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Some Hilbert spaces related with the Dirichlet space

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Abstract: We study the reproducing kernel Hilbert space with kernel k^d , where d is a positive integer and k is the reproducing kernel of the analytic Dirichlet space.

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MSC: 30H25, 47B35

1 Introduction

Consider the Dirichlet space \mathcal{D} on the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane. It can be defined as the Reproducing Kernel Hilbert Space (RKHS) having kernel

$$k_z(w) = k(w, z) = \frac{1}{\bar{z}w} \log \frac{1}{1 - \bar{z}w} = \sum_{n=0}^{\infty} \frac{(\bar{z}w)^n}{n+1}.$$

We are interested in the spaces \mathcal{D}_d having kernel k^d , with $d \in \mathbb{N}$. \mathcal{D}_d can be thought of in terms of function spaces on polydiscs, following ideas of Aronszajn [4]. To explain this point of view, note that the tensor d -power $\mathcal{D}^{\otimes d}$ of the Dirichlet space has reproducing kernel $k_d(z_1, \dots, z_d; w_1, \dots, w_d) = \prod_{j=1}^d k(z_j, w_j)$. Hence, the space of restrictions of functions in $\mathcal{D}^{\otimes d}$ to the diagonal $z_1 = \dots = z_d$ has the reproducing kernel k^d , and therefore coincides with \mathcal{D}_d .

We will provide several equivalent norms for the spaces \mathcal{D}_d and their dual spaces in Theorem 1.1. Then we will discuss the properties of these spaces. More precisely, we will investigate:

- \mathcal{D}_d and its dual space HS_d in connection with Hankel operators of Hilbert-Schmidt class on the Dirichlet space \mathcal{D} ;
- the complete Nevanlinna-Pick property for \mathcal{D}_d ;
- the Carleson measures for these spaces.

Concerning the first item, the connection with Hilbert-Schmidt Hankel operators served as our original motivation for studying the spaces \mathcal{D}_d .

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Note that the spaces \mathcal{D}_d live infinitely close to \mathcal{D} in the scale of weighted Dirichlet spaces $\tilde{\mathcal{D}}_s$, defined by the norms

$$\|\varphi\|_{\tilde{\mathcal{D}}_s}^2 = \int_{-\pi}^{+\pi} |\varphi(e^{it})|^2 \frac{dt}{2\pi} + \int_{|z|<1} |\varphi'(z)|^2 (1-|z|^2)^s \frac{dA(z)}{\pi}, \quad 0 \leq s < 1,$$

where $\frac{dA(z)}{\pi}$ is normalized area measure on the unit disc.

Notation: We use multiindex notation. If $n = (n_1, \dots, n_d)$ belongs to \mathbb{N}^d , then $|n| = n_1 + \dots + n_d$. We write $A \approx B$ if A and B are quantities that depend on a certain family of variables, and there exist independent constants $0 < c < C$ such that $cA \leq B \leq CA$.

Equivalent norms for the spaces \mathcal{D}_d and their dual spaces HS_d

Theorem 1.1. *Let d be a positive integer and let*

$$a_d(k) = \sum_{|n|=k} \frac{1}{(n_1+1)\dots(n_d+1)}.$$

Then the norm of a function $\varphi(z) = \sum_{k=0}^{\infty} \widehat{\varphi}(k)z^k$ in \mathcal{D}_d is

$$\|\varphi\|_{\mathcal{D}_d} = \left(\sum_{k=0}^{\infty} a_d(k)^{-1} |\widehat{\varphi}(k)|^2 \right)^{1/2} \approx [\varphi]_d, \quad (1)$$

where

$$[\varphi]_d = \left(\sum_{k=0}^{\infty} \frac{k+1}{\log^{d-1}(k+2)} |\widehat{\varphi}(k)|^2 \right)^{1/2}. \quad (2)$$

An equivalent Hilbert norm $\|[\varphi]\|_d \approx [\varphi]_d$ for φ in terms of the values of φ is given by

$$\|[\varphi]\|_d = |\varphi(0)|^2 + \left(\int_{\mathbb{D}} |\varphi'(z)|^2 \frac{1}{\log^{d-1}\left(\frac{1}{1-|z|^2}\right)} \frac{dA(z)}{\pi} \right)^{1/2}. \quad (3)$$

