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Compactness and hypercyclicity of co-analytic Toeplitz operators on de Branges-Rovnyak spaces

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Abstract: We study the compactness and the hypercyclicity of Toeplitz operators $T_{\bar{\varphi},b}$ with co-analytic and bounded symbols on de Branges-Rovnyak spaces $\mathcal{H}(b)$. For the compactness of $T_{\bar{\varphi},b}$, we will see that the result depends on the boundary spectrum of b . We will prove that there are non trivial compact operators of the form $T_{\bar{\varphi},b}$, with $\varphi \in H^\infty \cap C(\mathbb{T})$, if and only if $m(\sigma(b) \cap \mathbb{T}) = 0$. We will also show that, when b is non-extreme, then $T_{\bar{\varphi},b}$ is hypercyclic if and only if φ is non-constant and $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Keywords: Toeplitz operators, de Branges-Rovnyak spaces, compactness, hypercyclicity

MSC: 30J05, 30H10, 46E22, 47A16

1 Introduction

We shall mostly be discussing co-analytic Toeplitz operators $T_{\bar{\varphi}}$ with symbol $\bar{\varphi}$ where $\varphi \in H^\infty$, that are naturally defined on the de Branges-Rovnyak space into itself. These operators have been introduced by Lotto-Sarason in [13, Lemma 2.6], see also [14, Section II.7]. Some special cases have long ago appeared in literature for $\varphi \in L^\infty(\mathbb{T})$, most notably as standard Toeplitz operators $T_\varphi : H^2 \rightarrow H^2$ studied by A. Brown and P. Halmos in the paper [5] and as the adjoints of truncated Toeplitz operators A_φ^θ on model spaces K_θ introduced by Sarason in [15]. We will consider Toeplitz operators with different domains and different ranges. To avoid confusion, we adopt different notations. We will denote by $T_{\bar{\varphi}}$ the Toeplitz operator defined from H^2 into itself, by $T_{\bar{\varphi},b}$ the Toeplitz operator defined from $\mathcal{H}(b)$ into itself, and by $\mathbf{T}_{\bar{\varphi},b}$ the Toeplitz operator defined from $\mathcal{H}(b)$ into H^2 .

It turns out that de Branges-Rovnyak spaces, which are a family of subspaces $\mathcal{H}(b)$ of the Hardy space H^2 , parametrized by elements b of the closed unit ball of H^∞ are invariant under $T_{\bar{\varphi}}$, where $\varphi \in H^\infty$. We shall give the precise definition in section 2. In general $\mathcal{H}(b)$ is not closed in H^2 , but it carries its own norm $\|\cdot\|_{\mathcal{H}(b)}$ making it a Hilbert space. The spaces $\mathcal{H}(b)$ were introduced by de Branges and Rovnyak in the appendix of [6] and further studied in their book [7].

The general theory of $\mathcal{H}(b)$ -spaces generally splits into two cases, according to whether b is an extreme point or a non-extreme point of the unit ball of H^∞ . The dichotomy b extreme/non-extreme will also often appear in this paper. The general idea is that the extreme case has many features that are not far from the case of $b = \theta$ inner (the classical model space K_θ), while the non-extreme case has several properties that are similar to the case where $b = 0$ (the Hardy space H^2).

This paper treats two properties related to the restricted Toeplitz operators $T_{\bar{\varphi},b}$ when $\varphi \in H^\infty$. One of these properties is based on the particular operator $X_b = T_{\bar{z},b}$ that plays a central role in the theory and

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particularly in the model developed by de Branges and Rovnyak. Indeed, it serves as a model for a large class of contractions (see [8, Theorem 26.16]).

The first one concerns the compactness of the Toeplitz operator $T_{\bar{\varphi},b}$. More precisely, given an element b of the closed unit ball of H^∞ , that is extreme or non-extreme, we are interested in finding a necessary and sufficient condition for the operator $T_{\bar{\varphi},b}$ to be compact. Let us mention that Brown and Halmos [5] have shown that there is no compact Toeplitz operators $T_\varphi : H^2 \rightarrow H^2$ with symbol $\varphi \in L^\infty(\mathbb{T})$ (except the trivial case where $\varphi = 0$). It is not surprising that in the case where b is non-extreme, we reach a similar result for the restricted Toeplitz operator $T_{\bar{\varphi},b}$ with $\varphi \in H^\infty$.

On the other hand, Ahern and Clark [1] studied the compactness of truncated Toeplitz operators A_φ^θ , with continuous symbol φ on $\mathbb{T}$ and they got a necessary and sufficient condition based on the image of the spectrum of θ intersected with the unit circle. Recently, Garcia, Ross and Wogen [9] recovered this result with another proof. Using their method, we generalize the result of Ahern and Clark to the operator $T_{\bar{\varphi},b}$ with $\varphi \in C(\mathbb{T})$, and from this generalization we prove that there are non trivial compact operators of the form $T_{\bar{\varphi},b}$, with $\varphi \in H^\infty \cap C(\mathbb{T})$, if and only if $m(\sigma(b) \cap \mathbb{T}) = 0$.

Our second problem is related to the hypercyclicity of the restricted Toeplitz operator $T_{\bar{\varphi},b}$. Godefroy and Shapiro, using their Criterion [11], proved that if $\varphi \in H^\infty$, then $T_{\bar{\varphi}}$ is hypercyclic if and only if φ is non constant and $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. On the other hand, it is obvious that there are no hypercyclic Toeplitz operators with analytic symbols. For general symbols $\varphi \in L^\infty(\mathbb{T})$, few results are known; see a recent paper of A. Baranov and A. Lishanskii [2] and a paper of Shkarin [16] who studied the case where $\varphi(z) = a\bar{z} + b + cz$. We succeed to extend the result of Godefroy and Shapiro in our context of $\mathcal{H}(b)$ spaces, when b is non extreme. Whereas, when b is extreme, we give a necessary condition for the operator $T_{\bar{\varphi},b}$ to be non-hypercyclic. This condition is based on the point spectrum of the operator. It also turns out that this necessary condition is not sufficient.

The structure of the paper is the following. After a preliminary section with generalities about de Branges-Rovnyak spaces, Toeplitz operators and definitions of hypercyclic and frequently hypercyclic operators, we discuss compactness properties of the restricted Toeplitz operators $T_{\bar{\varphi},b}$ in Section 3. The last section is dedicated to the hypercyclicity of $T_{\bar{\varphi},b}$.

