Concr. Oper. 2019; 6:92–105 **DE GRUYTER**

Review Article **Open Access**

Sophie Grivaux*

The Blum-Hanson Property

https://doi.org/10.1515/conop-2019-0009
Received October 16, 2019; accepted December 9, 2019

Abstract: Given a (real or complex, separable) Banach space, and a contraction T on X, we say that T *has the Blum-Hanson property* if whenever $x, y \in X$ are such that $T^n x$ tends weakly to y in X as n tends to infinity, the means

$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k} x$$

tend to y in norm for *every* strictly increasing sequence $(n_k)_{k\geq 1}$ of integers. The space X itself has the Blum-Hanson property if every contraction on X has the Blum-Hanson property. We explain the ergodic-theoretic motivation for the Blum-Hanson property, prove that Hilbert spaces have the Blum-Hanson property, and then present a recent criterion of a geometric flavor, due to Lefèvre-Matheron-Primot, which allows to retrieve essentially all the known examples of spaces with the Blum-Hanson property. Lastly, following Lefèvre-Matheron, we characterize the compact metric spaces K such that the space $C(K)$ has the Blum-Hanson property.

Keywords: Blum-Hanson property; mean ergodic theorem along subsequences; strongly mixing dynamical systems; L^p spaces; composition operators on $C(K)$-spaces

MSC: 47A35

1 Introduction

These notes present the material for a mini-course on the Blum-Hanson property, given within the framework of the ACOTCA 2019 conference at the Université Paris-Est in Marne-la-Vallée (France) in June 2019. They were written down by Clément Coine. The mini-course consisted of three lectures of 45 minutes. The structure of this course is preserved in these notes, and the contents of the three lectures correspond to the contents of Sections 2, 3 and 4 respectively.

We will be concerned in this series of lectures with a property of contractions of bounded operators on Banach spaces, called *the Blum-Hanson property*. It originated in the work [8] of Blum and Hanson in the 60's who characterized a certain property of measure-preserving dynamical systems (strong mixing) in terms of a mean ergodic theorem along all subsequences. Just like in the ergodic theorem of von Neumann, this theorem of Blum and Hanson has an abstract formulation for contractions on a (real or complex, separable) Hilbert space, which is Theorem 1 below. It was proved by Akcoglu-Sucheston [3] and Jones-Kuftinec [11] independently.

Whenever (x_n) is a sequence of elements of a Banach space X, and $x \in X$, the notation

$$x_n \xrightarrow{\|.\|} x$$

***Corresponding Author: Sophie Grivaux:** CNRS, Univ. Lille, UMR 8524 - Laboratoire Paul Painlevé, F-59000 Lille, France, E-mail: sophie.grivaux@univ-lille.fr, http://math.univ-lille1.fr/~grivaux/

means that x_n tends to x in norm in X as n tends to infinity, while the notation $x_n \xrightarrow{w} x$ means that x_n tends weakly to x in X. We denote by $\mathcal{B}(X)$ the algebra of bounded operators on X, by B_X the closed unit ball of X, and by S_X its unit sphere.

Theorem 1. *[3, 11] Let H be a (real or complex) Hilbert space, and let $T \in \mathcal{B}(H)$ with $\|T\| \le 1$. If $x, y \in H$ are such that $T^n x \xrightarrow{w} y$, then*

$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k}x \xrightarrow{\|\cdot\|} y$$

for every strictly increasing sequence $(n_k)_{k\ge1}$ of (positive) integers.

This theorem motivates the following definition:

Definition 2. *Let X be a (real or complex) Banach space, and let $T \in \mathcal{B}(X)$. We say that T* has the BH property *(or simply* has BH*) if whenever $x, y \in X$ are such that $T^n x \xrightarrow{w} y$,*

$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k}x \xrightarrow{\|\cdot\|} y$$

for every strictly increasing sequence $(n_k)_{k\ge1}$ of integers. We say that X itself has the BH property if every contraction on X has the BH property.

In the first part of these lectures (Section 2), we will quickly present the theorem of Blum and Hanson and its ergodic-theoretic motivation. We will also prove Theorem 1 and give some examples of spaces which have BH. During the second lecture (Section 3), we will present and prove a recent criterion, due to Lefèvre-Matheron-Primot [15], proving that certain contractions (sometimes all contractions) on certain Banach spaces have BH. We will present some of its applications, as well as its limits. Finally, the last part of this mini-course (Section 4) will be devoted to the study of spaces which do not have the BH property.

2 Strongly mixing dynamical systems and the BH Theorem

Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $\phi : X \to X$ be a *measure-preserving transformation* of $(X, \mathcal{B}, \mu)$: this means that $\mu(\phi^{-1}(A)) = \mu(A)$ for every $A \in \mathcal{B}$. One associates to ϕ a canonical isometry U_ϕ on $L^2(X, \mathcal{B}, \mu)$, called the Koopman operator and defined as follows:

$$U_\phi : f \mapsto f \circ \phi, \quad f \in L^2(X, \mathcal{B}, \mu).$$

When ϕ is an invertible measure-preserving transformation, U_ϕ is a unitary operator. For a first reading in ergodic theory, we recommend the classical book [18] by Walters.

Von Neumann's mean ergodic theorem implies that

$$\frac{1}{N}\sum_{k=1}^{N} f \circ \phi^k = \frac{1}{N}\sum_{k=1}^{N} U_\phi^k f \xrightarrow{\|\cdot\|} P_{\ker(U_\phi - I)} f \ \text{ in } \ L^2(X, \mathcal{B}, \mu)$$

for every $f \in L^2(X, \mathcal{B}, \mu)$, where $P_{\ker(U_\phi - I)}$ denotes the orthogonal projection on the eigenspace $\ker(U_\phi - I)$ of U_ϕ in the space $L^2(X, \mathcal{B}, \mu)$.

The transformation ϕ is said to be *ergodic* when the only ϕ-invariant functions $f \in L^2(X, \mathcal{B}, \mu)$ are constant almost everywhere: $f \circ \phi = f$ μ-a.e. $\Rightarrow$ $f = c$ μ-a.e. Another way of saying this is that $\ker(U_\phi - I)$ is 1-dimensional. In this case,

$$\frac{1}{N}\sum_{k=1}^{N} f \circ \phi^k \xrightarrow{\|\cdot\|} \int_X f \, \mathrm{d}\mu \quad \text{for every } f \in L^2(X, \mathcal{B}, \mu),$$

and hence

$$\frac{1}{N}\sum_{k=1}^{N}\left\langle f\circ\phi^k, g\right\rangle \underset{N\to+\infty}{\longrightarrow} \left(\int_X f\mathrm{d}\mu\right)\overline{\left(\int_X g\mathrm{d}\mu\right)} \quad \text{for every } f,g\in L^2(X,\mathcal{B},\mu),$$

where $\langle\,.\,,\,.\,\rangle$ denotes the scalar product in $L^2(X,\mathcal{B},\mu)$. This is equivalent to the condition

$$\frac{1}{N}\sum_{k=1}^{N}\mu(\phi^{-k}(A)\cap B)\underset{N\to+\infty}{\longrightarrow}\mu(A)\mu(B) \quad \text{for every } A,B\in\mathcal{B},$$

and to the condition that if $A\in\mathcal{B}$ is such that $\phi^{-1}(A)=A$ up to a set of μ-measure 0, then $\mu(A)=0$ or $\mu(A)=1$.