Define now the holomorphic space HS_d by the norm:

$$\|\psi\|_{HS_d} = \left(\sum_{k=0}^{\infty} (k+1)^2 a_d(k) |\widehat{\psi}(k)|^2 \right)^{1/2}. \quad (4)$$

Then, $HS_d \equiv (\mathcal{D}_d)^*$ is the dual space of \mathcal{D}_d under the duality pairing of \mathcal{D} . Moreover,

$$\begin{aligned} \|\psi\|_{HS_d} &\approx [\psi]_{HS_d} := \left(\sum_{k=0}^{\infty} (k+1) \log^{d-1}(k+2) |\widehat{\psi}(k)|^2 \right)^{1/2} \approx \\ &[\psi]_{HS_d} := \left(|\psi(0)|^2 + \int_{\mathbb{D}} |\psi'(z)|^2 \log^{d-1}\left(\frac{1}{1-|z|^2}\right) \frac{dA(z)}{\pi} \right)^{1/2}. \end{aligned} \quad (5)$$

Furthermore, the norm can be written as

$$\|\psi\|_{HS_d}^2 = \sum_{(n_1, \dots, n_d)} |\langle e_{n_1} \dots e_{n_d}, \psi \rangle_{\mathcal{D}}|^2, \quad (6)$$

where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis of \mathcal{D} , $e_n(z) = \frac{z^n}{\sqrt{n+1}}$.

The remainder of this section is devoted to the proof of Theorem 1.1. The expression for $\|\varphi\|_{\mathcal{D}_d}$ in (1) follows by expanding $(k_z)^d$ as a power series. The equivalence $\|\varphi\|_{\mathcal{D}_d} \approx [\varphi]_d$, as well as $\|\varphi\|_{HS_d} \approx [\varphi]_{HS_d}$, are consequences of the following lemma. We denote by c, C positive constants which are allowed to depend on d only, whose precise value can change from line to line.

Lemma 1.2. *For each $d \in \mathbb{N}$ there are constants $c, C > 0$ such that for all $k \geq 0$ we have*

$$ca_d(k) \leq \frac{\log^{d-1}(k+2)}{k+1} \leq Ca_d(k).$$

Consequently, if $t \in (0, 1)$, then

$$c \left(\frac{1}{t} \log \frac{1}{1-t} \right)^d \leq \sum_{k=0}^{\infty} \frac{\log^{d-1}(k+2)}{k+1} t^k \leq C \left(\frac{1}{t} \log \frac{1}{1-t} \right)^d.$$

Proof of Lemma 1.2. We will prove the Lemma by induction on $d \in \mathbb{N}$. It is obvious for $d = 1$. Thus let $d \geq 2$ and suppose the lemma is true for $d - 1$. Also we observe that there is a constant $c > 0$ such that for all $k \geq 0$ and $0 \leq n \leq k$ we have

$$c \log^{d-2}(k+2) \leq \log^{d-2}(n+2) + \log^{d-2}(k-n+2) \leq 2 \log^{d-2}(k+2).$$

Then for $k \geq 0$

$$\begin{aligned} a_d(k) &= \sum_{n_1+\dots+n_d=k} \frac{1}{(n_1+1)\dots(n_d+1)} \\ &= \sum_{n=0}^k \frac{1}{n+1} \sum_{n_2+\dots+n_d=k-n} \frac{1}{(n_2+1)\dots(n_d+1)} \\ &\approx \sum_{n=0}^k \frac{1}{n+1} \frac{\log^{d-2}(k-n+2)}{k-n+1} \quad \text{by the inductive assumption} \\ &= \frac{1}{2} \sum_{n=0}^k \frac{\log^{d-2}(n+2) + \log^{d-2}(k-n+2)}{(n+1)(k-n+1)} \\ &\approx \log^{d-2}(k+2) \sum_{n=0}^k \frac{1}{(n+1)(k-n+1)} \quad \text{by the earlier observation} \\ &= \frac{\log^{d-2}(k+2)}{k+2} \sum_{n=0}^k \frac{1}{n+1} + \frac{1}{k-n+1} \\ &\approx \frac{\log^{d-1}(k+2)}{k+1}. \end{aligned}$$

□

Next, we prove the equivalence $[\varphi]_{HS_d} \approx [|\varphi|]_{HS_d}$ which appears in (5).