2 Preliminaries

2.1 Toeplitz operators and de Branges-Rovnyak spaces

We first recall some basic facts on Toeplitz operators on the Hardy space H^2 of the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Given $\varphi \in L^\infty(\mathbb{T}) = L^\infty(\mathbb{T}, m)$ where $\mathbb{T} = \partial\mathbb{D}$ and m is the normalized Lebesgue measure on $\mathbb{T}$, the corresponding Toeplitz operator $T_\varphi : H^2 \rightarrow H^2$ is defined by

$$T_\varphi f := P_+(\varphi f) \quad (f \in H^2),$$

where $P_+ : L^2(\mathbb{T}) \rightarrow H^2$ denotes the orthogonal projection of $L^2(\mathbb{T}) = L^2(\mathbb{T}, m)$ onto H^2 . Clearly T_φ is a bounded operator on H^2 with $\|T_\varphi\| = \|\varphi\|_{L^\infty(\mathbb{T})}$, moreover it is compact if and only if $\varphi = 0$ (Brown–Halmos, [5]). If $\varphi \in H^\infty$ the algebra of the analytic and bounded functions on $\mathbb{D}$, then T_φ is simply the operator of multiplication by φ and its adjoint is $T_{\bar{\varphi}}$. Consequently, if $\varphi, \psi \in H^\infty$, then $T_{\bar{\varphi}}T_{\bar{\psi}} = T_{\bar{\varphi}\bar{\psi}} = T_{\bar{\psi}}T_{\bar{\varphi}}$. Moreover, if $\varphi \in H^\infty$ we have

$$T_{\bar{\varphi}}k_\lambda = \overline{\varphi(\lambda)}k_\lambda, \tag{1}$$

where $k_\lambda(z) = (1 - \bar{\lambda}z)^{-1}$ is the reproducing kernel of H^2 (see [8, Section 12.4]).

If $\varphi \in L^\infty(\mathbb{T})$ satisfies $\|\varphi\|_\infty \leq 1$, then the corresponding Toeplitz operator T_φ is a contraction on the Hilbert space H^2 . The associated de Branges-Rovnyak space $\mathcal{H}(T_\varphi)$ is defined by

$$\mathcal{H}(T_\varphi) = (I - T_\varphi T_{\bar{\varphi}})^{1/2} H^2.$$

For simplicity, we denote the complementary space $\mathcal{H}(T_\varphi)$ by $\mathcal{H}(\varphi)$ (see [8, Section 17.3]). Therefore, the definition of an $\mathcal{H}(\varphi)$ -space uses the defect of the contraction T_φ [8]. Hence, no doubt, the Toeplitz operators are extremely important in this context. Our main concern is when φ is a nonconstant analytic function in the closed unit ball of H^∞ . In this case, by tradition, we use b instead of φ .

We recall an alternative and equivalent definition based on reproducing kernels. Namely, $\mathcal{H}(b)$ is the Hilbert space of analytic functions on $\mathbb{D}$ whose reproducing kernel is given by

$$k_\lambda^b(z) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \bar{\lambda}z}, \quad \lambda, z \in \mathbb{D}.$$

That is,

$$f(\lambda) = \langle f, k_\lambda^b \rangle_b, \quad \forall f \in \mathcal{H}(b), \forall \lambda \in \mathbb{D}.$$

For $b = 0$, we see that k_λ^b coincides with k_λ the reproducing kernels of H^2 , given by $k_\lambda(z) = (1 - \bar{\lambda}z)^{-1}$, whence $\mathcal{H}(0) = H^2$.

More generally when $\|b\|_\infty < 1$, $\mathcal{H}(b)$ coincides with the Hardy space H^2 with an equivalent norm.

For $b = \theta$, with θ an inner function (that is a function in the closed unit ball of H^∞ such that $|\theta(\zeta)| = 1$ almost everywhere on $\mathbb{T} = \partial\mathbb{D}$), the space $\mathcal{H}(\theta)$ is a closed subspace of H^2 , and we have

$$\mathcal{H}(\theta) = (\theta H^2)^\perp := \{f \in H^2 : \langle f, \theta g \rangle_2 = 0, \forall g \in H^2\}.$$

The space $\mathcal{H}(\theta)$ is also called the model space and is denoted by $K_\theta = \mathcal{H}(\theta)$. By Beurling's theorem, the spaces K_θ correspond to the lattice of closed, non trivial, invariant subspaces for the backward shift operator $S^* = T_{\bar{z}}$ on H^2 .

In the general case, the spaces $\mathcal{H}(b)$ are Hilbert spaces that are contained contractively in H^2 . Moreover, it is well-known that there are relations between the inner products of $\mathcal{H}(b)$ and its cousin $\mathcal{H}(\bar{b})$ since these relations are special cases of the Lotto–Sarason theorem [8, Theorem 16.18 and corollary 16.19]. For further reference, we restate this result below.

Theorem 2.1 ([8], Theorem 17.8). *Let $f \in H^2$. Then $f \in \mathcal{H}(b)$ if and only if $T_{\bar{b}}f \in \mathcal{H}(\bar{b})$ and*

$$\langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle T_{\bar{b}}f_1, T_{\bar{b}}f_2 \rangle_{\bar{b}}, \quad (f_1, f_2 \in \mathcal{H}(b)).$$

It is now a well-known fact that the general theory of $\mathcal{H}(b)$ -spaces splits into two cases, according to whether b is an extreme point or a non-extreme point of the unit ball of H^∞ (recall that, according to De Leeuw-Rudin's Theorem, b is a non-extreme point of the closed unit ball of H^∞ if and only if $\log(1 - |b|) \in L^1(\mathbb{T})$, in particular every inner function $b = \theta$ is an extreme point).

For example,

$$\forall \lambda \in \mathbb{D}, k_\lambda \in \mathcal{H}(b) \Leftrightarrow b \text{ is non-extreme}, \quad (2)$$

(see [8, Theorem 23.23 and corollary 25.8]).

Furthermore from the above characterization of a non-extreme point it follows that, if b is non-extreme, then there is an outer function a such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$ [14]. The function a is uniquely determined by b . We shall call (a, b) an euclidian pair. The following result gives a useful characterization of $\mathcal{H}(b)$ in this case.

Theorem 2.2 ([8], Theorem 23.8). *Let b be a non-extreme point of the closed unit ball of H^∞ , let (a, b) be an euclidian pair and let $f \in H^2$. Then $f \in \mathcal{H}(b)$ if and only if $T_b f \in T_{\bar{a}}(H^2)$. In this case, for $f_1, f_2 \in \mathcal{H}(b)$ there exists a unique $f_1^+, f_2^+ \in H^2$ such that $T_b f_i = T_{\bar{a}} f_i^+$ for $i = 1, 2$ and*

$$\langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle f_1^+, f_2^+ \rangle_2.$$

In particular, for each $f \in \mathcal{H}(b)$,

$$\|f\|_b^2 = \|f\|_2^2 + \|f^+\|_2^2.$$

An important operator in the theory of model spaces is the compression of Toeplitz operators on K_θ : for $\varphi \in L^\infty$ and θ an inner function, one defines the truncated Toeplitz operator A_φ^θ by

$$\begin{aligned} A_\varphi^\theta : K_\theta &\rightarrow K_\theta \\ f &\mapsto A_\varphi^\theta f = \mathbf{P}_\theta(T_\varphi f), \end{aligned}$$

with $\mathbf{P}_\theta$ the orthogonal projection of H^2 to K_θ . It turns out that when φ is in H^∞ , then K_θ is invariant for $T_{\bar{\varphi}}$ and the adjoint of the truncated Toeplitz operator with symbol φ is $(A_\varphi^\theta)^* = T_{\bar{\varphi}, \theta}$ [8, Section 14.7]. More generally, for $\varphi \in H^\infty$, $\mathcal{H}(b)$ is invariant for $T_{\bar{\varphi}}$. We will denote by

$$\begin{aligned} T_{\bar{\varphi}, b} : \mathcal{H}(b) &\rightarrow \mathcal{H}(b) \\ f &\mapsto P_+(\bar{\varphi}f), \end{aligned}$$

and we have, $\|T_{\bar{\varphi}, b}\|_{L(\mathcal{H}(b))} \leq \|\varphi\|_\infty$ [8]. In particular, $\mathcal{H}(b)$ is invariant for the backward shift $S^* = T_{\bar{z}}$. When b is a non-extreme point of the closed unit ball of H^∞ , it follows from (1) and (2) that

$$T_{\bar{\varphi}, b} k_\lambda = \overline{\varphi(\lambda)} k_\lambda, \quad \lambda \in \mathbb{D}. \quad (3)$$

