Concrete Operators

Research Article

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Hardy spaces of generalized analytic functions and composition operators

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Abstract: We present some recent results on Hardy spaces of generalized analytic functions on $\mathbb{D}$ specifying their link with the analytic Hardy spaces. Their definition can be extended to more general domains Ω. We discuss the way to extend such definitions to more general domains that depends on the regularity of the boundary of the domain $\partial\Omega$. The generalization over general domains leads to the study of the invertibility of composition operators between Hardy spaces of generalized analytic functions; at the end of the paper, we discuss invertibility and Fredholm property of the composition operator C_ϕ on Hardy spaces of generalized analytic functions on a simply connected Dini-smooth domain for an analytic symbol ϕ.

Keywords: Hardy spaces, Generalized analyticity, Composition operators

MSC: 30H10, 47B33, 30C62

1 Introduction

In this paper, we present some recent results on classes of generalized analytic functions from the point of view of function spaces and operator theory. Generalized analytic functions have been introduced in [28]. More recently, such functions have received a new interest with the work of V. Kravchenko in [19] where generalized analytic functions are used to solve some partial differential equations arising in Mathematical Physics (the Schrödinger equation for example). Indeed, general analytic functions are functions f defined on a domain Ω and are solutions in the distributional sense of the following $\overline{\partial}$-equation

$$\overline{\partial} f = \nu \overline{\partial f} \text{ or } \overline{\partial} w = \alpha \overline{w}, \tag{1}$$

with ν is in the Sobolev space $W^{1,r}(\Omega)$ and $\alpha \in L^r(\Omega)$ with $2 < r \leq \infty$. The first partial differential equation is called the conjugate Beltrami equation. When $\alpha = \nu = 0$, the functions f and w are analytic. Functions w satisfying the second partial differential equations are called pseudo-analytic functions. For some classes of coefficients α and ν, the equations in (1) are equivalent; this property has been shown in a work of Bers and Niremberg [8]. Pseudo-analytic functions are strongly related to analytic functions: a function $w \in G^p_\alpha(\Omega)$ can be written as $w = e^s F$ where $s \in W^{1,r}(\Omega)$ and F is analytic on Ω; this factorization property is called the Bers similarity principle and appeared in [8] and are explicitly studied in [4–6] for example.

A function f satisfying (1) is such that its real part and its imaginary part satisfy a generalized version of Cauchy-Riemann equations depending on a coefficient $\sigma \in W^{1,r}(\mathbb{D})$ related to ν. As in the analytic case, in a recent work [6], the authors considered the class of generalized analytic functions such that

$$\sup_{0<r<1} \|f\|_{L^p(r\mathbb{T})} < \infty \text{ respectively } \sup_{0<r<1} \|w\|_{L^p(r\mathbb{T})} < \infty, \tag{2}$$

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when $\Omega = \mathbb{D}$. This class of functions denoted by $H^p_\nu(\mathbb{D})$ and $G^p_\alpha(\mathbb{D})$ respectively are called Hardy spaces of generalized analytic functions referring to the class of Hardy spaces $H^p(\mathbb{D})$ (of analytic functions) when $\nu = 0$. As in the analytic case, the condition 2 guarantees the existence of a radial limit almost everywhere of f (w respectively) on $\mathbb{T}$. This property has a crucial role for the study of boundary value problems related to the partial differential equations in (1).

The H^p_ν and G^p_α can be defined when Ω is the annulus [13], smooth domains conformally equivalent to an annular domain [5], on $\mathbb{D}$ for a critical exponent $r = 2$ (corresponding to the case where the exponent is equal to the dimension of the domain) in [4].

A natural way to define the spaces H^p_ν and $G\alpha^p$ spaces on general domains is through the existence of a harmonic majorant for $|f|^p$ (respectively, $|w|^p$). When the domain Ω is regular enough (Dini-smooth domain see [5, 26]), the existence of a harmonic majorant is equivalent to the uniform boundedness of the L^p-norms of f (respectively w) over a sequence of rectifiable Jordan curves C_n of uniform bounded length where C_n is the boundary of a domain $\overline{D_n} \subset \Omega$:

$$\sup_{n\in\mathbb{N}} \|f\|_{L^p(C_n)} < \infty \text{ or } \sup_{n\in\mathbb{N}} \|w\|_{L^p(C_n)} < \infty.$$

where the integrals are considered with respect to arc-length measure $|dz|$. The last condition defines the Smirnov classes. When Ω is a simply connected domain such that $\partial\Omega$ is rectifiable, the Smirnov class can not be identified to the $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$ spaces defined through harmonic majorants.

In this paper, we present a short survey on some results on Hardy spaces of generalized analytic functions and we discuss their definition when Ω is a simply connected domain.

The paper is organized as follows: in the first Section, we focus on the Hardy spaces of pseudo-analytic functions defined on the unit disc, highlighting the link between the study of Hardy spaces of pseudo-analytic functions and the following boundary value problem:

$$\begin{cases} \operatorname{div}(\sigma\nabla u) = 0, & \text{on } \Omega \\ \quad u = \phi & \text{almost everywhere on } \partial\Omega \end{cases} \tag{3}$$

for certain classes of σ and ν. In the third Section, we give the definition of Hardy spaces of generalized analytic functions on simply-connected bounded domains. In the last Section, we consider the composition operator on $G^p_\alpha(\Omega)$ with Ω a Dini-smooth simply connected bounded domain generalizing to the case of $\alpha \in L^r(\Omega)$ extending some results obtained in [20].

Let us introduce some notations. We will denote by $\mathbb{D}$ the unit disc and $\mathbb{T}$ its boundary. The domain Ω we will consider is simply connected and bounded. The Lebesgue measure (on the complex plane or in the 1-dimensional case) will be denoted by m and $|dz|$ will denote the arc-length measure. We will define for $z = x + iy \in \mathbb{C}$ the following differential operators

$$\partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \quad \text{and} \quad \overline{\partial} = \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y),$$

where the derivatives are considered in the distributional sense: for ϕ a smooth compactly supported function on Ω,

$$\langle \partial f, \phi\rangle = -\int f\partial\phi, \quad \langle \overline{\partial} f, \phi\rangle = -\int f\overline{\partial}\phi.$$

The space $W^{1,r}(\Omega)$, $r > 2$ is the collection of functions defined on Ω such that f, ∂f and $\overline{\partial} f$ belong to $L^r(\Omega)$ and is equipped with the norm

$$\|f\|_{W^{1,r}(\Omega)} = \|f\|_{L^r(\Omega)} + \|\partial f\|_{L^r(\Omega)} + \|\overline{\partial} f\|_{L^r(\Omega)}.$$

We recall that when Ω is a domain such that its boundary is $\mathcal{C}^1$, the space $W^{1,r}(\Omega)$ embeds continuously in the space of Hölder smooth functions with exponent $1 - 2/p$ on Ω and thus functions in $W^{1,r}(\Omega)$ are continuous on $\overline{\Omega}$ [1, Chap 4].

We recall that a function g defined on $\mathbb{D}$ has a radial limit almost everywhere on $\mathbb{T}$ if there exists g^* defined almost everywhere on $\mathbb{T}$ such that

$$\lim_{r\to 1} g(r\zeta) = g^*(\zeta), \quad \text{for almost every } \zeta \in \mathbb{T}.$$

For $1 < p < \infty$, any function $f \in H^p(\mathbb{D})$ has a radial limit *a. e.* on $\mathbb{T}$ which we call the trace of f and is denoted by $\operatorname{tr} f$. For all $f \in H^p(\mathbb{D})$, we have that $\operatorname{tr}_{\mathbb{T}} f \in H^p(\mathbb{T})$ where $H^p(\mathbb{T})$ is the strict subspace of $L^p(\mathbb{T})$:

$$H^p(\mathbb{T}) = \left\{ h \in L^p(\mathbb{T}), \hat{h}(n) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})e^{-int}dt = 0,\ n < 0 \right\}.$$

We denote by $P_+ : L^p_{\mathbb{R}}(\mathbb{T}) \longrightarrow H^p(\mathbb{T})$ the Riesz projection which is bounded. The space $H^p(\mathbb{D})$ is isomorphic to $H^p(\mathbb{T})$ and $\|f\|_{H^p(\mathbb{D})} = \|\operatorname{tr}_{\mathbb{T}} f\|_{L^p(\mathbb{T})}$, which allows us to identify the two spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$.

Throughout this paper, even if it is not explicitly said, ν will be such that $|\nu(z)| \le k < 1$, $k \in (0,1)$ for $z \in \Omega$, σ is non-negative and p will be such that $1 < p < \infty$.

2 Hardy spaces of generalized analytic functions on $\mathbb{D}$

We denote by $\mathbb{D}$ the unit disc and by $\mathbb{T}$ the unit circle. Let $\nu \in W^{1,r}(\mathbb{D})$ with $r > 2$.

