Research Article

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Weak Berge Equilibrium in Finite Three-person Games: Conception and Computation

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Abstract: The paper proposes the concept of a weak Berge equilibrium. Unlike the Berge equilibrium, the moral basis of this equilibrium is the Hippocratic Oath "First do no harm". On the other hand, any Berge equilibrium is a weak Berge equilibrium. But, there are weak Berge equilibria, which are not the Berge equilibria. The properties of the weak Berge equilibrium have been investigated. The existence of the weak Berge equilibrium in mixed strategies has been established for finite games. The weak Berge equilibria for finite three-person non-cooperative games are computed.

Keywords: three-person game, non-cooperative game, Berge equilibrium, weak Berge equilibrium

1 Introduction

A wide class of economic, social and political processes are well described by the methods of game theory. Often, when making decisons, participants in such processes can not agree among themselves that they are modelled by using non-cooperative games. Certainly, the most well-known concept of a solution in the theory of non-cooperative games was proposed by John Nash in 1950 in [1]. For this work in 1994 he was awarded the Nobel Prize in Economics.

However, the application of the Nash equilibrium concept in the modelling of real socio-economic and political conflicts, in some cases, leads to paradoxical results, such as the "prisoner's dilemma". One of the first who has noticed this was Claude Berge in [2]. In this book, Berge proposed a new concept of equilibrium, according to which, players are divided into coalitions, while players of one coalition can work together to maximize the payoffs of players of another coalition. Apparently, a crushing review by Martin Shubik [3] on the book of Berge [2], led to the fact that Claude Berge switched his attention from game theory to other areas of mathematics. After decades, based on Berge's ideas, V.I. Zhukovsky [4, 5] and K.S. Weisman [6, 7] suggested a new altruistic concept of equilibrium which is called *Berge equilibrium*. In this concept, the players act on the principle of "One for all and all for one!" from Alexander Dumas's novel "The Three Musketeers". Another interpretation of the Berge equilibrium is [8] the Golden Rule of morality: "Treat others the way you want to be treated". The development of the Berge equilibrium concept is described in details in the review [9]. It is worth noting that the Berge equilibrium solves such well known paradoxes in game theory as the "Prisoner's Dilemma", "Battle of the sexes" and many others. Also the use of Berge equilibrium is possible to applications in economics [10].

At the same time, the Berge equilibrium concept has some drawbacks. One of these drawbacks is that the Berge equilibrium rarely exists in pure strategies. Moreover, in N-person games ($N \geq 3$) with a finite set of strategies, the Berge equilibrium may not exist in the class of mixed strategies. Such example was constructed, in particular, in [11]. The lack of Berge equilibrium might be caused by the fact that it is often impossible to follow the Golden Rule of morality in relation to all players at the same time. For example, if the goals of two players are opposite, then the third player will not be able to apply the Golden Rule to them simultaneously. In this case, increasing the payoff of one player, simultaneously reduces the payoff of the other.

In this paper, we introduce the concept of weak equilibrium according to Berge (the Weak Berge Equilibrium or the WBE). Unlike the Berge equilibrium, this concept is not based on the Gold Rule of morality. The weak Berge equilibrium concept is based on the Hippocratic oath "First do no harm!". Here, we will assume that, making a decision, each player adheres to the situation, one-sided deviation that can cause harm to one of the other players. Further, in Section 2, the concept of weak Berge equilibrium is formalized, some of its properties are studied and sufficient conditions for the existence of such an equilibrium in N-

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person games are given. In Section 3, a numerical WBE approximate search method based on [12–14] is proposed, and numerical simulation results are given for finite games of three person.