3 Compactness of $T_{\bar{\varphi}, b}$.

Ahern and Clark [1] have given a necessary and sufficient condition for the truncated Toeplitz operator A_φ^θ to be compact, when the symbol φ is continuous on the boundary. See also an alternative proof by Garcia-Ross-Wogen in [9]. The characterization of Ahern-Clark involves the notion of the spectrum of an inner function.

Recall that the spectrum of a function b in the closed unit ball of H^∞ [8, Section 5.2 and 22.6], denoted by $\sigma(b)$ is defined as follows

$$\sigma(b) = \left\{ \zeta \in \mathbb{T} : \liminf_{\substack{z \rightarrow \zeta \\ z \in \mathbb{D}}} |b(z)| < 1 \right\} \cup Z(b),$$

where $Z(b) = \{\lambda \in \mathbb{D} : b(\lambda) = 0\}$.

A generalization of Livsic-Moeller's result shows that b and every element in $\mathcal{H}(b)$ can be analytically continued across any arc $I \subset \mathbb{T} \setminus \text{clos}(\sigma(b))$, and $|b| = 1$ on I [8, Theorem 20.13].

In particular if $b = \theta$ is a non constant inner function, and since θ is unimodular a.e. on $\mathbb{T}$ then

$$\sigma(\theta) = \left\{ \zeta \in \mathbb{T} : \liminf_{\substack{z \rightarrow \zeta \\ z \in \mathbb{D}}} |\theta(z)| = 0 \right\} = \text{clos}(Z(\theta)) \cup \text{supp}(\nu),$$

where $Z(\theta) = \{\lambda \in \mathbb{D} : \theta(\lambda) = 0\}$ and ν is the measure representing the singular part of θ .

Now Ahern and Clark's result says:

Theorem 3.1 (Ahern-Clark, [1]). *Let $\varphi \in C(\mathbb{T})$, then A_φ^θ is compact if and only if $\varphi|_{\sigma(\theta) \cap \mathbb{T}} = 0$.*

The compactness property of the operators $T_{\bar{\varphi},b}$ will depend on the boundary spectrum of b and it is a consequence of the generalization of Ahern and Clark's result.

For this reason we begin by this generalization, and we study the compactness of the general operator $\mathbf{T}_{\bar{\varphi},b}$ with $\varphi \in C(\mathbb{T})$, using the same technique used by Garcia, Ross and Wogen [9] to prove the Ahern-Clark result on compactness of A_{φ}^{θ} .

3.1 Compactness of $\mathbf{T}_{\bar{\varphi},b}$.

Recall that the notation $\mathbf{T}_{\bar{\varphi},b}$ represents the Toeplitz operator defined from $\mathcal{H}(b)$ into H^2 .

Theorem 3.2. *Let b be a point of the closed unit ball of H^{∞} and let $\varphi \in C(\mathbb{T})$. Then the operator,*

$$\mathbf{T}_{\bar{\varphi},b} : \mathcal{H}(b) \rightarrow H^2 \\ f \mapsto P_+(\bar{\varphi}f) ,$$

is compact if and only if $\varphi|_{\sigma(b) \cap \mathbb{T}} = 0$.

Proof. ($\Leftarrow$) Suppose that $\varphi|_{\sigma(b) \cap \mathbb{T}} = 0$. Let $\varepsilon > 0$ and pick $\psi \in C(\mathbb{T})$; $\psi = 0$ on an open set containing $\text{clos}(\sigma(b) \cap \mathbb{T})$ and $\|\psi - \varphi\|_{\infty} < \varepsilon$. Since $\|\mathbf{T}_{\bar{\varphi},b} - \mathbf{T}_{\bar{\psi},b}\|_{L(\mathcal{H}(b), H^2)} \leq \|\psi - \varphi\|_{\infty} < \varepsilon$, it suffices to show that $\mathbf{T}_{\bar{\psi},b}$ is compact.

Let $K = \overline{\psi^{-1}(\mathbb{C} \setminus \{0\})}$ then $K \subset \mathbb{T} \setminus \text{clos}(\sigma(b))$. And consider $(f_n)_n$ a sequence of $\mathcal{H}(b)$ such that $(f_n)_n$ weakly converges to zero.

We know that for each $\zeta \in K$, the function

$$k_{\zeta}^b(z) = \frac{1 - \overline{b(\zeta)}b(z)}{1 - \bar{\zeta}z},$$

belongs to $\mathcal{H}(b)$ and for every $f \in \mathcal{H}(b)$,

$$f(\zeta) = \langle f, k_{\zeta}^b \rangle_b,$$

and

$$\|k_{\zeta}^b\|_b^2 = \frac{1 - |b(\zeta)|^2}{1 - |\zeta|^2} = |b'(\zeta)|.$$

(see [8, Theorem 21.1]).

In particular, since $(f_n)_n$ weakly converges to zero in $\mathcal{H}(b)$, we have $f_n(\zeta) = \langle f_n, k_{\zeta}^b \rangle_b \rightarrow 0$, as $n \rightarrow \infty$, and for every $n \in \mathbb{N}$, $\|f_n\|_b \leq C$.

Therefore, since b is analytic on a neighborhood of the compact set K we obtain

$$\forall \zeta \in K, |f_n(\zeta)| = |\langle f_n, k_{\zeta}^b \rangle_b| \leq \|f_n\|_b \|k_{\zeta}^b\|_b \leq C \sup_{\zeta \in K} \sqrt{|b'(\zeta)|} < \infty . \quad (4)$$

By the dominated convergence theorem, and using (4) it follows that

$$\|\mathbf{T}_{\bar{\psi},b} f_n\|_2^2 = \|P_+(\bar{\psi} f_n)\|_2^2 \leq \|\bar{\psi} f_n\|_2^2 = \int_{\mathbb{T}} |\psi|^2 |f_n|^2 d\zeta = \int_K |\psi|^2 |f_n|^2 d\zeta \rightarrow 0.$$

whence $\mathbf{T}_{\bar{\psi},b}$ is compact and therefore $\mathbf{T}_{\bar{\varphi},b}$ is compact.

($\Rightarrow$) Suppose that $\varphi \in C(\mathbb{T})$, $\zeta \in \sigma(b) \cap \mathbb{T}$ and $\mathbf{T}_{\bar{\varphi},b}$ is compact. Let

$$F_{\lambda}(z) = \frac{1 - |\lambda|^2}{1 - |b(\lambda)|^2} \left| \frac{1 - \overline{b(\lambda)}b(z)}{1 - \bar{\lambda}z} \right|^2,$$

which is the square of the absolute value of the normalized reproducing kernel for $\mathcal{H}(b)$. Observe that

$F_\lambda(z) \geq 0$.