Lemma 1.3. *Let $d \in \mathbb{N}$. Then*

$$\int_0^1 t^k \left(\frac{1}{t} \log \frac{1}{1-t} \right)^{d-1} dt \approx \frac{\log^{d-1}(k+2)}{k+1}, \quad k \geq d.$$

Given the Lemma, we expand

$$\begin{aligned} [|\psi|]_{HS_d}^2 &= |\widehat{\psi}(0)|^2 + \int_{\mathbb{D}} \left| \sum_{k=1}^{\infty} \widehat{\psi}(k) k z^{k-1} \right|^2 \log^{d-1} \frac{1}{1-|z|^2} \frac{dA(z)}{\pi} \\ &= |\widehat{\psi}(0)|^2 + \sum_{k=1}^{\infty} k^2 |\widehat{\psi}(k)|^2 \int_0^1 \log^{d-1} \frac{1}{1-t} t^{k-1} dt \end{aligned}$$

$$\begin{aligned} &\approx |\widehat{\psi}(0)|^2 + \sum_{k=1}^{\infty} k^2 |\widehat{\psi}(k)|^2 \frac{\log^{d-1}(k+2)}{k+1} \\ &\approx [\psi]_{HS_d}^2, \end{aligned}$$

obtaining the desired conclusion.

Proof of Lemma 1.3. The case $d = 1$ is obvious, leaving us to consider $d \geq 2$. We will also assume that $k \geq 2$. Then by Lemma 1.2 we have

$$\int_0^1 t^k \left(\frac{1}{t} \log \frac{1}{1-t} \right)^{d-1} dt \approx \int_0^1 t^k \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{n+1} t^n dt = \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} = S_1 + S_2,$$

where

$$\begin{aligned} S_1 &= \sum_{n=0}^{k-1} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \approx \frac{1}{k+1} \sum_{n=0}^{k-1} \frac{\log^{d-2}(n+2)}{n+1} \approx \frac{1}{k+1} \int_1^{k+2} \frac{\log^{d-2}(t)}{t} dt \\ &= \frac{1}{d-1} \frac{\log^{d-1}(k+2)}{k+1} \end{aligned}$$

and

$$\begin{aligned} S_2 &= \sum_{n=k}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \leq \sum_{n=k+1}^{\infty} \frac{\log^{d-2}(n+1)}{n^2} \leq \sum_{j=1}^{\infty} \sum_{n=k^j}^{k^{j+1}-1} \frac{\log^{d-2}(n+1)}{n^2} \\ &\leq \sum_{j=1}^{\infty} (j+1)^{d-2} \log^{d-2} k \sum_{n=k^j}^{k^{j+1}-1} \frac{1}{n^2} \leq \log^{d-2}(k+2) \sum_{j=1}^{\infty} (j+1)^{d-2} \int_{k^{j-1}}^{\infty} \frac{1}{x^2} dx \\ &= \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{k+1}{k^j-1} \leq \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{k+1}{(k-1)k^{j-1}} \\ &\leq \frac{\log^{d-2}(k+2)}{k+1} \sum_{j=1}^{\infty} (j+1)^{d-2} \frac{3}{2^{j-1}} = o\left(\frac{\log^{d-1}(k+2)}{k+1}\right). \quad \square \end{aligned}$$

Now, the duality between \mathcal{D}_d and HS_d under the duality pairing given by the inner product of \mathcal{D} is easily seen by considering $[\cdot]_d$ and $[\cdot]_{HS_d}$. They are weighted ℓ^2 norms and duality is established by means of the Cauchy-Schwarz inequality.

Next we will prove that $[\varphi]_d \approx [|\varphi|]_d$. This is equivalent to proving that the dual space of HS_d , with respect to the Dirichlet inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}}$, is the Hilbert space with the norm $[\cdot]_d$.

Let $d \in \mathbb{N}$ and set, for $0 < t < 1$, $w_d(t) = \left(\frac{1}{t} \log \frac{1}{1-t}\right)^d$ and, for $0 < |z| < 1$, $W_d(z) = w_d(|z|^2)$ and $W_d(0) = 1$.

