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A note on bi-contractive projections on spaces of vector valued continuous functions

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Abstract: This paper concerns the analysis of the structure of bi-contractive projections on spaces of vector valued continuous functions and presents results that extend the characterization of bi-contractive projections given by the first author. It also includes a partial generalization of these results to affine and vector valued continuous functions from a Choquet simplex into a Hilbert space.

Keywords: Contractive projections, Bi-contractive projections, Vector measures, Spaces of vector valued functions

MSC: 47B38, 46B04, 46E40

1 Introduction

We denote by $C(\Omega, E)$ the space of all continuous functions defined on a compact Hausdorff topological space Ω with values in a Banach space E, endowed with the norm $\|f\|_\infty = \max_{x\in\Omega} \|f(x)\|_E$. The continuity of a function $f \in C(\Omega, E)$ is understood to be "in the strong sense", meaning that for every $w_0 \in \Omega$ and $\epsilon > 0$ there exists $\mathcal{O}$, an open subset of Ω, containing w_0 such that $\|f(w) - f(w_0)\|_E < \epsilon$, for every $w \in \mathcal{O}$, see [9] or [16]. A contractive projection $P : C(\Omega, E) \to C(\Omega, E)$ is an idempotent bounded linear operator of norm 1. A contractive projection is called bi-contractive if its complementary projection $P^\perp$ $(= I - P)$ is contractive.

In this paper, we extend the characterization given in [6], of bi-contractive projections on $C(\Omega, E)$, with Ω a compact metric space, E a Hilbert space that satisfy an additional support disjointness property. Our characterization extends to all bi-contractive projections defined on $C(\Omega, E)$, with Ω a compact Haursdorff space and E a Hilbert space. See also [12] for a similar representation of the bi-contractive projections in $C(\Omega)$. More precisely, a bicontractive projection $P : C(\Omega, E) \to C(\Omega, E)$, is of one of the following forms:

1. $Pf = P_E \cdot f$, with $P_E : \Omega \to \mathrm{BC}(E)$ a strongly continuous function such that $P_E(x)$ is a bi-contractive projection on E and $(P_E \cdot f)(x) = P_E(x)(f(x))$, or
2. $Pf = \frac{f+U\cdot f\circ\tau}{2}$, with τ a degree 2 homeomorphism of Ω $(\tau^2 = id_\Omega)$ and $U : \Omega \to \mathrm{Isom}(E)$ a strongly continuous function such that, for every $x \in E$, $U(x)$ is a surjective isometry and $U(x)U(\tau(x)) = U(\tau(x))U(x) = Id_E$.

For a well studied class of bi-contractive projections on these spaces we refer the reader to [7] and [15].

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In Section 2 we give a brief description of the outline followed for the proof of the main result in [6]. We also explain the obstruct that lead to the artificial condition of restricting to "projections with the disjoint support property". In Section 3 we formulate the new extension and present some new arguments necessary for its proof, and we also include a few shorter proofs to some crucial lemmas formulated in [6] with interest of their own.

In Section 4 we partially extend the characterization of bi-contractive projections to spaces of vector valued affine maps on a Choquet simplex. We do not know how to carry our arguments forward to obtain a similar description even for a metrizable simplex.

2 Notation and background results

In this section we review the preliminaries needed to formulate our characterization for all the bi-contractive projections on $C(\Omega, E)$, with Ω a compact Hausdorff space and E a Hilbert space. We also give an outline of the proof given in [6]. We start with some general considerations about projections on a Banach space.

