Review Article

Open Access

Dan Timotin*

The invariant subspaces of $S \oplus S^*$

<https://doi.org/10.1515/conop-2020-0101>

Received April 9, 2020; accepted June 8, 2020

Abstract: Using the tools of Sz.-Nagy–Foias theory of contractions, we describe in detail the invariant subspaces of the operator $S \oplus S^*$, where S is the unilateral shift on a Hilbert space. This answers a question of Cămară and Ross.

Keywords: Invariant subspaces, unilateral shift, dual truncated shift

MSC: 47A15, 47A45, 47B37

1 Introduction

The recent series of papers [2–4] explore the class of so-called *dual truncated Toeplitz operators*, which act on a subspace of the usual Lebesgue space L^2 on the unit circle $\mathbb{T}$. In the preprint [1] Cămară and Ross discuss the invariant subspaces of one of these operators, the dual of the compressed shift. In their investigation they encounter the problem of determining the invariant subspaces of the operator $S \oplus S^*$, where S is the usual unilateral shift operator, acting as multiplication by the variable on the Hardy-Hilbert space H^2 , and they state it as an open question.

It turns out that the answer can be given through the Sz.-Nagy–Foias theory of characteristic functions of contractions on a Hilbert space [6]. That theory includes a general result about the relation between invariant subspaces of a contraction and regular factorizations of its characteristic function. In particular, we may use it in order to obtain an explicit description of all invariant subspaces of $S \oplus S^*$, giving thus a complete answer to the open question in [1].

The plan of the paper is the following. After some preliminaries, in Section 3 we provide a short presentation of the relevant part of the Sz.-Nagy–Foias theory. Section 4 contains the main result, the description of the invariant subspaces. Section 5 provides an example related to [1], while Section 6 details the most interesting class of invariant subspaces.

2 Preliminaries

We denote shortly $L^2 = L^2(\mathbb{T}, dm)$, where m is Lebesgue measure on the unit circle $\mathbb{T}$. Its subspace H^2 is the Hardy-Hilbert space of functions that can be analytically extended to the unit disk $\mathbb{D}$; then $H_-^2 = L^2 \ominus H^2$. The orthogonal projections in L^2 onto H^2 and H_-^2 will be denoted by P_+ and P_- respectively. The map $f \mapsto \tilde{f}$, with $\tilde{f}(z) = \overline{f(\bar{z})}$ is an involution on H^2 . An inner function $\theta \in H^2$ is characterized by $|\theta(e^{it})| = 1$ for almost all t . If θ is inner function, then $\tilde{\theta}$ is also inner.

We will also use Lebesgue and Hardy spaces defined on the unit circle with values in a Hilbert space $\mathcal{E}$; they will be denoted with $L^2(\mathcal{E})$ and $H^2(\mathcal{E})$ respectively.

*Corresponding Author: Dan Timotin: Institute of Mathematics Simion Stoilow of the Romanian Academy, Calea Griviței 21, Bucharest 010702, Romania, E-mail: dan.timotin@imar.ro

If S is the shift operator on H^2 , defined by $(Sf)(z) = zf(z)$, Beurling's Theorem states that the invariant subspaces of S are $\{0\}$ and the spaces θH^2 with θ inner. We denote $K_\theta = H^2 \ominus \theta H^2$; so the invariant subspaces for S^* are H^2 and K_θ for θ inner. The map C_θ defined by $C_\theta f = \theta \bar{z} \bar{f}$ is a conjugation on K_θ ; in particular,

$$C_\theta(K_\theta) = K_\theta. \quad (2.1)$$

It will be convenient in the sequel to consider, rather than S^* , the operator $S_\star$, acting on H^2 as the compression of multiplication by z to H^2_- . This is unitarily equivalent to S^* , and the precise unitary operator that implements this equivalence is $J : H^2 \rightarrow H^2_-$, $(Jf)(z) = \bar{z}f(\bar{z})$; we have $JS^* = S_\star J$. The invariant subspaces of $S_\star$ are then H^2_- together with $J(K_\theta) = J(H^2) \ominus J(\theta H^2) = H^2_- \ominus J(\theta H^2)$ for θ inner. Since

