DE GRUYTER Concr. Oper. 2018; 5: 35–41

Concrete Operators

Research Article

Joseph A. Cima* and William Derrick

The distribution function for a polynomial

https://doi.org/10.1515/conop-2018-0004
Received June 13, 2018; accepted October 16, 2018.

Abstract: This paper explores the continuity and differentiability properties for the distribution function for a polynomial.

Keywords: Polynomial, Distribution function

MSC: 28A99, 30E99

1 Introduction

It is the purpose of this article to discuss the basic properties of the distribution function (for definition see below) of a complex analytic polynomial on the unit circle in the complex plane. We discuss the continuity and differentiability properties of this function. Although some of these properties are probably known we have not been able to locate them in a convenient reference so we include some proofs for completeness and clarity of exposition. The setting here is perhaps the simplest (using polynomials) for distribution theory but it will suffice to show the difficulties in working with the distribution function. Further, in the last section we will discuss some of the generalizations of the case of the polynomials to analytic functions and mention motivations as to why the study of the polynomial cases are of interest to us.

2 Definitions, notation and supporting material

We use the notation $\mathbb{D}$ for the unit disc in the complex plane and $\mathbb{T}$ for its boundary, the unit circle. We use m for the Lebesgue measure on $\mathbb{T}$ and we use $e^{i\theta} = \zeta$, where $\theta \in [0, 2\pi)$ for points of $\mathbb{T}$.

Definition 2.1. *For λ a non-negative real number and p a non-trivial, complex, polynomial we set*

$$A(\lambda) = \{\zeta \in \mathbb{T} : |p(\zeta)| > \lambda\}.$$

Note that by properties of continuous functions, the set $A(\lambda)$ is an open subset of $\mathbb{T}$, which may be all of $\mathbb{T}$. and hence is a union of at most countably many open sub-arcs of $\mathbb{T}$.

Definition 2.2. *A function f defined on an interval $I = (a - \delta, a + \delta)$, $a \in \mathbb{R}$, $\delta > 0$ of the real line is said to be analytic on I if it possesses all its derivatives there and has a valid power series expansion about the points of I.*

***Corresponding Author: Joseph A. Cima:** Department of Mathematics, University of North Carolina, Chapel Hill, NC, USA, E-mail: cima@email.unc.edu
William Derrick: Department of Mathematics, University of Montana, Missoula, Montana, USA, E-mail: bill.derrick@yahoo.com

It is a well known fact that real analytic functions on such intervals possess a uniqueness property similar to that of analytic functions on domains in the plane. That is if f is zero on a countable set of points $\{x_j\} \in I$ with limit point in I then f is the zero function. Note that for p a complex polynomial $|p(\zeta)| = |p(e^{i\theta}|$ is a real analytic function of θ off its zero set.

For a given value of λ we use the notation

$$D(\lambda) = m(A(\lambda))$$

to denote the Lebesgue measure of $A(\lambda)$.

3 The continuity

We first list some elementary properties necessary for our discussion. Note that first if

$$\|p\|_\infty = \max\{|p(\zeta)| : \zeta \in \mathbb{T}\},$$

then for $\lambda > \|p\|_\infty$ the set $A(\lambda)$ is empty and so $m(A(\lambda)) = 0$ in that case, and this will not be of interest for our discussion. We write

$$A(\lambda) = \bigcup_{j=1}^{J(\lambda)} I_j(\lambda),$$

where the $I_j(\lambda)$ are disjoint arcs of $\mathbb{T}$. For $0 < \lambda < \mu < \|p\|_\infty$ we use the notation $A(\mu) \subset A(\lambda)$ for two purposes. First to denote that for each j there is a $k = k(j)$ for which the interval $I_j(\mu) \subset I_k(\lambda)$, and second considering $A(\mu)$ as a point set that each point in the union of the intervals of $A(\mu)$ is a point of the union of the intervals in $A(\lambda)$. Similarly, for $0 < \nu < \lambda < ||p||$ given $I_j(\lambda)$ there is a $k = k(j)$ for which $I_j(\lambda) \subset I_k(\nu)$ and each point in the union of the intervals of $A(\lambda)$ is a point in the intersection of the intervals for $A(\nu)$.