Ergodic systems are the basic building blocks for all measure-preserving systems (a good illustration of this is given by the Ergodic Decomposition Theorem, see for instance [1, Th. 2.2.9]); they satisfy Birkhoff's pointwise ergodic theorem: for every $f\in L^1(X,\mathcal{B},\mu)$,

$$\frac{1}{N}\sum_{k=1}^{N}f(\phi^k x)\underset{N\to+\infty}{\longrightarrow}\int_X f\mathrm{d}\mu \quad \text{for } \mu-\text{a.e. } x\in X.$$

which is classically rephrased as "the time means equal the space mean μ-a.e.".

The simplest examples of ergodic systems are the irrational rotations on the unit circle, but there are many more examples, in various contexts (see one of the references [18], [17] or [9]).

Let us go back to the definition of ergodicity in terms of Koopman operators: for all $f,g\in L^2(X,\mathcal{B},\mu)$,

$$\frac{1}{N}\sum_{k=1}^{N}\left\langle U_\phi^k f, g\right\rangle\underset{N\to+\infty}{\longrightarrow}\langle f,1\rangle\,\overline{\langle g,1\rangle}.$$

There are several natural reinforcements of this notion, where one requires a different kind of convergence of the quantities $\left\langle U_\phi^k f, g\right\rangle$ above. One of them is *strong mixing*:

Definition 3. *A measure-preserving transformation ϕ of $(X,\mathcal{B},\mu)$ is strongly mixing if for every $f,g\in L^2(X,\mathcal{B},\mu)$,*

$$\left\langle U_\phi^N f, g\right\rangle\underset{N\to+\infty}{\longrightarrow}\langle f,1\rangle\,\overline{\langle g,1\rangle}.$$

This is equivalent to the condition $\mu(\phi^{-N}(A)\cap B)\to\mu(A)\mu(B)$ for every $A,B\in\mathcal{B}$, i.e. to the condition that the events $\phi^{-N}(A)$ and B become asymptotically independent as N goes to infinity. Hence the terminology "strongly mixing".

Rotations of the unit circle are never strongly mixing. But endomorphisms of the tori $\mathbb{R}^n/\mathbb{Z}^n$ are strongly mixing as soon as they are ergodic. Endomorphisms of tori are given by $n\times n$ matrices with integer entries: to each such matrix

$$A=\begin{pmatrix}a_{11}&\dots&a_{1n}\\ \vdots&\ddots&\vdots\\ a_{n1}&\dots&a_{nn}\end{pmatrix} \quad \text{one associates the map } \Phi_A: \begin{array}{ll}\mathbb{R}^n/\mathbb{Z}^n & \longrightarrow \mathbb{R}^n/\mathbb{Z}^n\\ \begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} & \longmapsto A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\end{array}.$$

Here are the simplest examples of endomorphisms of tori: take $n=1$, $A=(p)$ with $p\in\mathbb{N}\setminus\{0,1\}$: one obtains the map

$$\begin{array}{ll}\mathbb{R}/\mathbb{Z} & \longrightarrow \mathbb{R}/\mathbb{Z}\\ x & \longmapsto px \bmod 1.\end{array}$$

Endomorphisms of tori preserve Haar measure (for example, check that for every semi-open arc I in $\mathbb{R}/\mathbb{Z}$, $\mu(\{x\,;\,px \bmod 1\in I\})=\mu(I)$). When Φ_A is surjective (which happens exactly when $\det(A)\neq 0$), Φ_A is ergodic if and only if A has no roots of unity as eigenvalues, if and only if Φ_A is strongly mixing. See [18] for details.

Going back to the theory of strongly mixing systems, we observe that ϕ is strongly mixing if and only if $U_\phi^N f \xrightarrow{w} 0$ for every $f \in L_0^2(X, \mathcal{B}, \mu)$, where

$$L_0^2(X, \mathcal{B}, \mu) := \left\{ f \in L^2(X, \mathcal{B}, \mu), \int_X f \, \mathrm{d}\mu = 0 \right\}$$

i.e. $U_\phi^N \to P_{\ker(U_\phi - I)}$ in the so-called Weak Operator Topology of $L^2(X, \mathcal{B}, \mu)$.

Here is the characterization of strongly mixing systems obtained by Blum and Hanson in 1960.

Theorem 4. *[8] The (measure-preserving) dynamical system $(X, \mathcal{B}, \mu; \phi)$ is strongly mixing if and only if for every strictly increasing sequence $(n_k)_{k\geq 1}$ of integers, we have*

$$\left\| \frac{1}{N} \sum_{k=1}^{N} U_\phi^{n_k} f - \int_X f d\mu \right\|_2 \underset{N \to +\infty}{\longrightarrow} 0 \text{ for every } f \in L^2(X, \mathcal{B}, \mu).$$

One can replace the norm $\|.\|_2$ by any norm $\|.\|_p$, $1 \leq p < +\infty$, in the statement of Theorem 4. But one cannot replace it by pointwise convergence. An example of a strongly mixing system $(X, \mathcal{B}, \mu; \phi)$ for which there exists a strictly increasing sequence $(n_k)_{k\geq 1}$ of integers and a function $f \in L^2(X, \mathcal{B}, \mu)$ such that

$$\liminf_{N \to +\infty} \frac{1}{N} \sum_{k=1}^{N} U_\phi^{n_k} f = 0 \quad \text{and} \quad \limsup_{N \to +\infty} \frac{1}{N} \sum_{k=1}^{N} U_\phi^{n_k} f = 1 \quad \mu-\text{a.e.}$$

was first given in [10], and then Krengel proved in [12] that there exists a universal strictly increasing sequence $(n_k)_{k\geq 1}$ of integers such that for every strongly mixing system $(X, \mathcal{B}, \mu; \phi)$, there exists a set $A \in \mathcal{B}$ with the property that

$$\liminf_{N \to +\infty} \frac{1}{N} \sum_{k=1}^{N} U_\phi^{n_k} \mathbb{1}_A = 0 \quad \text{and} \quad \limsup_{N \to +\infty} \frac{1}{N} \sum_{k=1}^{N} U_\phi^{n_k} \mathbb{1}_A = 1 \quad \mu-\text{a.e.},$$

where $\mathbb{1}_A$ denotes the indicator function of A.

As we already mentioned, the theorem of Blum and Hanson admits an abstract formulation valid for all contractions T on a Hilbert space H (Theorem 1 above), also called a *mean ergodic theorem along all subsequences*: if $T^n x \xrightarrow{w} y$, then

$$\left\| \frac{1}{N} \sum_{k=1}^{N} T^{n_k} x - y \right\| \to 0$$

for every strictly increasing sequence of integers $(n_k)_{k\geq 1}$. In the case where $n_k = k$ for all k, this is a classical mean ergodic theorem, valid on all Banach spaces X (see for instance [13, Ch.2, Th. 1.1]). In the same circle of ideas, recall that whenever T is a power-bounded operator on a reflexive Banach space X (i.e. $\sup ||T^n|| < +\infty$; this holds true in particular when T is a contraction), the averages

$$\frac{1}{N} \sum_{k=1}^{N} T^k x$$

converge in norm in X to a vector y belonging to $\ker(T - I)$.

Let us now prove Theorem 1.