$$\begin{aligned} J(\theta H^2) &= \{J(\theta f) : f \in H^2\} = \{\bar{z}\theta(\bar{z})f(\bar{z}) : f \in H^2\} \\ &= \{\bar{\theta}(z)\bar{z}\bar{f}(z) : f \in H^2\} = \bar{z}\bar{\theta}\bar{H}^2, \end{aligned}$$

we have

$$J(K_\theta) = \bar{z}\bar{H}^2 \ominus \bar{z}\bar{\theta}\bar{H}^2 = \bar{z}\bar{K}_\theta.$$

Our purpose in this paper will be the determination of the invariant subspaces of $S \oplus S_\star$; we will see below that this is a *model operator* in the sense of Sz.Nagy and Foias. The invariant subspaces of $S \oplus S^*$ are then immediately obtained by applying the operator J .

Some of these invariant subspaces of $S \oplus S_\star$ may easily be described; namely, the subspaces $X \oplus X'$, where $X \subset H^2$ is invariant to S , while $X' \subset H^2_-$ is invariant to $S_\star$. We will call them *splitting* invariant subspaces. The next lemma summarizes the above remarks.

Lemma 2.1. *The splitting invariant subspaces of $S \oplus S_\star$ acting on $H^2 \oplus H^2_-$ are of the form $X \oplus X'$, where X is either $\{0\}$ or θH^2 for some inner function θ , while X' is either H^2_- or $\bar{z}\bar{K}_{\theta'}$ for some inner function θ' .*

One may say that these are the obvious invariant subspaces of $S \oplus S_\star$. There is, however, a large variety of nonsplitting invariant subspaces, for whose determination we will have to bring into play the Sz.-Nagy–Foias theory of contractions [6].

We end the preliminaries with a lemma that will be helpful.

Lemma 2.2. *If θ is inner, then $P_-(\bar{\theta}H^2) = \bar{z}\bar{K}_\theta$.*

Proof. The decomposition $H^2 = K_\theta \oplus \theta H^2$ yields $\bar{\theta}H^2 = \bar{\theta}K_\theta \oplus H^2_-$, and so $P_-(\bar{\theta}H^2) = \bar{\theta}K_\theta$. But $\bar{\theta}K_\theta = \bar{z}\bar{K}_\theta$ follows from equality (2.1). $\square$

3 Sz.Nagy–Foias theory of contractions and invariant subspaces

The general reference for this section is the monograph [6]. Suppose $\Theta : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{E}^*)$ is an analytic function in the unit disc $\mathbb{D}$ with values in the algebra of bounded operators from $\mathcal{E}$ to $\mathcal{E}^*$, with $\|\Theta(z)\| \leq 1$ for all $z \in \mathbb{D}$; we will call it a *contractive analytic function*. Θ has boundary values almost everywhere on $\mathbb{T}$, that will be denoted by $\Theta(e^{it})$. A contractive analytic function is called *pure* if $\|\Theta(0)x\| < \|x\|$ for any $x \in \mathcal{E}$. Any contractive analytic function admits a decomposition in a direct sum $\Theta = \Theta_p \oplus \Theta_u$, where Θ_p is pure and Θ_u is a constant unitary operator; then Θ_p is called the *pure part* of Θ .

To a pure contractive analytic function corresponds a functional model, defined as follows. Denote $\Delta(e^{it}) = (I - \Theta(e^{it})^* \Theta(e^{it}))^{1/2}$. Then the *model space* is

$$\mathcal{H}_\Theta = (H^2(\mathcal{E}_\star) \oplus \overline{\Delta L^2(\mathcal{E})}) \ominus \{\Theta f \oplus \Delta f : f \in H^2(\mathcal{E})\}, \quad (3.1)$$

on which acts the *model operator* S_Θ , defined as the compression to $\mathcal{H}_\Theta$ of multiplication with e^{it} on both components of $H^2(\mathcal{E}_\star) \oplus \Delta L^2(\mathcal{E})$.