Then for $0 < \lambda_1 < \lambda_2 \le \|p\|_\infty$ and each j, we have $A(\lambda_2) \subset A(\lambda_1)$ and hence the distribution function $D(\lambda)$ is decreasing to zero. This implies, by well known properties of decreasing functions, [3] that D is differentiable almost everywhere in the sense of Lebesgue and hence is continuous almost everywhere. We will show that D is continuous at all points of $(0, \|p\|_\infty)$. Furthermore, except for one special case the graph of D has no flat spots. We have need of the following basic proposition.

Proposition 3.1. *Assume p is a polynomial and I is a subarc of the unit circle $\mathbb{T}$. Then $|p(\zeta)| = C > 0$ for all $\zeta \in I$ if and only if $p(z) = az^k$, for some $|a| = C$ and some nonnegative integer $k \in \mathbb{Z}_+$.*

Proof. The "if" part of the theorem is obvious. Assume the zeros of p are $\{d_k, k = 1, 2, \dots, K\}$. Factor p as

$$p(z) = B\prod_{k=1}^{K}(z - d_k)$$

with $|d_k| < 1, k = 1, 2, \dots L$, and $1 \le |d_k|, k = L+1, \dots, K)$, and $|p(\zeta)| = C$ on the interval I. Since $|p(\zeta)|$ is continuous on $\mathbb{T}$ and real analytic on $\mathbb{T}$, except perhaps at the finite number of zeroes of p, we have that $|p(\zeta)| = C$ for all $\zeta \in \mathbb{T}$. In particular, p has no zeros on $\mathbb{T}$. We wish to prove that there are no zeros of modulus larger than one and all the remaining d_j are all equal to zero. Let b be the finite Blaschke product

$$b(z) = \prod_{k=1}^{L} \frac{z - d_k}{1 - \overline{d}_k z}.$$

Set

$$R(z) = \frac{p(z)}{b(z)} = B\prod_{k=1}^{L}(1 - \overline{d}_k z)\prod_{L+1}^{K}(z - d_k).$$

R is a polynomial with $|R(\zeta)| = C$ on $\mathbb{T}$ and R is non-zero on $\overline{\mathbb{D}}$. Hence, $|R(0)| \le C$. Now if equality holds (i.e., $|R(0)| = C$) by the maximum modulus theorem there is a unimodular constant λ for which

$$R(z) = B\prod_{k=1}^{L}(1 - \overline{d_k} z)\prod_{k=L+1}^{K}(z - d_k) \equiv \lambda C.$$

But if there is a $d_k > 1$ this produces a contradiction. Hence, there are no zeroes with modulus larger than one and so

$$R(z) = B\prod_{1}^{J}(1 - \overline{d_j}z) \equiv \lambda C.$$

Again , if some $d_j \neq 0$, we choose $z = 1/\overline{d_j}$ to reach another contradiction.

Hence $d_j = 0, j = 1, 2, \dots, J$ implying the result that $p(z) = Bz^J$. This handles the case where $|R(0)| = C$.

If on the other hand, $|R(0)| < C$ we set $V(z) = 1/R(z)$ which is analytic on $\overline{D}$ and zero free. Then $|V(\zeta)| = 1/C$. We have in this case

$$\frac{1}{C} < \frac{1}{|R(0)|} = |V(0)| \leq \frac{1}{C},$$

which again is a contradiction. This establishes the proposition. □

We now state our first theorem.

Theorem 3.2. *For an analytic polynomial p the distribution function D is continuous on* $(0, \|p\|_\infty)$.

Proof. First, assume $\lambda_0 \in (0, \|p\|_\infty)$ and $\lambda_0 < \lambda < \|p\|_\infty$. Then $A(\lambda) \subset A(\lambda_0)$, and hence,

$$V \equiv \bigcup_{\lambda_0<\lambda<||p||} A(\lambda) \subset A(\lambda_0).$$

But clearly $V = A(\lambda_0)$ for if $|p(\zeta)| > \lambda_o$ then $\zeta \in A(\lambda)$ for any $\lambda \in (\lambda_0, \|p\|_\infty)$. We have

$$\begin{aligned} D(\lambda_0) &= \sum_{j=1}^{J(\lambda_0)} m(I_j(\lambda_o)) \\ &= \lim_{\lambda\to\lambda_0,\lambda>\lambda_0} \left(\sum_{j=1}^{J(\lambda)} |I_j(\lambda)|\right) \\ &= \lim_{\lambda\to\lambda_0,\lambda>\lambda_0} D(\lambda). \end{aligned}$$

Implying that D is continuous from the right.

