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Entropy bump conditions for fractional maximal and integral operators

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Abstract: We investigate weighted inequalities for fractional maximal operators and fractional integral operators. We work within the innovative framework of “entropy bounds” introduced by Treil–Volberg. Using techniques developed by Lacey and the second author, we are able to efficiently prove the weighted inequalities.

Keywords: Fractional integral operator, Fractional maximal operator, Weighted inequalities, Entropy bounds, Sparse operator

MSC: 42B20, 42B25

1 Introduction

We are concerned with two-weight inequalities for the fractional maximal and fractional integral operators. The goal is to find simple, A_p -like conditions for a pair of weights (non-negative, locally integrable functions) σ, w to ensure

$$\|T^\sigma : L^p(\sigma) \rightarrow L^q(w)\| < \infty, \quad (1)$$

where T denotes a fractional maximal or fractional integral operator, and $T^\sigma(f) := T(\sigma f)$.

One popular approach, initiated by Neugebauer in [12] and developed by Pérez in [13, 14], has been to slightly strengthen the A_p characteristic by introducing new factors. These new factors, known as bumps, have come in different forms. For example, Neugebauer requires that the weights $\sigma^{1+\epsilon}$ and $w^{1+\epsilon}$ belong to A_p , while Pérez requires that the two weights have finite Orlicz norm. The Orlicz approach is also taken by Cruz-Uribe and Moen in [2]. See the recent paper of Cruz-Uribe [3] and the references therein for more information.

In the context of Calderón–Zygmund operators, Treil–Volberg have recently introduced the notion of *entropy bounds* and are able to deduce stronger results than have been obtained using the Orlicz approach [18]. Lacey and the second author [9] simplified and extended the approach to the entropy conditions in the singular integral case. We use these same techniques to prove similar results for the fractional integral and fractional maximal operators. These results represent an extension of what is known, and can be proved by relatively simple techniques. In particular, we require that our weights satisfy certain “bump” or “separated bump” conditions (to be defined below.) It is not known to what extent these results are sharp. However, Treil and Volberg show that the bumps used here are - in many cases of interest - smaller than the Orlicz-based bumps.

Before stating the main theorems, we give some definitions. For $0 < \alpha < n$, the fractional maximal operator for functions defined on \mathbb{R}^n is

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$$M_\alpha f(x) := \sup_{Q \text{ a cube}} \frac{\mathbb{1}_Q(x)}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy,$$

and the fractional integral operator is

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^{n-\alpha}} dy.$$

In [11] Muckenhoupt and Wheeden give an A_p -like characteristic of a weight that is better suited for the fractional maximal and integral operators. The standard “two-weight” version of this characteristic is given by $[\sigma, w] := \sup_{Q \text{ a cube}} \sigma(Q)^{1/p'} w(Q)^{1/q} |Q|^{\alpha/n-1}$ (see for example [1–3, 14, 15]). Ideally we would like for (1) to hold when this is finite. However, it is well known that this condition is insufficient (see [1] for a counter example in the case of the Maximal operator).

This condition is enough, however, to deduce *weak-type* bounds for the maximal operator. We present an alternate proof of this well-known result in Section 1.1 as an example of the techniques used in the main theorems of this paper; see [1] for another proof. In particular, there holds:

Theorem 1.1. *With $[\sigma, w]$ defined as above, M_α the fractional maximal operator, and $1 \leq p \leq q \leq \infty$, there holds:*

$$\|M_\alpha(\sigma \cdot) : L^p(\sigma) \rightarrow L^{q,\infty}(w)\| \lesssim [\sigma, w].$$

Remark 1.2. *In an earlier draft of this paper, we claimed that the above inequality holds for the fractional integral operator as well. This is incorrect and we would like to thank Kabe Moen for pointing out this error.*

Since the finiteness of $[\sigma, w]$ is not enough to deduce strong bounds, we use two types of bumped conditions to deduce the strong estimates. The first set of conditions on the weights that we consider require a single bump (compare with the separated bumps to be discussed later). Set $\rho_\sigma(Q) := \frac{1}{\sigma(Q)} \int_Q M(\sigma \mathbb{1}_Q)$, and define ρ_w similarly, where M is the Hardy–Littlewood maximal operator. We deal first with the fractional maximal operator. In [13, 14], Pérez establishes bump conditions related to Theorems 1.3 and 1.4 using Orlicz norms.

Theorem 1.3. *Let σ and w be two weights, $1 < p \leq q < \infty$, and M_α be the fractional maximal operator. Let ϵ_q be a monotonic increasing function on $(1, \infty)$ that satisfies $\int_1^\infty \frac{dt}{t \epsilon_q^q(t)} = 1$. Define*

$$\beta(Q) := \frac{\sigma(Q)^{1/p'} w(Q)^{1/q}}{|Q|^{1-\alpha/n}} \rho_\sigma^{1/p}(Q) \epsilon_q(\rho_\sigma(Q)),$$

and set $[\sigma, w] := \sup_{Q \in \mathcal{D}} \beta(Q)$. Then

$$\|M_\alpha(f\sigma)\|_{L^q(w)} \lesssim [\sigma, w] \|f\|_{L^p(\sigma)}.$$

The corresponding theorem for the fractional integral operator is:

Theorem 1.4. *Let $1 \leq p \leq \infty$ and σ and w be two weights and let I_α be the fractional integral operator. Let ϵ_p be a monotonic increasing function on $(1, \infty)$ that satisfies $\int_1^\infty \frac{dt}{t \epsilon_p^p(t)} = 1$ and similarly for $\epsilon_{q'}$. Define:*

$$\beta(Q) := \frac{\sigma(Q)^{1/p'} w(Q)^{1/q}}{|Q|^{1-\alpha/n}} \rho_\sigma(Q)^{1/p} \epsilon_p(\rho_\sigma(Q)) \rho_w(Q)^{1/q'} \epsilon_{q'}(\rho_w(Q)),$$

and set $[\sigma, w] := \sup_{Q \in \mathcal{Q}} \beta(Q)$. Then

$$\|I_\alpha(f\sigma)\|_{L^q(w)} \lesssim C_{\alpha,n} [\sigma, w] \|f\|_{L^p(\sigma)}.$$

The $C_{\alpha,n}$ constant in the above and below theorems arise below in (7).

The condition in the next theorem is called a “separated bump” for obvious reasons. We use a bump defined in terms of the fractional maximal operator, namely

$$\varrho_\sigma^{\alpha,p,q}(Q) := \frac{\int_Q M_\alpha(\mathbb{1}_Q \sigma)^{q/p} dx}{\sigma(Q)^{q/p}},$$

or simply ϱ_σ or ϱ when clear. We have the following

Theorem 1.5. *Let σ and w be weights with densities, $1 < p \leq q < \infty$, and $\varepsilon_q, \varepsilon_{p'} : \mathbb{R}^+ \rightarrow \mathbb{R}$ be nonincreasing on $(0, 1)$ and nondecreasing on $(1, \infty)$ such that $\int_0^\infty \frac{dt}{t \varepsilon_q^{1/q}(t)}, \int_0^\infty \frac{dt}{t \varepsilon_{p'}^{1/p'}(t)} < \infty$. Define*

$$[[\sigma, w]]_{\alpha,p,q} := \sup_{Q \text{ a cube}} \left(|Q|^{\alpha/n} \langle \sigma \rangle_Q \right)^{q/p'} \langle w \rangle_Q \varrho_\sigma^{\alpha,p,q}(Q) \varepsilon_q \left(\varrho_\sigma^{\alpha,p,q}(Q) \right).$$

There holds:

$$\|I_\alpha^\sigma : L^p(\sigma) \rightarrow L^q(w)\| \lesssim C_{\alpha,n} \left([[\sigma, w]]_{\alpha,p,q}^{1/q} + [[w, \sigma]]_{\alpha,q',p'}^{1/p'} \right).$$

In Section 2, we give some preliminary information and lemmas that will be used below. In Section 3, we give a proof of the weak estimates. Sections 4 and 5 contain the proofs of the one–bump theorems for the fractional maximal and fractional integral operators. The proofs in these sections use the theory of sparse operators, discussed below, but avoid the explicit use of testing inequalities. Finally, Section 6 contains the proof of the separated bump theorem for the fractional integral operator. The proof uses both sparse operators and testing inequalities but is still elementary.

2 Preliminaries

In this section, we list several well–known results; we include some proofs because we could not find them in the literature. We start with some familiar definitions. For a measure μ , will write $\langle f \rangle_Q^\mu$ for $\frac{1}{\mu(Q)} \int_Q f$ and $\langle f \rangle_Q$ when μ is Lebesgue measure.

Definition 2.1. *A collection, \mathcal{D} of cubes is said to be a dyadic grid if:*

- (i) *The side length of every $Q \in \mathcal{D}$ equals 2^k for some $k \in \mathbb{Z}$.*
- (ii) *If $Q, R \in \mathcal{D}$ and $Q \cap R$ is not empty then either $Q \subset R$ or $R \subset Q$.*
- (iii) *If $\mathcal{D}_k = \{Q \in \mathcal{D} : \text{the side length of } Q \text{ equals } 2^k\}$, then $\mathbb{R}^n = \cup_{Q \in \mathcal{D}_k} Q$.*

Definition 2.2. *A subset S of a dyadic grid is said to be sparse if for every $P \in S$ there holds:*

$$\sum_{\substack{Q \in \mathcal{D}: Q \subsetneq P \\ Q \text{ is maximal}}} |Q| \leq \frac{1}{2} |P|.$$

Definition 2.3. *Given a measure μ on \mathbb{R}^n and a dyadic grid, \mathcal{D} , a sequence of positive numbers, $\{a_Q\}_{Q \in \mathcal{D}}$, is called a p, q –Carleson Sequence if for every $P \in \mathcal{D}$,*

$$\frac{1}{\mu(P)^{q/p}} \sum_{Q \in \mathcal{D}: Q \subset P} a_Q \lesssim 1. \tag{2}$$

The following is a variant of a Carleson Embedding Theorem. We are certain that Lemma 2.4 is contained in a paper, but we have not been able to find a reference. For the “continuous” version of this theorem, see [4].

Lemma 2.4. *Let μ be a measure on \mathbb{R}^n , \mathcal{D} be a dyadic grid, and $\{a_Q\}_{Q \in \mathcal{D}}$ be a p, q –Carleson Sequence. If $1 < p \leq q < \infty$, there holds:*

$$\sum_{Q \in \mathcal{D}} a_Q \left(\langle f \rangle_Q^\mu \right)^q \lesssim \|f\|_{L^p(\mu)}^q,$$

where the implied constant depends on p, q and the best constant in (2).