2 The concept of weak Berge equilibrium

Let us consider a non-cooperative N-person game in normal form:

$$\Gamma = \langle \mathbf{N}, \{X_i\}_{i\in\mathbf{N}}, \{f_i(x)\}_{i\in\mathbf{N}}\rangle, \quad (1)$$

where $\mathbf{N} = \{1, 2, \dots, N\}$ denotes the set of serial numbers of the players; the set of x_i strategies of the i-th player ($i \in \mathbf{N}$) is denoted by X_i, where $X_i \subseteq \mathbf{R}^{n_i}$. As a result of the players choosing their strategies, the strategy profile is $x = (x_1, \dots, x_N) \in X = X_1 \times X_2 \times \dots \times X_N \subseteq \mathbf{R}^n$ ($n = n_1 + n_2 + \dots + n_N$). On the set of strategy profiles X for each player i ($i \in \mathbf{N}$) the scalar payoff function $f_i(x) : X \to \mathbf{R}$ was defined. The value of $f_i(x)$ realized on the strategy profile chosen by the players $x \in X$ is called the payoff of the i-th player.

The game Γ is played as follows. Each player i ($i \in \mathbf{N}$), without entering into a coalition with other players, chooses his strategy $x_i \in X_i$. As a result of this choice, the strategy profile is $x = (x_1, \dots, x_N) \in X$. After that, each player i gets his payoff $f_i(x)$.

Thus, when making a decision, the player is forced to focus not only on his payoff function, but also on the possible choice of the other participants in the game.

Further, (y_i, x_{-i}) denotes the strategy profile $(x_i, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_N)$, which was obtained from the strategy profile x by replacing the strategy of the i-th player x_i on y_i.

The most popular concept of a solution in the theory of non-cooperative games is the Nash equilibrium.

Definition 1. *A strategy profile* $x^e = (x_1^e, \dots, x_N^e) \in X$ *is called a Nash equilibrium (NE) in the game* (1) *if for every* $x \in X$ *the system of inequalities*

$$f_i(x^e) \geq f_i(x_i, x_{-i}^e) \quad (i \in \mathbf{N}) \quad (2)$$

is true.

The Nash equilibrium strategy profile $x^e \in X$ is stable with respect to deviation of an individual player from his strategy which enters in x^e. Applying the concept of the Nash equilibrium, the player proceeds from his own selfish motives. He only cares about his payoff, without taking into account the interests of other players. However, this approach leads to a number of paradoxes, such as the Tucker problem in the classic game "Prisoner's Dilemma".

Example 1. Let us consider the Prisoner's Dilemma game. Two criminals are arrested on suspicion of a crime, but the police do not have direct evidence. Therefore, the police, have isolated them from each other, and offered them the same deal: if one testifies against the other, but he keeps silence, the first one is released for helping the investigation, and the second gets 10 years – the maximum term of imprisonment. If both are silent, their deed goes through a lighter article, and each of them are sentenced to a year in prison. If both testify against each other, each receives a minimum period of 2 years. Every prisoner can choose to keep silence or testify against another. However, none of them knows exactly what the other will do. The Nash equilibrium in this game dictates players to testify against each other, although silence will be more beneficial for them.

Thus, the players' egoism in the Prisoner's Dilemma leads them to the most unprofitable solution.

The opposite approach to the concept of equilibrium, based on altruism, is called the Berge equilibrium.

Definition 2. *A strategy profile* $x^B = (x_1^B, \dots, x_N^B) \in X$ *is called a Berge equilibrium (BE) in the game* (1), *if for each* $x \in X$ *the system of inequalities*

$$f_i(x^B) \geq f_i(x_i^B, x_{-i}) \quad (i \in \mathbf{N}) \quad (3)$$

is true.

The difference between Nash and Berge equilibria is that, in a Nash equilibrium, each player directs all efforts to increase individual payoff as much as possible. The antipode of (2) is (3), where each player strives to maximize the payoffs of the other players, ignoring its individual interests. Such an altruistic approach is intrinsic to kindred relations and occurs in religious communities. The elements of such altruism show up in charity, sponsorship, and so on.

In Example 1, players receive the best result if they use the Berge equilibrium, thus the Berge equilibrium solves the Tucker problem in the Prisoner's Dilemma.