Lemma 1.4. *Let $d \in \mathbb{N}$. Then*

$$\int_{1-\varepsilon}^1 w_d(t) dt \cdot \int_{1-\varepsilon}^1 \frac{1}{w_d(t)} dt \approx \varepsilon^2 \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. Write $\tilde{w}(t) = \left(\log \frac{1}{1-t}\right)^d$, and note that it suffices to establish the lemma for \tilde{w} in place of w_d . Let $\varepsilon > 0$. Then \tilde{w} is increasing in $(0, 1)$ and $\tilde{w}(1 - \varepsilon^{k+1}) = (k+1)^d \left(\log \frac{1}{\varepsilon}\right)^d$, hence

$$\int_{1-\varepsilon}^1 \tilde{w}(t) dt = \sum_{k=1}^{\infty} \int_{1-\varepsilon^k}^{1-\varepsilon^{k+1}} \tilde{w}(t) dt \leq \sum_{k=1}^{\infty} \tilde{w}(1 - \varepsilon^{k+1})(\varepsilon^k - \varepsilon^{k+1})$$

$$= \sum_{k=1}^{\infty} (k+1)^d \left(\log \frac{1}{\varepsilon}\right)^d \varepsilon^k (1-\varepsilon) \approx \varepsilon \left(\log \frac{1}{\varepsilon}\right)^d \frac{1}{(1-\varepsilon)^d}$$

For $1/\tilde{w}$ we just notice that it is decreasing and hence

$$\int_{1-\varepsilon}^1 \frac{1}{\tilde{w}(t)} dt \leq \frac{1}{\tilde{w}(1-\varepsilon)} \varepsilon = \frac{\varepsilon}{\left(\log \frac{1}{\varepsilon}\right)^d}$$

Thus as $\varepsilon \rightarrow 0$ we have

$$\varepsilon^2 \leq \int_{1-\varepsilon}^1 \tilde{w}(t) dt \int_{1-\varepsilon}^1 \frac{1}{\tilde{w}(t)} dt = O(\varepsilon^2).$$

□

For $0 < h < 1$ and $s \in [-\pi, \pi]$ let $S_h(e^{is})$ be the Carleson square at e^{is} , i.e.

$$S_h(e^{is}) = \{re^{it} : 1-h < r < 1, |t-s| < h\}.$$

A positive function W on the unit disc is said to satisfy the Bekollé-Bonami condition (B2) if there exists $c > 0$ such that

$$\int_{S_h(e^{is})} W dA \cdot \int_{S_h(e^{is})} \frac{1}{W} dA \leq ch^4$$

for every Carleson square $S_h(e^{is})$. If $d \in \mathbb{N}$ and if $W_d(z)$ is defined as above, then

$$\int_{S_h(e^{is})} W_d dA \cdot \int_{S_h(e^{is})} \frac{1}{W_d} dA = h^2 \int_{1-h}^1 w_d(t) dt \cdot \int_{1-h}^1 \frac{1}{w_d(t)} dt \approx h^4$$

by Lemma 1.4, at least if $0 < h < 1/2$. Observe that both W_d and $1/W_d$ are positive and integrable in the unit disc, hence it follows that the estimate holds for all $0 < h \leq 1$.

Thus W_d satisfies the condition (B2). Furthermore, note that if $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$ is analytic in the open unit disc, then

$$\int_{|z|<1} |f(z)|^2 w_d(|z|^2) \frac{dA(z)}{\pi} = \sum_{k=0}^{\infty} w_k |\hat{f}(k)|^2,$$

where $w_k = \int_0^1 t^k w_d(t) dt \approx \frac{\log^d(k+2)}{k+1}$.

A special case of Theorem 2.1 of Luecking’s paper [7] says that if W satisfies the condition (B2) by Bekollé and Bonami [5], then one has a duality between the spaces $L_a^2(W dA)$ and $L_a^2(\frac{1}{W} dA)$ with respect to the pairing given by $\int_{|z|<1} f \bar{g} dA$. Thus, we have

$$\begin{aligned} \int_{|z|<1} |g(z)|^2 \frac{1}{W_d(z)} dA &\approx \sup_{f \neq 0} \frac{\left| \int_{|z|<1} g(z) \overline{f(z)} \frac{dA(z)}{\pi} \right|^2}{\int_{|z|<1} |f(z)|^2 W_d(z) dA} = \sup_{f \neq 0} \frac{\left| \sum_{k=0}^{\infty} \frac{\hat{g}(k)}{(k+1)\sqrt{w_k}} \sqrt{w_k} \overline{\hat{f}(k)} \right|^2}{\sum_{k=0}^{\infty} w_k |\hat{f}(k)|^2} \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+1)^2 w_k} |\hat{g}(k)|^2 \end{aligned}$$

This finishes the proof of (5). It remains to demonstrate (6). We defer its proof to the next section.

