Research Article

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On one approach to the approximation of solutions to the direct kinematics problem of parallel manipulators

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Abstract: Any real continuous bounded function of many variables is representable as a superposition of functions of one variable and addition. Depending on the type of superposition, the requirements for the functions of one variable differ. The article investigated one of the options for the numerical implementation of such a superposition proposed by Sprecher. The superposition was presented as a three-layer Feedforward neural network, while the functions of the first's layer were considered as a generator of space-filling curves (Peano curves). The resulting neural network was applied to the problems of direct kinematics of parallel manipulators.

Keywords: parallel manipulators, Kolmogorov's superpositions, universal approximation, space-filling curve

1 Introduction

Recently, for the approximation of solutions of systems of nonlinear equations (many authors use the phrase “to solve systems of nonlinear equations”), machine learning methods are widely used, neural networks are mainly used. An example of such an application is the work in which neural networks are used to approximate solutions (to solve) the direct kinematics problem [1]. When developing a model of a neural network, we are forced to choose the architecture of the neural network, the number of layers, the connections between them, the type of activating

functions, etc. To simplify the development of the model, we decided to try using Kolmogorov's superpositions [2] and neural networks based on them.

In a series of papers [2–4], a theorem was formulated and proved on the representation of any real continuous bounded function of many variables by a superposition of functions of one variable and addition.

Theorem 1. For any integer $n \geq 2$ there exist continuous real functions $\psi^{pq}(x)$ defined on the unit segment $E^1 = [0; 1]$ such that each defined on the n -dimensional unit in the cube E^n , the continuous real function $f(x_1, \dots, x_n)$ can be represented as

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} \chi_q \left[\sum_{p=1}^n \psi^{pq}(x_p) \right], \quad (1)$$

where the functions $\chi_q(y)$ are real and continuous. It should be noted that, by their construction, functions ψ^{pq} are independent of the form of the function f , increase monotonically and are continuous.

This approach was further developed by Sprecher [5–9]. In [5] the theorem on the representation of a continuous function of several variables in a simpler form was proved (at the same time it was necessary to abandon the continuity of one of the functions included in the superposition).

Theorem 2. For any natural number n , $n \geq 2$, there exist real, monotonic increasing functions, $h^p(x)$, $1 \geq p \geq n$, dependent on n , and having the following properties:

– The function

$$\sum_{1 \leq p \leq n} h^p(x_p) \quad (2)$$

separates all points of the unit n -cube E_n , that is:

$$\sum_{1 \leq p \leq n} h^p(x_p) = \sum_{1 \leq p \leq n} h^p(y_p) \quad (3)$$

unless $x_p = y_p$ for all admitted values of p .

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- Each continuous function of n variables on a unit cube E_n can be represented as

$$f(x_1, \dots, x_n) = g \left[\sum_{1 \leq p \leq n} h^p(x_p) \right]. \quad (4)$$

Clearly, the function g will, in general, be discontinuous.

Later in [7], the Kolmogorov's superposition was reformulated in a different form. Functions ψ^{pq} that depend on the indices p and q are replaced by a single function ψ that does not depend on them.

Theorem 3. For each integer $n \geq 2$ there is a constant $\lambda \neq 0$ and a single monotonic function $\psi \in Lip(1)$ having the property that each function $f \in C(E^n)$ can be represented in the form

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} \chi_q \left[\sum_{p=1}^n \lambda^{p-1} \psi(x_p + q\epsilon) \right], \quad (5)$$

where functions χ_q are continuous and $\epsilon \neq 0$ is any constant, $C(E^n)$ stand for the space of real-valued continuous functions defined on E^n and $\phi \in Lip(\alpha)$ if there are constants $0 \leq \alpha \leq 1$ and A such that $|\phi(t_1) - \phi(t_2)| \leq A|t_1 - t_2|^\alpha$ for all points t_1 and t_2 in the domain of ϕ .

In [8–14], Kolmogorov's superposition was considered as a feedforward neural network.

Theorem 4. Every continuous function $f : E^n \rightarrow R$ with $n \geq 2$ can be implemented by a three-layer Feedforward neural network having n fanout processing elements in the x -input layer, $2n + 1$ processing elements in the second layer, and $2n + 1$ processing elements in the output layer. Furthermore, the first and second layers utilize a fixed processing element that is independent of n , and only the output layer depends on f .

It is worth mentioning a rather interesting opinion expressed in [15]: “Representation Properties of Networks: Kolmogorov's Theorem Is Irrelevant”. In some ways, we agree with the opinion of the authors of this work. Several numerical experiments we conducted showed that when applying Kolmogorov's superposition in an unchanged form (at least in the variant proposed in [8]), some problems arise. That is why in our work we consider the functions of the first layer (or inner function) as a generator of space-filling curves [16–20].

The proposed technique was tested on the example of approximating the solutions of direct kinematics problems for parallel manipulators structure [21, 22].