Property 1. *A Berge equilibrium in game (1) with* $\mathbf{N} = = \{1, 2\}$ *coincides with a Nash equilibrium if both players interchange their payoff functions and then apply the concept of Nash equilibrium to solve the game.*

In view of Property 1, all results concerning the Nash equilibrium in the two-player game are automatically transferred to the Berge equilibrium (of course, with an "interchange" of the payoff functions as described by Property 1).

Differences appear when $N \geq 3$. So, the Berge equilibrium may not exist in a finite 3-person games. The example of this is given in [11]. The following example is taken from [11].

Example 2. Let us consider the following three-player game in which each of the players has two pure strategies. Pure strategies of the first, the second, and the third player are denoted as A_1, A_2; B_1, B_2; C_1, C_2, respectively.

$$C_1 : \begin{matrix} & B_1 & B_2 \\ \begin{matrix} A_1 \\ A_2 \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} (2,1,0) & (1,1,1) \\ (2,0,1) & (1,0,2) \end{pmatrix}} \end{matrix},$$

$$C_2 : \begin{matrix} & B_1 & B_2 \\ \begin{matrix} A_1 \\ A_2 \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} (1,2,0) & (0,2,1) \\ (1,1,1) & (0,1,2) \end{pmatrix}} \end{matrix}.$$

The left-hand matrix refers to the pure strategy C_1 of the third player, while the right-hand matrix refers to his/her pure strategy C_2. Let us note that this game is a very special one: none of the players has any possibility to influence their own payoff, even if they use any of their pure or mixed strategies. On the contrary, players' payoffs depend exclusively on the choices of the remaining players.

One can easily check that the second and the third players' best support to any of the first player's (pure or mixed) strategies is a pair of pure strategies (B_1, C_1); the first and the third players' best support to any of the second player's (pure or mixed) strategies is a pair of pure strategies (A_1, C_2); and finally, the first and the second players' best support to any of (pure or mixed) strategies of the third player is a pair of pure strategies (A_2, B_2). This game has no Berge equilibria, neither in pure, nor in mixed strategies.

Next, we recall the concept of Pareto optimality, and then formalize the Weak Berge Equilibrium.

Definition 3. *The alternative x^* is a Pareto-maximal alternative in the N-criteria problem*

$$\langle X, \{f_i(x)\}_{i\in\mathbf{N}}\rangle,$$

if the system of N inequalities

$$f_i(x) \leq f_i(x^*) \quad (i \in \mathbf{N}),$$

with at least one strict inequality, is inconsistent.

The moral basis of the following definition is the Hippocratic Oath "First do no harm!".

Definition 4. *Let us call the strategy profile $x^w = (x_1^w, \ldots, x_n^w)$ a weak Berge equilibrium (WBE), if for each player i ($i \in \mathbf{N}$) strategy x_i^w is Pareto-maximal alternative in the $N-1$-criteria problem*

$$\Gamma_i = \langle X_i, \{f_j(x_i, x_{-i}^w)\}_{j\in\mathbf{N}\setminus\{i\}}\rangle.$$

Let us compare to the game Γ an auxiliary game

$$\tilde{\Gamma} = \langle \mathbf{N}, \{X_i\}_{i\in\mathbf{N}}, \{g_i(x)\}_{i\in\mathbf{N}}\rangle, \tag{4}$$

where the set of players $\mathbf{N}$ and the set of strategies X_i ($i \in \mathbf{N}$) are the same as in the game (1), and the payoff functions $g_i(x)$ have the form

$$g_i(x) = \sum_{j\in\mathbf{N}\setminus\{i\}} f_j(x). \tag{5}$$

Lemma 1. *The Nash equilibrium strategy profile in the game (4) is a weak Berge equilibrium strategy profile in the game (1).*

Proof. Let x^e is Nash equilibrium strategy profile in the game $\tilde{\Gamma}$, i.e.,

$$g_i(x_1^e, \ldots, x_{i-1}^e, x_i, x_{i+1}^e, \ldots, x_n^e) \leq g_i(x^e) \quad (i \in \mathbf{N}). \tag{6}$$