Let $P : E \to E$ be a contractive projection. It is easy to see that $E/ker(P)$ is isometric to $P(E)$. The adjoint projection $P^* : E^* \to E^*$ is also contractive and $P^*(E^*) = ker(P)^\perp$. This space is isometric to $P(E)^*$ via the mapping, $P^*(e^*) \to e^*|P(E)$. It may be noted that, in this identification, P^*-image of the unit ball of E^* is mapped onto the unit ball of $P(E)^*$, i.e. $P^*(E_1^*) = P(E)_1^*$. For a later use, we note that if $x_1^* = x_2^*$ on $P(X)$ then, for any $x \in X$, $P^*(x_1^*)(x) = x_1^*(P(x)) = x_2^*(P(x)) = P^*(x_2^*)(x)$. Hence $P^*(x_1^*) = P^*(x_2^*)$.

If P now denotes a contractive projection on $C(\Omega, E)$, $P : C(\Omega, E) \to C(\Omega, E)$, then P induces a contractive projection on the dual space: $P^* : C(\Omega, E)^* \to C(\Omega, E)^*$. Singer's Theorem (see [19]) identifies $C(\Omega, E)^*$ with the space of the bounded variation regular vector measures on the σ-algebra of the Borel subsets of Ω with values in E^* via an isometric isomorphism. We denote this space of vector measures by $\mathcal{M}(\Sigma_\Omega, E^*)$. We recall that this space is endowed with the norm $\|F\| = |F|(\Omega)$, where $|F|$ denotes the total variation of the vector measure F. We recall that the set of extreme points of the unit ball of $\mathcal{M}(\Sigma_\Omega, E^*)$ consists of Dirac measures of the form $\delta_\omega x^*$ for $\omega \in \Omega$ and x^* is an extreme point of the unit ball of E^*. Such a measure acts on $A \in \Sigma_\Omega$ by assigning x^* if $w \in A$, otherwise $\left(\delta_\omega x^*\right)(A) = 0$. Singer's identification implies that for any functional $\tau \in C(\Omega, E)^*$ there exists a unique vector measure F_τ in $\mathcal{M}(\Sigma_\Omega, E^*)$ such that $\tau(f) = \int_\Omega f dF_\tau$, for every $f \in C(\Omega, E)$. The integral is referred in [19] (p. 192) as the Gowurin integral. Moreover, $\|\tau\| = |F|(\Omega) = \|F\|$. When E^* has the Radon-Nikodým property (R. N. P.), given a vector measure F in $\mathcal{M}(\Sigma_\Omega, E^*)$, with total variation $|F|$, there exists a strongly measurable function $\frac{dF}{d|F|} : \Omega \to E^*$ such that

$$F(A) = \int_A \frac{dF}{d|F|} \, d|F|, \tag{1}$$

for every $A \in \Sigma_\Omega$, cf. [20] and [9]. Also $\|\frac{dF}{d|F|}\|_\infty \leq 1$. For F with $\|F\| = 1$, we assume without loss of generality that $\frac{dF}{d|F|}$ is everywhere defined and takes values in a separable subset of the unit sphere of E^*. In what follows, for a reflexive space E, we denote by ρ the duality map, i.e., $\rho : E^* \to 2^E$ defined by $\rho(e^*) = \{e \in E : \|e\| = 1 \text{ and } e^*(e) = \|e^*\|\}$. Thus $\rho(e^*)$ is a weakly compact convex subset of E, for details see p. 13 in [8]. We recall that any reflexive space can be renormed to be both smooth and strictly convex in the new norm (see Theorem VII.1.14 from [8]).

Lemma 2.1. *Let E be a reflexive Banach space and let $\rho : E^* \to 2^E$ be the duality map. Then there exists a strongly measurable function denoted by $\frac{d|F|}{dF} : \Omega \to E$ such that $\frac{dF}{d|F|}(w)(\frac{d|F|}{dF}(w)) = 1$ for all $w \in \Omega$.*

Proof. Since E is reflexive by Theorem 8 of [14], there exists a sequence of continuous functions $f_n : E^* \to E$ such that if $\sigma(e^*) = \lim_n f_n(e^*)$, then $\sigma(e^*) \in \rho(e^*)$. Hence σ is clearly a measurable function. We define $\frac{d|F|}{dF} = \sigma \circ \frac{dF}{d|F|}$, cf. [4], [10] or [9]. In order to obtain the strong measurability of this function we only need to verify that it has separable range. Since $\frac{dF}{d|F|}$ is separably valued and as f_n's are continuous we have that each