2 Construction space-filling curve

Space-filling curves, called Peano curves, are lines which pass arbitrarily closely to all points of a multi-dimensional volume. In other words, we can consider the filling curves as a map of the unit hypercube E^n onto a segment of non-negative integers $[0, 2^{pn} - 1]$. An example of such a space-filling curve (a Hilbert space-filling curve) can be seen in Figure 1. Consider the function of n -variables ($n \geq 2$) as

Figure 1: Hilbert space-filling curve, 32 x 32

$$f(x_1, \dots, x_n) = g \left[\sum_{1 \leq p \leq n} \lambda_p \psi(x_p) \right], \quad (6)$$

constant λ_p computed by the expression

$$\lambda_p = \begin{cases} 1, & p = 1 \\ \sum_{r=1}^{\infty} \gamma^{-(p-1)\beta(r)}, & p > 1 \end{cases} \quad (7)$$

parameter $\gamma \geq 2n + 1$, $\beta(r) = (n^r - 1)/(n - 1) = 1 + n + n_2 + \dots + n^{(r-1)}$. Define the internal function ψ as in [23]. For all terminating rational numbers $d_k \in I$, which have in the decimal notation to the base γ no more than k digits ($d_k = 0.i_1 i_2 \dots i_k$):

$$\psi_k(d_k) = \begin{cases} d_k, & k = 1 \\ \psi_{k-1}(d_k - \frac{i_k}{\gamma^k}) + \frac{i_k}{\gamma^{\beta(k)}}, & k > 1, \\ \frac{1}{2} \psi_k(d_k - \frac{i_k}{\gamma^k}) + \frac{1}{2} \psi_{k-1}(d_k + \frac{i_k}{\gamma^k}), & i_k < \gamma - 1 \\ \frac{1}{2} \psi_k(d_k - \frac{i_k}{\gamma^k}) + \frac{1}{2} \psi_{k-1}(d_k + \frac{i_k}{\gamma^k}), & i_k = \gamma - 1 \end{cases} \quad (8)$$

The form of this function can be seen in Figure 2. Now we

Figure 2: The form of the function $\psi(x)$ (blue) and, for clarity, the function $y = x$ (orange).

Figure 3: η space-filling curve, 30 x 30

impose a uniform grid on the n -dimensional cube E^n , calculate the values of the function $\eta = \sum_{1 \leq p \leq n} \lambda_p \psi(x_p)$ at grid nodes, and sort the nodes of the grid in an increasing order of the values of the function η . By connecting the grid nodes in series we get Figure 3 (for $n = 2$). We call the resulting curve - η space-filling curve. For approximation, we will use the following approach: each node of the filling curve is mapped to its coordinates and serial number, after which each point from the sample is mapped to the serial number of the nearest node of the curve. Moreover, g is approximated by a piecewise constant function. N is both the number of nodes of the curve and the number of partitions of the piecewise constant function g .

3 Parallel manipulators

Consider two instances of planar parallel manipulators. The first planar parallel manipulator [21] is shown in Figure 4 and consists of two links of variable length inter-

Figure 4: The planar parallel manipulator with two links

connected and fixed on a fixed platform through hinges. The system of kinematic equations (and inequalities) for a given manipulator can be written as follows:

$$\begin{cases} x^2 + y^2 = l_1^2, \\ (x - d)^2 + y^2 = l_2^2, \\ a_1 \leq l_1 \leq b_1, \\ a_2 \leq l_2 \leq b_2, \\ y \geq 0. \end{cases} \quad (9)$$

Parameters of the planar parallel manipulator shown in Figure 4 - $a_i = 5$, $b_i = 10$, $d = 8$. Note that the solution of the direct and inverse kinematics problems for this manipulator can be written out in an analytical form.

The second planar parallel manipulator (3-RPR) [22, 24–26] is shown in Figure 5 and consists of three variable-length links attached to a movable triangular platform and hinged on a fixed platform. The system of kinematic equations (and inequalities) for a given manipulator can be written as follows:

$$\begin{cases} x^2 + y^2 = l_1^2, \\ (x + d_1 \cos(\varphi) - c_2)^2 + (y + d_1 \sin(\varphi))^2 = l_2^2, \\ (x + d_2 \cos(\varphi + \beta) - c_3)^2 + \\ (y + d_2 \sin(\varphi + \beta) - c_4)^2 = l_3^2, \\ a_1 \leq l_1 \leq b_1, \\ a_2 \leq l_2 \leq b_2, \\ a_3 \leq l_3 \leq b_3, \\ \varphi = \pi/4. \end{cases} \quad (10)$$

Parameters of the planar parallel manipulator shown in Figure 5 - $a_i = 1$, $b_i = 6$, $c_2 = 6$, $c_3 = 3$, $c_4 = 5$, $d_1 = d_2 = d_3 = 2$, $\beta = \pi/3$. Note that in the general case, the direct kinematics problem has up to 6 solutions [27].