By Theorem 1.1 we have the following chain of inclusions:

$$\dots \hookrightarrow HS_{d+1} \hookrightarrow HS_d \hookrightarrow \dots \hookrightarrow HS_2 \hookrightarrow HS_1 = \mathcal{D} = \overline{\mathcal{D}}_1 \hookrightarrow \mathcal{D}_2 \hookrightarrow \dots \hookrightarrow \mathcal{D}_d \hookrightarrow \overline{\mathcal{D}}_{d+1} \hookrightarrow \dots$$

with duality w.r.t. \mathcal{D} linking spaces with the same index. It might be interesting to compare this sequence with the sequence of Banach spaces related to the Dirichlet spaces studied in [3]. Note that for $d \geq 3$ the reproducing kernel of HS_d is continuous up to the boundary. Hence functions in HS_d extend continuously to the closure of the unit disc, for $d \geq 3$.

Hilbert-Schmidt norms of Hankel-type operators

Let $\{e_n\}$ be the canonical orthonormal basis of \mathcal{D} , $e_n(z) = \frac{z^n}{\sqrt{n+1}}$. Equation (6) follows from the computation

$$\begin{aligned} \sum_{k=0}^{\infty} \sum_{|n|=k} |\langle e_{n_1 \dots e_{n_d}}, \psi \rangle|^2 &= \sum_{k=0}^{\infty} \sum_{|n|=k} \frac{1}{(n_1+1) \dots (n_d+1)} |\langle z^{n_1} \dots z^{n_d}, \psi \rangle|^2 \\ &= \sum_{k=0}^{\infty} \sum_{|n|=k} \frac{1}{(n_1+1) \dots (n_d+1)} |\langle z^k, \psi \rangle|^2 = \sum_{k=0}^{\infty} \sum_{|n|=k} \frac{(k+1)^2}{(n_1+1) \dots (n_d+1)} |\hat{\psi}(k)|^2 \\ &= \sum_{k=0}^{\infty} (k+1) a_d(k) |\hat{\psi}(k)|^2 \approx \sum_{k=0}^{\infty} \frac{\log^{d-1}(k+2)}{k+1} |\hat{\psi}(k)|^2. \end{aligned}$$

Polarizing this expression for $\|\cdot\|_{HS_d}$, the inner product of HS_d can be written

$$\langle \psi_1, \psi_2 \rangle_{HS_d} = \sum_{(n_1, \dots, n_d)} \langle \psi_1, e_{n_1} \dots e_{n_d} \rangle_{\mathcal{D}} \langle e_{n_1} \dots e_{n_d}, \psi_2 \rangle_{\mathcal{D}}.$$

Hence, for any $\lambda, \zeta \in \mathbb{D}$,

$$\begin{aligned} \langle k_\lambda, k_\zeta \rangle_{HS_d} &= \sum_{n \in \mathbb{N}^d} \langle k_\lambda, e_{n_1} \dots e_{n_d} \rangle_{\mathcal{D}} \langle e_{n_1} \dots e_{n_d}, k_\zeta \rangle_{\mathcal{D}} = \sum_{n \in \mathbb{N}^d} \overline{e_{n_1}(\lambda) \dots e_{n_d}(\lambda)} e_{n_1}(\zeta) \dots e_{n_d}(\zeta) \\ &= \left(\sum_{i=0}^{\infty} \overline{e_i(\lambda)} e_i(\zeta) \right)^d = k_\lambda(\zeta)^d = \langle k_\lambda^d, k_\zeta^d \rangle_{\mathcal{D}_d}. \end{aligned}$$

That is,

Proposition 1.5. *The map $U : k_\lambda \mapsto k_\lambda^d$ extends to a unitary map $HS_d \rightarrow \mathcal{D}_d$.*

When $d = 2$, HS_2 contains those functions b for which the Hankel operator $H_b : \mathcal{D} \rightarrow \overline{\mathcal{D}}$, defined by $\langle H_b e_j, \overline{e_k} \rangle_{\overline{\mathcal{D}}} = \langle e_j e_k, b \rangle_{\mathcal{D}}$, belongs to the Hilbert-Schmidt class.