Proof of Theorem 1. Since $Ty = y$, we can assume without loss of generality that $y = 0$. Thus, we suppose that $T^n x \xrightarrow{w} 0$. We have

$$\left\| \frac{1}{N} \sum_{k=1}^{N} T^{n_k} x \right\|^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \Re e \left\langle T^{n_i} x, T^{n_j} x \right\rangle.$$

An important ingredient in all the existing proofs of Theorem 1 is the following fact.

Fact 5. *Let $(c_{ij})_{i,j\geq 1}$ be a bounded sequence of nonnegative numbers. If $c_{ij} \to 0$ as $|i-j| \to +\infty$, then*

$$\frac{1}{N^2}\sum_{i,j=1}^{N} c_{ij} \to 0 \ \ as \ \ N \to +\infty.$$

Proof. Let $M = \sup\limits_{i,j} c_{ij}$. The proof relies on a Cesáro type argument. Let $\varepsilon > 0$ and $K \in \mathbb{N}$ be such that $0 \leq c_{ij} < \varepsilon$ for every pair (i,j) of integers with $|i-j| \geq K$. We have

$$\begin{aligned}\frac{1}{N^2}\sum_{i,j=1}^{N} c_{ij} &\leq \frac{1}{N^2}\sum_{|i-j|<K} c_{ij} + \frac{1}{N^2}\sum_{|i-j|\geq K} c_{ij}\\ &\leq \frac{1}{N^2}(2K+1)N + \varepsilon.\end{aligned}$$

□

Once this fact is observed, there are several ways of proving Theorem 1. Probably the most elegant argument is the one presented in [15, Sec. 6.1], which relies on the existence of spectral measures for contractions on complex Hilbert spaces. We prefer to follow here the elementary approach from [3] (see [13, Ch.8, Th. 1.3]), which runs as follows:

Note that the sequence $(\|T^n x\|)_{n\geq 1}$ is decreasing so that the limit $\lim\limits_{n\to+\infty} \|T^n x\|$ exists. Hence, given $\varepsilon > 0$, there exists $K \geq 0$ such that, for all $k \geq K$ and for all $i \geq 0$,

$$0 \leq \|T^k x\|^2 - \|T^{k+i}x\|^2 < \varepsilon^2 \quad \text{and} \quad \left|\left\langle T^k x, x\right\rangle\right| \leq \varepsilon.$$

One now observes the following fact: if $S \in \mathcal{B}(H)$ and $u \in H$ are such that $\|S\| \leq 1$ and $0 \leq \|u\|^2 - \|Su\|^2 < \varepsilon^2$, then

$$|\langle u, y\rangle - \langle Su, Sy\rangle| \leq \varepsilon\|y\| \quad \text{for every } y \in H.$$

Indeed,

$$\begin{aligned}|\langle u, y\rangle - \langle Su, Sy\rangle| = \left|\left\langle (I - S^*S)u, y\right\rangle\right| &\leq \|u - S^*Su\|\|y\|\\ &\leq (\|u\|^2 - \|Su\|^2)^{1/2}\|y\|\\ &\leq \varepsilon\|y\|.\end{aligned}$$

Apply this with $S = T^i$ and $u = T^k x$, $y = x$ to get

$$\left|\left\langle T^k x, x\right\rangle - \left\langle T^{i+k}x, T^i x\right\rangle\right| \leq \varepsilon\|x\|$$

and hence

$$\left|\left\langle T^{i+k}x, T^i x\right\rangle\right| \leq (1 + \|x\|)\varepsilon$$

for every $i \geq 0$, $k \geq K$. So

$$\left|\left\langle T^j x, T^i x\right\rangle\right| \leq (1 + \|x\|)\varepsilon$$

for every $0 \leq i < j$ with $j - i \geq K$. Hence Fact 5 can be applied, and this proves Theorem 1. □

More generally, it now makes sense to investigate whether Theorem 1 can be extended to contractions on other Banach spaces, or at least to certain classes of contractions. Here is a quick list of what is known and what is not known:

1. By [16, Ex. 4.1], there exist power bounded operators on H which do not have BH. Consequently, there exist reflexive spaces which do not have the BH property (see the beginning of Section 4 for a bit more on this example).
2. $\ell_1(\mathbb{N})$ has Schur's property (every weakly convegent sequence in $\ell_1(\mathbb{N})$ is norm-convergent) and hence trivially has BH.

3. $\ell_p(\mathbb{N})$, for $1 < p < \infty$, has BH, see [16, Th. 2.5]. This is one of the important recent results on the BH property, which motivated the work of Lefèvre-Matheron-Primot [15] which is the object of the next section.
4. By [4], positive contractions on spaces $L^p(\Omega, \mathcal{F}, \mu)$, $1 < p < +\infty$, where $(\Omega, \mathcal{F}, \mu)$ is a standard probability space, have BH. It is unknown whether *all* contractions on $L^p(\Omega, \mathcal{F}, \mu)$ have BH, i.e. whether $L^p(\Omega, \mathcal{F}, \mu)$ has BH for $1 < p \neq 2 < +\infty$. This is one of the major open questions concerning the BH property, see [5].
5. The spaces $L^1(\Omega, \mathcal{F}, \mu)$ have BH by [3, Th. 2.1].
6. If K is a compact metric space, the space $C(K)$ has BH if and only if K has finitely many accumulation points. See [14, Th. 1.1] and Theorem 10 below.

We will present in the next lecture a criterion from [15] which allows to retrieve all positive results on the BH property thanks to a rather geometric argument, involving the asymptotic behavior at infinity of a certain "modulus of smoothness".

3 A geometric criterion for the BH property

Let us begin by fixing some notation. Let X be a real, separable Banach space, $\mathcal{C} \subset X$ a convex cone (that is, $\mathcal{C}$ is a non-empty convex set such that $t\,\mathcal{C} \subset \mathcal{C}$ for every $t \geq 0$). Let us set

$$\mathrm{WN}(B_X \cap \mathcal{C}) = \left\{ (x_n)_n \subset B_X \cap \mathcal{C}\,;\; x_n \xrightarrow{w} 0 \right\}.$$

In other words, the set $\mathrm{WN}(B_X \cap \mathcal{C})$ consists of all weakly null sequences in $B_X \cap \mathcal{C}$. For every $x \in \mathcal{C}$ and every $t \geq 0$, define

$$r_{\mathcal{C}}(x, t) = \sup_{(x_n)\in \mathrm{WN}(B_X \cap \mathcal{C})} \overline{\lim_{n\to+\infty}} \|x + tx_n\|.$$

In the rest of the paper, we will use the following terminology: if $x \in X$ and $T \in \mathcal{B}(X)$ with $\|T\| \leq 1$, we say that T *satisfies the BH property at* x if the weak convergence $T^n x \xrightarrow{w} 0$ implies that for every strictly increasing sequence $(n_k)_{k\geq 1}$ of integers,

$$\left\| \frac{1}{N} \sum_{k=1}^{N} T^{n_k} x \right\| \underset{N\to+\infty}{\longrightarrow} 0.$$

Theorem 6. *[15] Suppose that for every $x \in \mathcal{C}$,*

$$\overline{\lim_{t\to+\infty}}\, r_{\mathcal{C}}(x, t) - t \leq 0.$$

Then every operator $T \in \mathcal{B}(X)$ with $\|T\| \leq 1$ and $T(\mathcal{C}) \subset \mathcal{C}$ satisfies the BH property at every point $x \in \mathcal{C}$.