If $\mathcal{H}$ is a Hilbert space, a completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is a linear operator that satisfies $\|T\| \leq 1$, and there is no reducing subspace of T on which it is unitary. The defect of T is the operator $D_T = (I - T^*T)^{1/2}$, and the defect space is $\mathcal{D}_T = \overline{D_T\mathcal{H}}$. It is shown in [6] that any completely nonunitary contraction T is unitarily equivalent to $\mathbf{S}_{\Theta_T}$, where Θ_T is the pure contractive analytic function with values in $\mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*})$ defined by

$$\Theta_T(z) = -T + D_{T^*}(I - zT^*)^{-1}D_T|_{\mathcal{D}_T}.$$

Note that the domain of $\Theta_T(z)$ is $\mathcal{D}_T$.

The invariant subspaces of $\mathbf{S}_{\Theta}$ are in correspondence with the *regular* factorizations of Θ , that we will define in the sequel. Suppose $\mathcal{F}$ is a third Hilbert space and $\Theta_1 : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{F})$, $\Theta_2 : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{F}, \mathcal{E}_*)$ are two other contractive analytic functions such that $\Theta = \Theta_2\Theta_1$. If $\Delta(e^{it}) = (I - \Theta(e^{it})^*\Theta(e^{it}))^{1/2}$ for $i = 1, 2$, then the map

$$\Delta f \mapsto \Delta_2\Theta_1 f \oplus \Delta_1 f, \quad f \in \Delta L^2(\mathcal{E})$$

is isometric, and may thus be completed to an isometry

$$Z : \overline{\Delta L^2(\mathcal{E})} \rightarrow \overline{\Delta_2 L^2(\mathcal{F})} \oplus \overline{\Delta_1 L^2(\mathcal{E})}.$$

The factorization $\Theta = \Theta_2\Theta_1$ is called *regular* if Z is unitary.

The relation between invariant subspaces and regular factorization is summed up in the next theorem, which follows from [6, Theorem VII.1.1], [6, Theorem VII.4.3], and the remark following it.

Theorem 3.1. *To any regular factorization $\Theta = \Theta_2\Theta_1$ corresponds an invariant subspace of $\mathbf{S}_{\Theta}$, defined by the formula*

$$Y = \{\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(\mathcal{F}), v \in \overline{\Delta L^2(\mathcal{E})}\} \ominus \{\Theta w \oplus \Delta w : w \in H^2(\mathcal{E})\}. \quad (3.2)$$

The characteristic function of $\mathbf{S}_{\Theta}|_Y$ is the pure part of Θ_1 .

Conversely, any invariant subspace Y determines a regular factorization $\Theta = \Theta_2\Theta_1$, such that Y is given by (3.2).

If $\Theta = \Theta'_2\Theta'_1$, with $\Theta_1 : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{F}')$, $\Theta_2 : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{F}', \mathcal{E}_*)$ produces through (3.2) the same subspace Y , then there exists $\Omega \in \mathcal{L}(\mathcal{F}, \mathcal{F}')$ unitary, such that $\Theta'_1 = \Omega\Theta_1$, $\Theta'_2 = \Theta_2\Omega^*$.

In general, the main difficulty in the application of Theorem 3.1 is the identification of the regular factorizations of a given contractive analytic function. Fortunately, this can be done explicitly in the case that interests us.