To check the continuity from the left we assume $\lambda < \lambda_0$. Then $A(\lambda) \supset A(\lambda_0)$, and we set

$$M = \bigcap_{\lambda<\lambda_o} A(\lambda).$$

We have $A(\lambda_0) \subset M$, and we claim $M \setminus A(\lambda_0)$ is a finite set. For if $\zeta \in M \setminus A(\lambda_0)$ then there is θ so that $|p(e^{i\theta})| = \lambda_0$. But since $|p|$ is real analytic this can only occur for a finite set of points say $\zeta_k = e^{i\theta_k}$, with $\theta_1, \theta_2, \dots, \theta_q$. Thus

$$M = A(\lambda_0) \cup \{\zeta_1, \dots, \zeta_q\}$$

and so

$$m(M) = m(A(\lambda_o)) = \lim_{\lambda\leftarrow\lambda_0} D(\lambda).$$

This establishes the claim. □

4 Differentiability of the distribution function

As noted earlier the distribution function $D(\lambda)$ is a non-negative, decreasing function on the open interval $(0, \|p\|_\infty)$ and as such has a derivative at "most points" (in the sense of Lebesgue) of this interval. However, for polynomials we have the following stronger result.

Theorem 4.1. *The distribution function $D(\lambda)$ of a polynomial is differentiable except at a finite set of points of its domain.*

Proof. Writing $p(z) = U(z) + iV(z)$, where U and V are real valued, we have

$$|p(z)| = \sqrt{U^2(z) + V^2(z)}.$$

For $\zeta = e^{i\theta} \in \mathbb{T}$ we denote $|p(\zeta)| = S(\theta)$ and observe that S is real analytic off the zero set of p. So if the zeroes of p on $\mathbb{T}$ are

$$M = \{\zeta_k = e^{i\theta_k} : k = 1, \dots K\},$$

we consider S on the set $\mathbb{T} \setminus M$. Note if p is a monomial then the function $D(\lambda)$ is constant on $(0, \|p\|_\infty)$ and so the conclusion of the stated theorem is obvious. Next we consider the zero set of S. The function $S^2(\theta)$ is real analytic on $\mathbb{T}$ and

$$\frac{dS^2(\theta)}{d\theta} = 2S(\theta)S'(\theta) = -2\Im(\overline{p(\zeta)}p'(\zeta)).$$

We claim S' can not vanish on a countable set of points of its domain. If not, there is an arc, say $I \subset \mathbb{T}$, and points $\{\zeta_k : k = 1, \dots \infty\} \subset I$ for which $S'(\zeta_k) = 0$. But the function

$$g(\theta) \equiv \Im(\overline{p(\zeta)}p'(\zeta))$$

is a trigonometric polynomial function of θ and hence is real analytic on $\mathbb{T}$. But $g(\theta_k) = 0$ implies $g(\theta) \equiv 0$ on the arc I. But $2S(\zeta)S'(\zeta) \equiv 0$ on $\mathbb{T}$ implies that $S'(\theta) \equiv 0$ on the arc I so that $S(\theta)$ is a constant on I. By Proposition 3.1 this implies the p is a monomial and we have already discussed this case. So now let

$$Z' = \{\zeta_k : S'(\zeta_k) = 0\}, \quad Z = \{S(\theta_k) = \lambda_k\}.$$

Then excluding the finite set Z we show that $D(\lambda)$ is differentiable for $\lambda \in (0, \|p\|_\infty) \setminus Z$, an open set. Fix λ_0 in the open set $(0, \|p\|_\infty) \setminus Z$ and write

$$A(\lambda_0) = \bigcup_{j=1}^{J(\lambda_0)} I_j(\lambda_0).$$

Writing

$$I_j(\lambda_0) = (e^{i\alpha_j(\lambda_0)}, e^{i\beta_j(\lambda_0)}),$$

with $\alpha_j(\lambda_0) < \beta_j(\lambda_0) < \alpha_{j+1}(\lambda_0)$ and for λ near λ_0

$$A(\lambda) = \bigcup_{j=1}^{J(\lambda)} I_j(\lambda),$$

we see that $J(\lambda) = J(\lambda_0)$. This will follow from the fact that S is strictly monotone in an open neighborhood of each point $\lambda \in (0, \|p\|_\infty) \setminus Z$.