2.1 Generalized analyticity, generalized Cauchy-Riemann equations and the Dirichlet problem

Let f be an analytic function on $\mathbb{D}$. It is well-known that f is solution of the $\bar{\partial}$-equation $\bar{\partial} f = 0$ on $\mathbb{D}$. Conversely, by Weyl's Lemma [15, Corollary 24.10], any function f solution in the distributional sense of $\bar{\partial} f = 0$ is analytic on $\mathbb{D}$.

Now, we consider the following $\bar{\partial}$-equation in the distributional sense on $\mathbb{D}$:

$$\bar{\partial} f = \nu \overline{\partial f} \tag{4}$$

with $\nu \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$ and $r > 2$. The following Lemma sums up some properties of functions solutions of (4) introduced in [6] in $\mathbb{D}$ for $\nu \in W^{1,\infty}(\mathbb{D})$.

Lemma 2.1. *If f solves* (4), *$u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ satisfy the generalized Cauchy-Riemann equations as follows:*

$$\begin{cases} \partial_x v = -\sigma\ \partial_y u \\ \partial_y v = \ \ \sigma\ \partial_x u \end{cases}$$

in the distributional sense on $\mathbb{D}$, where $\sigma = \frac{1-\nu}{1+\nu} \in W^{1,r}(\mathbb{D})$. Moreover, we have that

$$\nabla.(\sigma \nabla u) = 0 \ \ \text{and} \ \ \nabla.\left(\frac{1}{\sigma}\nabla v\right) = 0, \tag{5}$$

in the distributional sense. Conversely, if u and v are two functions solution in the distributional sense of (5), *then $f = u + iv$ is solution of* (4).

In view of Equations (5), we can define the σ-harmonicity on $\mathbb{D}$ also called generalized harmonic functions. The definitions extend the notion of harmonic functions related to analytic functions.

Definition 2.2. *Let $u : \mathbb{D} \longrightarrow \mathbb{R}$ be a measurable function. The function u is said to be σ-harmonic if $u \in L^p(\mathbb{D})$ and u is solution in the distributional sense in $\mathbb{D}$ of*

$$\nabla.(\sigma \nabla u) = 0.$$

A measurable function $v : \mathbb{D} \longrightarrow \mathbb{R}$ is called a generalized harmonic conjugate of u if $v \in L^p(\mathbb{D})$ is solution in the distributional sense in $\mathbb{D}$ of

$$\nabla.\left(\frac{1}{\sigma}\nabla v\right) = 0,$$

and $f = u + iv$ is solution of (4).

We introduce another $\overline{\partial}$-equation connected to the conjugate Beltrami equations:

$$\overline{\partial} w = \alpha \overline{w}, \tag{6}$$

with $\alpha \in L^r(\mathbb{D})$. Functions solutions in the distributional sense on $\mathbb{D}$ of

$$\overline{\partial} B(z) = g(z)B(z) + h(z)\overline{B(z)}, \tag{7}$$

are called pseudo-analytic functions and can be expressed as functions solution of (6) (see [4, Section 3]). The relation between Equation (6) and (4) is given in the following Lemma.

Lemma 2.3. *Let $\alpha \in L^r(\mathbb{D})$ and $\nu \in W^{1,r}(\mathbb{D})$ be such that*

$$\alpha = -\frac{\overline{\partial}\nu}{1-\nu^2}.$$

If f is solution of (4), then $w = \frac{f-\nu\overline{f}}{\sqrt{1-\nu^2}} = J_\nu(f)$ solves (6).

Solutions of Equation (6) have the advantage of satisfying the Bers similarity principle [4, 6].

Proposition 2.4. *Let $\alpha \in L^r(\mathbb{D})$ and $w \in L^\gamma(\mathbb{D})$ with $\gamma > \frac{r-1}{r}$ solution of (6), then there exists $s \in W^{1,r}(\mathbb{D})$ and F analytic on $\mathbb{D}$ such that $w = e^s F$ and there exists $C > 0$ such that*

$$\|s\|_{W^{1,r}(\mathbb{D})} \lesssim \|\alpha\|_{L^r(\mathbb{D})}.$$

Proof. Let $w \in L^\gamma(\mathbb{D})$ be such that $\overline{\partial} w = \alpha\overline{w}$. We consider the function $A(z) = \alpha\frac{\overline{w(z)}}{w(z)}$ if $z \in \mathbb{D}$ and $w(z) \neq 0$ and $A(z) = 0$ otherwise. As $\|A\|_{L^r(\mathbb{D})} = \|\alpha\|_{L^r(\mathbb{D})}$, we have $A \in L^r(\mathbb{C})$. We define the function s on $\mathbb{D}$ by

$$s(z) = \frac{1}{2i\pi}\int_{\mathbb{D}} \frac{A(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

If we denote by $\mathcal{C}$ the Cauchy operator $\mathcal{C}(f)(z) = \frac{1}{2i\pi}\int_{\mathbb{C}} \frac{A(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}$ for $f \in L^r(\mathbb{C})$, bounded from $L^r(\mathbb{C})$ to $L^r(\mathbb{C})$ [17], then s is the restriction to $\mathbb{D}$ of the Cauchy operator applied to $A\chi_{\mathbb{D}}$. We denote by J the restriction operator from $L^r(\mathbb{C})$ onto $L^r(\mathbb{D})$. As the Cauchy operator is such that $\overline{\partial}\mathcal{C}(A) = A$, we have that

$$\overline{\partial}(J(\mathcal{C}(A\chi_{\mathbb{D}})) = J(\overline{\partial}\mathcal{C}(A)) = J(\chi_{\mathbb{D}}A) = \alpha\frac{\overline{w}}{w} \text{ or } 0 \text{ if } w(z) = 0.$$

It follows that $s \in L^r(\mathbb{D})$ and when $w(z) \neq 0$, $\overline{\partial} s = \alpha\frac{\overline{w}}{w} \in L^r(\mathbb{D})$. Now, $\partial J\mathcal{C}(A\chi_{\mathbb{D}}) = J\partial\mathcal{C}(A\chi_{\mathbb{D}}) = JS(A\chi_{\mathbb{D}})$, where S is the Beurling operator [17] defined by the singular integral

$$Sf(z) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|\zeta-z|>\varepsilon} \frac{A(\zeta)}{(\zeta-z)^2} d\zeta \wedge d\overline{\zeta}.$$

By continuity of the Cauchy operator and the Beurling operator on $L^r(\mathbb{C})$, it follows that $s \in W^{1,r}(\mathbb{D})$ and

$$\begin{aligned}\|s\|_{W^{1,r}(\mathbb{D})} &= \|s\|_{L^r(\mathbb{D})} + \|\partial s\|_{L^r(\mathbb{D})} + \|\overline{\partial} s\|_{L^r(\mathbb{D})}\\ &= \|J\mathcal{C}(A\chi_{\mathbb{D}})\|_{L^r(\mathbb{D})} + \|JS(A\chi_{\mathbb{D}})\|_{L^r(\mathbb{D})} + \|\alpha\|_{L^r(\mathbb{D})}\\ &\lesssim \|A\|_{L^r(\mathbb{D})} + \|A\chi_{\mathbb{D}}\|_{L^r(\mathbb{D})} + \|\alpha\|_{L^r(\mathbb{D})}\\ &\lesssim \|\alpha\|_{L^r(\mathbb{D})}.\end{aligned}$$

Now, let $F = e^{-s}w$. We first prove that F is analytic on $\mathbb{D}$ showing that $\overline{\partial}(e^{-s}w) = 0$. Let $(s_n)_n$ be a sequence of smooth compactly supported functions on $\mathbb{C}$ such that $s_{n|_{\mathbb{D}}}$ converges in $W^{1,r}(\mathbb{D})$ to s. Let ϕ be a smooth compactly supported function on $\mathbb{D}$. Note that e^{-s_n} is a smooth compactly supported function. We have

$$\langle\overline{\partial} F, \phi\rangle = -\langle F, \overline{\partial}\phi\rangle = \langle e^{-s}w, \phi\rangle = \langle \lim_{n\to\infty} e^{-s_n}w, , \overline{\partial}\phi\rangle.$$