With regard to (5), the inequality (6) can be rewritten as

$$\sum_{j\in\mathbf{N}\setminus\{i\}} f_j(x_i, x_{-i}^e) \leq \sum_{j\in\mathbf{N}\setminus\{i\}} f_j(x^e) \quad (i \in \mathbf{N}). \tag{7}$$

Suppose x^e is not a WBE strategy profile, then there exists a number i for which the system of inequalities is consistent with

$$f_j(x_i, x_{-i}^e \geq f_j(x^e) \quad (j \in \mathbf{N} \setminus \{i\}), \tag{8}$$

of which at least one inequality is strict.

Adding inequalities (8), we obtain

$$\sum_{j\in\mathbf{N}\setminus\{i\}} f_j(x_i, x_{-i}^e) > \sum_{j\in\mathbf{N}\setminus\{i\}} f_j(x^e) \quad (i \in \mathbf{N}),$$

that contradicts (7). □

Similarly, the following lemma can be obtained.

Lemma 2. *Let us suppose that in the auxiliary game (4) the payoff function is*

$$g_i(x) = \sum_{j\in\mathbf{N}\setminus\{i\}} \alpha_j f_j(x) \quad \forall \alpha_j \geqslant 0.$$

Then the Nash equilibrium strategy profile in the game (4) is the weak Berge equilibrium strategy profile in the original game (1).

Remark 1. To construct a WBE strategy profile in the game (1), we can use the following *algorithm*:

1. compose the auxiliary game $\tilde{\Gamma}$;
2. construct the strategy profile x^e which is the Nash equilibrium strategy profile in the auxiliary game $\tilde{\Gamma}$;
3. the found strategy profile x^e will be the WBE strategy profile in the original game Γ.

As an example, consider the game "Snowdrift" which is proposed in [15].

Example 3. Let us consider the three-player Snowdrift game which is shown in Table 1. The history of the game lies in the fact that A, B and C are the drivers of three cars, that got stuck in a snowdrift at night, each of them has a shovel. If a solution is found for any one car, others can use it. Every driver chooses to dig or wait (in the hope that someone else will dig, or that a snowplow will come to the place of incident). Digging will cost 6 points, which are divided equally between those who perform the work; provided that there is at least one digger. If the players dug out by themselves of a snowdrift, then each player will get 4 points. Thus, if all three players dig, then everyone will get 2 points. If two players dig, they will get one point each, and the third player will earn 4 points. If one player digs, then his payoff will be negative −2, and the payoffs of the remaining two players will be 4 points each. In the case that the players do not dig, but wait until the morning when the utilities arrive and clear the snow, their payoff will be zero.

Table 1: The three-player Snowdrift game.

C – to wait		
$A\backslash B$	to wait	to dig
to wait	(0,0,0)	(4,−2,4)
to dig	(−2,4,4)	(1,1,4)

C – to dig		
$A\backslash B$	to wait	to dig
to wait	(4,4,−2)	(4,1,1)
to dig	(1,4,1)	(2,2,2)

Here, A, B, C are the 3-dimensional matrices, which determine the payoffs of the players will be

$$A: \quad A_1 = \begin{pmatrix} 0 & 4 \\ -2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4 & 4 \\ 1 & 2 \end{pmatrix};$$

$$B: \quad B_1 = \begin{pmatrix} 0 & -2 \\ 4 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 4 & 1 \\ 4 & 2 \end{pmatrix};$$

$$C: \quad C_1 = \begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The Nash equilibrium (NE) here will be (wait, wait, wait) with payoffs (0, 0, 0).

