of $\{\cdot,\cdot\}_{\omega_{I_S}}$).

Proposition 4.16. *For all $f, g \in C^\infty(M /\!\!/\!\!/ K)$ we have $\{f, g\} \in C^\infty(M /\!\!/\!\!/ K)$.*

Proof. First note that $(M, K, \mu_\mathbb{R})$ is a Hamiltonian manifold as in §2.3, so Sjamaar–Lerman's original theorem holds, i.e. $M /\!\!/_{\mu_\mathbb{R}} K$ is endowed with a Poisson $\mathbb{R}$-algebra $C^\infty(M /\!\!/_{\mu_\mathbb{R}} K)$ defined analogously. By definition, the inclusion $M /\!\!/\!\!/ K \hookrightarrow M /\!\!/_{\mu_\mathbb{R}} K$ is smooth with respect to $C^\infty(M /\!\!/\!\!/ K)$ and $C^\infty(M /\!\!/_{\mu_\mathbb{R}} K)$.

Now, let $f, g \in C^\infty(M /\!\!/\!\!/ K)$, so that $f \circ \pi = F|_{\mu^{-1}(0)}$ and $g \circ \pi = G|_{\mu^{-1}(0)}$ for some $F, G \in C^\infty(M)^K$. Let $\tilde{\pi} : \mu_\mathbb{R}^{-1}(0) \to M /\!\!/_{\mu_\mathbb{R}} K$ be the quotient map. Then $F|_{\mu_\mathbb{R}^{-1}(0)} = \tilde{f} \circ \tilde{\pi}$ and $G|_{\mu_\mathbb{R}^{-1}(0)} = \tilde{g} \circ \tilde{\pi}$, for some $\tilde{f}, \tilde{g} \in C^\infty(M /\!\!/_{\mu_\mathbb{R}} K)$. Note that $\tilde{f}|_{M/\!\!/\!\!/K} = f$ and $\tilde{g}|_{M/\!\!/\!\!/K} = g$. Let $x \in M /\!\!/\!\!/ K$, let S be the stratum containing x, and let $\tilde{S}$ be the stratum of $M /\!\!/_{\mu_\mathbb{R}} K$ containing x. Then by construction of the hyperkähler structure on S (see §2.5), S is a Kähler submanifold of $\tilde{S}$. In particular, $\{f, g\}(x) = \{f|_S, g|_S\}_S(x) = \{\tilde{f}|_{\tilde{S}}, \tilde{g}|_{\tilde{S}}\}_{\tilde{S}}(x) = \{\tilde{f}, \tilde{g}\}(x)$. Hence, $\{f, g\} = \{\tilde{f}, \tilde{g}\}|_{M/\!\!/\!\!/K} \in C^\infty(M /\!\!/\!\!/ K)$ since $\{\tilde{f}, \tilde{g}\} \in C^\infty(M /\!\!/_{\mu_\mathbb{R}} K)$. □

Acknowledgements: I thank Andrew Dancer for useful discussions, and the anonymous referees for valuable comments and suggestions. This work was supported by a Moussouris Scholarship from the University of Oxford, a Postdoctoral Fellowship at the University of Toronto, a PGS-D and a PDF scholarship from the Natural Sciences and Engineering Research Council of Canada (NSERC), and a postdoctoral scholarship from the Fonds de Recherche du Québec Nature et technologies (FRQNT).

Conflict of interest: Author state no conflict of interest.

References

[1] Abraham, R., Marsden, J.E., *Foundations of mechanics.* Second edition, revised and enlarged. With the assistance of Tudor Ratiu and Richard Cushman. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978.

[2] Bulois, M., *On the normality of the null-fiber of the moment map for θ- and tori representations.* J. Algebra 507 (2018), 502–524.

[3] Coddington, E.A., Levinson, N., *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955

[4] Dancer, A., Swann, A., *Hyper-Kähler metrics associated to compact Lie groups.* Math. Proc. Cambridge Philos. Soc. **120** (1996), no. 1, 61–69.

[5] Dancer, A., Swann, A., *The geometry of singular quaternionic Kähler quotients.* Internat. J. Math. **8** (1997), no. 5, 595–610.

[6] Duistermaat, J.J., Kolk, J.A.C., *Lie groups.* Universitext, Springer-Verlag, Berlin, 2000.

[7] Goresky, M., MacPherson, R., *Stratified Morse theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 14, Springer-Verlag, Berlin, 1988.

[8] Greb, D., *Compact Kähler quotients of algebraic varieties and geometric invariant theory.* Adv. Math. **224** (2010), 401–431.

[9] Greb, D., *Projectivity of analytic Hilbert and Kähler quotients.* Trans. Amer. Math. Soc. **362** (2010), no. 6, 3243–3271.

[10] Greb, D., *Complex-analytic quotients of algebraic G-varieties.* Math. Ann. **363** (2015), 77–100.

[11] Guillemin, V., Sternberg, S., *Geometric quantization and multiplicities of group representations.* Invent. Math. **67** (1982), no. 3, 515–538.