Proof. We will treat \mathcal{D} as a discrete measure space with measure ν where $\nu(Q) = a_Q$. We show that the operator T with rule $(Tf)(Q) = \langle f \rangle_Q^\mu$ satisfies $\|Tf\|_{L^q(\nu)}^q \lesssim \|f\|_{L^p(\mu)}^q$. The objective then is to show that for every $\lambda > 0$, there holds:

$$\lambda^q \nu(\{Tf > \lambda\}) \lesssim (\lambda^p \mu(Mf > \lambda))^{q/p}, \tag{3}$$

where M is the dyadic maximal function. The lemma follows from (3) since the dyadic maximal function is bounded for $p > 1$:

$$\|Tf\|_{L^q(\nu)}^q \simeq \sum_{k \in \mathbb{Z}} 2^{kq} \nu(\{Tf > 2^k\}) \lesssim \left(\sum_{k \in \mathbb{Z}} 2^{kp} \mu(\{Mf > 2^k\}) \right)^{q/p} \simeq \|Mf\|_{L^p(\mu)}^{q/p}.$$

We now turn to proving (3). Fix $\lambda > 0$, and let \mathcal{D}_λ be the maximal elements $Q \in \mathcal{D}$ such that $\langle f \rangle_Q^\mu > \lambda$ (such maximal cubes exist since $f \in L^p(\mu)$). Using the Carleson property of the sequence $\{a_Q\}_{Q \in \mathcal{D}}$, there holds:

$$\lambda^q \nu(\{Tf > \lambda\}) = \lambda^q \sum_{P \in \mathcal{D}_\lambda} \sum_{Q \in \mathcal{D}_\lambda: Q \subset P} a_Q \leq \sum_{P \in \mathcal{D}_\lambda} (\lambda^p \mu(P))^{q/p} \leq (\lambda^p \mu(\{Mf > \lambda\}))^{q/p}.$$

The last inequality follows by the disjointness of the $P \in \mathcal{D}_\lambda$ and the fact that $q/p \geq 1$. □

For a given dyadic grid, \mathcal{D} , define the dyadic fractional maximal operator:

$$M_\alpha^\mathcal{D} f(x) := \sup_{Q \in \mathcal{D}} \mathbb{1}_Q(x) |Q|^{\alpha/n} \langle f \rangle_Q$$

and the dyadic fractional integral operator:

$$I_\alpha^\mathcal{D} f(x) := \sum_{Q \in \mathcal{D}} |Q|^{\alpha/n} \langle f \rangle_Q \mathbb{1}_Q(x).$$

The following lemma is well-known and shows that fractional maximal and fractional integral operators can be estimated pointwise by sums of dyadic operators. For the proof of the fractional integral estimate see [2]; the proof of the estimate for the fractional maximal operator is obvious given the fact that for every cube, Q , there is a cube, P_Q in a dyadic grid such that $Q \subset P_Q$ and $|P_Q| \leq 3^n |Q|$:

Lemma 2.5. *Let M_α be the fractional maximal operator and I_α be the fractional integral operator. There is a collection of 3^n dyadic grids such that the following point-wise equivalences hold for all non-negative f :*

$$M_\alpha f \simeq \sum_{k=1}^{3^n} M_\alpha^{\mathcal{D}^k} f \quad \text{and} \quad I_\alpha f \simeq \sum_{k=1}^{3^n} I_\alpha^{\mathcal{D}^k} f.$$

Remark 2.6. *When proving the estimates below for the dyadic fractional maximal operator, it is more convenient to deal with the following truncated version:*

$$\mathbb{1}_{Q_0}(x) \sup_{Q \in \mathcal{D}: Q \subset Q_0} |Q|^{\alpha/n} \langle f \rangle_Q \mathbb{1}_Q(x). \tag{4}$$

We then prove estimates that are independent of Q_0 and appeal to the monotone convergence theorem to conclude the desired results. Assuming that f is finite almost everywhere (which will always be the case for us), we can further simplify matters. We start by building a stopping collection, \mathcal{S} . Initialise $\{Q_0\} \rightarrow \mathcal{S}$, and in the recursive stage, if $P \in \mathcal{S}$ is minimal, add to \mathcal{S} all maximal children Q of P such that $|Q|^{\alpha/n} \langle f \rangle_Q > 4|P|^{\alpha/n} \langle f \rangle_P$. For a cube $Q \subset Q_0$, let Q^S denote the \mathcal{S} -parent of Q . Similarly, let $ch(\mathcal{S})$ denote the maximal \mathcal{S} -descendants of \mathcal{S} . Finally, let $E_Q = Q \setminus ch(Q)$. A simple computation shows that for every $S \in \mathcal{S}$,

$$\sum_{Q \in ch(S)} |S| \leq \frac{1}{2} |S| \quad \text{and} \quad |S| \leq 2 |E_S|.$$