We will now compile an auxiliary game, the payoff matrices will be:
for the first player

$$A^* = B + C: \quad A_1^* = \begin{pmatrix} 0 & 2 \\ 8 & 5 \end{pmatrix}, \quad A_2^* = \begin{pmatrix} 2 & 2 \\ 5 & 4 \end{pmatrix};$$

for the second player

$$B^* = A + C: \quad B_1^* = \begin{pmatrix} 0 & 8 \\ 2 & 5 \end{pmatrix}, \quad B_2^* = \begin{pmatrix} 2 & 5 \\ 2 & 4 \end{pmatrix};$$

for the third player

$$C^* = A + B: \quad C_1^* = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}, \quad C_2^* = \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix}.$$

The Nash equilibrium in the auxiliary game with matrices A^*, B^*, C^* will be (dig, dig, dig), respectively, the weak Berge equilibrium (WBE) in the original game will also be (dig, dig, dig) with payoffs (2, 2, 2).

Obviously, in this example, the WBE is more profitable for all players than the NE.

Remark 2. In this game, the Berge equilibrium (BE) coincides with the WBE.

Following to Lemma 1, it is possible to obtain sufficient conditions for the existence of the WBE under the usual restrictions for the game theory.

Theorem 1. *In a non-cooperative N-person game Γ with a finite set of strategies, a weak Berge equilibrium strategy profile in mixed strategies exists.*

Theorem 2. *If in a non-cooperative N-person game Γ, the sets of strategies X_i are convex compacts, and the payoff functions $f_i(x)$ are continuous in the aggregate of variables, then in the game Γ a weak Berge equilibrium strategy profile in mixed strategies exists.*

3 The WBE in a finite three-person game

This section consists of two parts. In subsection 3.1, a finite 3-person game is formalized and the 3LP-method for solving the finite 3-persons game is described. In subsection

3.2, the test results for computing the WBE in the finite 3-person game are presented.

3.1 The 3LP-method for solving the finite 3-person game

Let us consider a non-cooperative three-person game.

$$\Gamma_3 = \langle \{1,2,3\}, \{X_i\}_{i=1,2,3}, \{f_i(x)\}_{i=1,2,3} \rangle.$$

The strategy profile $x^w = (x_1^w, x_2^w, x_3^w)$ is the WBE strategy profile, if and only if

1) the strategy x_1^w is the Pareto maximum alternative in the two-criterial problem

$$\langle X_1, \{f_2(x_1, x_2^w, x_3^w), f_3(x_1, x_2^w, x_3^w)\}\rangle;$$

2) the strategy x_2^w is the Pareto maximum alternative in the two-criterial problem

$$\langle X_2, \{f_1(x_1^w, x_2, x_3^w), f_3(x_1^w, x_2, x_3^w)\}\rangle;$$

3) the strategy x_3^w is the Pareto maximum alternative in the two-criterial problem

$$\langle X_3, \{f_1(x_1^w, x_2^w, x_3), f_2(x_1^w, x_2^w, x_3)\}\rangle.$$

Let us compose an axillary game for the game Γ_3

$$\tilde{\Gamma}_3 = \langle \{1,2,3\}, \{X_i\}_{i=1,2,3}, \{g_i(x)\}_{i=1,2,3} \rangle,$$

where, according to (5)

$$g_1(x) = f_2(x) + f_3(x), \qquad (9)$$
$$g_2(x) = f_1(x) + f_3(x),$$
$$g_3(x) = f_1(x) + f_2(x).$$

The Nash equilibrium strategy profile in $\tilde{\Gamma}_3$ will be the WBE strategy profile in the original game, Γ_3.