$f_n \circ \frac{dF}{d|F|}$ is a separably valued function. Therefore $\frac{d|F|}{dF}$ takes values in a separable subset of the unit sphere of E. Hence it is a strongly measurable function. □

We identify conditions under which P, at every function in $C(\Omega, E)$, is almost everywhere constant, when projected along the range of an extreme element of $P^* C(\Omega, E)_1^*$. The subscript 1 in $C(\Omega, E)_1^*$ denotes the closed unit ball. This is formulated in the theorem.

Theorem 2.2 (see [3] and [6]). *Let E be a Banach space such that E^* has the R. N. P. If P is a contractive projection on $C(\Omega, E)$ and F is an extreme point of $P^*(C(\Omega, E)_1^*)$, then, for every $f \in C(\Omega, E)$, the function $\langle Pf, \frac{dF}{d|F|}\rangle : \Omega \to \mathbb{C}$, given by*

$$\left\langle Pf, \frac{dF}{d|F|}\right\rangle (x) = \frac{dF}{d|F|}(x)\,((Pf)(x)),$$

is constant almost everywhere on the support of $|F|$.

The proof is omitted since it is given in [6]. Applying Theorem 2.2 and a vector valued formulation of the Tietze Extension Theorem due to Dugundji (see [11]) it was shown, in [6], that the support of each extreme point of $P^*(C(\Omega, E)_1^*)$ is either one point or two points. This implies that the extreme points are of the form $e^* \delta_x$ or $\frac{e_1^* \delta_x + e_2^* y}{2}$ with $x, y \in \Omega$ and e^*, e_1^* and e_2^* are extreme points of the unit ball E_1^*. We shall refer to them as extreme points of type I and II, respectively. If S denotes the set of points in Ω that belong to the support of some extreme point, then it was also shown in [6] that $Pf(x) = 0$, for every f and $x \notin \overline{S}$, provided the range space E is uniformly convex. In addition, for Hilbert-ranged continuous functions and a bicontractive projections P on this setting, such that all the aforementioned extreme points are of type I, then $Pf(x) = P_E(f(x))$ for every $f \in C(\Omega, E)$ and $x \in \Omega$. We observe that the proof of this result relies on the inner product structure of the range space. The main result followed provided that distinct extreme points have disjoint supports.

3 Bi-contractive projections and vector-valued Tietze extension theorem

In this section we give a characterization for all bi-contractive projections on $C(\Omega, E)$ with Ω a compact Hausdorff space and E a Hilbert space. One of the main tools is the vector valued Tietze Extension Theorem. We give a proof, that uses different ideas from those employed in [11] and [2]. We also provide simpler and shorter arguments for some lemmas in [6]. We formulate the statement and give the respective proof for the more general characterization of bi-contractive projections, without any further support disjointness condition.

We start with basic definitions and results on M-ideals, which we now recall from Chapter I of [13]. This is the main tool for our new proof of the Tietze Extension Theorem.

A closed subspace of a Banach space X, $J \subset X$, is said to be an M-ideal if there is a linear projection $Q : X^* \to X^*$ such that $ker(Q) = J^\perp$ and $\|Q(x^*)\| + \|x^* - Q(x^*)\| = \|x^*\|$ for all $x^* \in X^*$. It is known that J^* is isometric to range of Q and $X^* = J^\perp \oplus_1 J^*$. An important property of an M-ideal that we will be needing is that it is a proximinal subspace, i.e., $d(x, J) = \|x - j\|$ for some $j \in J$. For a closed set $A \subset \Omega$, the space $M_A(\Omega, E) = \{f \in C(\Omega, E) : f(A) = 0\}$ is an M-ideal. See Corollary 10.2 ([5]). For any Borel set $B \subset \Omega$, let $\mathcal{M}(B, E^*) = \{F \in M(\Omega, E^*) : supp|F| \subset B\}$.