Since by the dominated convergence theorem, we have that $\langle \lim_{n\to\infty} e^{-s_n} w, , \overline{\partial}\phi\rangle = \lim_{n\to\infty}\langle e^{-s_n} w, \overline{\partial}\phi\rangle$, it follows that

$$\begin{aligned}
\langle \overline{\partial} F, \phi\rangle &= -\lim_{n\to\infty}\langle w, e^{-s_n}\overline{\partial}\phi\rangle \\
&= -\lim_{n\to\infty}\langle w, \overline{\partial}(e^{-s_n}\phi) - \overline{\partial}(e^{-s_n})\phi\rangle \\
&= -\lim_{n\to\infty}\langle w, \overline{\partial}(e^{-s_n}\phi)\rangle - \lim_{n\to\infty}\langle w, \overline{\partial}(s_n)e^{-s_n}\phi\rangle \\
&= \lim_{n\to\infty}\langle \overline{\partial} w, e^{-s_n}\phi\rangle - \lim_{n\to\infty}\langle \overline{\partial}(s_n)e^{-s_n} w, \phi\rangle \\
&= \lim_{n\to\infty}\langle \overline{\partial} w, e^{-s_n}\phi\rangle - \lim_{n\to\infty}\langle \overline{\partial}(s_n)e^{-s_n} w, \phi\rangle \\
&= \lim_{n\to\infty}\langle \alpha\overline{w}e^{-s_n}, \phi\rangle - \langle \alpha\frac{\overline{w}}{w}e^{-s} w, \phi\rangle \\
&= \langle \alpha\overline{w}e^{-s}, \phi\rangle - \langle \alpha\overline{w}e^{-s}, \phi\rangle \\
&= 0.
\end{aligned}$$

By Weyl's lemma, we conclude that F is analytic. □

Remark 2.5. *Note that Proposition 2.4 implies that $w \in C(\mathbb{D})$. Indeed, writting $w = e^s F$, with $s \in W^{1,r}(\mathbb{D})$ and F analytic on $\mathbb{D}$, we have that F is continuous on $\mathbb{D}$ and it follows from the Sobolev embedding of $W^{1,r}(\mathbb{D})$ into $C(\overline{\mathbb{D}})$ that w is continuous on $\mathbb{D}$.*

Remark 2.6. *There exists a "reverse" Bers similarity principle established in [4]: for a given $F \in H^p(\mathbb{D})$ and $\alpha \in L^r(\mathbb{D})$, there exists $s \in W^{1,r}(\mathbb{D})$ such that $e^s F \in G^p_\alpha(\mathbb{D})$ with $Retr_{\mathbb{T}}(s) = 0$ on $\mathbb{T}$ and $\|s\|_{W^{1,r}(\mathbb{D})} \le C\|\alpha\|_{L^r(\mathbb{D})}$. This result permits to build some examples of functions in $G^p_\alpha(\mathbb{D})$. The "reverse" Bers similarity principle can be extended easily to the case of a bounded Dini-smooth domain (see [20]).*

The theory of pseudo-analytic functions and its development in the recent works [6, 18, 22] are motivated by the following Dirichlet problem: let ϕ, $\psi \in L^p_{\mathbb{R}}(\mathbb{T})$ with $1 < p < \infty$, we search a function u such that

$$\begin{cases} \nabla.(\sigma\nabla u) = 0 & \text{on } \mathbb{D} \\ u = \phi & \text{almost everywhere on } \mathbb{T}. \end{cases} \tag{8}$$

To solve such problem, we need to define the boundary value of u when $\nabla.(\sigma\nabla u) = 0$. An approach is to consider another Dirichlet problem that can be presented as a complex version of the Dirichlet problem 8.

$$\begin{cases} \overline{\partial} f = \nu\overline{\partial f} & \text{on } \mathbb{D} \\ \operatorname{Re} f = \psi & \text{almost everywhere on } \mathbb{T}. \end{cases} \tag{9}$$

To solve Problem 9, we use the link between Equations (4) and (6) given in Lemma 2.3. We thus consider the following problem related to Equation (6)

$$\begin{cases} \overline{\partial} w = \alpha\overline{w} & \text{on } \mathbb{D} \\ \operatorname{Re} w = \phi & \text{almost everywhere on } \mathbb{T}. \end{cases} \tag{10}$$

We will start solving Problem 10 taking advantages of the factorization in Proposition 2.4 and properties of analytic functions.

2.2 Radial limit and Hardy condition

In this subsection, we define a condition of existence of a radial limit for pseudo-analytic functions. We will deduce by Lemma 2.4 a condition for the existence of a radial limit for functions solution of the conjugate Beltrami equation.

Proposition 2.7. *Let $w,\ f \in L^{\gamma}(\mathbb{D})$ with $\gamma > \frac{r-1}{r}$, $1 < p < \infty$. We suppose that w and f are solution of Equation* (6) *and Equation* (4) *respectively.*

(i) *If*

$$\sup_{0<r<1} \|w\|_{L^p(r\mathbb{T})} < \infty, \tag{11}$$

then w has a radial limit almost everywhere on $\mathbb{T}$ and its radial limit denoted by $tr_{\mathbb{T}}\, w$ belongs to $L^p(\mathbb{T})$.

(ii) *If f is solution of Equation* (4) *and if f satisfies* (11) *then, f has a radial limit almost everywhere on $\mathbb{T}$ in $L^p(\mathbb{T})$.*

Proof. We start showing (i). Applying Proposition 2.4 to w, one can write $w = e^s F$ with $s \in W^{1,r}(\mathbb{D})$ and F is analytic. Since $s \in \mathcal{C}(\overline{\mathbb{D}})$ (see Remark), s is bounded on $\mathbb{T}$. Moreover, a function F satisfying the condition (11) has a radial limit (see [12, Chap. 2]) almost everywhere on $\mathbb{T}$ denoted by $\mathrm{tr}_{\mathbb{T}}\, F \in L^p(\mathbb{T})$. It follows that w has a radial limit almost everywhere denoted by $\mathrm{tr}_{\mathbb{T}}\, w \in L^p(\mathbb{T})$.

(ii) Let w be the function defined by $w = \frac{f - \nu \bar{f}}{\sqrt{1-\nu^2}}$. According to Lemma 2.3, we have that w solves Equation (6) with $\alpha = -\frac{\bar{\partial} \nu}{1-\nu^2}$ and w satisfies (11). The conclusion follows from (i). □

Remark 2.8. *Suppose that $w = e^s F$ with $s \in W^{1,r}(\mathbb{D})$ and F analytic on $\mathbb{D}$ as in Proposition 2.4. When w satisfies the Hardy condition* (11)*, F satisfies as well the Hardy condition which means that $F \in H^p(\mathbb{D})$. Moreover, if $w = e^s F$ with $s \in W^{1,r}(\mathbb{D})$ and F is analytic on $\mathbb{D}$, then $w \in G^p_\alpha(\mathbb{D})$ if and only if $F \in H^p(\mathbb{D})$.*

2.3 Hardy spaces of generalized analytic functions

Let $1 < p < \infty$. We define the Hardy spaces of pseudo-analytic functions that have been introduced in [6] for the unit disc.

Definition 2.9.

(i) *We define the Hardy space of pseudo-analytic function (also called generalized Hardy space) and denoted by $G^p_\alpha(\mathbb{D})$ the collection of functions w solutions of* (6) *and satisfying the Hardy condition*

$$\|w\|_{G^p_\alpha(\mathbb{D})} := \sup_{0<r<1} \|w\|_{L^p(r\mathbb{T})} < \infty, \tag{12}$$

(ii) *Similarly, the generalized Hardy space $H^p_\nu(\mathbb{D})$ is the collection of functions f solutions of* (4) *and satisfying the Hardy condition*

$$\|f\|_{H^p_\nu(\mathbb{D})} := \sup_{0<r<1} \|f\|_{L^p(r\mathbb{T})} < \infty. \tag{13}$$

Remark 2.10. *The generalized Hardy spaces $G^p_\alpha(\mathbb{D})$ and $H^p_\nu(\mathbb{D})$ equipped with the respective norms $\|.\|_{G^p_\alpha(\mathbb{D})}$ and $\|.\|_{H^p_\nu(\mathbb{D})}$ are real Banach spaces. When $\nu \equiv 0$ ($\alpha \equiv 0$ respectively), we have $H^p_\nu(\mathbb{D}) = H^p(\mathbb{D})$ ($G^p_\alpha(\mathbb{D}) = H^p(\mathbb{D})$ respectively) where $H^p(\mathbb{D})$ is the (analytic) Hardy space [16].*

Remark 2.11. *The mapping J_ν such that $J_\nu(f) = \frac{f-\nu\bar{f}}{\sqrt{1-\nu^2}}$ for $f \in H^p_\nu(\mathbb{D})$ and introduced in Lemma 2.3 defines an isomorphism from $H^p_\nu(\mathbb{D})$ onto $G^p_\alpha(\mathbb{D})$. As ν is continuous on $\mathbb{T}$ (by the Sobolev embedding of $W^{1,r}(\mathbb{D})$ into $\mathcal{C}(\overline{\mathbb{D}})$), J_ν can be extended to an isomorphism from $tr\,(H^p_\nu(\mathbb{D}))$ onto $tr\,(G^p_\alpha(\mathbb{D}))$, the respective trace spaces of $H^p_\nu(\mathbb{D})$ and $G^p_\alpha(\mathbb{D})$.*

The following proposition gives some properties for G^p_α-functions and H^p_ν-functions proved in [6] and generalized in [5].

Proposition 2.12.