Below, a finite non-cooperative 3-person game Γ_3 is defined with three sets X, Y, Z of strategies of the first, second, and third player respectively, where $X = \{x = (x_1, \dots, x_m)^T \in \mathbf{R}^m \ : \ x^T e_m = 1,\ x \geq 0_m\}$, $Y = \{y = (y_1, \dots, y_n)^T \in \mathbf{R}^n \ : \ y^T e_n = 1,\ y \geq 0_n\}$, $Z = \{z = (z_1, \dots, z_l)^T \in \mathbf{R}^l \ : \ z^T e_l = 1,\ z \geq 0_l\}$, $\omega = (x, y, z) \in \mathbf{R}^{m+n+l}$, together with their payoff functions as follows

$$f_x(\omega) = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l a_{ijk} x_i y_j z_k,$$
$$f_y(\omega) = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l b_{ijk} x_i y_j z_k,$$
$$f_z(\omega) = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l c_{ijk} x_i y_j z_k.$$

Here, one has (a_{ijk}), (b_{ijk}), (c_{ijk}) is the players' 3-dimensional payoff tables (without any loss of generality one can assume that all the entries of those tables are positive real numbers); the vector $\omega^T = (x^T, y^T, z^T)$, $\omega \in \Omega = X \times Y \times Z \subset \mathbf{R}_+^{m+n+l}$. Next, for $p = m, n, l$, we define the vectors $0_p = (0, \dots, 0)^T \in \mathbf{R}_+^p$, $e_p = (1, \dots, 1)^T \in \mathbf{R}^p$, as well as $\mathbf{R}_+^p$ is the nonnegative orthant of the Euclidean space $\mathbf{R}^p$. The symbol ^{T} denotes the operation of transposition of a vector (matrix).

Following the algorithm in remark 1, we construct the functions (9).

$$g_x(\omega) = f_y(\omega) + f_z(\omega) = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l (b_{ijk} + c_{ijk}) x_i y_j z_k,$$
$$g_y(\omega) = f_x(\omega) + f_z(\omega) = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l (a_{ijk} + c_{ijk}) x_i y_j z_k,$$
$$g_z(\omega) = f_x(\omega) + f_y(\omega) = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l (a_{ijk} + b_{ijk}) x_i y_j z_k.$$

Introduce the Nash function $G(\omega) = \delta_x(\omega) + \delta_y(\omega) + \delta_z(\omega)$, where

$$\delta_x(\omega) = \max_{x' \in X} g(x', y, z) - g(\omega),$$
$$\delta_y(\omega) = \max_{y' \in Y} g(x, y', z) - g(\omega),$$
$$\delta_z(\omega) = \max_{z' \in Z} g(x, y, z') - g(\omega).$$

The function $G(\omega)$ is an analogue of the Nash function defined for the bi-matrix games [16]. As the above–defined payoff functions are linear with the respect to each variable x, y, z (when the other two variables are fixed), the auxiliary game $\tilde{\Gamma}_3$ is convex, hence the set of Nash points Ω^* is non-empty (but not necessarily convex).

Since $G(\omega) \geq 0$ for all $\omega \in \Omega$, and $G(\omega) = 0$ if, and only if ω is the Nash equilibrium of the game $\tilde{\Gamma}_3$, one can find the Nash equilibrium strategy profile of game $\tilde{\Gamma}_3$ as the global minimum (equalling zero) of the function $G(\omega)$ on Ω.

Now we turn to the approximately numerical method for the construction of the WBE in Γ_3. In [12] the algorithm for approximately solving finite non-cooperative three-person games (3LP) was proposed. The testing results illustrating the efficiency of applying this algorithm can be found in [13, 14].

Next we denote $\tilde{a}_{ijk} = b_{ijk} + c_{ijk}$, $\tilde{b}_{ijk} = a_{ijk} + c_{ijk}$, $\tilde{c}_{ijk} = a_{ijk} + b_{ijk}$ and $d_{ijk} = \tilde{a}_{ijk} + \tilde{b}_{ijk} + \tilde{c}_{ijk} = 2(a_{ijk} + b_{ijk} + c_{ijk})$.