We use the same notation as above for the intervals $I_j(\lambda)$, namely,

$$I_j(\lambda) = (e^{i\alpha_j(\lambda)}, e^{i\beta_j(\lambda)}).$$

We claim the $\overline{I_j(\lambda_0)} = [e^{i\alpha_j(\lambda_0)}, e^{i\beta_j(\lambda_0)}]$ are pairwise disjoint. For if not we have a value of $j \in \{1, 2, \dots, J(\lambda_0)\}$ for which

$$\overline{I_j(\lambda_o)} \bigcap \overline{I_{j+1}(\lambda_0)} = \{e^{i\beta_j(\lambda)}\}$$

and $S(\beta_j(\lambda_0)) = \lambda_0 = S(\alpha_{j+1}(\lambda_0))$.

This implies that S has a minimum at $e^{i\beta_j(\lambda_0)}$ and since S is differentiable at this point $S'(e^{i\beta_j(\lambda_0)}) = 0$. This is a contradiction. Hence, there is a positive distance between the closed arcs $\overline{I_j(\lambda_0)}$. We know $S'(\theta) \neq 0$ for $e^{i\theta} \in \overline{I_j(\lambda_o)}$ and so is strictly monotone at the end points of the interval. With out loss of generality assume S is strictly increasing at $\alpha_j(\lambda_0)$ and strictly decreasing at $\beta_j(\lambda_0)$. For λ sufficiently close to λ_o we have

$$A(\lambda) = \bigcup_{j=1}^{J(\lambda)} I_j(\lambda), \quad I_j(\lambda) = (e^{i\alpha_j(\lambda)}, e^{i\beta_j(\lambda)}),$$

with $J(\lambda) = J(\lambda_o)$, and $\alpha_j(\lambda) < \beta_j(\lambda) < \alpha_{j+1}(\lambda)$. Then

$$m(I_j(\lambda)) = |I_j(\lambda)| = (\beta_j(\lambda) - \alpha_j(\lambda).$$

With out loss of generality we may assume $j = 1$. The numbers λ can be chosen with $\lambda > \lambda_0$ and so close to λ that the associated

$$I_1(\lambda) \equiv (e^{i\alpha_1(\lambda)}, e^{i\beta_1(\lambda)}) \subsetneq I_1(\lambda_0))$$

and $S(\theta) > \lambda$ for $e^{i\theta} \in I_1(\lambda)$. Similarly, λ can be chosen with $\lambda < \lambda_0$ so close to λ_0 that the associated

$$I_1(\lambda) = (e^{i\alpha_1(\lambda)}, e^{i\beta_1(\lambda)} \supsetneq I_1(\lambda_0)$$

and $S(\theta) > \lambda$ for $e^{i\theta} \in I_1(\lambda_0)$.

If we consider the difference quotient for D, it involves

$$\begin{aligned} D(\lambda) - D(\lambda_0) &= \sum_{j=1}^{J(\lambda_0)} |I_j(\lambda)| - \sum_{j=1}^{J(\lambda_0)} |I_j(\lambda_0)| \\ &= \sum_{j=1}^{J(\lambda_0)} (\beta_j(\lambda) - \alpha_j(\lambda)) - \sum_{j=1}^{J(\lambda_0)} (\beta_j(\lambda_0) - \alpha_j(\lambda_0)) \\ &= \sum_{j=1}^{J(\lambda_0)} [(\beta_j(\lambda) - \beta_j(\lambda_0)) - (\alpha_j(\lambda) - \alpha_j(\lambda_0))]. \end{aligned}$$

We consider one of these terms, say for $j = 1$, and show the appropriate limit exists. The limit in question (for $j = 1$) is

$$\frac{(\beta_1(\lambda) - \beta_1(\lambda_0)) - (\alpha_1(\lambda) - \alpha_1(\lambda_0))}{\lambda - \lambda_0}.$$

Let us consider the expansion of S at $\alpha_1(\lambda_0)$,

$$\begin{aligned} S(\theta) &= S(\alpha_1(\lambda_0)) + S'(\alpha_1(\lambda_0)(\theta - \alpha_1(\lambda_0) + O((\theta - \alpha_1(\lambda_0)^2) \\ &= \lambda_0 + S'(\alpha_1(\lambda_0))(\theta - \alpha_1(\lambda_0)) + O((\theta - \alpha_1(\lambda_0))^2). \end{aligned}$$