Since $\zeta \in \sigma(b) \cap \mathbb{T}$ then there is a sequence λ_n in $\mathbb{D}$ such that $\lambda_n \rightarrow \zeta$ and $|b(\lambda_n)| \rightarrow c$ with $c < 1$ (by the definition of the spectrum of b already mentioned). Suppose that $\zeta = e^{i\alpha}$ and note that if $|t - \alpha| \geq \delta$, then

$$F_{\lambda_n}(e^{it}) \leq C_\delta \frac{1 - |\lambda_n|^2}{1 - |b(\lambda_n)|^2},$$

for some absolute constant $C_\delta > 0$. Thus since $|b(\lambda_n)| \rightarrow c$ with $c < 1$, we get that

$$\sup_{|t-\alpha| \geq \delta} F_{\lambda_n}(e^{it}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Write,

$$\begin{aligned} & \varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{\|k_{\lambda_n}^b\|_b^2} \langle k_{\lambda_n}^b, \mathbf{T}_{\bar{\varphi}, b} k_{\lambda_n}^b \rangle_2 \\ &= \varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{\|k_{\lambda_n}^b\|_b^2} \langle k_{\lambda_n}^b, P_+(\bar{\varphi} k_{\lambda_n}^b) \rangle_2 \\ &= \varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{\|k_{\lambda_n}^b\|_b^2} \langle k_{\lambda_n}^b, \bar{\varphi} k_{\lambda_n}^b \rangle_2 \quad (\text{see [8, lemma 4.8]}) \\ &= \varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{\|k_{\lambda_n}^b\|_b^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{it}) |k_{\lambda_n}^b(e^{it})|^2 dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\zeta) F_{\lambda_n}(e^{it}) dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{it}) F_{\lambda_n}(e^{it}) dt \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} (\varphi(\zeta) - \varphi(e^{it})) F_{\lambda_n}(e^{it}) dt \\ &= \frac{1}{2\pi} \int_{|t-\alpha| \leq \delta} (\varphi(\zeta) - \varphi(e^{it})) F_{\lambda_n}(e^{it}) dt + \frac{1}{2\pi} \int_{|t-\alpha| \geq \delta} (\varphi(\zeta) - \varphi(e^{it})) F_{\lambda_n}(e^{it}) dt. \end{aligned}$$

The first integral can be made small by the continuity of φ . Once $\delta > 0$ is fixed the second term goes to zero since $\sup_{|t-\alpha| \geq \delta} F_{\lambda_n}(e^{it}) \rightarrow 0$ as $n \rightarrow \infty$. In addition

$$\begin{aligned} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt &= \frac{1 - |\lambda_n|^2}{1 - |b(\lambda_n)|^2} \int_{-\pi}^{\pi} \frac{|1 - \overline{b(\lambda_n)} b(e^{it})|^2}{|1 - \overline{\lambda_n} e^{it}|^2} dt \\ &\geq \frac{1 - |\lambda_n|^2}{1 - |b(\lambda_n)|^2} (1 - |b(\lambda_n)|)^2 \int_{-\pi}^{\pi} \frac{1}{|1 - \overline{\lambda_n} e^{it}|^2} dt \\ &= \frac{(1 - |\lambda_n|^2)(1 - |b(\lambda_n)|)^2}{(1 - |b(\lambda_n)|)(1 + |b(\lambda_n)|)} \frac{1}{|1 - |\lambda_n|^2|} \\ &= \frac{1 - |b(\lambda_n)|}{1 + |b(\lambda_n)|} \geq \frac{1 - c}{2} > 0. \end{aligned}$$

Furthermore, on one hand

$$\frac{\|k_{\lambda_n}^b\|_2}{\|k_{\lambda_n}^b\|_b} \leq 1.$$

And on the other hand the sequence $(\frac{k_{\lambda_n}^b}{\|k_{\lambda_n}^b\|_b})_n$ converges weakly to 0, because $|\lambda_n| \rightarrow 1$ and $|b(\lambda_n)| \rightarrow c$ with $c < 1$. Indeed, using that

$$\|k_{\lambda_n}^b\|_b^2 = \frac{1 - |b(\lambda_n)|^2}{1 - |\lambda_n|^2}.$$

We deduce that for $f \in H^\infty \cap \mathcal{H}(b)$,

$$| \langle f, \frac{k_{\lambda_n}^b}{\|k_{\lambda_n}^b\|_b} \rangle_b | = \frac{|f(\lambda_n)| \sqrt{1-|\lambda_n|^2}}{\sqrt{1-|b(\lambda_n)|^2}} \leq \frac{\|f\|_\infty \sqrt{1-|\lambda_n|^2}}{\sqrt{1-|b(\lambda_n)|^2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Furthermore, $H^\infty \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$, since for every $\lambda \in \mathbb{D}$, $k_\lambda^b \in H^\infty \cap \mathcal{H}(b)$. Thus the sequence $\frac{k_{\lambda_n}^b}{\|k_{\lambda_n}^b\|_b}$ converges weakly to 0 in $\mathcal{H}(b)$, with $\mathbf{T}_{\bar{\varphi},b}$ considered compact. We deduce that

$$\begin{aligned} \frac{1}{\|k_{\lambda_n}^b\|_b^2} | \langle k_{\lambda_n}^b, \mathbf{T}_{\bar{\varphi},b} k_{\lambda_n}^b \rangle_2 | &\leq \frac{\|k_{\lambda_n}^b\|_2 \|\mathbf{T}_{\bar{\varphi},b} k_{\lambda_n}^b\|_2}{\|k_{\lambda_n}^b\|_b \|k_{\lambda_n}^b\|_b} \\ &\leq \frac{\|\mathbf{T}_{\bar{\varphi},b} k_{\lambda_n}^b\|_2}{\|k_{\lambda_n}^b\|_b} \leq \frac{\|\mathbf{T}_{\bar{\varphi},b} k_{\lambda_n}^b\|_b}{\|k_{\lambda_n}^b\|_b} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

After all these computations, we see that

$$\varphi(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt - \frac{1}{\|k_{\lambda_n}^b\|_b^2} \langle k_{\lambda_n}^b, \mathbf{T}_{\bar{\varphi},b} k_{\lambda_n}^b \rangle_2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

with

$$\int_{-\pi}^{\pi} F_{\lambda_n}(e^{it}) dt > 0 \quad \text{and} \quad \frac{1}{\|k_{\lambda_n}^b\|_b^2} | \langle k_{\lambda_n}^b, \mathbf{T}_{\bar{\varphi},b} k_{\lambda_n}^b \rangle_2 | \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Finally, $\varphi(\zeta) = 0$.

□

We now present consequences of this result.

3.2 Compactness of $\mathbf{T}_{\bar{\varphi},b}$

Recall that the notation $T_{\bar{\varphi},b}$ represents the Toeplitz operator defined from $\mathcal{H}(b)$ into itself.