Set the iteration counter $t = 0$. As a starting strategy, one can use any pair of the players' pure strategies (the total number of such pairs is $mn + ml + nl$); for example, fix a pair of strategies $\{y^{(0)}, z^{(0)}\}$ with the components $y_1^{(0)} = 1$, $y_j^{(0)} = 0$ $(j = 2, \dots, n)$, $z_1^{(0)} = 1$, $z_k^{(0)} = 0$ $(k = 2, \dots, l)$, and solve successively (for $t = 0, 1, \dots$) the triple problem $P_x(x^{(t+1)}, y^{(t)}, z^{(t)})$, $P_y(x^{(t+1)}, y^{(t+1)}, z^{(t)})$, $P_z(x^{(t+1)}, y^{(t+1)}, z^{(t+1)})$, where

$$P_x(x, y', z'): \quad \begin{array}{l} \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\sum_{k=1}^{l} d_{ijk} y'_j z'_k\right) x_i - \beta - \gamma \to \max_{x,\beta,\gamma}, \\ \sum_{i=1}^{m}\left(\sum_{k=1}^{l} \tilde{b}_{ijk} z'_k\right) x_i - \beta \le 0, \quad j = 1, \dots, n, \\ \sum_{i=1}^{m}\left(\sum_{j=1}^{n} \tilde{c}_{ijk} y'_j\right) x_i - \gamma \le 0, \quad k = 1, \dots, l, \\ x^T e_m = 1, \quad x \ge 0_m, \quad \beta, \gamma \in \mathbf{R}^1_+. \end{array}$$

If $x^\star$ is an optimal solution of this problem, then we set $x' := x^\star$; next we solve:

$$P_y(x', y, z'): \quad \begin{array}{l} \sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{l} d_{ijk} x'_i z'_k\right) y_j - \alpha - \gamma \to \max_{y,\alpha,\gamma}, \\ \sum_{j=1}^{n}\left(\sum_{k=1}^{l} \tilde{a}_{ijk} z'_k\right) y_j - \alpha \le 0, \quad i = 1, \dots, m, \\ \sum_{j=1}^{n}\left(\sum_{i=1}^{m} \tilde{c}_{ijk} x'_i\right) y_j - \gamma \le 0, \quad k = 1, \dots, l, \\ y^T e_n = 1, \quad y \ge 0_n, \quad \alpha, \gamma \in \mathbf{R}^1_+. \end{array}$$

Again, if $y^\star$ is an optimal plan for the above problem, then put $y' := y^\star$, and continue solving:

$$P_z(x', y', z): \quad \begin{array}{l} \sum_{k=1}^{l}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ijk} x'_i y'_j\right) z_k - \alpha - \beta \to \max_{z,\alpha,\beta}, \\ \sum_{k=1}^{l}\left(\sum_{i=1}^{m} \tilde{b}_{ijk} x'_k\right) z_k - \alpha \le 0, \quad j = 1, \dots, n, \\ \sum_{k=1}^{l}\left(\sum_{j=1}^{n} \tilde{c}_{ijk} y'_j\right) z_k - \beta \le 0, \quad i = 1, \dots, m, \\ z^T e_l = 1, \quad z \ge 0_l, \quad \alpha, \beta \in \mathbf{R}^1_+. \end{array}$$

Now that $z^\star$ is an optimal solution of that problem, we denote $z' := z^\star$.

The optimal objective function values $G_t = G(\omega^{(t+1)})$ are monotone non-increasing by t. The iteration process continues until the value G_t stabilizes, that is, for some $t^\star$, the difference $G_{t^\star} - G_{t^\star+1}$ becomes small enough. In addition, if $G_{t^\star} = 0$, it means that an (exact) Nash point has been found. If the value $G_{t^\star}$ is positive but small enough, an approximate solution of the game is reported. Otherwise, select a new pair of the initial strategies and start the process again (probably, having altered the order of the solved problems P_x, P_y, P_z).

3.2 Test results for the 3LP-algorithms for finding the WBE

We tested the algorithms finding the weak Berge equilibrium in the finite 3-persons games by making use of the personal computer with the processor Intel(R) Core(TM) i5-3427U (CPU @ 1.80GHz 2.300 GHz, memory 4.00 GB, 4 cores). The test codes were written in the MatLab. A series of 10 games was solved for each triple n, m, l.