[12] Guillemin, V., Sternberg, S., *A normal form for the moment map.* Differential geometric methods in mathematical physics (Jerusalem, 1982), 161–175, Math. Phys. Stud., 6, Reidel, Dordrecht, 1984.

[13] Heinzner, P., *Geometric invariant theory on Stein spaces.* Math. Ann. **289** (1991), no. 4, 631–662.

[14] Heinzner, P., Huckleberry, A., *Kählerian potentials and convexity properties of the moment map.* Invent. Math. **126** (1996), no. 1, 65–84.
[15] Heinzner, P., Huckleberry, A., *Analytic Hilbert quotients.* Several complex variables (Berkeley, CA, 1995–1996), 309–349, Math. Sci. Res. Inst. Publ., 37, Cambridge Univ. Press, Cambridge, 1999.
[16] Heinzner, P., Huckleberry, A., *Kählerian structures on symplectic reductions.* Complex analysis and algebraic geometry, 225–253, de Gruyter, Berlin, 2000.
[17] Heinzner, P., Huckleberry, A., Loose, F., *Kählerian extensions of the symplectic reduction.* J. Reine Angew. Math. **455** (1994), 123–140.
[18] Heinzner, P., Loose, F., *Reduction of complex Hamiltonian G-spaces.* Geom. Funct. Anal. **4** (1994), no. 3, 288–297.
[19] Heinzner, P., Migliorini, L., Polito, M., *Semistable quotients.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 2, 233–248.
[20] Heinzner, P., Schwarz, G.W., *Cartan decomposition of the moment map.* Math. Ann. **337** (2007), no. 1, 197–232.
[21] Hitchin, N.J., Karlhede, A., Lindström, U., Roček, M., *Hyperkähler metrics and supersymmetry.* Comm. Math. Phys. **108** (1987), no. 4, 535–589.
[22] Huebschmann, J., *Singular Poisson-Kähler geometry of stratified Kähler spaces and quantization.* Geometry and quantization, 27–63, Trav. Math., 19, Univ. Luxemb., Luxembourg, 2011.
[23] Kaledin, D., *Singular hyperkähler quotients.* PhD thesis, Massachusetts Institute of Technology. 1995.
[24] Kempf, G., Ness, L., *The length of vectors in representation spaces.* Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), pp. 233–243, Lecture Notes in Math., 732, Springer, Berlin, 1979.
[25] Kirwan, F.C., *Cohomology of quotients in symplectic and algebraic geometry.* Mathematical Notes, 31. Princeton University Press, Princeton, NJ, 1984.
[26] Kronheimer, P., *A hyperkähler structure on the cotangent bundle of a complex Lie group.* MSRI preprint (arXiv:math/0409253), 1988.
[27] Lee, J.M., *Introduction to smooth manifolds.* Second edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013.
[28] Losev, I. V., *Symplectic slices for actions of reductive groups.* Mat. Sb. **197** (2006), no. 2, 75–86; translation in Sb. Math. **197** (2006), no. 1–2, 213–24
[29] Marle, C.-M., *Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique.* Rend. Sem. Mat. Univ. Politec. Torino **43** (1985), no. 2, 227–251.
[30] Marsden, J., Weinstein, A., *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. **5** (1974), 121–130.
[31] Mather, J., *Notes on topological stability.* Bull. Amer. Math. Soc. (N.S.) **49** (2012), no. 4, 475–506.
[32] Mayrand, M., *Stratified hyperkähler spaces from semisimple Lie algebras.* Transform. Groups 25 (2020), no. 1, 191–215.
[33] Mayrand, M., *Kempf–Ness type theorems and Nahm equations.* J. Geom. Phys. 136 (2019), 138–155.
[34] Nakajima, H., *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras.* Duke Math. J. 76 (1994), no. 2, 365–416.
[35] Nakajima, H., *Lectures on Hilbert schemes of points on surfaces.* University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999.
[36] Palais, R.S., *On the existence of slices for actions of non-compact Lie groups*. Ann. of Math. (2) **73** (1961), 295–323.
[37] Cannas da Silva, A., *Lectures on symplectic geometry.* Lecture Notes in Mathematics 1764, Springer-Verlag, Berlin, 2001.
[38] Sjamaar, R., *Holomorphic slices, symplectic reduction and multiplicities of representations.* Ann. of Math. (2) **141** (1995), no. 1, 87–129.
[39] Sjamaar, R., Lerman, E., *Stratified symplectic spaces and reduction.* Ann. of Math. (2) **134** (1991), no. 2, 375–422.
[40] Snow, D.M., *Reductive group actions on Stein spaces.* Math. Ann. **259** (1982), no. 1, 79–97.
[41] Teleman, A., *Symplectic stability, analytic stability in non-algebraic complex geometry.* Internat. J. Math. **15** (2004), 183–209.
[42] Weinstein, A., *Symplectic manifolds and their Lagrangian submanifolds.* Advances in Math. **6** (1971), 329–346.
[43] Whitney, H., *Tangents to an analytic variety.* Ann. of Math. (2) **81** (1965), 496–549.