Corollary 3.3. *Let b be a point of the closed unit ball of H^∞ such that $m(\sigma(b) \cap \mathbb{T}) > 0$. Let $\varphi \in C(\mathbb{T}) \cap H^\infty$. Then the operator*

$$\begin{aligned} T_{\bar{\varphi},b} &: \mathcal{H}(b) \rightarrow \mathcal{H}(b) \\ f &\mapsto P_+(\bar{\varphi}f) \end{aligned}$$

is compact if and only if $\varphi = 0$.

Proof. Assume that $T_{\bar{\varphi},b}$ is compact. Then $\mathbf{T}_{\bar{\varphi},b}$ is also compact and by Theorem 3.2,

$$\varphi(\sigma(b) \cap \mathbb{T}) = 0.$$

However, since $\varphi \in H^\infty$ with $m(\sigma(b) \cap \mathbb{T}) > 0$ then $\varphi = 0$.

□

Corollary 3.4. *Let b be a point of the closed unit ball of H^∞ . The following assertions are equivalent:*

1. $\exists \varphi \in H^\infty \cap C(\mathbb{T})$, $\varphi \neq 0$ such that $T_{\bar{\varphi},b}$ is compact.
2. $m(\sigma(b) \cap \mathbb{T}) = 0$.

Proof. (2) $\Rightarrow$ (1) First note that (2) implies that b is an inner function. Indeed, assume on the contrary that b is not inner. Then, the set

$$E = \{\zeta \in \mathbb{T} : |b(\zeta)| \neq 1\}$$

has a positive Lebesgue measure. Moreover, it turns out that $E \subset \text{clos}(\sigma(b)) \cap \mathbb{T}$. Indeed, if $\zeta \in \mathbb{T} \setminus \text{clos}(\sigma(b))$ then b admits an analytic continuation across a neighborhood $D(\zeta, r) = \{w : |w - \zeta| < r\}$ of ζ with $|b| \equiv 1$ on the arc $D(\zeta, r) \cap \mathbb{T}$.

In particular, $|b(\zeta)| = 1$ and $\zeta \in \mathbb{T} \setminus E$. We deduce that

$$0 < m(E) \leq m(\text{clos}(\sigma(b)) \cap \mathbb{T}).$$

Now according to Rudin's theorem, we can find a function $\varphi \in \mathcal{A}(\mathbb{D}) = \text{Hol}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$, $\varphi \neq 0$ such that $\varphi(\sigma(b) \cap \mathbb{T}) = 0$. Then we apply Ahern-Clark's result (see Theorem 3.1) to get that $T_{\bar{\varphi}, b} = (A_{\bar{\varphi}}^b)^*$ is compact.

(1) $\Rightarrow$ (2) Follows from Corollary 3.3. $\square$

In the case where b is a non-extreme point of the closed unit ball of H^∞ , we can get a more general result without the hypothesis that the symbol φ is continuous.

Theorem 3.5. *Let b be a non-extreme point of the closed unit ball of H^∞ and let $\varphi \in H^\infty$. Then the operator*

$$T_{\bar{\varphi}, b} : \begin{array}{ccc} \mathcal{H}(b) & \rightarrow & \mathcal{H}(b) \\ f & \mapsto & P_+(\bar{\varphi}f) \end{array}$$

is compact if and only if $\varphi = 0$.

Proof. Let a be the unique outer function such that (a, b) is an euclidian pair. Note that since b is non-extreme then $k_z \in \mathcal{H}(b)$, for all $z \in \mathbb{D}$ (see (2)).

Suppose that $T_{\bar{\varphi}, b}$ is compact. Notice that for every $(\lambda_n)_n \subset \mathbb{D}$ such that $|\lambda_n| \rightarrow 1$, the sequence $(\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_b})_n$ converges weakly to 0 in $\mathcal{H}(b)$.

Indeed, let $f \in \mathcal{H}(b)$ such that f and $f^+ \in H^\infty$. Recall that f^+ is defined in Theorem 2.2. Then, using that

$$T_{\bar{b}} k_{\lambda_n} = \overline{b(\lambda_n)} k_{\lambda_n} = T_{\bar{a}} \left(\frac{\overline{b(\lambda_n)}}{a(\lambda_n)} k_{\lambda_n} \right) \text{ (see (3))},$$

we see that

$$k_{\lambda_n}^+ = \frac{\overline{b(\lambda_n)}}{a(\lambda_n)} k_{\lambda_n}.$$

Whence by Theorem 2.2, we have

$$\begin{aligned} \langle f, \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_b} \rangle_b &= \langle f, \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_b} \rangle_2 + \langle f^+, \frac{k_{\lambda_n}^+}{\|k_{\lambda_n}\|_b} \rangle_2 \\ &= \left(f(\lambda_n) + \frac{b(\lambda_n)}{a(\lambda_n)} f^+(\lambda_n) \right) \frac{1}{\|k_{\lambda_n}\|_b}. \end{aligned}$$

On the other hand, it is known that in the non-extreme case:

$$\|k_{\lambda_n}\|_b^2 = \frac{1}{1 - |\lambda_n|^2} \left(1 + \frac{|b(\lambda_n)|^2}{|a(\lambda_n)|^2} \right)$$

(see [8, Corollary 23.25]). Hence

$$\frac{|b(\lambda_n)|^2}{|a(\lambda_n)|^2 \|k_{\lambda_n}\|_b^2} = \frac{(1 - |\lambda_n|^2) |b(\lambda_n)|^2}{|a(\lambda_n)|^2 + |b(\lambda_n)|^2} \leq 1 - |\lambda_n|^2.$$

Using this inequality and the inequality

$$\|k_{\lambda_n}\|_b^2 \geq \frac{1}{1 - |\lambda_n|^2},$$

we deduce that

$$\begin{aligned} \left| \left\langle f, \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_b} \right\rangle_b \right| &\leq (|f(\lambda_n)| + |f^+(\lambda_n)|) \sqrt{1 - |\lambda_n|^2} \\ &\leq (\|f\|_\infty + \|f^+\|_\infty) \sqrt{1 - |\lambda_n|^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Furthermore, the set $\{f \in \mathcal{H}(b); f \& f^+ \in H^\infty\}$ is dense in $\mathcal{H}(b)$, since $\{k_\lambda^b; \lambda \in \mathbb{D}\} \subset \{f \in \mathcal{H}(b); f \& f^+ \in H^\infty\}$. Indeed, for every $\lambda \in \mathbb{D}$,

$$(k_\lambda^b)^+ = (k_\lambda - \overline{b(\lambda)} b k_\lambda)^+ = k_\lambda^+ - \overline{b(\lambda)} (b k_\lambda)^+ = \overline{b(\lambda)} a k_\lambda \in H^\infty,$$

with

$$(b k_\lambda)^+ = \frac{k_\lambda}{a(\lambda)} - a k_\lambda$$

(see [8, Theorem 23.23]).

Hence, the sequence $(\frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_b})_n$ converges weakly to 0 in $\mathcal{H}(b)$.