Since S is strictly monotone increasing near $\alpha_1(\lambda_0)$ given λ sufficiently close to λ_0 there is a unique value of $\theta = \theta(\lambda)$ for which $S(\theta(\lambda)) = \lambda$. So it follows that

$$\begin{aligned} \lim_{\lambda\to\lambda_0} \frac{\alpha_1(\lambda) - \alpha_1(\lambda_o)}{\lambda - \lambda_0} &= \lim_{\lambda\to\lambda_0} \frac{\theta(\lambda) - \alpha_1(\lambda_0)}{(\theta(\lambda) - \alpha_1(\lambda_o))(S'(\alpha_1(\lambda_0)) + O((\theta(\lambda) - \alpha_1(\lambda_0))} \\ &= \frac{1}{S'(\alpha_1(\lambda_o)}. \end{aligned}$$

For the term $\beta_1(\lambda) - \beta_1(\lambda_0)$ we use the expansion of S at $\beta_1(\lambda_o)$ to obtain

$$\begin{aligned} \lim_{\lambda\to\lambda_0} \frac{\beta_1(\lambda) - \beta_1(\lambda_0)}{\lambda - \lambda_0} &= \lim_{\lambda\to\lambda_0} \frac{\theta(\lambda) - \beta_1(\lambda_0)}{(\theta(\lambda) - \beta_1(\lambda_0))[S'(\beta_1(\lambda_0) + O((\theta(\lambda) - \beta_1(\lambda_0))]} \\ &= \frac{1}{S'(\beta_1(\lambda_0))}. \end{aligned}$$

Hence, for the first term

$$\lim_{\lambda\to\lambda_o} \frac{[(\beta_1(\lambda) - \beta_1(\lambda_o)) - (\alpha_1(\lambda) - \alpha_1(\lambda_o)]}{\lambda - \lambda_o} = \frac{1}{S'(\beta_1(\lambda_o))} - \frac{1}{S'(\alpha_1(\lambda_o)}.$$

Obviously then for $\lambda \in \mathbb{T} \setminus Z$,

$$\lim_{\lambda\to\lambda_0} \frac{D(\lambda) - D(\lambda_0)}{\lambda - \lambda_0} = \sum_{j=1}^{J(\lambda_0)} [\frac{1}{S'(\beta_j(\lambda)} - \frac{1}{S'(\alpha_j(\lambda)}].$$

This completes the proof. □

5 Examples and motivations

It is clear that the results above are valid with the assumptions that p is again a polynomial and instead of the unit circle we may use any circle $\Gamma = \{\zeta : |\zeta - A| = R\}$, where A is a point in the complex plane. This is clear since setting

$$\phi(z) = \frac{z - A}{R} = w,$$

the pullback polynomial $P(w) = p(z(w))$ on the unit circle, we observe that the measure of the pull back of intervals on Γ is a constant multiple of the original intervals. Further, since we use no special polynomial properties of p (that is we use only analyticity on the closed disc [2]) that the results hold for functions f analytic on the closed disc. A natural step is to ask if the results hold for functions continuous on the closed disc and analytic on the disc (disc algebra functions)or functions in H^∞ (the bounded analytic functions on the disc)? The following examples show that the theorems as stated are close to sharp. Example 2 below was shown to us by our colleague Prof. W. Wogen and we thank him for it.

Example 5.1. *Let p be the polynomial $p(z) = 2 + z$. It is obvious that $D(\lambda) = 1 = 2\pi/2\pi$ for $0 < \lambda < 1$. For $1 < \lambda < 2$,*

$$D(\lambda) = \frac{2\pi - 2\arccos(\frac{\lambda^2-5}{4})}{2\pi}.$$

It is clear that

$$D'(\lambda) = \frac{1}{\sqrt{1 - (\frac{\lambda^2-5}{4})^2}}$$

is not defined at $\lambda = 1$. Hence $D(\lambda)$ is continuous on the open interval $(0, 3)$ but is not differentiable at $\lambda = 1$.