1) *Let $w \in G^p_\alpha(\mathbb{D})$. There exists $C_\alpha > 0$ such that*

$$\|tr\, w\|_{L^p(\mathbb{T})} \le \|w\|_{G^p_\alpha(\mathbb{D})} \le C_\alpha \|tr\, w\|_{L^p(\mathbb{T})}.$$

2) *Let* $f \in H^p_\nu(\mathbb{D})$. *There exists* $C_\nu > 0$ *such that*

$$\|trf\|_{L^p(\mathbb{T})} \leq \|f\|_{H^p_\nu(\mathbb{D})} \leq C_\nu \|trf\|_{L^p(\mathbb{T})}.$$

Proof. To prove the right-side inequality of 1), we consider $w \in G^p_\alpha(\mathbb{D})$ and use the Bers similarity principle to write that $w = e^s F$ with $s \in W^{1,r}(\mathbb{D})$ such that $\Re(s) = 0$ on $\mathbb{T}$, s bounded on $\mathbb{D}$ by $\|\alpha\|_{L^r(\mathbb{D})}$ and $F \in H^p(\mathbb{D})$. Then, we have

$$\|w\|_{G^p_\alpha(\mathbb{D})} \leq C_\alpha \|F\|_{H^p(\mathbb{D})} = C_\alpha \|\mathrm{tr}\, F\|_{L^p(\mathbb{T})} = C_\alpha \|\mathrm{tr}\, w\|_{L^p(\mathbb{T})}.$$

The left-side inequality of 1) and 2) follow from Fatou's lemma. Now, if $f \in H^p_\nu(\mathbb{D})$ then

$$\begin{aligned}\|f\|_{H^p_\nu(\mathbb{D})} &\leq \|J_\nu\|\, \|w\|_{G^p_\alpha(\mathbb{D})}\\ &\leq C(\nu)\, \|J_\nu\| \|\mathrm{tr}\, w\|_{L^p(\mathbb{T})}\\ &\leq C(\nu)\, \|J_\nu\|\, \|J_\nu^{-1}\|\, \|\mathrm{tr}\, f\|_{L^p(\mathbb{T})}\end{aligned}$$

where $w = J_\nu(f) \in G^p_\alpha(\mathbb{D})$, $C(\nu) := C_\alpha$ and $J_\nu^{-1}(\mathrm{tr}\,(w)) = (\mathrm{tr}\,(f))$ according to Remark 2.11, which proves the right-side of the inequality in 2). □

One can also define the generalized Hardy spaces through harmonic majorants. As for the analytic Hardy spaces, this definition is equivalent to the definition given in Definition 2.9.

Definition 2.13.
- *The space* $G^p_\alpha(\mathbb{D})$ *is the space of functions w solution in the distributional sense of* (6) *such that there exists a harmonic function* $u : \mathbb{D} \longrightarrow [0, +\infty)$ *such that* $|w(z)|^p \leq u(z)$ *for* $z \in \mathbb{D}$. *The space* $G^p_\alpha(\mathbb{D})$ *is equipped with the norm*

$$\|w\|_{G^p_\alpha(\mathbb{D})} = \inf\left\{u(0)^{1/p},\ u : \mathbb{D} \longrightarrow [0, +\infty)\ \textit{such that}\ |w(z)|^p \leq u(z),\ z \in \mathbb{D}\right\}.$$

- *The space* $H^p_\nu(\mathbb{D})$ *is the space of functions f solution in the distributional sense of* (4) *such that there exists a harmonic function* $u : \mathbb{D} \longrightarrow [0, +\infty)$ *such that* $|f(z)|^p \leq u(z)$ *for* $z \in \mathbb{D}$. *The space* $H^p_\nu(\mathbb{D})$ *is equipped with the norm*

$$\|f\|_{H^p_\nu(\mathbb{D})} = \inf\left\{u(0)^{1/p},\ u : \mathbb{D} \longrightarrow [0, +\infty)\ \textit{such that}\ |f(z)|^p \leq u(z),\ z \in \mathbb{D}\right\}.$$

Remark 2.14. *Definition 2.9 and Definition 2.13 are equivalent; indeed, the Hardy condition and the existence of a harmonic majorant are equivalent. Precisely, suppose that w satisfies the Hardy condition* (11). *By Proposition 2.4, there exists* $s \in W^{1,r}(\mathbb{D})$ *and F analytic on* $\mathbb{D}$ *such that* $w = e^s F$ *and* $|w(z)| \leq C_\alpha |F(z)|$ *for* $z \in \mathbb{D}$. *By Remark 2.5, F satisfies the Hardy condition. It follows from Theorem 6.7 in [16] that* $|F|^p$ *has a harmonic majorant U and* $|w|^p$ *has a harmonic majorant* $u = C_\alpha U$.

Conversely, if $|w|^p$ *has a harmonic majorant, then we have*

$$\sup_{0<r<1} \int_{r\mathbb{T}} |w(\zeta)|^p d\zeta \leq \sup_{0<r<1} \int_{r\mathbb{T}} u(\zeta) d\zeta = u(0) < \infty.$$

The same property holds for $H^p_\nu(\mathbb{D})$ *using the bijection from* $G^p_\alpha(\mathbb{D})$ *onto* $H^p_\nu(\mathbb{D})$ *in Lemma 2.3.*

2.4 Solving the Dirichlet problem

We solve the different Dirichlet problems introduced in Subsection 2.1 starting with Problem 10. Let $\phi \in L^p_\mathbb{R}(\mathbb{T})$. We search a function $w : \mathbb{D} \longrightarrow \mathbb{R}$ such that $\bar{\partial} w = \alpha \overline{w}$ on $\mathbb{D}$ and Re $w = \phi$ on $\mathbb{T}$. We have the following result [6]. We only give some elements for the proof of such results ; for more details, we refer to [6].

Theorem 2.15. *Let* $\alpha \in L^r(\mathbb{D})$ *with* $r > 2$, $\phi \in L^p_\mathbb{R}(\mathbb{T})$ *and* $c \in \mathbb{R}$. *There exists a unique function* $w \in G^p_\alpha(\mathbb{D})$ *such that* $\mathrm{Re}\, tr_\mathbb{T}\, w = \phi$ *almost everywhere on* $\mathbb{T}$ *and* $\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Im}\, tr_\mathbb{T} w(e^{i\theta}) d\theta = c$.

Ideas of the proof. For $w \in G^p_\alpha(\mathbb{D})$, there is $g \in H^p(\mathbb{D})$ such that

$$w(z) = \mathcal{C}(\mathrm{tr}_{\mathbb{T}}\, g)(z) + T_\alpha(w)(z), \quad z \in \mathbb{D}.$$

where

$$\mathcal{C}(\mathrm{tr}_{\mathbb{T}}\, g)(z) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{\mathrm{tr}_{\mathbb{T}}\, g(\zeta)}{\zeta - z} d\zeta \quad \text{and} \quad T_\alpha(w)(z) = \frac{1}{2i\pi} \int_{\mathbb{D}} \frac{\alpha \overline{w}(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}.$$

see [6, Thm 4.4.1.1]. The operator T_α is compact from $L^p(\mathbb{D})$ to $W^{1,p}(\mathbb{D})$ and $I - T_\alpha$ is injective from $G^p_\alpha(\mathbb{D})$ to $H^p(\mathbb{D})$. It follows that $I - T_\alpha$ is invertible. Let $g \in H^p(\mathbb{D})$ be such that $\mathrm{tr}_{\mathbb{T}}\, g = \phi + i\mathcal{H}\phi + ic$, where $\mathcal{H}$ denotes the harmonic conjugation on $L^p_{\mathbb{R}}(\mathbb{T})$; more precisely, $\mathcal{H}\phi$ is the function $v \in L^p_{\mathbb{R}}(\mathbb{T})$ such that $\phi + iv \in H^p(\mathbb{T})$ and

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{Im}\, v(e^{i\theta}) d\theta = 0.$$

Now for $\phi \in L^p_{\mathbb{R}}(\mathbb{T})$ and $c \in \mathbb{R}$, $w_{\phi,c}$ denotes the unique function in $G^p_\alpha(\mathbb{D})$ such that $P_+(\mathrm{tr}_{\mathbb{T}}\, w_{\phi,c}) = \mathrm{tr}_{\mathbb{T}}\, g$, where P_+ is the Riesz projection from $L^p_{\mathbb{R}}(\mathbb{T})$ onto $H^p(\mathbb{T})$. We consider the mapping

$$\begin{aligned} B : L^p_{\mathbb{R}}(\mathbb{T}) \times \mathbb{R} &\longrightarrow L^p_{\mathbb{R}}(\mathbb{T}) \times \mathbb{R} \\ (\phi, c) &\longmapsto \left(\mathrm{Re}(\mathrm{tr}_{\mathbb{T}}\, T_\alpha(w_{\phi,c})), \mathrm{Im}\left(\tfrac{1}{2\pi} \textstyle\int_0^{2\pi} \mathrm{tr}_{\mathbb{T}}\, w_{\phi,c}(e^{i\theta}) d\theta\right) - c\right). \end{aligned}$$

Then, $(\phi, c) \longmapsto T_\alpha(w_{\phi,c})$ maps $L^p_{\mathbb{R}}(\mathbb{T}) \times \mathbb{R}$ on $W^{1,p}_{\mathbb{R}}(\mathbb{D})$ and $(\phi, c) \longmapsto \mathrm{Re}(\mathrm{tr}_{\mathbb{T}}\, T_\alpha(w_{\phi,c}))$ maps $L^p_{\mathbb{R}}(\mathbb{T}) \times \mathbb{R}$ on $W^{1,1-1/p}_{\mathbb{R}}(\mathbb{T})$ (Re $(\mathrm{tr}_{\mathbb{T}}\, u) \in W^{1-1/p,p}_{\mathbb{R}}(\mathbb{T})$ if $u \in W^{1,p}(\mathbb{D})$, see [1, Thereom 7.39]). By the Rellich-Kondrachov theorem [10, Theorem 9.16]), it follows that the first component B_1 of B is compact on $L^p_{\mathbb{R}}(\mathbb{T})$. The second component B_2 of B is real-valued and so is compact. Thus, B is compact and $I + B$ is invertible and the conclusion holds.