Now by compactness of $T_{\bar{\varphi}, b}$ it follows that for every $(\lambda_n)_n \subset \mathbb{D}$ such that $|\lambda_n| \rightarrow 1$,

$$\|T_{\bar{\varphi}, b} \frac{k_{\lambda_n}}{\|k_{\lambda_n}\|_b}\|_b \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5)$$

But $T_{\bar{\varphi}, b} k_{\lambda_n} = \overline{\varphi(\lambda_n)} k_{\lambda_n}$ (see (3)). Thus (5) is equivalent to

$$\forall (\lambda_n)_n \subset \mathbb{D}; \lim_{n \rightarrow \infty} |\lambda_n| = 1, |\varphi(\lambda_n)| \rightarrow 0.$$

Which implies that $\varphi = 0$. □

The proof of Theorem 3.5 obviously doesn't work in the case when b is an extreme point of the closed unit ball of H^∞ , since in that case, the Cauchy kernels k_λ do not belong to $\mathcal{H}(b)$ when $b(\lambda) \neq 0$, $\lambda \in \mathbb{D}$.

4 Hypercyclicity of $T_{\bar{\varphi}, b}$

4.1 Hypercyclic and frequently hypercyclic operators

Let X be a complex infinite-dimensional separable Banach space. An operator $T \in L(X)$ is said to be hypercyclic if there is some vector $x \in X$ such that the orbit

$$O(x, T) := \{T^n(x); n \in \mathbb{N}\}$$

is dense in X . Such a vector x is said to be hypercyclic for T , and the set of all hypercyclic vectors for T is denoted by $HC(T)$.

Moreover we say that T is frequently hypercyclic, if there exists a vector $x \in X$ such that for every non-empty open subset U of X , the set $N(x, U) = \{n \geq 0; T^n(x) \in U\}$ of instants when the iterates of x under T visit U has positive lower density, i.e.

$$\underline{\text{dens}}(N(x, U)) = \liminf_{N \rightarrow \infty} \frac{\text{card}(N(x, U) \cap [1, N])}{N} > 0.$$

We refer the reader to the recent book [4] for more information on these topics.

Frequent hypercyclicity is a much stronger notion than hypercyclicity, and some operators are hypercyclic without being frequently hypercyclic: an example is the Bergman backward shift [3].

Let us complete this section by recalling two criterions for hypercyclicity and frequent hypercyclicity that we will use to study the hypercyclicity properties of the Toeplitz operator $T_{\bar{\varphi}, b}$.

We start with the Godefroy-Shapiro Criterion [11], according to which a bounded operator having a large supply of eigenvectors associated to eigenvalues of modulus strictly larger than 1 and strictly smaller than 1 is hypercyclic.

Theorem 4.1 (Godefroy-Shapiro Criterion, [11]). *Let $T \in L(X)$. Suppose that $\bigcup_{|\lambda| < 1} \text{Ker}(T - \lambda)$ and $\bigcup_{|\lambda| > 1} \text{Ker}(T - \lambda)$ both span a dense subspace of X . Then T is hypercyclic.*

Then it was shown by S. Grivaux that an operator T which has "sufficiently many" eigenvectors associated to eigenvalues of modulus 1 is automatically frequently hypercyclic.

Theorem 4.2 (S. Grivaux, [12]). *Let T be a bounded operator on X . Suppose that there exists a sequence $(u_i)_{i \geq 1}$ of vectors of X having the following properties:*

1. *For each $i \geq 1$, u_i is an eigenvector of T associated to an eigenvalue λ_i of T with $|\lambda_i| = 1$ and the λ_i 's all distinct;*
2. *$\text{span}\{u_i; i \geq 1\}$ is dense in X ;*
3. *For any $i \geq 1$ and any $\varepsilon > 0$, there exists an $n \neq i$ such that $\|u_n - u_i\| < \varepsilon$.*

Then T is frequently hypercyclic (in particular hypercyclic).

It is also a natural question, given a family of hypercyclic operators to ask if they have a common hypercyclic vector. The following result gives a sufficient condition for a family of multiples of an operator to have a dense G_δ -set of common hypercyclic vectors.

Theorem 4.3 (Shkarin, [17]). *Let X be a separable Fréchet space, $T \in L(X)$, $0 \leq a < c \leq \infty$. Assume also that for all $\alpha, \beta \in \mathbb{R}$ such that $a < \alpha < \beta < c$ there exists a dense subset E of X and a map $S : E \rightarrow E$ such that $TSx = x$, $\alpha^{-n}T^n x \rightarrow 0$ and $\beta^n S^n x \rightarrow 0$ for each $x \in E$. Then*

$$\cap HC(\lambda T : c^{-1} < |\lambda| < a^{-1}),$$

is a dense G_δ -set in X .

We finish by giving an example of a hypercyclic Toeplitz operator.

Rolewicz's result [4] in 1960, says that the operator $\lambda S^* = T_{\lambda \bar{z}} : H^2 \rightarrow H^2$ for every $\lambda \in \mathbb{C}$, $|\lambda| > 1$, is hypercyclic, this was shown using Kitai's Criterion (a particular case of the Hypercyclicity Criterion) [10]. This result of Rolewicz was generalized by Godefroy-Shapiro [11] in 1991.

Theorem 4.4 (Godefroy-Shapiro). *Let $\varphi \in H^\infty$. The operator $T_{\bar{\varphi}} : H^2 \rightarrow H^2$ is hypercyclic if and only if φ is non-constant and $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.*

4.2 Hypercyclicity of $T_{\bar{\varphi},b}$.

Following the approach of Godefroy-Shapiro, we generalize Theorem 4.4 to operators $T_{\bar{\varphi},b}$ when b is a non-extreme point of the closed unit ball of H^∞ and $\varphi \in H^\infty$, even we get a better result. And according to this generalization we noticed that in the non-extreme case, for every $|\lambda| > 1$ the operator $\lambda X_b = \lambda T_{\bar{\varphi},b}$ is frequently hypercyclic (in particular hypercyclic). On the contrary when b is extreme, λX_b is never hypercyclic.

Theorem 4.5. *Let b be a non-extreme point of the closed unit ball of H^∞ and let $\varphi \in H^\infty$. Then the operator*

$$\begin{aligned} T_{\bar{\varphi},b} : \mathcal{H}(b) &\rightarrow \mathcal{H}(b) \\ f &\mapsto T_{\bar{\varphi},b}f = P_+(\bar{\varphi}f). \end{aligned}$$

is frequently hypercyclic if and only if φ is non-constant and $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Proof. Let φ be a non constant analytic function on $\mathbb{D}$ and assume that $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. Let $\zeta_0 \in \varphi(\mathbb{D}) \cap \mathbb{T}$. Since $\varphi(\mathbb{D})$ is an open set of $\mathbb{C}$ (open mapping theorem), there exists $r > 0$ such that $D(\zeta_0, r) \subset \varphi(\mathbb{D})$ and let Γ be a closed arc contained in $D(\zeta_0, r) \cap \mathbb{T} \subset \varphi(\mathbb{D})$. Consider an exhaustive sequence of compacts $(K_n)_n$ associated to $\mathbb{D}$. Then