That is, the stopping collection \mathcal{S} is sparse. Additionally, the E_Q are pairwise disjoint and for almost every $x \in Q_0$ there is some Q with $x \in E_Q$ (this follows from the fact that $f = \infty$ on a set of measure zero). Thus, we may further reduce (4) to:

$$\mathbb{1}_{Q_0}(x) \sup_{Q \in \mathcal{D}: Q \subset Q_0} |Q|^{\alpha/n} \langle f \rangle_Q \mathbb{1}_Q(x) = \sum_{Q \in \mathcal{S}: Q \subset Q_0} |Q|^{\alpha/n} \langle f \rangle_Q \mathbb{1}_{E_Q}(x). \tag{5}$$

We also note that if $\{E_Q\}_{Q \in \mathcal{D}}$ is any collection of pairwise disjoint sets such that $E_Q \subset Q$, then $\sum_{Q \in \mathcal{D}} |Q|^{\alpha/n} \langle f \rangle_Q \mathbb{1}_{E_Q}(x) \leq M_\alpha f(x)$.

There is a similar reduction for the dyadic fractional integral operator. Again, we may reduce matters to:

$$\mathbb{1}_{Q_0}(x) \sum_{Q \in \mathcal{D}: Q \subset Q_0} |Q|^{\alpha/n} \langle f \rangle_Q \mathbb{1}_Q(x). \tag{6}$$

We now create the stopping family by initialising $\{Q_0\} \rightarrow \mathcal{S}$ and in the recursive stage, if $P \in \mathcal{S}$ is minimal, add to \mathcal{S} all maximal children Q of P such that $\langle f \rangle_Q > 4\langle f \rangle_P$. Note that we are stopping on averages, not fractional averages. Again, simple computations show that \mathcal{S} is sparse. For fixed $x \in Q_0$, and fixed $S \in \mathcal{S}$, the sequence $\{|Q|^{\alpha/n} \mathbb{1}_Q(x)\}_{Q^S=S}$ is geometric and so

$$\sum_{Q^S=S} |Q|^{\alpha/n} \mathbb{1}_Q(x) \simeq C_{\alpha,n} |S|^{\alpha/n} \mathbb{1}_S(x). \tag{7}$$

Therefore, the sum in (6) can be estimated as:

$$\sum_{S \in \mathcal{S}} \sum_{Q^S=S} |Q|^{\alpha/n} \langle f \rangle_Q \mathbb{1}_Q(x) \lesssim \sum_{S \in \mathcal{S}} \langle f \rangle_S \sum_{Q^S=S} |Q|^{\alpha/n} \mathbb{1}_Q(x) \lesssim \sum_{S \in \mathcal{S}} |S|^{\alpha/n} \langle f \rangle_S \mathbb{1}_S(x). \tag{8}$$

Therefore, in all estimates below, for fixed f , we can replace the operator of interest with one from the right hand side of (5) or (8); our estimates will be independent of sparse collection \mathcal{S} and root Q_0 . \square

We have the following well-known testing conditions for dyadic operators, originally due to Sawyer. See [5, 8, 16].

Lemma 2.7. *Let $1 < p \leq q < \infty$, let \mathcal{D} be a dyadic grid and let $\mathcal{S} \subset \mathcal{D}$ be sparse. Let T be the operator given by $Tf = \sum_{Q \in \mathcal{S}} |Q|^{\alpha/n} \langle f \rangle_Q \mathbb{1}_Q$. Define:*

$$\beta_1 := \sup_{P \in \mathcal{S}} \frac{1}{\sigma(P)^{q/p}} \int_P \left| \sum_{Q \in \mathcal{S}: Q \subset P} |Q|^{\alpha/n} \langle \sigma \rangle_Q \mathbb{1}_Q(x) \right|^q w(x) dx,$$

$$\beta_2 := \sup_{P \in \mathcal{S}} \frac{1}{w(P)^{p'/q'}} \int_P \left| \sum_{Q \in \mathcal{S}: Q \subset P} |Q|^{\alpha/n} \langle w \rangle_Q \mathbb{1}_Q(x) \right|^{p'} \sigma(x) dx.$$

Then:

$$\|T_\sigma : L^p(\sigma) \rightarrow L^q(w)\| \lesssim \beta_1 + \beta_2.$$

3 Proof of Theorem 1.1

By Lemma 2.5, Theorem 1.1 follows from the following lemma.

Lemma 3.1. *Let $1 \leq p \leq q < \infty$ and σ and w be two weights. Let \mathcal{D} be a dyadic grid, and let M_α the dyadic fractional integral operator. Define:*

$$\beta(Q) = \frac{\sigma(Q)^{1/p'} w(Q)^{1/q} |Q|^{\alpha/n}}{|Q|}.$$

Set $[\sigma, w] := \sup_{Q \in \mathcal{D}} \beta(Q)$, then

$$\lambda^q w(\{M_\alpha f > \lambda\}) \lesssim [\sigma, w]^q \|f\|_{L^p(\sigma)}^q. \quad (9)$$