Here are some straightforward remarks on the functions $r_{\mathcal{C}}(x, \,.\,)$.

– the map $t \mapsto r_{\mathcal{C}}(x, t)$ is 1-Lipschitz on $[0, +\infty)$, so that the function $r_{\mathcal{C}}(x, t) - t$ is decreasing and the quantity $\lim_{t\to+\infty} (r_{\mathcal{C}}(x, t) - t)$ exists in $\mathbb{R} \cup \{-\infty\}$;

– if $\mathrm{WN}(B_X \cap \mathcal{C})$ contains a sequence $(x_n) \subset S_X$, then $r_{\mathcal{C}}(x, t) \geq t - \|x\|$, so that $\lim_{t\to+\infty} r_{\mathcal{C}}(x, t) - t \geq -\|x\|$;

– if moreover $\mathcal{C}$ is symmetric, i.e. if $t\,\mathcal{C} \subset \mathcal{C}$ for every $t \in \mathbb{R}$, then $r_{\mathcal{C}}(x, t) \geq t$, so that $\lim_{t\to+\infty} r_{\mathcal{C}}(x, t) - t \geq 0$. Indeed, if $(x_n) \in \mathrm{WN}(B_X \cap \mathcal{C})$, $(-x_n) \in \mathrm{WN}(B_X \cap \mathcal{C})$ and

$$\begin{aligned} r_{\mathcal{C}}(x, t) &\geq \sup_{(x_n)\in \mathrm{WN}(B_X \cap \mathcal{C})} \overline{\lim_{n\to+\infty}} \left(\frac{1}{2}\|x + tx_n\| + \frac{1}{2}\|x - tx_n\|\right) \\ &\geq t \sup_{(x_n)\in \mathrm{WN}(B_X \cap \mathcal{C})} \overline{\lim_{n\to+\infty}} \|x_n\| = t. \end{aligned}$$

– the function $t \mapsto r_{\mathcal{C}}(x, t)$ is increasing on $[0, +\infty)$.

Some applications and examples: if one wishes to show, thanks to Theorem 6, that a given space X has the BH property, one applies it to $\mathcal{C} = X$ (which is indeed a symmetric convex cone).

1. $X = \ell_p(\mathbb{N})$, $1 < p < +\infty$: in this case, $r_{B_X}(x, t) = (\|x\|^p + t^p)^{1/p}$ for every $x \in X$ and every $t \geq 0$. Thus
$$r_{B_X}(x, t) - t \underset{t \to +\infty}{\sim} \frac{\|x\|^p}{p} \frac{1}{t^{p-1}}$$
for every $x \neq 0$, and this tends to 0 as $t \to +\infty$. Hence X has the BH property.
2. $X = c_0(\mathbb{N})$: in this case $r_{B_X}(x, t) = \max(\|x\|, t) = t$ if $t \geq \|x\|$. So X has the BH property, see [7].
3. $X = L^p(\Omega, \mathcal{F}, \mathbb{P})$, $1 < p < +\infty$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard probability space, for instance the interval $[0, 1]$ with Lebesgue measure: as we already mentioned at the end of Section 2, it is unknown whether X has BH for $p \neq 2$. We observe next, following [15, Sec. 6.4] that Theorem 6 does not apply in this case:

Proposition 7. *For every* $1 < p \neq 2 < +\infty$, $\lim\limits_{t \to +\infty} r_{B_{L^p}}(x, t) - t > 0$.

Proof. Let $a, b > 0$ with $a \neq b$ and let $\lambda \in (0, 1)$. Let $(\xi_n)_n$ be a sequence of independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}(\xi_n = a) = \lambda$, $\mathbb{P}(\xi_n = -b) = 1 - \lambda$, $\mathbb{E}(\xi_n) = 0$ and $\|\xi_n\|_p = 1$. The last two conditions place constraints on the parameters a, b and λ. We must have
$$\lambda a - (1 - \lambda) b = 0 \text{ and } \lambda a^p + (1 - \lambda) b^p = 1.$$
The sequence $(\xi_n)_n$ tends weakly to 0 in $L^p(\Omega, \mathcal{F}, \mathbb{P})$. Indeed, $\xi_n \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ for each n, and since the ξ_n's are independent and satisfy $\mathbb{E}(\xi_n) = 0$, they are orthogonal in $L^2(\Omega, \mathcal{F}, \mathbb{P})$: $\mathbb{E}(\xi_n \xi_m) = \mathbb{E}(\xi_n)\mathbb{E}(\xi_m) = 0$ if $m \neq n$. Moreover, $\|\xi_n\|_\infty \leq \max(a, b)$, so the sequence $(\xi_n)_n$ is bounded in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, and thus $\xi_n \xrightarrow{w} 0$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. An approximation argument, using the fact that $\sup_n \|\xi_n\|_\infty < +\infty$, then shows that $\xi_n \xrightarrow{w} 0$ in $L^p(\Omega, \mathcal{F}, \mathbb{P})$.

Now,
$$\begin{aligned} \|\mathbb{1} + t\xi_n\|_p^p &= \lambda |1 + ta|^p + (1 - \lambda)|1 - tb|^p \\ &= \lambda (1 + ta)^p + (1 - \lambda)(tb - 1)^p \text{ for } t \geq \frac{1}{b}. \end{aligned}$$
Since $\|\xi_n\|_p = 1$,
$$r_{B_{L^p}}(\mathbb{1}, t) \geq \big(\lambda(1 + ta)^p + (1 - \lambda)(tb - 1)^p\big)^{\frac{1}{p}} \text{ for } t \geq \frac{1}{b}.$$
The fact that $\lambda a^p + (1 - \lambda) b^p = 1$ and straightforward computations show that
$$\lim_{t \to +\infty} r_{B_{L^p}}(\mathbb{1}, t) - t \geq \lambda a^{p-1} - (1 - \lambda) b^{p-1}.$$
Since $\lambda a = (1 - \lambda) b$, the right-hand side is equal to $\lambda a(a^{p-2} - b^{p-2})$. If $p = 2$, this term is equal to 0 and if $p \neq 2$, the parameters can be chosen in such a way that this term is positive (take $a < b$ if $p < 2$ and $a > b$ if $p > 2$). □

On the other hand, Theorem 6 can be applied to show that positive contractions on $L^p(\Omega, \mathcal{F}, \mathbb{P})$ have the BH property at every positive $f \in L^p(\Omega, \mathcal{F}, \mathbb{P})$: we apply the theorem to $\mathcal{C} = L^p_+(\Omega, \mathcal{F}, \mathbb{P})$, where $L^p_+(\Omega, \mathcal{F}, \mathbb{P}) = \{f \in L^p(\Omega, \mathcal{F}, \mathbb{P})\,;\, f \geq 0 \text{ a.e. on } \Omega\}$.