$$\varphi^{-1}(\Gamma) = \bigcup_{n \geq 1} \varphi^{-1}(\Gamma) \cap K_n,$$

since $\varphi^{-1}(\Gamma)$ is uncountable, there is, indeed, $n_0 \geq 1$ such that $\text{Card}(\varphi^{-1}(\Gamma) \cap K_{n_0}) = +\infty$. We can therefore construct a sequence $(\lambda_n)_n$ in $\varphi^{-1}(\Gamma) \cap K_{n_0}$ with $\lambda_n \neq \lambda_\ell$, $n \neq \ell$. Now $\lambda_n \in K_{n_0}$ -compact, so there is a subsequence $(\lambda_{n_\ell})_\ell$ such that $\lambda_{n_\ell} \rightarrow \lambda \in K_{n_0} \subset \mathbb{D}$, $\ell \rightarrow +\infty$. Since b is non-extreme, reproducing kernels of H^2 , k_λ for all $\lambda \in \mathbb{D}$, are elements of $\mathcal{H}(b)$ and they are eigenvectors of $T_{\bar{\varphi},b}$, of eigenvalues $\bar{\varphi}(\lambda)$ (see (3)). Thus the sequence of reproducing kernels of H^2 associated to the subsequence $(\lambda_{n_\ell})_\ell$ will be dense in $\mathcal{H}(b)$ (since λ is the unique accumulation point of this subsequence):

$$\text{Span}(k_{\lambda_{n_\ell}} : n_\ell) = \mathcal{H}(b)$$

and $|\varphi(\lambda_{n_\ell})| = 1$. Moreover this sequence of eigenvectors satisfies the property of "continuity" which is the third condition of Grivaux's criterion (see Theorem 4.2) because the application $\mu \rightarrow k_\mu$ is continuous and the subsequence is convergent. Hence the Toeplitz operator $T_{\bar{\varphi},b}$ is frequently hypercyclic.

Conversely, assume that $T_{\bar{\varphi},b}$ is frequently hypercyclic then it is in particular hypercyclic (so that φ is certainly non constant). And assume that $\varphi(\mathbb{D}) \cap \mathbb{T} = \emptyset$. Since $\mathbb{D}$ is connected with φ continuous on $\mathbb{D}$, $\varphi(\mathbb{D})$ is connected, hence $\varphi(\mathbb{D}) \subset \mathbb{D}$ or $\varphi(\mathbb{D}) \subset \mathbb{C} \setminus \mathbb{D}$. If $\varphi(\mathbb{D}) \subset \mathbb{D}$ then $\forall z \in \mathbb{D}$, $|\varphi(z)| < 1$, it implies that $\|\varphi\|_\infty \leq 1$ and $\|T_{\bar{\varphi},b}\| \leq \|\varphi\|_\infty \leq 1$, whence $T_{\bar{\varphi},b}$ is non-hypercyclic (absurd). If $\varphi(\mathbb{D}) \subset \mathbb{C} \setminus \mathbb{D}$ then $\forall z \in \mathbb{D}$, $|\varphi(z)| > 1$. In this case, $\frac{1}{\varphi} \in H^\infty$ and $T_{\frac{1}{\bar{\varphi}},b}$ is non-hypercyclic since $\|T_{\frac{1}{\bar{\varphi}},b}\| \leq \|\frac{1}{\bar{\varphi}}\|_\infty \leq 1$. Seeing that, $T_{\bar{\varphi},b}T_{\frac{1}{\bar{\varphi}},b} = T_{\frac{1}{\bar{\varphi}},b}T_{\bar{\varphi},b} = I$, then $T_{\frac{1}{\bar{\varphi}},b} = (T_{\bar{\varphi},b})^{-1}$, consequently $T_{\bar{\varphi},b}$ is non-hypercyclic (indeed an invertible operator is hypercyclic if and only if its inverse is hypercyclic [4, page 3]). We get also a contradiction. $\square$

Remark 4.6. *Note that when $b = 0$, we recover Theorem 4.4 of Godefroy and Shapiro.*

In the particular case when $\varphi(z) = z$, corresponding to operator $X_b : \mathcal{H}(b) \rightarrow \mathcal{H}(b)$, $X_b(f) = S^*f$, we get the following result:

Corollary 4.7. *Let b be a non-extreme point of the closed unit ball of H^∞ . Let the operator*

$$\begin{aligned} X_b : \mathcal{H}(b) &\rightarrow \mathcal{H}(b) \\ f &\mapsto S^*f \end{aligned}$$

For all $|\lambda| > 1$, λX_b is frequently hypercyclic (in particular hypercyclic).

As we saw in the previous Corollary, for all $|\lambda| > 1$, λX_b is hypercyclic, so this naturally raises the question of finding a common hypercyclic vector for $(\lambda X_b)_{|\lambda|>1}$. We will apply Shkarin's Theorem 4.3 but we need to introduce another operator on $\mathcal{H}(b)$.

It is well known that $\mathcal{H}(b)$ is invariant under the unilateral forward shift operator S if and only if b is non-extreme [8, Corollary 20.20]. In that case, the mapping

$$\begin{aligned} S_b &: \mathcal{H}(b) \rightarrow \mathcal{H}(b) \\ f &\mapsto Sf = zf \end{aligned}$$

gives a well-defined operator. Moreover S_b is bounded on $\mathcal{H}(b)$ with $\|S_b\| = \sqrt{1 + |a(0)|^{-2}} \|S^* b\|_b^2$ (see [8, Section 24.1]) In particular, we see that except in the case when $b = 0$ (corresponding to $\mathcal{H}(b) = H^2$), the operator S_b has a norm strictly greater than 1.

Theorem 4.8. *Let b be a non-extreme point of the closed unit ball of H^∞ , and let*

$$\begin{aligned} X_b &: \mathcal{H}(b) \rightarrow \mathcal{H}(b) \\ f &\mapsto S^* f \end{aligned}$$

Then,

$$\mathcal{G} = \bigcap HC(\lambda X_b; \|S_b\| < |\lambda| < \infty),$$

is a dense G_δ -set of $\mathcal{H}(b)$.

Proof. We would like to apply Shkarin's Theorem 4.3 with $a = 0$, $c = \|S_b\|^{-1}$ and $E = \mathcal{P}$, with $\mathcal{P}$ the set of analytic polynomials, dense in $\mathcal{H}(b)$ [8, Theorem 23.13]. Let

$$S_b : \mathcal{P} \rightarrow \mathcal{P}; \quad S_b p = zp.$$

It is clear that $X_b S_b = I$. For all $0 < \alpha < \beta < \|S_b\|^{-1}$, and for all $p \in \mathcal{P}$, we have on one hand $\alpha^{-n} X_b^n p \rightarrow 0$ as $n \rightarrow \infty$, since from a certain rank $n_0 = \deg(p) + 1$, $X_b^{n_0} p = 0$, and on the other hand, $\|\beta^n S_b^n p\|_b \leq (\beta \|S_b\|)^n \|p\|_b \rightarrow 0$ as $n \rightarrow \infty$. Hence, using Theorem 4.3, we conclude that $\mathcal{G}$ is a dense G_δ -set of $\mathcal{H}(b)$. $\square$

Remark 4.9. *It remains the question of whether we can replace in the previous Theorem the lower bound*

$$\|S_b\| < |\lambda| \quad \text{by} \quad 1 < |\lambda|.$$

In other word, is the set

$$\bigcap HC(\lambda X_b; 1 < |\lambda| < \infty)$$

a dense G_δ -set of $\mathcal{H}(b)$?