If $w \in G^p_\alpha(\mathbb{D})$ is solution of Problem (10), then $f = \frac{w + \nu \overline{w}}{\sqrt{1-\nu^2}} \in H^p_\nu(\mathbb{D})$ is solution of Problem 9 with $\psi = \sigma^{-1/2}\phi$. □

Theorem 2.16. *Let $\nu \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$ with $r > 2$. There exists a unique function $f \in H^p_\nu(\mathbb{D})$ such that* $\mathrm{Re}\, tr f = \psi$ *almost everywhere on $\mathbb{T}$.*

One can deduce the following theorem.

Theorem 2.17. *Let $\sigma \in W^{1,r}_{\mathbb{R}}(\mathbb{D})$ with $r > 2$. There exists a unique function $u : \mathbb{D} \longrightarrow \mathbb{R}$ with* $\sup_{0<r<1} \|u\|_{L^p(r\mathbb{T})} < \infty$ *such that $\nabla.(\sigma.\nabla u) = 0$ and $u = \psi$ almost everywhere on $\mathbb{T}$.*

3 Hardy spaces of generalized analytic functions on simply-connected domains

In this section, we define the generalized Hardy spaces for functions defined on a simply-connected domain.

Let $\Omega \subset \mathbb{C}$ be a bounded simply-connected domain. The most natural way to define $G^p_\alpha(\Omega)$ is to use the harmonic majorants.

Definition 3.1.

- *The space $G^p_\alpha(\Omega)$ is the space of functions w solution in the distributional sense of* (6) *such that $|w|^p$ has a harmonic majorant. The space $G^p_\alpha(\Omega)$ is equipped with the norm*

$$\|w\|_{G^p_\alpha(\Omega)} = \inf\left\{u(z_0)^{1/p},\ u : \Omega \longrightarrow [0, +\infty) \ \text{such that}\ |w(z)|^p \le u(z),\ z \in \Omega\right\},$$

  where $z_0 \in \Omega$.

- *The space $H^p_\nu(\Omega)$ is the space of functions f solution in the distributional sense of* (4) *such that $|f|^p$ has a harmonic majorant. The space $H^p_\nu(\Omega)$ is equipped with the norm*

$$\|f\|_{H^p_\nu(\Omega)} = \inf\left\{u(z_0)^{1/p},\ u:\Omega\longrightarrow[0,+\infty)\ \text{such that}\ |f(z)|^p \le u(z),\ z\in\Omega\right\}.$$

Remark 3.2. *As in the previous subsection, another choice of z_0 leads to an equivalent norm (Harnack inequality).*

The Bers similarity principle in Proposition 2.4 can be extended to the case of a bounded domain Ω (not necessary simply-connected) see [4, Lem. 3.1]. Indeed, the Cauchy operator $\mathcal{C}$ restricted to the bounded domain Ω defines a bounded operator from $L^r(\Omega)$ to $W^{1,r}(\Omega)$. Taking $s = \mathcal{C}(A)_{|\Omega}$ and defining $F = e^{-s}w$, we obtain that $w = e^s F$ where F is analytic (the same argument as for $\mathbb{D}$ holds). Moreover, we have that $\|s\|_{W^{1,r}(\Omega)} \le C\|\alpha\|_{L^r(\Omega)}$.

Let $\phi:\mathbb{D}\longrightarrow\Omega$ be a conformal map. It is known that the regularity of ϕ depends on the regularity of the domain. We will consider two classes of simply-connected domains: the class of Dini-smooth domains and the class of domains bounded by a rectifiable Jordan curve.

3.1 Generalized Hardy spaces on Dini-smooth domains

This class of Hardy spaces have been defined and studied in [14]. The definition has been extended to finitely connected Dini-smooth domains in [5]. A simply-connected domain Ω is a Dini-smooth domain if and only if its boundary is a Jordan curve with non-singular Dini-smooth parametrization. A function f is said to be Dini-smooth if its derivative is Dini-continuous in the sense that the modulus of continuity ω_f is such that

$$\int_0^\varepsilon \frac{\omega_f(t)}{t}\,dt < \infty, \quad \text{for some } \varepsilon > 0.$$

In this case, if $\phi:\mathbb{D}\longrightarrow\Omega$ is a conformal map, ϕ extends continuously in an homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ and the derivative ϕ' extends continuously on $\overline{D}$ with $\phi'(z)\neq 0$, see [26, Thm 10.2].

Definition 3.3.

1. *Let $\alpha\in L^r(\Omega)$. We define the space $G^p_\alpha(\Omega)$ as the collection of functions $w:\Omega\longrightarrow\mathbb{C}$ solution of* (6) *and for which there is a sequence of domains D_n where the boundary ∂D_n is a rectifiable Jordan curve of uniformly bounded length and*
$$\sup_{n\in\mathbb{N}}\|w\|_{L^p(\partial D_n)} < \infty.$$
2. *Let $\nu\in W^{1,r}(\Omega)$. We define similarly the space $H^p_\nu(\Omega)$ as the collection of functions $f:\Omega\longrightarrow\mathbb{C}$ in $L^p_{loc}(\Omega)$, solutions of* (4) *and for which there is a sequence of domains D_n where the boundary ∂D_n is a rectifiable Jordan curve of uniformly bounded length and*
$$\sup_{n\in\mathbb{N}}\|f\|_{L^p(\partial D_n)} < \infty.$$

Remark 3.4. *Note that $\nu\in W^{1,r}(\Omega)$ if and only if $\nu\circ\phi\in W^{1,r}(\mathbb{D})$. Indeed, if $\nu\in W^{1,r}(\Omega)$ then $\|\nu\circ\phi\|_{L^r(\mathbb{D})} = \int_{\mathbb{D}}|\nu(z)|^r|(\phi^{-1'})(z)|^2\,dA(z)$. As $\phi^{-1'}$ is continuous from Ω onto $\mathbb{D}$ and $\nu\in W^{1,r}(\Omega)$, we obtain that $\nu\circ\phi\in L^r(\mathbb{D})$. By the chain rule, we have*

$$\partial(\nu\circ\phi) = [(\partial\nu)\circ\phi]\phi' \ \text{and}\ \overline{\partial}(\nu\circ\phi) = [(\overline{\partial}\nu)\circ\phi]\overline{\phi'}.$$

By the boundedness of ϕ' on $\mathbb{D}$, we obtain

$$\|\partial(\nu\circ\phi)\|^r_{L^r(\mathbb{D})} = \int_{\mathbb{D}}|\partial\nu\circ\phi|^r|\phi'|^{r-2}|\phi'|^2\,dA(z)$$

$$\lesssim \int_{\mathbb{D}} |\partial\nu \circ \phi|^r |\phi'|^2 dA(z)$$
$$\lesssim \|\partial\nu\|^r_{L^r(\Omega)}.$$

In the same way, we show that $\|\bar\partial(\nu \circ \phi)\|_{L^r(\Omega)} \lesssim \|\bar\partial\nu\|_{L^r(\Omega)}$ *which proves that* $\nu \circ \phi \in H^p_\nu(\mathbb{D})$. *If we suppose that* $\nu \circ \phi \in W^{1,r}(\mathbb{D})$, *then the same arguments above hold with* $\phi^{-1'}$ *instead of* ϕ' *since* $\phi^{-1'}$ *is bounded on* Ω.