Theorem 3.1. *Let Ω be a compact Hausdorff space and let E be a closed subset of Ω. Let X be a Banach space. Then every continuous function in $C(E, X)$ has a norm preserving extension to Ω.*

Proof. First, we observe that since $M_E(\Omega, X) = \{f \in C(\Omega, X) : f(E) = 0\}$ is an M-ideal of $C(\Omega, X)$, $M(\Omega, X^*) = M_E(\Omega, X)^\perp \oplus_1 M_E(\Omega, X)^*$. The additive property of a measure also implies that

$$C(\Omega, X)^* = M_E(\Omega, X)^\perp \oplus_1 \mathcal{M}(\Omega \setminus E, X^*).$$

We define the function

$$\Phi : C(\Omega, X)/M_E(\Omega, X) \to C(E, X) \ \ s.t. \ \ \Phi([f]) = f|_E.$$

For any $f \in C(\Omega, X)$, clearly $d(f, M_E(\Omega, X)) \geq \|f|E\|$. Also by an application of the Krein-Milman theorem there exists an extreme point τ of the unit ball of $M_E(\Omega, X)^\perp$ such that $d(f, M_E(\Omega, X)) = \tau(f)$. Because of the L-decomposition of $C(\Omega, X)^*$, τ is also an extreme point of the unit ball of the space of measures and hence $\tau = \delta(\omega)x^*$. Let $x \in X$ be such that $x^*(x) = 1$. If $\omega \notin E$, let $g \in C(\Omega)$ be such that $g(E) = 0$ and $g(\omega) = 1$. Now $gx \in M_E(\Omega, X)$ so that $\tau(gx) = 0$ where as $\tau(gx) = (\delta(\omega)x^*)(gx) = 1$. This contradiction shows that $\omega \in E$. Therefore $d(f, M_E(\Omega, X)) = x^*(f(\omega))$ so that $d(f, M_E(\Omega, X)) \leq \|f(e)\| \leq \|f|E\|$. Thus Φ is an isometry.

We show that Φ is onto. If Φ was not onto, there would exist $h \in C(E, X)$ such that $\Phi([f]) \neq h$, for every $[f] \in C(\Omega, X)/M$. Hahn-Banach's Theorem implies the existence of a norm 1 functional τ in $C(E, X)^*$ such that $\tau(h) = \|h\|$ and τ vanishes on all functions in $C(E, X)$ that have a continuous extension to the whole Ω. Then τ is identified by $F \in \mathcal{M}(E, X^*)$ such that $\int_E f dF = 0$ for all $f \in C(\Omega, X)$. If $F' \in \mathcal{M}(\Omega, X^*)$ is an extension vanishing off E, then $\int f dF' = 0$, for all $f \in C(\Omega, X)$. So by uniqueness of Singer's representation theorem, $F' = 0$. This is a contradiction since F is not zero. Therefore Φ is onto. For $g \in C(E, X)$, $\|g\| = 1$, let $f \in C(\Omega, X)$ be such that $f|E = g$ and $d(f, M_E(\Omega, X)) = 1$. Since $M_E(\Omega, X)$ is an M-ideal, there is a $h \in M_E(\Omega, X)$ such that $\|f - h\| = 1$ (see [13], Chapter I). Thus $f - h$ is the required norm-preserving extension of g. This completes the proof. □

We denote by $Supp_F$ the support of the vector measure F.