Analogous interpretations can be given for $d \geq 3$, but then function spaces on polydiscs are involved. We consider the case $d = 3$, which is representative. Consider first the operator $T_b : \mathcal{D} \rightarrow \overline{\mathcal{D}} \otimes \overline{\mathcal{D}}$ defined by

$$\langle T_b f, \overline{g} \otimes \overline{h} \rangle_{\overline{\mathcal{D}} \otimes \overline{\mathcal{D}}} = \langle fgh, b \rangle_{\mathcal{D}}.$$

The formula uniquely defines an operator, whose action is

$$\begin{aligned} T_b f(z, w) &= \langle T_b f, \overline{k_z k_w} \rangle_{\overline{\mathcal{D}} \otimes \overline{\mathcal{D}}} \\ &= \langle f k_z k_w, b \rangle_{\mathcal{D}} \\ &= \sum_{n, m, j} \hat{f}(j) \frac{\overline{z}^n}{n+1} \frac{\overline{w}^m}{m+1} \langle \zeta^{n+m+j}, b \rangle_{\mathcal{D}} \\ &= \sum_{n, m, j} \hat{f}(j) \overline{\hat{b}(n+m+j)} \frac{n+m+j+1}{(n+1)(m+1)} \overline{z}^n \overline{w}^m \end{aligned}$$

Then, the Hilbert-Schmidt norm of T_b is:

$$\sum_{l, m, n} |\langle T_b e_l, e_m e_n \rangle_{\overline{\mathcal{D}} \otimes \overline{\mathcal{D}}}|^2 = \sum_{l, m, n} |\langle e_l e_m e_n, b \rangle_{\mathcal{D}}|^2 = \|b\|_{HS_3}^2.$$

Similarly, we can consider $U_b : \mathcal{D} \otimes \mathcal{D} \rightarrow \overline{\mathcal{D}}$ defined by

$$\langle U_b(f \otimes g), \overline{h} \rangle_{\overline{\mathcal{D}}} = \langle fgh, b \rangle_{\mathcal{D}}.$$

The action of this operator is given by

$$U_b(f \otimes g)(\bar{z}) = \sum_{l,m,n=0}^{\infty} \widehat{f}(l)\widehat{g}(m) \frac{(l+m+n+1)\widehat{b}(l+m+n)}{n+1} \bar{z}^n.$$

The Hilbert-Schmidt norm of U_b is still $\|b\|_{HS_3}$.

Carleson measures for the spaces \mathcal{D}_d and HS_d

The (B2) condition allows us to characterize the Carleson measures for the spaces \mathcal{D}_d and HS_d . Recall that a nonnegative Borel measure μ on the open unit disc is Carleson for the Hilbert function space H if the inequality

$$\int_{|z|<1} |f|^2 d\mu \leq C(\mu) \|f\|_H^2$$

holds with a constant $C(\mu)$ which is independent of f . The characterization [2] shows that, since the (B2) condition holds, then

Theorem 1.6. For $d \in \mathbb{N}$, a measure $\mu \geq 0$ on $\{|z| < 1\}$ is Carleson for \mathcal{D}_d if and only if for $|a| < 1$ we have:

$$\int_{\tilde{S}(a)} \log^{d-1} \left(\frac{1}{1-|z|^2} \right) (1-|z|^2) \mu(S(z) \cap S(a))^2 \frac{dx dy}{(1-|z|^2)^2} \leq C_1(\mu) \mu(S(a)),$$

where $S(a) = \{z : 0 < 1 - |z| < 1 - |a|, |\arg(z\bar{a})| < 1 - |a|\}$ is the Carleson box with vertex a and $\tilde{S}(a) = \{z : 0 < 1 - |z| < 2(1 - |a|), |\arg(z\bar{a})| < 2(1 - |a|)\}$ is its “dilation”.

The characterization extends to HS_2 , with the weight $\log^{-1} \left(\frac{1}{1-|z|^2} \right)$. Since functions in HS_d are continuous for $d \geq 3$, all finite measures are Carleson measures for these spaces. Once we know the Carleson measures, we can characterize the multipliers for \mathcal{D}_d in a standard way.

The complete Nevanlinna-Pick property for \mathcal{D}_d

Next, we prove that the spaces \mathcal{D}_d have the Complete Nevanlinna-Pick (CNP) Property. Much research has been done on kernels with the CNP property in the past twenty years, following seminal work of Sarason and Agler. See the monograph [1] for a comprehensive and very readable introduction to this topic. We give here a definition which is simple to state, although perhaps not the most conceptual. An irreducible kernel $k : X \times X \rightarrow \mathbb{C}$ has the CNP property if there is a positive definite function $F : X \rightarrow \mathbb{D}$ and a nowhere vanishing function $\delta : X \rightarrow \mathbb{C}$ such that:

$$k(x, y) = \frac{\bar{\delta}(x)\delta(y)}{1 - F(x, y)}$$

whenever x, y lie in X . The CNP property is a property of the kernel, not of the Hilbert space itself.