We investigated 2 cases: an independent matrices and a mutually dependent matrices. In the first case (independent matrices) we used pseudo-random counters to generate independently the elements of the tables a_{ijk}, b_{ijk}, c_{ijk} $(1 \le i \le m, 1 \le j \le n, 1 \le k \le l)$.

For the game with mutually dependent matrices, we first used a pseudo-random counters to generate independently the elements of the auxiliary tables a'_{ijk}, b'_{ijk}, c'_{ijk} $(1 \le i \le m, 1 \le j \le n, 1 \le k \le l)$. At the second stage, we constructed the mutually dependent payoff tables by the formulas

$$a_{ijk} = a'_{ijk} - \lambda \frac{b'_{ijk} + c'_{ijk}}{2} + 1,$$
$$b_{ijk} = b'_{ijk} - \lambda \frac{a'_{ijk} + c'_{ijk}}{2} + 1,$$
$$c_{ijk} = c'_{ijk} - \lambda \frac{a'_{ijk} + b'_{ijk}}{2} + 1$$

for all $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le l$, where $0 < \lambda \le \frac{1}{2}$ is a covariance coefficient.

We solved games up to the dimension $dim = m = n = k = 100$. For comparison, using 3LP-algorithm, we calculated the Nash equilibrium for the same games.

Table 2 reports the results of the 3LP-algorithm solving the set of test games (5 series with 10 instances in each) with independent matrices. The algorithm switched to the next initial pair of strategies after having made dim iterations.

Here in Table 2, the following notation is used: $dim = m = n = k$ are the game's sizes (dimension); NE — the number of initial (starting) point when searching for a Nash equilibrium; WBE — number of starting points when searching for a weak Berge equilibrium; tNE — the total amount of time to search a Nash equilibrium for the series of 10 games (sec); $tWBE$ — the total amount of time to search a weak Berge equilibrium for the series of 10 games (sec).

Also in Table 3, for mutually dependent cases, the following notation is used: $dim = m = n = k$ are the game's sizes (dimension); WBE — number of starting points when searching for a weak Berge equilibrium; $tWBE$ — the total amount of time to search a weak Berge equilibrium for the

Table 2: The results of solving 5 series of games of ten problems with independent matrices

dim	NE	WBE	tNE	tWBE
20	327	85	745.85	129.88
40	230	59	539.28	99.34
60	169	40	404.43	88.1
80	129	28	373.14	92.03
100	159	41	904.85	162.62

series of 10 games (sec); itn - the total number of steps of 3LP algorithm. Covariance coefficient $\lambda = 0,4$ was used in the calculation of Table 3.

For mutually dependent cases, we have given the results only for the WBE, so when calculating the NE for these problems took an unacceptable time or they were not solved at all.

Table 3: The results of solving 5 series of games of ten problems with mutually dependent matrices

dim	itn	WBE	tWBE
20	3173	479	1106.49
40	6532	814	2101.15
60	12826	1415	5134.66
80	10306	1017	5564.54
100	16725	1527	13049.09

It is easy to see from the reported results (see Table 2 and Table 3), that the reciprocal dependence of the payoff matrices affect much to solve a problem by the 3LP-algorithm. The reciprocal dependence sufficiently increases the complexity of problems.

It is also clear that, the search WBE is much faster than the searching NE. This is most likely due to a pure strategy weak Berge equilibrium existing more often than a pure strategy Nash equilibrium.

4 Conclusion

In this paper, we formalized conception of the weak Berge equilibrium. The WBE follows the Hippocratic oath "First do no harm!". This conception is more often realized in practice than the Berge equilibrium conception. In contrast to the NE, the WBE always exists for every finite N-person games. As an example, we found the WBE in the finite three-person games using 3LP-algorithm.

In our opinion, the properties of the weak Berge equilibrium should be studied in more detail. In the future, the authors plan to study equilibrium in linear-quadratic games. Also we plan to develop a numerical method for finite 3-person games based on the Germeier convolution [17].

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