Proposition 3.2. *Let Ω be a compact Hausdorff space and E be a reflexive Banach space. Let P be a bi-contractive projection on $C(\Omega, E)$ and F an extreme point in $P^*(C(\Omega, E)^*_1)$. Then the support of F is either $\{x\}$ or $\{x, y\}$, with $x, y \in \Omega$.*

Lemma 3.3. *Let Ω be a compact Hausdorff space and E a reflexive Banach space. Let P be a bi-contractive projection on $C(\Omega, E)$ and F an extreme point in $P^*(C(\Omega, E)^*_1)$. Then, for every $x \in Supp_F$ and an open subset of Ω, $\mathcal{O}$, containing x, $|F|(\mathcal{O}) \geq \frac{1}{2}$.*

The proofs for these two previous results rely on the existence of continuous functions with values in a Banach space having some specific properties, see [6]. Such functions are shown to exist by Theorem 3.1, see also [11] and [2].

Remark 3.4. *The extreme points of $P^*(C(\Omega, E)^*_1)$ are of one of the following forms: $e^*\, \delta_x$ or $\frac{e_1^*\, \delta_x + e_2^*\, \delta_y}{2}$, with $x, y \in \Omega$ and e^*, e_1^* and e_2^* are unit vectors of E^*_1.*

We denote by $\mathcal{S}$ the set of all points x in Ω such that, for some $e^* \in S_{E^*}$, either $e^*\, \delta_x$ is an extreme point of $P^*(C(\Omega, E)^*_1)$ or there exist $y \in \Omega$, u^* and v^* in S_{E^*} such that $\frac{u^*\, \delta_x + v^*\, \delta_y}{2}$ is an extreme point of $P^*(C(\Omega, E)^*_1)$. Each extreme point is determined by its symbols. In the first case the functional is defined by x, e^* and in the second case by x, y, u^*, v^*.

The next lemma formulates an extension of Lemma 4.6 in [6]. It deals with spaces of continuous functions $C(\Omega, E)$, where Ω denotes a compact Hausdorff space and E a smooth and reflexive Banach space, while the aforementioned lemma was for compact metric space Ω and uniformly convex range space E . The argument given here uses Theorem 3.1 and is shorter than the one presented in [6].

Lemma 3.5. *Let E be a reflexive and smooth Banach space. Any two extreme points of $P^*(C(\Omega, E)^*_1)$ that coincide on $P(C(\Omega, E))$ have equal symbols.*

Proof. We first observe that E reflexive and smooth implies that E^* is also reflexive and E^* is strictly convex. Given two extreme points that coincide on $P(C(\Omega, E))$, say $u^*\, \delta_x$ and $\frac{w_0^* \delta_{y_0} + w_1^* \delta_{y_1}}{2}$, as noted in the introduction, $u^*\, \delta_x = \frac{w_0^* \delta_{y_0} + w_1^* \delta_{y_1}}{2}$. Applying this equation to the Borel set $\{x, y_0, y_1\}$ yields $x = y_0 = y_1$ and $u^* = \frac{w_0^* + w_1^*}{2}$.

The strict convexity of E^* implies $u^* = w_0^* = w_1^*$. Let u be a unit vector such that $u^*(u) = 1$. Then applying to a continuous function f of norm one, such that $f(x) = u$ and $f(y_0) = f(y_1) = 0$, implies that

$x \in \{y_0, y_1\}$. Suppose $x = y_0$. Again applying the same equation to a f such that $f(x) = 0$ and $f(y_1) = u$, we get a contradiction . Such functions exist due to Theorem 3.1. The other cases are similarly handled. □

Lemma 3.6. *Let Ω be a compact Hausdorff space and E a reflexive Banach space. Let P be a bi-contractive projection on $C(\Omega, E)$. Then for every $x \notin \overline{S}$ and for every $f \in C(\Omega, E)$, $Pf(x) = 0$.*

Proof. Let $f \in C(\Omega, E)$, $\|f\| = 1$, $x \notin \overline{S}$, with $P(f)(x) \neq 0$. Construct $g \in C(\Omega, E)$ and of norm 1 such that $g = f$ on S and $g(x) = \frac{P(f)(x)}{\|P(f)(x)\|}$. Now if $P(f) \neq P(g)$, there is an extreme point τ of $P^*(C(\Omega, E)^*)_1)$ (see the remarks in the introduction), such that $\tau(P(f)) \neq \tau(P(g))$. Note that $P^*(\tau) = \tau$.