Theorem 1.7. There are norms on \mathcal{D}_d such that the CNP property holds.

Proof. A kernel $k : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ of the form $k(w, z) = \sum_{k=0}^{\infty} a_k (\bar{z}w)^k$ has the CNP property if $a_0 = 1$ and the sequence $\{a_n\}_{n=0}^{\infty}$ is positive and log-convex:

$$\frac{a_{n-1}}{a_n} \leq \frac{a_n}{a_{n+1}}.$$

See [1], Theorem 7.33 and Lemma 7.38. Consider $\eta(x) = \alpha \log \log(x) - \log(x)$, with real α . Then, $\eta''(x) = \frac{\log^2(x) - \alpha \log(x) - \alpha}{x^2 \log^2(x)}$, which is positive for $x \geq M_\alpha$, depending on α . Let now

$$a_n = \frac{\log^{d-1}(M_d(n+1))}{\log(M_d) \cdot (n+1)} \approx \frac{1}{n+1} + \frac{\log^{d-1}(n+1)}{n+1} \quad (7)$$

Then, the sequence $\{a_n\}_{n=0}^\infty$ provides the coefficients for a kernel with the CNP property for the space \mathcal{D}_d . \square

The CNP property has a number of consequences. For instance, we have that the space \mathcal{D}_d and its multiplier algebra $M(\mathcal{D}_d)$ have the same interpolating sequences. Recall that a sequence $Z = \{z_n\}_{n=0}^\infty$ is *interpolating* for a RKHS H with reproducing kernel k^H if the weighted restriction map $R : \varphi \mapsto \left\{ \frac{\varphi(z_n)}{k^H(z_n, z_n)^{1/2}} \right\}_{n=0}^\infty$ maps H boundedly onto ℓ^2 ; while Z is interpolating for the multiplier algebra $M(H)$ if $Q : \psi \mapsto \{\psi(z_n)\}_{n=0}^\infty$ maps $M(H)$ boundedly onto ℓ^∞ . The reader is referred to [1] and to the second chapter of [8] for more on this topic.

It is a reasonable guess that the *universal interpolating sequences* for \mathcal{D}_d and for its multiplier space $M(\mathcal{D}_d)$ are characterized by a Carleson condition and a separation condition, as described in [8] (see the Conjecture at p. 33). See also [6], which contains the best known result on interpolation in general RKHS spaces with the CNP property. Unfortunately we do not have an answer for the spaces \mathcal{D}_d .

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References

- [1] J. Agler, J. E. McCarthy, Pick interpolation and Hilbert function spaces. Graduate Studies in Mathematics, 44 (American Mathematical Society, Providence, RI, 2002), xx+308 pp. ISBN: 0-8218-2898-3.
- [2] N. Arcozzi, R. Rochberg, E. Sawyer, Carleson measures for analytic Besov spaces, Rev. Mat. Iberoamericana 18 (2002), no. 2, 443-510.
- [3] N. Arcozzi, R. Rochberg, E. Sawyer, B. D. Wick, Function spaces related to the Dirichlet space, J. Lond. Math. Soc. (2) 83 (2011), no. 1, 1-18.
- [4] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc. 68, (1950) 337-404.
- [5] D. Bekollé, A. Bonami, Inégalités à poids pour le noyau de Bergman, C. R. Acad. Sci. Paris Sér. A-B 286 (1978), no. 18, A775-A778 (in French).
- [6] B. Boe, An interpolation theorem for Hilbert spaces with Nevanlinna-Pick kernel, Proc. Amer. Math. Soc. 133 (2005), no. 7, 2077-2081.
- [7] D. H. Luecking, Representation and duality in weighted spaces of analytic functions, Indiana Univ. Math. J. 34 (1985), no. 2, 319-336.
- [8] K. Seip, Interpolation and sampling in spaces of analytic functions, University Lecture Series, 33 (American Mathematical Society, Providence, RI, 2004), xii+139 pp. ISBN: 0-8218-3554-8.