In the case where b is extreme, the operator X_b is no longer hypercyclic, which shows a significant difference in the $\mathcal{H}(b)$ space theory following that $\log(1 - |b|)$ is integrable or not on $\mathbb{T}$. The proof of this result requires basic facts on the spectrum of hypercyclic operators, which we now briefly recall.

Let X be a complex Banach space, and let $T \in \mathcal{L}(X)$ be hypercyclic. Then $\sigma_p(T^*) = \emptyset$ and every connected component of the spectrum of T intersects the unit circle (see [4, Page 11]).

Theorem 4.10. *Let b be an extreme point of the closed unit ball of H^∞ and let*

$$\begin{aligned} X_b &: \mathcal{H}(b) \rightarrow \mathcal{H}(b) \\ f &\mapsto S^* f \end{aligned}$$

Then for every complex number λ , λX_b is not hypercyclic.

Proof. For all $|\lambda| \leq 1$, $\|\lambda X_b\| \leq 1$ hence λX_b is not hypercyclic. Now take $\lambda \in \mathbb{C}$, $|\lambda| > 1$. By [8, Corollary 26.3], we have

$$\sigma_p(\bar{\lambda} X_b^*) = \bar{\lambda} \sigma_p(X_b^*) = \{\bar{\lambda} \beta : \beta \in \mathbb{D} \text{ and } b(\beta) = 0\}.$$

By the preceding equality, we notice that, if b has a Blaschke factor then $\sigma_p(\bar{\lambda}X_b^*) \neq \emptyset$, and thus λX_b is not hypercyclic. Now if b does not admit a Blaschke factor we get from [8, Corollary 26.4] that $\sigma(X_b) \subset \mathbb{T}$. That implies, since $|\lambda| > 1$, $\sigma(\lambda X_b) \cap \mathbb{T} = (\lambda\sigma(X_b)) \cap \mathbb{T} = \emptyset$. Therefore one of the connected component of $\sigma(\lambda X_b)$ do not intersect the unit circle, hence λX_b is not hypercyclic. We conclude that for every complex number λ , λX_b is not hypercyclic. $\square$

In the case when b is extreme, it has not been possible to reach the non-hypercyclicity of the Toeplitz operator $T_{\bar{\varphi},b}$. However we give a necessary (but not sufficient) condition for such an operator to be non-hypercyclic.

Proposition 4.11. *Let b be an extreme point of the closed unit ball of H^∞ , and let $\varphi \in H^\infty$. If the Toeplitz operator $T_{\bar{\varphi},b}$ is hypercyclic then $\sigma_p(T_{\bar{\varphi},b}) = \emptyset$.*

Proof. From [18] it follows that $T_{\bar{\varphi},b}$ is complex symmetric, and therefore, if it has an eigenvalue, the adjoint also has an eigenvalue and in this case $T_{\bar{\varphi},b}$ is not hypercyclic. $\square$

In particular, if b has a blaschke factor, then $T_{\bar{\varphi},b}$ is not hypercyclic, as shown in the following result.

Corollary 4.12. *If $\lambda \in \mathbb{D}$ such that $b(\lambda) = 0$ then $T_{\bar{\varphi},b}$ is not hypercyclic*

Proof. Suppose that $\lambda \in \mathbb{D}$ such that $b(\lambda) = 0$, then $k_\lambda = k_\lambda^b \in \mathcal{H}(b)$. Moreover $T_{\bar{\varphi},b}k_\lambda = \overline{\varphi(\lambda)}k_\lambda$ (see (3)). Hence $\overline{\varphi(\lambda)} \in \sigma_p(T_{\bar{\varphi},b})$. Thus by Proposition 4.11, $T_{\bar{\varphi},b}$ is not hypercyclic. $\square$

Remark 4.13. *It turns out that the necessary condition in proposition 4.11 is not sufficient. Indeed, if b has no Blaschke factor, then $\sigma_p(X_b)$ is empty, though by Theorem 4.10, we know that X_b is not hypercyclic.*

Remark 4.14. *Notice that*

$$\sigma_p(T_{\bar{\varphi},b}) = \emptyset \Leftrightarrow \forall \lambda \in \mathbb{C}, K_{(\varphi-\lambda)_i} \cap \mathcal{H}(b) = \{0\},$$

with $\varphi - \lambda = (\varphi - \lambda)_i(\varphi - \lambda)_e$, and $(\varphi - \lambda)_i$ and $(\varphi - \lambda)_e$ are respectively the inner and outer part of $\varphi - \lambda$.

Proof. Let $\lambda \in \mathbb{C}$. Then $\bar{\lambda} \in \sigma_p(T_{\bar{\varphi},b})$ if and only if there exists $f \in \mathcal{H}(b)$, $f \neq 0$ such that

$$T_{\bar{\varphi},b}f = T_{\bar{\varphi}}f = \bar{\lambda}f.$$

This equation is equivalent to

$$T_{\overline{\varphi-\lambda}}f = 0 \Leftrightarrow T_{(\overline{\varphi-\lambda})_e(\overline{\varphi-\lambda})_i}f = 0 \Leftrightarrow T_{(\overline{\varphi-\lambda})_i}f = 0,$$

because when a is outer $T_{\bar{a}}$ is one-to-one. The last equation is equivalent to

$$f \in K_{(\varphi-\lambda)_i} \cap \mathcal{H}(b),$$

where $K_{(\varphi-\lambda)_i}$ denotes the model space associated to $(\varphi - \lambda)_i$. Thus $\bar{\lambda} \in \sigma_p(T_{\bar{\varphi},b})$ if and only if $K_{(\varphi-\lambda)_i} \cap \mathcal{H}(b) \neq \{0\}$. $\square$

Remark 4.15. *If b is extreme and outer, then $\sigma_p(T_{\bar{\varphi},b}) = \emptyset$.*

Proof. Let $\lambda \in \mathbb{C}$.

If $(\varphi - \lambda)_i \equiv cte$ then $K_{(\varphi-\lambda)_i} = \{0\}$ (see [8, Theorem 18.2]), which implies that $K_{(\varphi-\lambda)_i} \cap \mathcal{H}(b) = \{0\}$.

On the other hand if $(\varphi - \lambda)_i \neq cte$, and if $f \in K_{(\varphi-\lambda)_i}$ then f is not a cyclic vector for S^* (since $\text{span}\{S^{*n}f : n \geq 0\} \subset K_{(\varphi-\lambda)_i} \neq H^2$ because $S^*K_{(\varphi-\lambda)_i} \subset K_{(\varphi-\lambda)_i}$, see section 2.1 and [8, Section 1.10]). Using [8, Theorem 25.17], it implies $f \in K_\Theta$ where $\Theta = b_i$ is the inner part of b . But since b is considered outer, then $b_i \equiv cte$,

thus $K_\Theta = \{0\}$. Hence $K_{(\varphi-\lambda)_i} \cap \mathcal{H}(b) = \{0\}$.

Which gives by Remark 4.14 that $\sigma_p(T_{\varphi,b}) = \emptyset$. □

Remark 4.16. In the case where b is extreme and outer, it would be interesting to know if $T_{\varphi,b}$ is hypercyclic or not.

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