Suppose $\tau = \frac{e_1^*\delta_{y_1} + e_2^*\delta_{y_2}}{2}$ for some $y_i \in \Omega$. By definition $y_i \in S$. So $g(y_i) = f(y_i)$. Therefore $\tau(P(f)) = P^*(\tau)(f) = \tau(f) = \frac{e_1^*(f(y_1)) + e_2^*(f(y_2))}{2} = \frac{e_1^*(g(y_1)) + e_2^*(g(y_2))}{2} = \tau(P(g))$. This is a contradiction. Therefore $P(f) = P(g)$ leading to the further contradiction that $I - P$ is a contraction.

The other possibility for τ is similarly handled. This completes the proof. □

For each point x in $\mathcal{S}$, we associate the subspace of E^*, A_x^*, spanned by all elements u^* in E_1^* such that $u^*\,\delta_x$ is an extreme point of $P^*(C(\Omega, E)_1^*)$. Since $P^*(u^*\,\delta_x) = u^*\,\delta_x$ then for every $f \in C(\Omega, E)$ we have $u^*(Pf(x)) = u^*(f(x))$. Linearity of P^* implies that $\varphi(Pf(x)) = \varphi(f(x))$, for every $f \in C(\Omega, E)$, and for every element $\varphi \in A_x^*$. If E^* is a smooth space, given a functional in A_x^*, u^*, there exists a unique $u \in S_E$ such that $u^*(u) = \|u^*\|$. We denote by A_x the closed subspace of E spanned by all such vectors.

We now consider the case when E is a Hilbert space. Clearly we have a smooth, strictly convex reflexive space. So we can apply the earlier analysis of bi-contractive projections.

Proposition 3.7. *Let Ω be a compact Hausdorff topological space and E a Hilbert space. Let P be a bi-contractive projection on $C(\Omega, E)$. If the set $\mathcal{S}$, determined by P, consists of points in Ω associated with an extreme point of type 1, then there exists a strongly continuous function from Ω into the set of all bi-contractive projections on E, i.e. $P_E : \Omega \to BC(E)$, such that*

$$Pf(x) = P_E(x)(f(x)),$$

for every $f \in C(\Omega, E)$ and $x \in \Omega$.

Let E be a reflexive and strictly convex space. We next show that any two distinct extreme points of $P^*(C(\Omega, E)_1^*)$ with P a bi-contractive projection have disjoint support.

Proposition 3.8. *Let Ω be a compact Hausdorff topological space and E a reflexive and strictly convex space. Let P be a bi-contractive projection on $C(\Omega, E)$. Then any two distinct extreme points of $P^*(C(\Omega, E)_1^*)$ have disjoint support.*

Proof. We have shown that any extreme point is either of the form $u^*\delta_x$ (support reduces to a single point in Ω) or $\frac{u^*\delta_x + v^*\delta_y}{2}$ (the support consists of two points in Ω).

We assume that $\frac{u^*\delta_x + v^*\delta_y}{2}$ and $\frac{u_1^*\delta_x + v_1^*\delta_z}{2}$ are extreme points of $P^*(C(\Omega, E)_1^*)$, with $z \neq x \neq y \neq z$. Since these functionals are fixed points of P^*, we have

$$u^*[Pf(x)] + v^*[Pf(y)] = u^*[f(x)] + v^*[f(y)]$$

and

$$u_1^*[Pf(x)] + v_1^*[Pf(z)] = u_1^*[f(x)] + v_1^*[f(z)].$$

Theorem 2.2 implies that

$$u^*[Pf(x)] = v^*[Pf(y)] \text{ and } u_1^*[Pf(x)] = v_1^*[Pf(z)].$$

Therefore these equations imply

$$P^*\left(u^*\delta_x\right) = \frac{u^*\delta_x + v^*\delta_y}{2} \text{ and } P^*\left(u_1^*\delta_x\right) = \frac{u_1^*\delta_x + v_1^*\delta_z}{2}. \quad (2)$$