Let $(f_n)_n \subset B_{L^p} \cap \mathcal{C}$, $f_n \xrightarrow{w} 0$: since $f_n \geq 0$ for every n, (f_n) converges in probability to 0, i.e. $\mathbb{P}(|f_n| > \varepsilon) \to 0$ for every $\varepsilon > 0$. Hence, for any $f \in L^p(\Omega, \mathcal{F}, \mathbb{P})$,
$$\varlimsup_{n \to +\infty} \|f + f_n\|_p \leq \Big(\|f\|_p^p + \varlimsup_{n \to +\infty} \|f_n\|_p^p\Big)^{1/p}. \tag{1}$$
Indeed, for a fixed $\varepsilon > 0$, write
$$\|f + f_n\|_p^p = \int\limits_{\{|f_n| \leq \varepsilon\}} |f + f_n|^p \, \mathrm{d}\mathbb{P} + \int\limits_{\{|f_n| > \varepsilon\}} |f + f_n|^p \, \mathrm{d}\mathbb{P}.$$

For the first term of the right-hand side, note that

$$\int_{\{|f_n|\le\varepsilon\}} |f+f_n|^p \,\mathrm{d}\mathbb{P} \le \int_{\{|f_n|\le\varepsilon\}} (|f|+\varepsilon)^p \,\mathrm{d}\mathbb{P} \le \||f|+\varepsilon\|_p^p.$$

For the second term, we have, by the triangle inequality

$$\begin{aligned}\left(\int_{\{|f_n|>\varepsilon\}} |f+f_n|^p \,\mathrm{d}\mathbb{P}\right)^{1/p} &\le \left(\int_{\{|f_n|>\varepsilon\}} |f|^p \,\mathrm{d}\mathbb{P}\right)^{1/p} + \left(\int_{\{|f_n|>\varepsilon\}} |f_n|^p \,\mathrm{d}\mathbb{P}\right)^{1/p}\\ &\le \left(\int_{\{|f_n|>\varepsilon\}} |f|^p \,\mathrm{d}\mathbb{P}\right)^{1/p} + \|f_n\|_p.\end{aligned}$$

Since $|f|^p$ is integrable and $\mathbb{P}(|f_n|>\varepsilon)\to 0$, we have $\left(\int_{\{|f_n|>\varepsilon\}} |f|^p \,\mathrm{d}\mathbb{P}\right)^{1/p} \to 0$ as $n\to+\infty$. Hence

$$\overline{\lim_{n\to+\infty}} \|f+f_n\|_p \le \left(\||f|+\varepsilon\|_p^p + \overline{\lim_{n\to+\infty}} \|f_n\|_p^p\right)^{1/p}.$$

Since ε can be chosen arbitrarily small, this proves (1). This inequality means that the supports of f and f_n become asymptotically disjoint as n goes to infinity. Thus $\lim_{t\to+\infty} r_{\mathcal{C}}(f,t)-t = 0$ for every $f\in L^p(\Omega,\mathcal{F},\mathbb{P})$. Theorem 6 yields that T has BH at every point $f\in L^p_+$.

Proof of Theorem 6. Let $x\in\mathcal{C}$. In order to make the notation lighter, we write $x_n = T^n x$, $n\ge 0$. We thus suppose that $x_n \xrightarrow{w} 0$. We will prove successively several equivalent formulations of the BH property for the sequence (x_n), which will ultimately yield the result. Without loss of generality, we suppose that $\|x\|=1$.

We claim that the following assertions are equivalent:

1. For every strictly increasing sequence $(n_k)_{k\ge1}$ of integers,
$$\left\|\frac{1}{N}\sum_{k=1}^{N} x_{n_k}\right\| \underset{N\to+\infty}{\longrightarrow} 0.$$
2. Denote by FIN the class of all finite subsets of $\mathbb{N}$:
$$\frac{1}{|A|}\left\|\sum_{n\in A} x_n\right\| \longrightarrow 0 \text{ as } |A|\to+\infty,\ A\in \mathrm{FIN}.$$
3. For every $s\in\mathbb{N}$, write $\mathrm{FIN}(s)=\{A\in\mathrm{FIN},\ |A|=s\}$ and
$$G(s)=\sup_{A\in\mathrm{FIN}(s)}\left\|\sum_{n\in A} x_n\right\|:$$
then
$$\frac{G(s)}{s}\to 0 \text{ as } s\to+\infty.$$
4. For every $d\in\mathbb{N}$, write $\mathrm{FIN}(s,d)=\left\{A\in\mathrm{FIN}(s)\,;\ \forall i\ne j\in A,\ |i-j|\ge d\right\}$ and
$$G_d(s)=\sup_{A\in\mathrm{FIN}(s,d)}\left\|\sum_{n\in A} x_n\right\|.$$
Write also $F(s)=\inf_{d\in\mathbb{N}} G_d(s)=\lim_{d\to+\infty} G_d(s)$. Then
$$\frac{F(s)}{s}\to 0 \text{ as } s\to+\infty.$$

– The equivalence between (1) and (2) is essentially obvious: (2) $\Rightarrow$ (1) is clear. In the converse direction, suppose that (2) is not true:

$$\exists \varepsilon > 0, \exists (A_k)_k \subset \mathrm{FIN}, |A_k| \to +\infty, \frac{1}{|A_k|}\left\|\sum_{n\in A_k} x_n\right\| \geq \varepsilon.$$

Then make the sets A_k disjoint, and enumerate a suitable infinite subsequence of (A_k) as (n_k).
– The equivalence between (2) and (3) is not difficult either.
– (3) $\Rightarrow$ (4) is obvious since for every $d \in \mathbb{N}$, $F(s) \leq G_d(s) \leq G(s)$.
– Let us prove (4) $\Rightarrow$ (1). Our assumption is that $\frac{F(s)}{s} \to 0$, so

$$\forall \varepsilon > 0, \exists s_0, \forall s \geq s_0, \exists d_s \in \mathbb{N}, G_{d_s}(s) \leq \varepsilon s$$

i.e. for all $A \in \mathrm{FIN}(s, d_s)$, $\|\sum_{n\in A} x_n\| \leq \varepsilon s$.

Writing $d_0 := d_{s_0}$, this implies in particular that for all $A \in \mathrm{FIN}(s_0, d_0)$ we have

$$\left\|\sum_{n\in A} x_n\right\| \leq \varepsilon s_0.$$

Let us now fix $(n_k)_k$, and let $N \geq 1$. Let l be such that $l s_0 d_0 \leq N < (l+1) s_0 d_0$. We claim that it is possible to partition the interval $[1, N]$ as

$$[1, N] = \bigcup_{\substack{1\leq i\leq l\\ 1\leq j\leq d_0}} B_{i,j} \bigcup B$$

where $B_{i,j} \in \mathrm{FIN}(s_0, d_0)$ and $|B| < s_0 d_0$. Indeed, let

$$B_{i,j} = \left\{(i-1)s_0 d_0 + j, (i-1)s_0 d_0 + j + d_0, \ldots, (i-1)s_0 d_0 + j + (s_0 - 1)d_0\right\}.$$

Then $|B_{i,j}| = s_0$, $B_{i,j} \in \mathrm{FIN}(s_0, d_0)$ and we let $B = [1, N] \setminus \bigcup_{\substack{1\leq i\leq l\\ 1\leq j\leq d_0}} B_{i,j}$.

For $i = 1$ we have

$$\begin{aligned}
B_{1,1} &= \left\{1, 1 + d_0, \ldots, 1 + (s_0 - 1)d_0\right\}\\
B_{1,2} &= \left\{2, 2 + d_0, \ldots, 2 + (s_0 - 1)d_0\right\}\\
&\vdots\\
B_{1,d_0} &= \{d_0, 2d_0, \ldots, s_0 d_0\}
\end{aligned}$$

so $\bigcup_{1\leq j\leq d_0} B_{1,j} = [1, s_0 d_0]$.