A consequence is that $\alpha \in L^r(\Omega)$ *if and only if* $(\alpha \circ \phi)\overline{\phi'} \in L^r(\mathbb{D})$. *Indeed, if* $\alpha \in L^r(\Omega)$ *then, the function* $A = \mathcal{C}(\alpha)_{|\Omega} \in W^{1,r}(\Omega)$ *is such that* $\bar\partial A = \alpha$. *Now, if* $B = A \circ \phi$ *then, from what preceeds, it follows that* $B \in W^{1,r}(\mathbb{D})$ *and* $\bar\partial B = [(\bar\partial A) \circ \phi]\overline{\phi'} = (\alpha \circ \phi)\overline{\phi'} \in L^r(\mathbb{D})$. *The converse sense can be proved in the same way.*

The results obtained for $G^p_\alpha(\mathbb{D})$ and $H^p_\nu(\mathbb{D})$ can be extended to Dini-smooth domains using the composition with a conformal map; this property depends on the regularity of ϕ' and specially its regularity from $\mathbb{T}$ to $\partial\Omega$. The following proposition and its proof illustrate it.

Proposition 3.5.
1. *We have that* $w \in G^p_\alpha(\Omega)$ *if and only if* $w \circ \phi \in G^p_{(\alpha\circ\phi)\overline{\phi'}}(\mathbb{D})$. *Similarly,* $f \in H^p_\nu(\Omega)$ *if and only if* $f \circ \phi \in H^p_{\nu\circ\phi}(\mathbb{D})$.
2. *The definitions 3.1 and 3.3 are equivalent.*

Proof. We start proving (1). Let $w \in G^p_\alpha(\Omega)$. Then, $w \circ \phi$ is solution in the distributional sense of the equation

$$\bar\partial(w \circ \phi) = (\bar\partial(w) \circ \phi)\,\overline{\phi'} = [(\alpha \circ \phi)\overline{\phi'}]\overline{w \circ \phi}.$$

By the Definition 3.1 of $G^p_\alpha(\mathbb{D})$, $|w|^p$ has a harmonic majorant U on $\mathbb{D}$. Taking $u = U \circ \phi$, u is harmonic on $\mathbb{D}$ and for $z \in \mathbb{D}$,

$$|w(z)|^p = |w(\phi(z))|^p \le U \circ \phi(z),$$

which proves that w has a harmonic majorant and the conclusion follows. We prove the reverse sense in the same way with $\widetilde{w} = w \circ \phi \in G^p_{(\alpha\circ\phi)\overline{\phi'}}(\mathbb{D})$ and ϕ^{-1} instead of ϕ.

Now, $f \in H^p_\nu(\Omega)$ if and only if $w = \frac{f-\nu\bar f}{\sqrt{1-\nu^2}} \in G^p_\alpha(\Omega)$ with $\alpha = \frac{-\bar\partial\nu}{1-\nu^2}$ if and only if $w \circ \phi = \left(\frac{f-\nu\bar f}{\sqrt{1-\nu^2}}\right) \circ \phi \in G^p_{(\alpha\circ\phi)\overline{\phi'}}(\mathbb{D})$ if and only if $w \circ \phi = \left(\frac{f\circ\phi-\nu\circ\phi\overline{f\circ\phi}}{\sqrt{1-\nu^2\circ\phi}}\right) \in G^p_{(\alpha\circ\phi)\overline{\phi'}}(\mathbb{D})$ with $(\alpha \circ \phi)\overline{\phi'} = \frac{-\bar\partial(\nu\circ\phi)}{1-\nu^2\circ\phi}$ if and only if $f \circ \phi H^p_{\nu\circ\phi}(\mathbb{D})$.

It remains to prove 2. Since each function $w \in G^p_\alpha(\Omega)$ can be expressed as $e^s F$ with $s \in W^{1,r}(\Omega)$ bounded and F analytic, it is equivalent to prove for an analytic function F on Ω that

$$\sup_{n\in\mathbb{N}} \|F\|_{L^p(\partial D_n)} < \infty \iff \quad |F|^p \quad \text{has a harmonic majorant.}$$

The condition on the left is equivalent to $(F \circ \phi)(\phi')^{1/p} \in H^p(\mathbb{D})$ (see [12, Chap 10]) where ϕ is a conformal map from $\mathbb{D}$ onto Ω. By Remark 2.14, it is equivalent to $|F \circ \phi|^p|\phi'|$ has a harmonic majorant on $\mathbb{D}$. Since Ω is a Dini-smooth domain, ϕ' is bounded and non zero on $\mathbb{D}$. It follows that $|F \circ \phi|^p|$ has a harmonic majorant on $\mathbb{D}$ which is equivalent to the existence of a harmonic majorant for $|F|^p$ on Ω. □

Remark 3.6. *Note that a function* $w \in G^p_\alpha(\Omega)$ *has a trace on* $L^p(\partial\Omega)$ *when* $\alpha \in L^r(\Omega)$ *with* $r > 2$. *Indeed, according to the Bers similarity principle see [4, Lemma 3.1], there exists* $s \in W^{1,r}(\Omega)$ *with* $r > 2$ *and* $F \in H^p(\Omega)$ *such that* $w = e^s F$. *As* $e^s \in W^{1,r}(\Omega)$ *(see [4, page 6]),* e^s *has a trace on* $\partial\Omega$ *and is continuous on* $\overline{\Omega}$ *(by the Sobolev embedding theorem, see [1, Theorem 4.12]). Moreover, any function* $F \in H^p(\Omega)$ *has a trace on* $\partial\Omega$ *when* Ω *is a Dini-smooth domain (see [5, Lemma D.1]) and* $tr_{\partial\Omega} F \in L^p(\partial\Omega)$. *It follows that* w *has a trace on* $\partial\Omega$ *and* $tr_{\partial\Omega}\, w \in L^p(\partial\Omega)$.

3.2 Generalized Hardy spaces on general domains and Smirnov classes of pseudo-analytic functions

We consider in this subsection a simply-connected bounded domain Ω such that $\partial\Omega$ is rectifiable (its 1-dimensional Hausdorff measure is finite) [26, Thm 10.11].

For such a domain, if $\phi : \mathbb{D} \longrightarrow \Omega$ is a conformal map, ϕ' is not anymore continuous on $\overline{D}$ but ϕ' belongs to the (analytic) Hardy space $H^1(\mathbb{D})$. As a consequence, ϕ extends continuously from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$ (see [12, Thm 3.11]). The function ϕ' being less regular than in the case of Dini-smooth domains, one can define another class of spaces of pseudo-analytic functions through the uniform boundedness of the L^p norms of w on sequence of curves defined on Ω. These new classes are called Smirnov classes of pseudo-analytic functions and generalize the Smirnov classes of analytic functions (see [12, Chap. 10]) to pseudo-analytic functions. Such spaces do not coincide to G^p_α and H^p_ν defined as in Definition 3.1.

Moreover, the composition by ϕ of the function ν when $\nu \in W^{1,r}(\Omega)$ does not belong anymore to $W^{1,r}(\Omega)$ but $\nu \circ \phi \in W^{1,2}(\mathbb{D})$. Note that the case of a coefficient $\widetilde{\nu} = \nu \circ \phi in W^{1,2}(\mathbb{D})$ is considered as a critical case (r is equal to the dimension of $\mathbb{D}$). In this case, functions in $W^{1,2}(\mathbb{D})$ may not be bounded: indeed, the Sobolev embedding into the space of Hölder continuous functions no longer holds. In the sequel, we give the definition of Smirnov spaces of pseudo-analytic functions. A complete study of the Smirnov classes $\mathcal{F}^p_\alpha$ and $\mathcal{H}^p_\nu$ spaces has been done in [22], [18] and more recently in [7].

Definition 3.7. *Let* $1 < p < \infty$ *and* $\alpha \in L^r(\Omega)$.

$$\mathcal{F}^p_\alpha(\Omega) = \Big\{ w : \Omega \longrightarrow \mathbb{C} : \overline{\partial} w = \alpha \overline{w} \text{ in the distributional sense on } \Omega$$

$$\|w\|^p_{\mathcal{F}^p_\alpha(\Omega)} := \left(\sup_{0<\rho<1} \int_{\Gamma_\rho} |w(z)|^p |dz| \right) < \infty \Big\},$$

$\Gamma_\rho = \psi(\mathbb{T}_\rho)$ where $\psi : \mathbb{D} \longrightarrow \Omega$ is a conformal map, $|dz|$ denotes the arc-length measure.

Remark 3.8. *If* $\alpha \equiv 0$, *then* $\mathcal{F}^p_\alpha(\Omega) = \mathcal{H}^p(\Omega)$ *(the Smirnov class of analytic functions) where*

$$\mathcal{H}^p(\Omega) = \left\{ F : \Omega \longrightarrow \mathbb{C} \text{ analytic on } \Omega \text{ and } \left(\sup_{0<\rho<1} \int_{\Gamma_\rho} |F(z)|^p |dz| \right) < \infty \right\},$$

The choice of a different conformal map leads to an equivalent norm.