Let u, v, v_1 be unit vectors such that $u^*(u) = 1 = v^*(v) = v_1^*(v_1)$. We select a continuous function f on $\{x, y, z\}$ (of norm 1) such that $f(x) = u$, $f(y) = v$ and $f(z) = v_1$. An application of Theorem 3.1 implies the existence of a continuous function $\tilde{f}$ of norm 1 that extends f to Ω. For simplicity of notation we shall denote this extension by f. Hence, the first equation in (2), applied to f, yields $u^*[Pf(x)] = 1$ and the strict convexity of the range space implies that $Pf(x) = u$. Therefore the second equation in (2), applied to f, yields

$$u_1^*[u] = \frac{u_1^*[u] + v_1^*[v_1]}{2},$$

thus $u_1^*(u) = 1$. This implies that $u_1 = u$. We now consider two extreme points $\frac{u^*\delta_x + v_0^*\delta_y}{2}$ and $\frac{u^*\delta_x + v_1^*\delta_z}{2}$. As before let $v_0^*(v_0) = 1$. For any $f \in C(\Omega, E)$, Theorem 2.2 implies that

$$u^*[Pf(x)] = v_0[Pf(y)] \text{ and } u^*[Pf(x)] = v_1[Pf(z)]$$

On the other hand we have

$$u^*[Pf(x)] + v_0^*[Pf(y)] = u^*[f(x)] + v_0^*[f(y)]$$

and

$$u^*[Pf(x)] + v_1^*[Pf(z)] = u^*[f(x)] + v_1^*[f(z)].$$

Therefore we conclude that $v_0^*[f(y)] = v_1^*[f(z)]$, for all $f \in C(\Omega, E)$. This implies by arguments similar to the ones above, that $y = z$ and $v_0 = v_1$. The other cases are similar. This completes the proof. □

We formulate our main result. We denote the surjective isometries on E by $\mathrm{Isom}(E)$ and the bi-contractive projections on E by $\mathrm{BP}(E)$.

Theorem 3.9. *Let Ω be a compact Hausdorff topological space and E a Hilbert space. Let P be a bi-contractive projection on $C(\Omega, E)$. Then*

1. *There exists a strongly continuous map $P_E : \Omega \to BP(E)$ such that*

$$Pf(x) = P_E(x)(f(x)),$$

for every $f \in C(\Omega, E)$ and $x \in \Omega$, or

2. *There exist a homeomorphism τ of Ω with $\tau^2 = \mathrm{id}_\Omega$ and a strongly continuous map $R : \Omega \to \mathrm{Isom}(E)$ with $R(x)R(\tau(x)) = R(\tau(x))R(x) = Id_E$ such that*

$$Pf(x) = \frac{f(x) + R(\tau(x))f(\tau(x))}{2},$$

for every $f \in C(\Omega, E)$ and $x \in \Omega$.

The proof is as given in [6], since Proposition 3.8 now ensures that bi-contractive projections have disjoint support property when E is a Hilbert space.

Remark 3.10. *It is an easy computation to check that every operator of the form listed in the statement of the theorem is indeed a bi-contractive projection.*

4 Bi-contractive projections on spaces of vector valued affine maps

The result formulated in Theorem 3.9 can be partially extended to spaces $A(K, H)$ of all affine and vector valued continuous functions from a Choquet simplex K into a Hilbert space $\mathcal{H}$, equipped with the supremum norm. We start by proving a generalization of Tietze Extension Theorem in this context. We will be using results from convexity theory from E. M. Alfsen's monograph, [1].