Proof. Let \mathcal{D}_λ be the maximal elements of \mathcal{D} contained in Q_0 such that $|Q|^{\alpha/n} \langle f \sigma \rangle_Q > \lambda$. Since $\langle f \sigma \rangle_Q = \langle f \rangle_Q^\sigma \langle \sigma \rangle_Q$, there holds:

$$\lambda^q w(\{M_\alpha f > \lambda\}) \leq \sum_{Q \in \mathcal{D}_\lambda} \lambda^q w(Q) \leq \sum_{Q \in \mathcal{D}_\lambda} |Q|^{\frac{q\alpha}{n}} \langle \sigma \rangle_Q^q w(Q) (\langle f \rangle_Q^\sigma)^q \leq [\sigma, w]^q \sum_{Q \in \mathcal{D}_\lambda} \sigma(Q)^{\frac{q}{p}} (\langle f \rangle_Q^\sigma)^q.$$

Given the disjointness of the sets $Q \in \mathcal{D}_\lambda$, (9) is immediate for $p = 1$. For $p > 1$, notice the sequence $\{\sigma(Q)^{q/p}\}_{Q \in \mathcal{D}_\lambda}$ is p, q -Carleson with respect to the measure σ . \square

4 Proof of Theorem 1.3

By Lemma 2.5, Theorem 1.3 follows from the following lemma. We remark that while the following proof does not make explicit use of the Sawyer Maximal testing inequalities in [17], the proof does use some of the same ideas.

Lemma 4.1. *Let $1 < p \leq q < \infty$, and let σ and w be two weights. Given a dyadic grid \mathcal{D} , let M_α be the dyadic fractional maximal operator. Let ϵ_q be a monotonic increasing function on $(1, \infty)$ that satisfies $\int_1^\infty \frac{dt}{t \epsilon_q^q(t)} = 1$. Define*

$$\beta(Q) := \frac{\sigma(Q)^{1/p'} w(Q)^{1/q}}{|Q|^{1-\alpha/n}} \rho_\sigma^{1/p}(Q) \epsilon_q(\rho_\sigma(Q)),$$

Set $[\sigma, w] := \sup_{Q \in \mathcal{Q}} \beta(Q)$, then

$$\|M_\alpha f \sigma\|_{L^q(w)} \lesssim [\sigma, w] \|f\|_{L^p(\sigma)}.$$

Proof. Let S be any sparse subset of \mathcal{D} . By Remark 2.6 and Lemma 2.7 we need to verify

$$\int_{Q_0} \left| \sum_{Q \in S: Q \subset Q_0} |Q|^{\alpha/n} \langle f \sigma \rangle_Q \mathbb{1}_{E_Q}(x) \right|^q w(x) dx \lesssim [\sigma, w]^q \|f\|_{L^p(\sigma)}^q. \quad (10)$$

Let $\mathcal{Q}_k := \{Q \in S, Q \subset Q_0 : [\sigma, w] 2^{-k} \leq \beta(Q) \leq [\sigma, w] 2^{-k+1}\}$. We will show

$$\int_{Q_0} \left| \sum_{Q \in \mathcal{Q}_k} |Q|^{\alpha/n} \langle f \sigma \rangle_Q \mathbb{1}_{E_Q}(x) \right|^q w(x) dx \lesssim (2^{-k})^q [\sigma, w]^q \|f\|_{L^p(\sigma)}^q. \quad (11)$$

Taking q^{th} roots and summing over k will imply (10).

Using the identity $\langle f \sigma \rangle_Q = \langle \sigma \rangle_Q \langle f \rangle_Q^\sigma$ and the pairwise disjointness of the sets E_Q , (11) will follow from:

$$\sum_{Q \in \mathcal{Q}_k} \frac{|Q|^{q\alpha/n} \sigma(Q)^q w(Q)}{|Q|^q} (\langle f \rangle_Q^\sigma)^q \lesssim (2^{-k})^q [\sigma, w]^q \|f\|_{L^p(\sigma)}^q.$$

Thus, by the Carleson Embedding Theorem (Lemma 2.4), it is enough to verify:

$$\frac{1}{\sigma(P)^{q/p}} \sum_{Q \in \mathcal{Q}_k: Q \subset P} \frac{|Q|^{q\alpha/n} \sigma(Q)^q w(Q)}{|Q|^q} \lesssim (2^{-k})^q [\sigma, w]^q,$$

for all $P \in \mathcal{Q}_k$. Using the fact that $\beta(Q) \simeq 2^{-k} [\sigma, w]$ for $Q \in \mathcal{Q}_k$ we estimate:

$$\sum_{Q \in \mathcal{Q}_k: Q \subset P} \frac{|Q|^{q\alpha/n} \sigma(Q)^q w(Q)}{|Q|^q} = \sum_{Q \in \mathcal{Q}_k: Q \subset P} \frac{|Q|^{q\alpha/n} \sigma(Q)^{q/p'} w(Q)}{|Q|^q} \sigma(Q)^{q/p}$$

$$\simeq (2^{-k})^q [\sigma, w]^q \sum_{Q \in \mathcal{Q}_k: Q \subset P} \frac{1}{\rho_\sigma(Q)^{q/p} \epsilon_q^q(\rho_\sigma(Q))} \sigma(Q)^{q/p}.$$

We want to show that the sum above is dominated by $\sigma(P)^{q/p}$. To this end, set $\mathcal{S}_r = \{Q \in \mathcal{Q}_k, Q \subset P : 2^{r-1} \leq \rho_\sigma(Q) \leq 2^r\}$. Thus, the sum above is dominated by

$$\sum_{r=0}^{\infty} \frac{1}{2^{rq/p} \epsilon_q^q(2^r)} \sum_{Q \in \mathcal{S}_r} \sigma(Q)^{q/p}.$$