In the same fashion, $\bigcup_{\substack{1\leq i\leq l\\ 1\leq j\leq d_0}} B_{i,j} = [1, l s_0 d_0]$, so $|B| < s_0 d_0$.

Write now $A_{i,j} = \left\{n_k\ ;\ k \in B_{i,j}\right\}$ and $A = \{n_k\ ;\ k \in B\}$: $A_{i,j} \in \mathrm{FIN}(s_0, d_0)$, $|A| < s_0 d_0$, so

$$\left\|\sum_{n\in A_{i,j}} x_n\right\| \leq \varepsilon s_0 \text{ and } \left\|\sum_{n\in A} x_n\right\| < s_0 d_0.$$

Hence

$$\left\|\sum_{k\in[1,N]} x_{n_k}\right\| \leq l d_0(\varepsilon s_0) + s_0 d_0$$

so that

$$\frac{1}{N}\left\|\sum_{k\in[1,N]} x_{n_k}\right\| \le \varepsilon \underbrace{\frac{ld_0 s_0}{N}}_{\le 1} \quad + \quad \underbrace{\frac{s_0 d_0}{N}}_{<\varepsilon \text{ if } N>\frac{1}{\varepsilon}s_0 d_0} .$$

It follows that $\frac{1}{N}\left\|\sum_{k\in[1,N]} x_{n_k}\right\| \to 0$, and we are done.

We can now conclude the proof of the theorem. We need the following fact.

Fact 8. *For every $s \in \mathbb{N}$, we have $F(s+1) \le r_{\mathcal{C}}(x, F(s))$.*

Proof of Fact 8. The definition of $F(s+1)$ is

$$F(s+1) = \lim_{d\to+\infty} \sup_{A\in \mathrm{FIN}(s+1,d)} \left\|\sum_{n\in A} x_n\right\| .$$

Hence there exists $(A_d)_d$, $A_d \in \mathrm{FIN}(s+1,d)$, with

$$\left\|\sum_{n\in A_d} x_n\right\| \to F(s+1)$$

as $d \to +\infty$. Write $A_d = \{n_{1,d} < n_{2,d} < \ldots < n_{s+1,d}\}$, with $n_{j,d} - n_{i,d} \ge d$ for every pair (i,j) of indices with $j > i$. Then, because T is a contraction, we have

$$\left\|\sum_{n\in A_d} x_n\right\| = \left\|\sum_{j=1}^{s+1} T^{n_{j,d}} x\right\| \le \left\|x + \sum_{j=2}^{s+1} T^{n_{j,d}-n_{1,d}} x\right\| = \left\|x + \sum_{j=2}^{s+1} x_{n_{j,d}-n_{1,d}}\right\| .$$

Set

$$z_d := \sum_{j=2}^{s+1} x_{n_{j,d}-n_{1,d}},$$

and observe that:

- $z_d \in \mathcal{C}$ (because $\mathcal{C}$ is a convexe cone);
- $z_d \xrightarrow{w} 0$ as $d \to +\infty$ (s is fixed, $x_n \xrightarrow{w} 0$, and $n_{j,d} - n_{1,d} \ge d$ for all $j = 2, \ldots, s+1$);
- $B_d = \{n_{j,d}\ ;\ 2 \le j \le s+1\}$ belongs to $\mathrm{FIN}(s,d)$.

We have $F(s+1) = \lim_{d\to+\infty} \|x + z_d\| \le r_{\mathcal{C}}(x, \overline{\lim}\,\|z_d\|)$. Now, $\overline{\lim}\|z_d\| = \overline{\lim}\left\|\sum_{n\in B_d} x_n\right\| \le F(s)$. Since the function $t \mapsto r_{\mathcal{C}}(x,t)$ is increasing, $F(s+1) \le r_{\mathcal{C}}(x, F(s))$ and this proves Fact 8. □

Using Fact 8, we have $F(s+1) - F(s) \le r_{\mathcal{C}}(x, F(s)) - F(s)$. If F were increasing, we would be able to deduce that $F(s+1) - F(s)$ tends to 0, and hence by the Cesàro theorem that

$$\frac{F(s)}{s} \to 0.$$

Since F is not necessarily increasing, we replace F by the increasing function $\tilde{F}$ defined by $\tilde{F}(s) = \max(F(1), \ldots, F(s))$, $s \in \mathbb{N}$, and check that the same inequality as the one given in Fact 8 holds true for $\tilde{F}$. This concludes the proof of Theorem 6. □

An improved version of Theorem 6, which is also perhaps more natural, can be of use in certain situations.

Theorem 9. *[14] Let $T \in \mathcal{B}(X)$, with $\|T\| \le 1$ and $T(\mathcal{C}) \subset \mathcal{C}$. Suppose that for every $x \in \mathcal{C}$,*

$$\inf_{k\in\mathbb{N}} \lim_{t\to+\infty} (r_{\mathcal{C}}(T^k x, t) - t) \le 0.$$

Then T satisfies the BH property at every $x \in \mathcal{C}$.

Theorem 9 can be used to show that the space $c = \left\{(u_k)_k \in \mathbb{R}^{\mathbb{N}}, \lim_{k\to+\infty} u_k \text{ exists}\right\}$, endowed with the norm $\|.\|_\infty$, has the BH property ([14]).

4 Spaces which do not have the BH property

Our aim is now to investigate spaces which do *not* have the BH property. We will present in particular a characterization, due to Lefèvre and Matheron [14], of the compact metric spaces K which are such that $C(K)$ has the BH property.

Examples of spaces without the BH property:

1. The space $C(\mathbb{T}^2)$ does not have the BH property. More precisely, there exists a continuous map $\theta : \mathbb{T}^2 \to \mathbb{T}^2$ such that the associated composition operator $C_\theta : f \mapsto f \circ \theta$ on $C(\mathbb{T}^2)$ fails the BH property. See [2].
2. By [16, Ex. 4.1], there exists a power-bounded operator T on the Hilbert space H which fails the BH property. Renorm H by setting
$$|||x||| = \sup_{n\geq 0} \|T^n x\|, \quad x \in H.$$
The norm $|||.|||$ is an equivalent norm on H, and thus $(H, |||.|||)$ is a (super-)reflexive space. The operator T is a contraction on $(H, |||.|||)$, and it fails the BH property.
3. In [15, Prop. 6.1], it is proved that if K is an uncountable metric space, $C(K)$ does not have the BH property: this result follows from the fact that $C(\mathbb{T}^2)$ does not have the BH property, combined with the so-called Milutin Lemma and the linear version, due to Borsuk, of Tietze extension theorem. See below for more details.

Theorem 10. *[14] Let K be a compact metric space. Then $C(K)$ has the BH property if and only if K has finitely many accumulation points.*

Recall that $s \in K$ is an accumulation point of K if $V\setminus\{s\} \neq \emptyset$ for every open neighborhood V of s in K.

Proof. If K is a compact metric space, we denote by K' the set of its accumulation points. This set K' is nonempty as soon as K is infinite, and K' is also a compact metric space.

The easy part of the proof is to show that if K' is finite, let us say $K' = \{a_1, \dots, a_N\}$, then $C(K)$ has BH. There exist disjoint compact sets $K_1, \dots, K_N$ such that $K_i' = \{a_i\}$ for every $i = 1, \dots, N$ and $K = \bigcup_{1\leq i\leq N} K_i$.