Let $\partial_e K$ denote the set of extreme points of K. We recall that (Theorem II.3.12 in [1]) for any compact set $E \subset \partial_e K$, for any $f \in C(E)$, there is an affine continuous function on K which is a norm preserving extension of f. As a consequence we note for any $k \in \partial_e K \setminus E$, there is a function $a \in A(K)$ with $a(E) = 0$ and $a(k) = 1$. Also, $\overline{conv}(E)$ is a split face of K (see Theorem II.6.22 in [1]). If $J = \{a \in A(K) : a(E) = 0\}$, then J is an M-ideal in $A(K)$ (see Example I.1.4(c) of [13]). In view of Theorem II.3.12 ([1]), we have that $[a] \to a|E$ is a surjective isometry of $A(K)/J$ and $C(E)$.

Proposition 4.1. *Let K be a Choquet simplex and let E be a compact subset of $\partial_e K$. Let X be a Banach space. Then every continuous X-valued function on E has an affine norm preserving extension to K.*

Proof. It is known that when K is a simplex, $A(K,X)$ can be identified with the injective tensor product $A(K) \otimes_\epsilon X$. (See [17]). We proceed as in the proof for Theorem 3.1, we set $M_E(K,X) = \{a \in A(K,X) : a(E) = 0\}$. Since this set can be identified with $J \oplus_\epsilon X$, by Proposition VI.3.1 in [13], it is an M-ideal in $A(K,E)$. Let $\Phi : A(K,X)/M_E(K,X) \to C(E,X)$ be the restriction map. As before we have the identification $A(K,X)^* = M_E(K,X)^\perp \oplus_1 M_E(K,X)^*$. We also recall from Theorem VI.1.3 in [13], that any extreme point of the unit ball of $A(K,X)^*$ is of the form $\delta_k x^*$ for some $k \in \partial_e K$ and x^* is an extreme point of the unit ball of X^*. Since the norm of a continuous affine function is determined by its values at $\partial_e K$, it now follows from our remarks before the proposition and the proof of Theorem 3.1 that Φ is an isometry.

To show that Φ is onto we next recall vector valued integral representation and its uniqueness in the context of Choquet simplexes from [18] (these are analogues of Singer's theorem for $C(\Omega, X)$). A X^*-valued measure on the Borel σ-field of K, is called a boundary measure if the total variation measure $|F|$ is a boundary measure as in the context of convexity theory. It is easy to see that any $F \in M(E, X^*)$ extends to a boundary measure on K. If Φ is not onto, we get a non-zero boundary measure F with $supp(F) \subset E$ such that $\int_K adF = \int_E adF = 0$ for all $a \in A(K,E)$. By the uniqueness of representing measures, $F = 0$. This is a contradiction. Since $M_E(K,X)$ is an M-ideal in $A(K,X)$, we have that functions in $C(E,X)$ have norm-preserving extensions to $A(K,X)$. □

Let X be a Banach space and let $P : X \to X$ be a bi-contractive projection. Let $J \subset X$ be a closed subspace and $P(J) \subset J$. Define $P' : X/J \to X/J$ by $P'([x]) = [P(x)]$. This is a well defined linear projection. $\|P'([x])\| = inf\{\|P(x) - j\| : j \in J\} \leq inf\{\|P(x) - P(j)\| : j \in J\} \leq inf\{\|x - j\| : j \in J\} = \|[x]\|$. The symmetry of the argument shows that P' is a bi-contractive projection.

Theorem 3.9 can now be reformulated.

Theorem 4.2. *Let K be a Choquet simplex, $E \subset \partial_e K$ be a compact set and $\mathcal{H}$ a Hilbert space. Let P be a bi-contractive projection on $A(K,\mathcal{H})$ such that $P(M_E(K,\mathcal{H})) \subset M_E(K,\mathcal{H})$. Then P' can be described using bi-contractive projections of $C(E,\mathcal{H})$.*

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