Example 5.2. *Let Ω be a simply connected domain in $|z| > 1$ bounded by a smooth curve $\Gamma = \gamma_1 \cup \gamma_2$, where γ_1 is an open arc in the unit circle and γ_2 lies in $|z| > 1$. Let $q(z)$ be a Riemann mapping from the unit disc onto Ω, with open (non-trivial) arc (e^{iA}, e^{iB}) mapping one to one and continuously onto γ_1. It is well known (Caratheodory's theorem) that q is in the disc algebra, that is it is a homeomorphism from the closed disc $\overline{\mathbb{D}}$ onto the closure of Ω. In fact, q is a diffeomorphism. We have $D(\lambda) = 1$ for $0 < \lambda < 1$. For $\lambda > 1$ the function $D(\lambda)$ has a jump. Hence, $D(\lambda)$ is not differentiable at $\lambda = 1$. Hence, Theorem 1 can not be improved to functions in the disc algebra (or even certain functions smooth up to the boundary). Note that by minor modifications to Ω one can make a function in the disc algebra for which the distribution function is continuous but not differentiable at a finite number of points.*

Example 5.3. *Choose values $0 < \theta_1 < \theta_2 < \theta_3 < \cdots < 2\pi$, with $\lim_n \theta_n = 2\pi$, and form the disjoint intervals on the unit circle $I_n = (e^{i\theta_n}, e^{i\theta_{n+1}})$. Define a bounded function $f(\zeta) = \frac{1}{2^n}$ for $\zeta \in I_n$. Now form the Herglotz integral*

$$H(z) = \int_{\mathbb{T}} \frac{\zeta - z}{\zeta + z} f(\zeta)\, dm(\zeta).$$

Taking the real part gives

$$\Re(H(z)) = \int_{T} P(z, \zeta) f(\zeta)\, dm(\zeta)$$

the Poisson integral of f on $\mathbb{T}$. This is a harmonic function with values $\frac{1}{2^n}$ on I_n. Taking the function

$$G(z) = e^{H(z)}$$

yields a bounded, analytic function which has a countable number of jump discontinuities in its distribution function.

6 Open problems

We think the following problems are worth consideration.

Problem 6.1. *Given a Jordan analytic curve in the plane and a polynomial, do either of the two theorems above hold in this setting?*

Very likely the answer may come from a Riemann mapping solution of the disc onto the domain bounded by the curve.

Problem 6.2. *Given a set, say $\mathbb{P}(n)$, of analytic polynomials of degree n (e.g. classical sets of orthogonal polynomials of degree n, etc.) compute the "semi-norm" expression or growth of*

$$\sup_{p\in\mathbb{P}(n)} \sup_{0<\lambda<\|p\|_\infty} \lambda D(\lambda, p),$$

as a function of n.

The study of distribution functions is really set more generally in the setting of a measure space $(X, d\mu)$ and a measurable function on the space [1]. That is one defines

$$D(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\}).$$

We list two rather significant results from the literature and refer the reader to the references as to their applications.

In this setting the "semi-norm" expression

$$\sup_{\lambda>0} \lambda D(\lambda),$$

makes sense, and if this expression is finite we say that the function f is in weak $L^1(X, d\mu)$. A significant result concerning this more general distribution expression is the following well known theorem.

Theorem 6.3. *Assume $(X, d\mu)$ is a measure space and $p \in (0, \infty)$. Assume f is a measurable function then*

$$\int_X |f(x)|^p \, d\mu(x) = p \int_o^\infty y^{p-1} D(y)) dy.$$

A further result due to John and Niremburg [4], which arises in harmonic analysis and partial differential equations is the following. We refer the reader to the reference [4] for the terminology appearing in the statement of the result.

Theorem 6.4. *For all f of Bounded Mean Oscillation ($f \in BMO(\mathbf{R}^n)$) in $\mathbf{R}^n$ and for all cubes Q in R^n (in standard position i.e., sides parallel to the coordinate axes) and all $\alpha > 0$ we have*

$$|\{x \in R^n : |f(x) - \frac{\int_Q f(y) dy}{|Q|}| > \alpha\}| \le e|Q|e^{-A\alpha/\|f\|},$$

where $\|f\|$ is the norm of f in $BMO(R^n)$, with $A = (2^n e)^{-1}$.

References

[1] G.B. Folland, *Real analysis - Modern techniques and their applications*, Second Edition John-Wiley and Sons Inc., New York, 1999.
[2] T. W. Gamelin, *Complex Analysis*, Undergraduate texts in mathematics, Springer-Verlag, New York, 2001
[3] W. Rudin, *Principles of mathematical analysis*, Second Edition, McGraw-Hill, New York, 1964.
[4] L. Grafakos, *Modern Fourier Analysis*, Third Edition, Graduate Texts in Mathematics, 250, Springer Verlag, 2014.