Indeed, let $V_1, \dots, V_n$ be open neighborhoods of $a_1, \dots, a_N$ respectively, such that their closures $\overline{V_1}, \dots, \overline{V_n}$ are pairwise disjoint. Let $\widetilde{K_i} = \overline{V_i} \cap K$. Then $\widetilde{K_i}' = \{a_i\}$ and $K\setminus \cup_{1\leq i\leq N} \widetilde{K_i}$ is finite. Set $K_i = \widetilde{K_i}$ for $2 \leq i \leq N$ and $K_1 = \widetilde{K_1} \cup \left(K\setminus \cup_{1\leq i\leq N} \widetilde{K_i}\right)$.
Then
$$C(K) = \bigoplus_{\ell_\infty} C(K_i) = \underbrace{c \underset{\ell_\infty}{\oplus} \cdots \underset{\ell_\infty}{\oplus} c}_{N \text{ times}}.$$
We have seen that c satisfies the assumption of Theorem 9; it is easy to check that the direct ℓ_∞-sum of finitely many copies of c also satisfies it, and hence has BH.

Conversely, let K be such that K' is infinite. So $K'' \neq \emptyset$. The simplest of these compact sets are the ones where K'' is reduced to one point. We study this case first. Let S be a compact metric space such that
$$S'' = \{s_{\infty,\infty}\}$$
$$S' = \{s_{\infty,k}\ ;\ k \in \mathbb{N}\}, \text{ where } s_{\infty,k} \to s_{\infty,\infty} \text{ as } k \to +\infty$$
$$S = \underbrace{\{s_{i,k}\ ;\ i, k \in \mathbb{N}\} \cup \{s_{\infty,k}\ ;\ k \in \mathbb{N}\}}_{:=S_k} \cup \{s_{\infty,\infty}\}$$

where all the points $s_{i,k}$ are distinct, $s_{i,k} \underset{i\to+\infty}{\longrightarrow} s_{\infty,k}$ for every $k \in \mathbb{N}$ and S_k tends to $s_{\infty,\infty}$ as $k \to +\infty$ in the sense that any neighborhood of $s_{\infty,\infty}$ contains the sets S_k for all but finitely many k's.

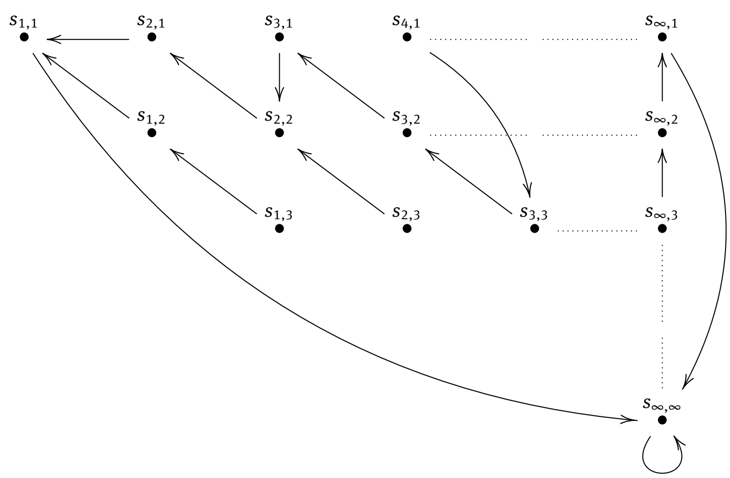
Indeed, if $S'' = \{s_{\infty,\infty}\}$, S necessarily has this form: there exists $(s_{\infty,k})_k$ such that $S' = \left\{s_{\infty,k}\,;\, k \in \mathbb{N}\right\}$, where $s_{\infty,k} \to s_{\infty,\infty}$ as $k \to +\infty$. Let V_k be a neighborhood of $s_{\infty,k}$ in S, with the sets $\overline{V_k}$, $k \in \mathbb{N}$, disjoint, $\operatorname{diam}(\overline{V_k}) < 2^{-k}$, and $s_{\infty,\infty} \notin \overline{V_k}$. It is clear that the sets $\overline{V_k}$ tend to $s_{\infty,\infty}$ in the sense above. Let $S_k = \overline{V_k} \cap S$: $S'_k = \left\{s_{\infty,k}\right\}$ and hence there exists $(s_{i,k})_i$ such that $S_k = \left\{s_{i,k}\,;\, i \in \mathbb{N}\right\} \cup \left\{s_{\infty,k}\right\}, s_{i,k} \underset{i\to+\infty}{\longrightarrow} s_{\infty,k}$. The set $S\backslash \left(\bigcup_{k\geq 1} S_k \bigcup \{s_{\infty,\infty}\}\right)$ is finite, so we add these few points to S_1, for instance, and we are done.

Proposition 11. *There exists a continuous map $\theta : S \to S$ such that the contraction $C_\theta : f \mapsto f \circ \theta$ on $C(S)$ does not have the BH property. Hence $C(S)$ does not have the BH property.*

Proof. Define $\theta : S \to S$ by setting

$$\begin{cases} \theta(s_{i,k}) = s_{i,k-1} & \text{if } k \geq 2 \\ \theta(s_{i,1}) = s_{i-1,i-1} & \text{if } i \geq 2 \\ \theta(s_{1,1}) = s_{\infty,\infty} & \\ \theta(s_{\infty,k}) = s_{\infty,k-1} & \text{if } k \geq 2 \\ \theta(s_{\infty,1}) = s_{\infty,\infty} & \\ \theta(s_{\infty,\infty}) = s_{\infty,\infty}. & \end{cases}$$

Visually, the map θ acts as follows:



The map θ is clearly continuous on S. Moreover, the orbit of any point $s \in S$ under the map θ attains $s_{\infty,\infty}$ in a finite number of steps:

$$\forall s \in S,\ \#\left\{n \in \mathbb{N}\,;\, \theta^n(s) \neq s_{\infty,\infty}\right\} < +\infty.$$

After reaching $s_{\infty,\infty}$, the orbit remains stationary in $s_{\infty,\infty}$. Set $V_1 = S\backslash S_1$ (recall that $S_1 = \{s_{i,1}\,;\, i \in \mathbb{N}\} \cup \{s_{\infty,1}\}$), which is a clopen neighborhood of $s_{\infty,\infty}$ in S. For every $N \geq 1$,

$$\#\left\{n \in \mathbb{N}\,;\, \theta^n(s_{N,1}) \notin V_1\right\} = \#\left\{n \in \mathbb{N}\,;\, \theta^n(s_{N,1}) \in S_1\right\} = N.$$

For every $u \in C(S)$, $C_\theta^n(u)(s) = u(\theta^n(s)) \underset{n\to+\infty}{\longrightarrow} u(s_{\infty,\infty})$. Hence $C_\theta^n u \xrightarrow{w} u(s_{\infty,\infty})\mathbb{1}$.

Let $f \in C(S)$ be such that $f \equiv 1$ on S_1 and $f \equiv 0$ on $V_1 = S\backslash S_1$. Then $C_\theta^n f \xrightarrow{w} 0$. For every $N \geq 1$, consider the set $I_N = \left\{n \in \mathbb{N}\,;\, \theta^n(s_{N,1}) \in S_1\right\}$. Then $|I_N| = N$ and

$$\frac{1}{|I_N|}\left\|\sum_{n\in I_N} C_\theta^n f\right\|_\infty \geq \frac{1}{N}\left|\sum_{n\in I_N} f(\theta^n(s_{N,1}))\right| = 1.$$

Hence

$$\frac{1}{|A|}\left\|\sum_{n\in A} C_\theta^n f\right\|_\infty \not\to 0 \quad \text{as } |A| \to +\infty,$$

and we have proved that C_θ does not have the BH property. □

The remaining part of the argument is rather general, and allows to pass from the space $C(S)$ to any space $C(K)$ with K' infinite. It relies on several observations.