Appealing to the summability condition on ϵ_q , it suffices to show that

$$\sum_{Q \in \mathcal{S}_r} \sigma(Q)^{q/p} \leq 2^{qr/p} \sigma(P)^{q/p}. \tag{12}$$

Let \mathcal{S}_r^* be the maximal elements in \mathcal{S}_r . Observe that for fixed $S^* \in \mathcal{S}_r^*$, and for any $P \subset S^*$, there holds:

$$\left(\int_{E_Q} \langle \mathbb{1}_{S^*} \sigma \rangle_Q \mathbb{1}_Q \right)^{q/p} \leq \left(\int_{E_Q} \sup_{P \in \mathcal{D}: P \subset S^*} \langle \mathbb{1}_{S^*} \sigma \rangle_P \mathbb{1}_P \right)^{q/p}.$$

Since the sets E_Q are pairwise disjoint, $|Q| \simeq |E_Q|$, and $\int_{S^*} \sup_{P \in \mathcal{D}: P \subset S^*} \langle \mathbb{1}_{S^*} \sigma \rangle_P \leq \sigma(S^*) \rho_\sigma(S^*) \simeq 2^r \sigma(S^*)$ for $S^* \in \mathcal{S}_r^*$, we estimate

$$\begin{aligned} \sum_{Q \in \mathcal{S}_r} \sigma(Q)^{q/p} &\leq \sum_{S^* \in \mathcal{S}_r^*} \sum_{Q \subset S^*} \left(\int_{E_Q} \sup_{P \in \mathcal{D}} \langle \mathbb{1}_{S^*} \sigma \rangle_P \mathbb{1}_P \right)^{q/p} \\ &\leq \sum_{S^* \in \mathcal{S}_r^*} \left(\int_{S^*} \sup_{P \in \mathcal{D}} \langle \mathbb{1}_{S^*} \sigma \rangle_P \mathbb{1}_P \right)^{q/p} \\ &\lesssim 2^{qr/p} \sum_{S^* \in \mathcal{S}_r^*} \sigma^{q/p}(S^*). \end{aligned}$$

Using the disjointness of the sets $S^* \in \mathcal{S}_r^*$, the sum in the last line above is dominated by $\sigma(P)^{q/p}$, completing the proof. \square

5 Proof of Theorem 1.4

By Lemma 2.5, Theorem 1.4 follows from the following lemma.

Lemma 5.1. *Let $1 < p \leq q < \infty$, and let σ and w be two weights. Given a dyadic grid \mathcal{D} , let $I_\alpha^\mathcal{D}$ be the dyadic fractional integral operator. Let ϵ_p be a monotone increasing function on $(1, \infty)$ such that $\int_1^\infty \frac{dt}{t \epsilon_p(t)} = 1$, and similarly for $\epsilon_{q'}$. Define*

$$\beta(Q) := \frac{\sigma(Q)^{1/p'} w(Q)^{1/q} |Q|^{\alpha/n}}{|Q|} \rho_\sigma(Q)^{1/p} \epsilon_p(\rho_\sigma(Q)) \rho_w(Q)^{1/q'} \epsilon_{q'}(\rho_w(Q)).$$

Set $[\sigma, w] := \sup_{Q \in \mathcal{Q}} \beta(Q)$, then

$$\|I_\alpha^\mathcal{D}(f\sigma)\|_{L^q(w)} \lesssim [\sigma, w] \|f\|_{L^p(\sigma)}.$$

Proof. We proceed by duality. Let $f \in L^p(\sigma)$ and $g \in L^{q'}(w)$. Below we use the identity: $\langle f\sigma \rangle_Q = \langle f \rangle_Q^\sigma \langle \sigma \rangle_Q$, where $\langle f \rangle_Q^\sigma := \sigma(Q)^{-1} \int_Q f \sigma$. Using the definition of $[\sigma, w]$, there holds:

$$\left\langle \sum_{Q \in \mathcal{Q}} |Q|^{\alpha/n} \langle f\sigma \rangle_Q \mathbb{1}_Q, gw \right\rangle_{L^2(dx)} = \sum_{Q \in \mathcal{Q}} \langle f \rangle_Q^\sigma \langle g \rangle_Q^w |Q|^{\alpha/n} \langle \sigma \rangle_Q w(Q)$$

$$\begin{aligned}
&= \sum_{Q \in \mathcal{Q}} \langle f \rangle_Q^\sigma \sigma(Q)^{\frac{1}{p}} \langle g \rangle_Q^w w(Q)^{\frac{1}{q'}} \frac{\sigma(Q)^{\frac{1}{p'}} w(Q)^{\frac{1}{q}}}{|Q|^{1-\frac{\alpha}{n}}} \\
&\lesssim [\sigma, w] \sum_{Q \in \mathcal{Q}} \frac{\langle f \rangle_Q^\sigma \sigma(Q)^{\frac{1}{p}}}{\rho_\sigma(Q)^{\frac{1}{p}} \epsilon_p(\rho_\sigma(Q))} \frac{\langle g \rangle_Q^w w(Q)^{\frac{1}{q'}}}{\rho_w(Q)^{\frac{1}{q'}} \epsilon_{q'}(\rho_w(Q))}.
\end{aligned}$$