4 Composition operators on Hardy spaces of generalized analytic functions

Let Ω_1, Ω_2 be two bounded Dini-smooth domains in $\mathbb{C}$, $\alpha \in L^r(\Omega_2)$. Let $\phi : \Omega_1 \longrightarrow \Omega_2$ be an analytic function such that $\phi \in W^{1,\infty}_{\Omega_2}(\Omega_1)$. We extend some results on composition operators obtained in [20] where the authors consider the case $\alpha \in L^\infty(\Omega_2)$. The results on boundednesss, invertibility and Fredholm property about the composition operator C_ϕ can be extended to the composition operator $\widetilde{C}_\phi : H^p_\nu(\Omega_2) \to H^p_{\nu\circ\phi}(\Omega_1)$ when $\nu \in W^{1,r}(\Omega_2)$ using the isomorphism J_ν for the same class of symbols ϕ.

4.1 Boundedness of the composition operator

We denote by C_ϕ the composition operator on $G^p_\alpha(\Omega_2)$ defined by $C_\phi(w) = w \circ \phi$ for $w \in G^p_\alpha(\Omega_2)$. Let $\alpha_\phi = (\alpha \circ \phi)\overline{\phi'}$. By Remark 3.4, $\alpha_\phi \in L^r(\Omega_1)$.

Proposition 4.1. *The composition operator $C_\phi : G^p_\alpha(\Omega_2) \to G^p_{\alpha_\phi}(\Omega_1)$ is continuous.*

The proof is the same as in [20].

Proof. Let $w \in G^p_\alpha(\Omega_2)$. We have that $w \circ \phi : \Omega_1 \longrightarrow \mathbb{C}$ is measurable and

$$\bar\partial(w \circ \phi) = [(\bar\partial w) \circ \phi]\bar\partial(\bar\phi) = (\alpha \circ \phi)\overline{\phi'}(\overline{w \circ \phi}),$$

where the equalities are considered in the sense of distributions. Now, if u is any harmonic majorant of $|w|^p$ in Ω_2, then $u \circ \phi$ is a harmonic majorant of $|w \circ \phi|^p$ in Ω_1, which proves that $C_\phi(w) \in G^p_{\alpha_\phi}(\Omega_1)$. Moreover, by the Harnack inequality applied in Ω_2,

$$\left\|C_\phi(w)\right\|_{G^p_{\alpha_\phi}(\Omega_1)} \le u(\phi(z_0))^{1/p} \le C\,u(z_0)^{1/p},$$

for $z_0 \in \Omega_2$ as in Definition 2.13, and where the constant C depends on Ω_2, z_0 and $\phi(z_0)$ but not on u. Taking the infimum over all harmonic functions $u \ge |w|^p$ in Ω_2, we obtain that

$$\left\|C_\phi(w)\right\|_{G^p_{\alpha_\phi}(\Omega_1)} \lesssim \|w\|_{G^p_\alpha(\Omega_2)}.$$ □

4.2 Invertible and Fredholm composition operators

In this subsection, we give a characterization of invertible and Fredholm composition operators. To this aim, we need to introduce the real-valued evaluation map on $G^p_\alpha(\Omega)$ with $\alpha \in L^r(\Omega)$ when Ω is a bounded Dini-smooth domain as defined in [20] for $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$ with $\nu \in W^{1,\infty}(\Omega)$ and $\alpha \in L^\infty(\Omega)$ respectively.

For $z \in \Omega$, let E^ν_z and F^ν_z be the real-valued evaluation maps at z defined on $G^p_\alpha(\Omega)$ by

$$E^\alpha_z(w) := \operatorname{Re} w(z) \text{ and } F^\alpha_z(w) := \operatorname{Im} w(z) \text{ for all } w \in G^p_\alpha(\Omega).$$

Proposition 4.2. *For $z \in \Omega$, the evaluation maps E^α_z and F^α_z are continuous on $G^p_\alpha(\Omega)$.*

Proof. Let $w \in G^p_\alpha(\Omega)$ and $z \in \Omega$. By the equivalence of Definition 3.1 and 3.3, we have that

$$|\operatorname{Re} w(z)| \le |w(z)| \le u_w(z)^{1/p} \lesssim u_w(z_0)^{1/p} \lesssim \|w\|_{G^p_\alpha(\Omega)},$$

where $z_0 \in \Omega$ and u_w is a harmonic majorant of $|w|^p$ on Ω. The same holds for Im w. □

Remark 4.3. *Observe that for $w \in G^p_\alpha(\Omega)$, we have $\|F^\alpha_z\|_{(G^p_\alpha(\Omega))'} = \|E^\alpha_z\|_{(G^p_\alpha(\Omega))'}$ where $(G^p_\alpha(\Omega))'$ denotes the dual space of $G^p_\alpha(\Omega)$. Indeed, the equality $|Im\, w(z)| = |Re\,(-iw(z))|$ leads to*

$$\|F^\alpha_z\|_{(G^p_\alpha(\Omega))'} = \sup_{\substack{w\in G^p_\alpha(\Omega)\\ \|tr\, w\|_{L^p(\partial\Omega)}\le 1}} |Im\, w(z)| = \sup_{\substack{w\in G^p_\alpha(\Omega)\\ \|tr\, w\|_{L^p(\partial\Omega)}\le 1}} |Re\, w(z)| = \|E^\alpha_z\|_{(G^p_\alpha(\Omega))'}. \tag{14}$$

We denote by A^α_z the linear functional defined by $A^\alpha_z = E^\alpha_z + iF^\alpha_z : w \in G^p_\alpha(\Omega) \longmapsto w(z)$ and $k_z \in H^q(\Omega)$ (q is the conjugate exponent of p) such that $F(z) = \langle F, k_z\rangle_{L^2(\partial\Omega)}$.

Lemma 4.4. *We have that*

$$\|A^\alpha_z\|_{(G^p_\alpha(\Omega))'} \sim C_\alpha \|k_z\|_{H^q(\Omega)}.$$

Moreover, we have that $\|E^\alpha_z\|_{(G^p_\alpha(\Omega))'} \sim C\|A^\alpha_z\|_{(G^p_\alpha(\Omega))'}$.

Proof. Let $w \in G^p_\alpha(\Omega)$ with $\|\operatorname{tr} w\|_{L^p(\partial\Omega)} \le 1$. According to the Bers similarity principle, we write $w = e^s F$ with $s \in W^{1,r}(\Omega)$ with $\operatorname{Re} s = 0$ on $\partial\Omega$ and $F \in H^p(\Omega)$. Note that $\|F\|_{H^p(\Omega)} = \|\operatorname{tr} F\|_{L^p(\partial\Omega)} = \|\operatorname{tr} w\|_{L^p(\partial\Omega)} \le 1$. Then, we have

$$|A^\alpha_z(w)| = |w(z)| = |e^{s(z)}F(z)| \le C_\alpha |F(z)| \le C_\alpha \|k_z\|_{H^q(\Omega)},$$

as $\|F\|_{H^p(\Omega)} \le 1$. Taking the supremum over the functions $w \in G^p_\alpha(\Omega)$ such that $\|\mathrm{tr}\, w\|_{L^p(\partial\Omega)} \le 1$, we obtain that

$$\|A^\alpha_z\|_{(G^p_\alpha(\Omega))'} \le \|k_z\|_{H^q(\Omega)}.$$

For the reverse inequality, we use the "reverse" Bers similarity principle (see [20, Thm 4]). Let $F \in H^p(\Omega)$ such that $\|\mathrm{tr}\, F\|_{L^p(\partial\Omega)} \le 1$ and $s \in W^{1,r}(\Omega)$ with $\mathrm{Re}\, s = 0$ on $\partial\Omega$ such that $w = e^s F \in G^p_\alpha(\Omega)$ where $\|\mathrm{tr}\, w\|_{L^p(\partial\Omega)} \le 1$. We have that

$$|F(z)| \le |e^{-s(z)} w(z)| \le C_\alpha |w(z)| \le C_\alpha \|A^\alpha_z\|_{(G^p_\alpha(\Omega))'}.$$

It follows that

$$\|k_z\|_{H^q(\Omega)} = \sup_{\substack{F\in H^p(\Omega)\\ \|F\|_{H^p(\Omega)}\le 1}} |F(z)| \le C_\alpha \|A^\alpha_z\|_{(G^p_\alpha(\Omega))'}. \qquad \square$$

The evaluation maps have an interesting behavior under the action of the adjoint of a composition operator as in the analytic case:

Lemma 4.5. *For all $z \in \Omega$, $C^*_\phi(E^{\alpha_\phi}_z) = E^\alpha_{\phi(z)}$ and $C^*_\phi(F^{\alpha_\phi}_z) = F^\alpha_{\phi(z)}$.*

Proof. Let $f \in H^p_\nu(\Omega_2)$. Then

$$\langle C^*_\phi(E^{\alpha_\phi}_z), f\rangle = \langle E^{\alpha_\phi}_z, C_\phi(f)\rangle = \langle E^{\alpha_\phi}_z, f\circ\phi\rangle = \mathrm{Re}\, f(\phi(z)) = \langle E^\alpha_{\phi(z)}, f\rangle,$$

and the argument is analogous for F^α_z. $\square$

The following theorem characterizes the invertible composition operators. It extends the result obtained in [20] for $\alpha \in L^r(\Omega_2)$. Even if the following proof follows the same steps as in [20], some of the arguments are different from those used in [20].