Figure 5: The planar parallel manipulator with three links (3-RPR)

Figure 6: Blue line - $\bar{x}(\eta)$, red line - $\bar{y}(\eta)$

Therefore, to obtain an unambiguous solution, the restriction $\varphi = \pi/4$ was introduced and the only assembly was taken as the workspace. The inverse kinematics problem has a simple analytical solution.

4 Numerical experiment

The approximation of the function $f(x_1, \dots, x_n)$ was carried out as follows:

- a sample is created consisting of the values of the variables $x_1, \dots, x_n$ and $f(x_1, \dots, x_n)$;
- to each node of the space-filling curve its coordinates and serial number are matched, after which each row

Figure 7: Blue line - $\bar{x}(\eta)$, red line - $\bar{y}(\eta)$ Figure 8: Blue line - $\bar{x}(\eta)$, red line - $\bar{y}(\eta)$

from the selection is assigned the number of the nearest node of the curve. The value of the function $g(\eta)$ at the nodes of the curve is calculated as the average of the values of function f at the corresponding points from the sample.

- function $g(\eta)$ is represented as a piecewise-linear function. Moreover, if necessary, the range of acceptable values $x_1, \dots, x_n$ is mapped to a unit cube E_n . For a planar parallel manipulator with two links, the solution to the direct kinematics problem was approximated - $x(l_1, l_2), y(l_1, l_2)$. We randomly select the points x, y , solving the inverse kinematics problem we obtain the values l_1, l_2 , if these values satisfy the constraints $a_i \geq l_i \geq b_i$, the point is included in the sample. A sample was obtained consisting of 2369990 points. The dimension of the space-filling curve was taken 90×90 . The number of nodes of the space-filling curve is 8100, respectively. As a result, we obtain two functions $\bar{x}(\eta), \bar{y}(\eta)$, which are shown in Figures 6-9. As you can see, the functions $\bar{x}(\eta), \bar{y}(\eta)$ have a clearly defined “fractal” structure. We will estimate the approximation error as the maximum of the Euclidean distance between the points (x, y) and $(\bar{x}, \bar{y})$. At 8100 nodes of the space-filling curve, the approximation error is 0.072. It should be noted that

Figure 9: Blue line - $\bar{x}(\eta)$, red line - $\bar{y}(\eta)$

Table 1: Approximation with a planar parallel manipulator with two links

Fold's number	Train error	Test error
1	0.075	0.075
2	0.076	0.071
3	0.075	0.075

at this stage there are no differences from piecewise-constant approximation over a uniform grid, since order relations are not used in any way. The next step - using Ordinary least squares Linear Regression [28] we get a more compact representation of the functions $(\bar{x}, \bar{y})$. To do this, we divide 8100 nodes of the space-filling curve into groups of 9 elements (see Figure 9) and restore 900 linear regressions. For training and validation of results we applied K-Folds cross-validator [28]. The results are shown in Table 1.

As a result, the approximation error decreases only to 0.075. The effect of overfitting is not observed. If necessary, it is possible to apply multi-level grouping.

For a planar parallel 3-RPR manipulator, the solution to the direct kinematics problem was approximated - $x(l_1, l_2, l_3), y(l_1, l_2, l_3)$. Since the system of kinematic equations may not have a unique solution and there are singularity regions, points x, y belonging to the same assembly region were added to the sample. A sample of 247346 points was obtained. The dimensions of the filling curve were taken $K \times K \times K$, where $K = 20, 30, 40, 50, 60, 70, 80, 90$. The approximation results can be seen in Table 2. The form of the functions $\bar{x}(\eta), \bar{y}(\eta)$ can be seen in Figure 10. It should be noted that the proposed approach (like neural networks of a different architecture) is not by far the best tool for direct kinematics problems, it is also necessary to consider analytical and numerical methods of solution [25–27, 29].

Table 2: Approximation with a 3-RPR manipulator

K	Each node error	Piecewise-linear error
20	0.233	0.233
30	0.140	0.146
40	0.108	0.111
50	0.082	0.087
60	0.069	0.075
70	0.059	0.062
80	0.054	0.055
90	0.047	0.052

Figure 10: blue line - $\bar{x}(\eta)$, red line - $\bar{y}(\eta)$

5 Conclusions

One of the options for the numerical implementation of Kolmogorov's superposition, in its non-continuous version, was considered. Internal functions (functions of the first layer) were considered as a generator of space-filling curves (Peano curves). The form of these curves was obtained. A method for training a neural network and its more compact presentation has been developed. The resulting neural network was applied to the problems of direct kinematics of parallel manipulators. In the future, we plan to conduct a study of the "fractal" properties of the external function $g(\eta)$. We plan to consider various options for a compact representation of this function, and also try to switch to a continuous version of the numerical implementation of Kolmogorov's superposition.

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