By Hölder's inequality, it suffices to show that

$$\left(\sum_{Q \in \mathcal{S}} \frac{\sigma(Q)}{\rho_\sigma(Q) \epsilon_p(\rho_\sigma(Q))} (\langle f \rangle_Q^\sigma)^p \right)^{1/p} \quad \text{and} \quad \left(\sum_{Q \in \mathcal{S}} \frac{w(Q)^{p'/q'}}{\rho_w(Q)^{p'/q'} \epsilon_{q'}(\rho_w(Q))} (\langle g \rangle_Q^w)^{q'} \right)^{1/p'}$$

are dominated by $\|f\|_{L^p(\sigma)}$ and $\|g\|_{L^{q'}(w)}$, respectively. Since $p \leq q$, it follows that $q' \leq p'$, so by the Carleson Embedding Theorem (Lemma 2.4), it suffices to show:

$$\sum_{Q \in \mathcal{S}: Q \subset P} \frac{\sigma(Q)}{\rho_\sigma(Q) \epsilon_p(\rho_\sigma(Q))} \lesssim \sigma(Q_0) \quad \text{and} \quad \sum_{Q \in \mathcal{S}: S \subset P} \frac{w(Q)^{p'/q'}}{\rho_w(Q)^{p'/q'} \epsilon_{q'}(\rho_w(Q))} \langle g \rangle_Q^{wq'} \lesssim w^{p'/q'}(Q_0)$$

for all $Q_0 \in \mathcal{S}$. But the proof of each of these estimates is similar to those in Lemma 4.1 and we omit the details \square

6 Proof of Theorem 1.5

From Remark 2.6 and Lemma 2.7, it is enough to show

$$\int_{Q_0} \left| \sum_{Q \in \mathcal{Q}: Q \subset Q_0} |Q|^{\alpha/n} \langle \sigma \rangle_Q \mathbb{1}_Q(x) \right|^q w(x) dx \lesssim [[\sigma, w]]_{\alpha, p, q} \sigma(Q_0)^{q/p}$$

for any sparse collection \mathcal{Q} and $Q_0 \in \mathcal{Q}$ (the dual testing condition follows identically). For the remainder, fix a root Q_0 and let \mathcal{Q} be a sparse collection of cubes contained in Q_0 . Fix α, p, q in the respective appropriate range; we'll ignore these fixed indices where there is no confusion. It remains to show

$$\left\| \sum_{Q \in \mathcal{Q}} |Q|^{\alpha/n} \langle \sigma \rangle_Q \mathbb{1}_Q \right\|_{L^q(w, Q_0)} \lesssim [[\sigma, w]]^{1/q} \sigma(P)^{1/p}.$$

For $Q \in \mathcal{Q}$, define

$$\beta(Q) := \left(|Q|^{\alpha/n} \langle \sigma \rangle_Q \right)^{q/p'} \langle w \rangle_Q \epsilon_q(Q) \epsilon_q(\rho_\sigma(Q)).$$

For integers a and r , set $\mathcal{Q}^{a,r} := \{Q \in \mathcal{Q} : \beta(Q) \simeq 2^a, \rho(Q) \simeq 2^r\}$; notice $\mathcal{Q}^{a,r}$ is empty for a large enough. Construct a stopping family \mathcal{S} for the σ fractional averages: initialize \mathcal{S} by adding the cube Q_0 . In the recursive stage, if $Q \in \mathcal{S}$ is minimal and $P \in \mathcal{Q}^{a,r}$, $P \subset Q$ is maximal with $|P|^{\alpha/n} \langle \sigma \rangle_P > 4|Q|^{\alpha/n} \langle \sigma \rangle_Q$ add P to the collection \mathcal{S} . Denote by Q^S the \mathcal{S} -parent of Q . Partition $\mathcal{Q}^{a,r}$ into $\mathcal{Q}_k^{a,r}$, those cubes in $\mathcal{Q}^{a,r}$ such that $|Q|^{\alpha/n} \langle \sigma \rangle_Q \simeq 2^{-k} |Q^S|^{\alpha/n} \langle \sigma \rangle_{Q^S}$. We temporarily denote $\mathcal{Q}_k^{a,r}$ by \mathcal{Q}' . We will show

$$\left\| \sum_{Q \in \mathcal{Q}'} |Q|^{\alpha/n} \langle \sigma \rangle_Q \mathbb{1}_Q \right\|_{L^q(w)} \lesssim 2^{-k} \left[\sum_{S \in \mathcal{S}} |S|^{q\alpha/n} \langle \sigma \rangle_S^q w(S) \right]^{1/q}, \quad (13)$$

where summing over $k \geq -2$ gives

$$\left\| \sum_{Q \in \mathcal{Q}^{a,r}} |Q|^{\alpha/n} \langle \sigma \rangle_Q \mathbb{1}_Q \right\|_{L^q(w)} \lesssim \left[\sum_{S \in \mathcal{S}} |S|^{q\alpha/n} \langle \sigma \rangle_S^q w(S) \right]^{1/q}. \quad (14)$$

Define for each $S \in \mathcal{S}$

$$\Phi_S := \sum_{Q \in \mathcal{Q}': Q^S = S} |Q|^{\alpha/n} \langle \sigma \rangle_Q \mathbb{1}_Q \quad \text{and} \quad \Phi_{S,\ell} := \Phi_S \mathbb{1}_{\{\Phi_S \simeq \ell 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S\}}.$$