Theorem 4.6. *Assume that Ω_1, Ω_2 are bounded Dini-smooth domains. Then, the composition operator $C_\phi : G^p_\alpha(\Omega_2) \to G^p_{\alpha_\phi}(\Omega_1)$ is invertible if, and only if, ϕ is a bijection from Ω_1 onto Ω_2.*

Proof. If ϕ is invertible, then $C_{\phi^{-1}} = (C_\phi)^{-1}$. Assume conversely that C_ϕ is invertible. Since C_ϕ is one-to-one with closed range, for all $L \in (G^p_{\alpha_\phi}(\Omega_1))'$, one has

$$\left\|C^*_\phi L\right\|_{(G^p_\alpha(\Omega_2))'} \gtrsim \|L\|_{(G^p_{\alpha_\phi}(\Omega_1))'}. \tag{15}$$

If $z_1, z_2 \in \Omega_1$ are such that $\phi(z_1) = \phi(z_2)$. Then, by Lemma 4.5,

$$C^*_\phi(E^{\alpha_\phi}_{z_1}) = E^\alpha_{\phi(z_1)} = E^\alpha_{\phi(z_2)} = C^*_\phi(E^{\alpha_\phi}_{z_2}).$$

Since C^*_ϕ is invertible, it follows that $E^{\alpha_\phi}_{z_1} = E^{\alpha_\phi}_{z_2}$. Similarly, $F^{\alpha_\phi}_{z_1} = F^{\alpha_\phi}_{z_2}$. As for any $\alpha \in L^r(\Omega_1)$, the space $G^p_\alpha(\Omega_1)$ separates points (indeed, the Hardy space $H^p(\Omega_1)$ separates points. If $F \in H^p(\Omega_1)$ separates points, then by the reverse Bers similarity principle, one can find $s \in W^{1,r}(\Omega_1)$ such that $w = e^s F \in G^p_{\alpha_\phi}(\Omega_1)$ is such that $w(z_1) \ne w(z_2)$). We conclude that $z_1 = z_2$, and ϕ is univalent.

Now, suppose that ϕ is not surjective. Then, $\partial\phi(\Omega_1) \cap \Omega_2 \ne \varnothing$ (see [20]). Let $a \in \partial\phi(\Omega_1) \cap \Omega_2$ and $(z_n)_{n\in\mathbb{N}}$ be a sequence of Ω_1 such that

$$\phi(z_n) \underset{n\to\infty}{\longrightarrow} a.$$

Up to a subsequence, there exists $z \in \overline{\Omega_1}$ such that $z_n \underset{n\to\infty}{\longrightarrow} z$. Note that $z \in \partial\Omega_1$, otherwise $\phi(z) = a$ which is impossible (indeed, since $a \in \partial\phi(\Omega_1)$ and $\phi(\Omega_1)$ is open, thus $a \notin \phi(\Omega_1)$). Now, we have that

$$\|E^\alpha_{\phi(z_n)}\|_{(G^p_\alpha(\Omega_2))'} \le \|A^\alpha_{\phi(z_n)}\|_{(G^p_\alpha(\Omega_2))'} \le C_\alpha \|k_{\phi(z_n)}\|_{H^q(\Omega_2)},$$

where $\|k_{\phi(z_n)}\|_{H^q(\Omega_2)}$ is the norm of the evaluation map on $H^p(\Omega_2)$.

Let ψ_2 be a conformal map from $\mathbb{D}$ onto Ω_2. As Ω_2 is a Dini-smooth domain, the function ψ extends continuously from $\overline{\mathbb{D}}$ onto $\overline{\Omega_2}$ (see [24, Thm 14.19]). As the norm of the evaluation map on $H^q(\Omega_2)$ at $\phi(z_n)$ is equivalent to the norm of the evaluation map on $H^q(\mathbb{D})$ at $\psi_2^{-1}(\phi(z_n))$ which is equal to

$$\left(\frac{1}{1-|\psi_2^{-1}(\phi(z_n))|}\right)^{1/q},$$

(see [11, Cor. 2.14]) and tends to $\left(\frac{1}{1-|\psi_2^{-1}(a)|}\right)^{1/q}$. It follows that

$$\sup_{n\in\mathbb{N}} \|E^{\alpha}_{\phi(z_n)}\|_{(G^p_\alpha(\Omega_2))'} < \infty.$$

We claim that

$$\|k_{z_n}\|_{H^q(\Omega_1)} \underset{n\to\infty}{\longrightarrow} +\infty.$$

Indeed, if ψ_1 is a conformal map from $\mathbb{D}$ onto Ω_1, we have that $\psi_1^{-1}(\partial\Omega_1) = \mathbb{T}$ and $\|k_{z_n}\|_{H^q(\Omega_1)}$ is equivalent to $\left(\frac{1}{1-|\psi_1^{-1}(z_n)|^2}\right)^{1/q}$ which tends to $+\infty$ as n tends to $+\infty$ since $\psi_1^{-1}(z_n)$ tends to $\psi_1^{-1}(z) \in \mathbb{T}$. By Lemma 4.4, it follows that

$$\|E^{\alpha_\phi}_{z_n}\|_{(G^p_{\alpha_\phi}(\Omega_1))'} \underset{n\to\infty}{\longrightarrow} +\infty.$$

Thus, we have that

$$\frac{\|C^*_\phi(E^{\alpha_\phi}_{z_n})\|_{(G^p_\alpha(\Omega_2))'}}{\|E^{\alpha_\phi}_{z_n}\|_{(G^p_{\alpha_\phi}(\Omega_1))'}} = \frac{\|E^{\alpha}_{\phi(z_n)}\|_{(G^p_\alpha(\Omega_2))'}}{\|E^{\alpha_\phi}_{z_n}\|_{(G^p_{\alpha_\phi}(\Omega_1))'}} \underset{n\to\infty}{\longrightarrow} 0,$$

which contradicts (15). We conclude that ϕ is surjective. □

To describe the Fredholm composition operators, we use the following Lemma established by McCluer in [21] for Hilbert spaces of analytic functions defined on a domain of $\mathbb{C}^N$ with $N = 1$.

Lemma 4.7. *If there exists a sequence $(g_n)_n$ in $G^2_{\alpha_\phi}(\Omega_1)$ with $\|g_n\|_{(G^2_{\alpha_\phi}(\Omega_1))'} = 1$ such that*

$$\|C^*_\phi(g_n)\|_{(G^2_{\alpha_\phi}(\Omega_1))'} \underset{n\to\infty}{\longrightarrow} 0.$$

Then, $C_\phi : G^2_\alpha(\Omega_2) \longrightarrow G^2_{\alpha_\phi}(\Omega_1)$ is not Fredholm.

Proposition 4.8. *The composition operator $C_\phi : G^2_\alpha(\Omega_2) \longrightarrow G^2_{\alpha_\phi}(\Omega_1)$ is Fredholm if, and only if ϕ is a bijection from Ω_1 onto Ω_2.*

Proof. If ϕ is a bijection, then by Theorem 4.6, C_ϕ is invertible and thus C_ϕ is Fredholm. Now, we suppose that C_ϕ is Fredholm. If ϕ is not univalent, using the same arguments as in [9] or [21], there is an infinite linearly independent set of differences between $A^{\alpha_\phi}_{z_n}$ where the $z_n \in \Omega_1$ are distincts, denoted by $\mathcal{S}$ such that $\mathcal{S} \subset \ker(C^*_\phi)$ which contradicts that C_ϕ is Fredholm.

If ϕ is not surjective then, by the proof of Theorem 4.6, it follows that there exists $(z_n)_{n\in\mathbb{N}}$ a sequence in Ω_1 such that

$$\phi(z_n) \underset{n\to\infty}{\longrightarrow} a,$$

where $a \in \partial\phi(\Omega_1) \cap \Omega_2$. For $g_n = \frac{E^{\alpha_\phi}_{z_n}}{\|E^{\alpha_\phi}_{z_n}\|_{(G^2_{\alpha_\phi}(\Omega_1))'}}$, then we have $\|g_n\|_{G^2_{\alpha_\phi}(\Omega_1))'} = 1$ and using the same arguments as in proof of Theorem 4.6, we have that

$$\|C^*_\phi(g_n)\| = \frac{\|E^{\alpha}_{\phi(z_n)}\|_{(G^2_\alpha(\Omega_2))'}}{\|E^{\alpha_\phi}_{z_n}\|_{(G^2_{\alpha_\phi}(\Omega_1))'}} \underset{n\to\infty}{\longrightarrow} 0,$$

which contradicts that C_ϕ is Fredholm. □

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