Fact 12. *If K' is infinite, it contains a set S of the form above.*

Proof. Since K' is infinite, $K'' \neq \emptyset$. Let $s_{\infty,\infty} \in K''$. Let $(V_k)_k$ be a neighborhood basis of $s_{\infty,\infty}$ with $\operatorname{diam}(\overline{V_k}) < 2^{-k}$ and $V_k \backslash \overline{V_{k+1}} \neq \emptyset$ for every k. For each k, choose $s_{\infty,k} \in K' \cap V_k$ with $s_{\infty,k} \neq s_{\infty,\infty}$. Without loss of generality, we can suppose that $s_{\infty,k} \in V_k \backslash \overline{V_{k+1}}$. Let then $(s_{i,k}) \subset K \cap \left(V_k \backslash \overline{V_{k+1}}\right)$ (which is an open neighborhood of $s_{\infty,k}$) be such that $s_{i,k} \underset{i}{\to} s_{\infty,k}$ with all the $s_{i,k}$ distinct and distinct from $s_{\infty,k}$.

Finally, let $S_k := \left\{s_{i,k}\,;\, 1 \le i \le +\infty\right\}$: $S_k \subset V_k$ and hence the sets S_k tend to $s_{\infty,\infty}$. The set $S = \left\{s_{i,k}\,;\, 1 \le i, k \le +\infty\right\}$ then satisfies all the required properties. □

Theorem 13 (Borsuk's linear isomorphic extension theorem)**.** *Let K be a compact metric space, and let E be a closed subset of K. There exists a bounded operator $J : C(E) \to C(K)$ such that $J\mathbb{1} = \mathbb{1}$, $\|J\| = 1$, and $J(f)_{|E} = f$ for every $f \in C(E)$.*

In particular, $J : C(E) \to X := J(C(E))$ is an isometry, and X is 1-complemented in $C(K)$.

To see that X is 1-complemented in $C(K)$, observe that $P : C(K) \to X$ defined by $Pg = J(g_{|E})$, $g \in C(K)$, is a projection of $C(K)$ onto X.

Theorem 13 is a linear version of Tietze's extension theorem which states that for every $f \in C(E)$, there exists $g \in C(K)$ with $\|g\|_\infty = \|f\|_\infty$ such that $g_{|E} = f$. A proof of Theorem 13 can be found for instance in [6, Th. 4.4.4].

As a consequence of Fact 12 and Theorem 13, we obtain that $C(S)$ is isometric to a 1-complemented subspace X of $C(K)$.

Fact 14. *Let X be a 1-complemented subspace of a Banach space Z. If X fails the BH property, so does Z.*

Proof. Let $P : Z \to X$ be a projection of Z onto X, with $\|P\| = 1$. Let $T \in \mathcal{B}(X)$ with $\|T\| \le 1$, and let $x_0 \in X$ be such that $T^n x_0 \xrightarrow{w} 0$, but there exists a strictly increasing sequence $(n_k)_k$ of integers such that

$$\limsup_{N\to+\infty} \frac{1}{N}\left\|\sum_{k=1}^{N} T^{n_k} x_0\right\| > 0.$$

Set $\widetilde{T} = T \circ P : Z \to X \subset Z$: then $\widetilde{T} \in \mathcal{B}(Z)$, $\|\widetilde{T}\| \le 1$, and $\widetilde{T}^n z = T^n(Pz)$ for every $n \ge 1$ and $z \in Z$. Hence $\widetilde{T}^n x_0 \xrightarrow{w} 0$. But the vector $x_0 \in X \subset Z$ satisfies $Px_0 = x_0$ and

$$\limsup_{N\to+\infty} \frac{1}{N}\left\|\sum_{k=1}^{N} \widetilde{T}^{n_k} x_0\right\| > 0.$$

□

By Proposition 11, $C(S)$ fails BH, and hence $C(K)$ fails BH as well. This concludes the proof of Theorem 10. □

Acknowledgement: We are grateful to Pascal Lefèvre for a careful reading of a first version of these notes, and for several interesting comments. This work was supported in part by the project FRONT of the French National Research Agency (grant ANR-17-CE40-0021) and by the Labex CEMPI (ANR-11-LABX-0007-01)

References

[1] J. Aaronson, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs **50**, American Mathematical Society (1997).

[2] M. A. Akcoglu, J. P. Huneke and H. Rost, *A couterexample to Blum-Hanson theorem in general spaces*, Pacific J. Math. **50** (1974), 305–308.

[3] M. A. Akcoglu and L. Sucheston, *On operator convergence in Hilbert space and in Lebesgue space*, Period. Math. Hungar. **2** (1972), 235–244.

[4] M. A. Akcoglu and L. Sucheston, *Weak convergence of positive contractions implies strong convergence of averages*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete (Probab. Theory Relat. Fields) **32** (1975), 139–145. J. Convex Anal. **14** (2007), 249–269.

[5] A. Bellow, *An L_p-inequality with application to ergodic theory*, Houston J. Math. **1** (1975), 153–159.

[6] F. Albiac and N. J. Kalton, *Topics in Banach space theory*, Graduate Text in Mathematics 233, Springer, 2006.

[7] J.-M. Augé, *Quelques problèmes de dynamique linéaire dans les espaces de Banach*, PhD thesis, Université Bordeaux 1 (2012).

[8] J. R. Blum and D. L. Hanson, *On the mean ergodic theorem for subsequences*, Bull. Amer. Math. Soc. **66** (1960), 308–311.

[9] M. Einsiedler and T. Ward, *Ergodic theory with a view towards number theory*, Graduate Texts in Mathematics **259**, Springer-Verlag (2011).

[10] N. Friedman and D. S. Ornstein, *On mixing and partial mixing*, Illinois J. Math. **16** (1972), 61–68.

[11] L. K. Jones and V. Kuftinec, *A note on the Blum-Hanson theorem*, Proc. Amer. Math. Soc. **30** (1971), 202–203.

[12] U. Krengel, *On the individual ergodic theorem for subsequences*, Ann. Math. Statist. **42** (1971), 1091–1095.

[13] U. Krengel, *Ergodic theorems*, with a supplement by Antoine Brunel, De Gruyter Studies in Mathematics **6** (1985).

[14] P. Lefèvre and É. Matheron, *The Blum-Hanson property for $\mathcal{C}(K)$ spaces*, Pacific J. Math. **282** (2016), no. 1, 203–212.

[15] P. Lefèvre, É. Matheron and A. Primot, *Smoothness, asymptotic smoothness and the Blum-Hanson property*, Israel J. Math. **211** (2016), no. 1, 271–309.

[16] V. Müller and Y. Tomilov, *Quasi-similarity of power-bounded operators and Blum-Hanson property*, J. Funct. Anal. **246** (2007), 385–399.

[17] K. Petersen, *Ergodic theory*, corrected reprint of the 1983 original, Cambridge Studies in Advanced Mathematics **2**, Cambridge University Press (1989).

[18] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics **79**, Springer-Verlag (1982).