Since $\sum_{S \in \mathcal{S}} \Phi_{S,\ell}$ is geometric for fixed $\ell \in \mathbb{Z}^+$, Hölder's inequality yields

$$\left(\sum_{\ell \geq 1} \sum_{S \in \mathcal{S}} \Phi_{S,\ell} \right)^q \lesssim \sum_{\ell \geq 1} \ell^{2q/q'} \left(\sum_{S \in \mathcal{S}} \Phi_{S,\ell} \right)^q \simeq \sum_{\ell \geq 1} \ell^{2q/q'} \sum_{S \in \mathcal{S}} \Phi_{S,\ell}^q. \tag{15}$$

It is apparent that we need the following distributional estimate.

Lemma 6.1. *There holds*

$$w \left\{ \Phi_S > \lambda 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S \right\} \lesssim 2^{-\lambda} w(S).$$

Proof. The inequality is immediate in the case w is Lebesgue measure from sparseness of \mathcal{Q} . Notice that we have for $Q \in \mathcal{Q}'$ with $Q^S = S$,

$$\langle w \rangle_Q \simeq \frac{2^a}{2^r \varepsilon_q(2^r)} (2^{-k} \langle \sigma \rangle_S |S|^{\alpha/n})^{-q/p'} =: \tau_S,$$

where the equivalence is independent of S . Denote by \mathcal{Q}^* the maximal cubes in \mathcal{Q}' . Since the $\{\Phi_S > \lambda 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S\}$ is the union of the maximal cubes $P \in \mathcal{Q}'$ with $P^S = S$ and $\inf_{x \in P} \Phi_S(x) > \lambda 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S$, hence a disjoint union, it follows that

$$\begin{aligned} w \left\{ \Phi_S > \lambda 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S \right\} &\simeq \tau_S \left| \left\{ \Phi_S > \lambda 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S \right\} \right| \\ &\lesssim \tau_S \left(2^{-(\lambda-1)} \sum_{Q^* \in \mathcal{Q}^*} |Q^*| \right) \\ &\simeq 2^{-\lambda} \sum_{Q^* \in \mathcal{Q}^*} w(Q^*). \end{aligned}$$

The collection \mathcal{Q}^* is disjoint, so the proof is complete. □

Since $\{\Phi_{S,\ell} > \lambda 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S\} = \{\Phi_{S,\ell} > \lambda' 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S\}$ is constant for $0 < \lambda, \lambda' < \frac{\ell}{2}$ and $\{\Phi_{S,\ell} > \lambda 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S\}$ is empty for $\lambda > \ell$, we have

$$\begin{aligned} \int_{\mathcal{Q}_0} \Phi_{S,\ell}^q dw &= 2^{-kq} |S|^{q\alpha/n} \langle \sigma \rangle_S^q \int_0^\infty q \lambda^{q-1} w \{ \Phi_{S,\ell} > \lambda 2^{-k} |S|^{\alpha/n} \langle \sigma \rangle_S \} d\lambda \\ &\lesssim 2^{-kq} |S|^{q\alpha/n} \langle \sigma \rangle_S^q \left[\left(\frac{\ell}{2} \right)^q 2^{-\ell/2} w(S) + \frac{\ell}{2} q \ell^{q-1} 2^{-\ell/2} w(S) \right] \\ &\simeq 2^{-kq} |S|^{q\alpha/n} \langle \sigma \rangle_S^q \left[\ell^q 2^{-\ell/2} w(S) \right], \end{aligned}$$

where the second inequality is the application of Lemma 6.1. Recalling (15), this gives (13).

For each S define E_S to be S less the members of \mathcal{S} properly contained in S . Let \mathcal{S}^* be the maximal elements of \mathcal{S} . Since $\beta(S) \simeq 2^a$ and $\varrho(S) \simeq 2^r$ for all $S \in \mathcal{S}$, the right hand side of (14) is equivalent to

$$\left(\frac{2^a}{2^r \varepsilon_q(2^r)} \sum_{S \in \mathcal{S}} \left(|S|^{\alpha/n} \langle \sigma \rangle_S \right)^{q/p} |S| \right)^{1/q} \lesssim \left[\frac{2^a}{2^r \varepsilon_q(2^r)} \left(\sum_{S^* \in \mathcal{S}^*} \sum_{S^* \supseteq S \in \mathcal{S}} \int_{E_S} M_\alpha(\mathbb{1}_{S^*} \sigma)^{\frac{q}{p}} dx \right) \right]^{1/q}$$

$$\begin{aligned} &\simeq \left[\frac{2^a}{\varepsilon_q(2^r)} \left(\sum_{S^* \in \mathcal{S}^*} \sigma(S^*)^{a/p} \right) \right]^{1/q} \\ &\lesssim (2^{1/q})^a \frac{1}{\varepsilon_q^{1/q}(2^r)} \sigma(Q_0)^{1/p}. \end{aligned}$$

The first inequality above follows from $|S| \simeq |E_S| = \int_{E_S} dx$, and the third from $p \leq q$. Summing over integers $r \geq 0$ evokes the integrability condition on ε_q ; summing over relevant integers a completes the proof.

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