Let us denote by $0_{m \rightarrow n}$ the zero contractive analytic function considered as acting from $\mathbb{C}^m$ to $\mathbb{C}^n$. The functional model associated to $\Theta = 0_{1 \rightarrow 1}$ is the space

$$(H^2 \oplus L^2) \ominus \{0 \oplus f : f \in H^2\} = H^2 \oplus H_-^2,$$

on which the model operator is precisely $S \oplus S_*$. Noting that $\Delta(e^{it}) = 1$ for all t , we obtain

$$\begin{aligned} Y &= \{\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(\mathcal{F}), v \in \overline{\Delta_1 L^2}\} \ominus (\{0\} \oplus H^2) \\ &= P_{H^2 \oplus H_-^2}(\{\Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(\mathcal{F}), v \in \overline{\Delta_1 L^2}\}) \\ &= \{\Theta_2 u \oplus P_-(Z^{-1}(\Delta_2 u \oplus v)) : u \in H^2(\mathcal{F}), v \in \overline{\Delta_1 L^2}\}, \end{aligned}$$

Theorem 3.1 yields then the next corollary.

Corollary 3.2. (i) *To any regular factorization $0_{1 \rightarrow 1} = \Theta_2\Theta_1$, where $\Theta_1 : \mathbb{D} \rightarrow \mathcal{L}(\mathbb{C}, \mathcal{F})$, $\Theta_2 : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{F}, \mathbb{C})$, corresponds an invariant subspace of $\mathbf{S}_{\Theta}$, defined by the formula*

$$Y = \{\Theta_2 u \oplus P_-(Z^{-1}(\Delta_2 u \oplus v)) : u \in H^2(\mathcal{F}), v \in \overline{\Delta_1 L^2}\}, \quad (3.3)$$

where

$$Z : L^2 \rightarrow \overline{\Delta_2 L^2(\mathcal{F})} \oplus \overline{\Delta_1 L^2}, \quad Zv = \Delta_2\Theta_1 v \oplus \Delta_1 v. \quad (3.4)$$

The characteristic function of $T_Y := S \oplus S_*|Y$ is the pure part of Θ_1 .

(ii) Conversely, any invariant subspace determines a regular factorization $0_{1 \rightarrow 1} = \Theta_2 \Theta_1$.

(iii) If another factorization $0 = \Theta'_2 \Theta'_1$, with $\Theta_1 : \mathbb{D} \rightarrow \mathcal{L}(\mathbb{C}, \mathcal{F}')$, $\Theta_2 : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{F}', \mathbb{C})$, produces by (3.3) the same invariant subspace Y , then there exists $\Omega \in \mathcal{L}(\mathcal{F}, \mathcal{F}')$ unitary, such that $\Theta'_1 = \Omega \Theta_1$, $\Theta'_2 = \Theta_2 \Omega^*$.

In order to obtain a concrete description of the invariant subspaces in Corollary 3.2, we have to know the regular factorizations of the function $0_{1 \rightarrow 1}$. This can be obtained from another result of Sz.-Nagy and Foias, namely [6, Proposition VII.3.5], which describes all regular factorizations of a scalar contractive analytic function. Applying it to the null function yields the next statement.

Lemma 3.3. *The regular factorizations of the function $\Theta = 0_{1 \rightarrow 1}$ are of the following three types, corresponding to $\dim \mathcal{F} = 0, 1$ or 2 :*

1. $\dim \mathcal{F} = 0$. There is a unique possibility: $0_{1 \rightarrow 1} = 0_{0 \rightarrow 1} 0_{1 \rightarrow 0}$.
2. $\dim \mathcal{F} = 1$. Then $0 = \Theta_2 \Theta_1$ with Θ_i scalar functions, and there are two cases:

(2.1) $\Theta_1 = 0_{1 \rightarrow 1}$, Θ_2 inner.

(2.2) $\Theta_2 = 0_{1 \rightarrow 1}$, Θ_1 inner.

3. $\dim \mathcal{F} = 2$. Then $\Theta_1 = \begin{pmatrix} \theta_{11} \\ \theta_{12} \end{pmatrix}$, $\Theta_2 = \begin{pmatrix} \theta_{21} & \theta_{22} \end{pmatrix}$, where

$$|\theta_{11}|^2 + |\theta_{12}|^2 = 1, \quad |\theta_{21}|^2 + |\theta_{22}|^2 = 1, \quad \theta_{11}\theta_{21} + \theta_{12}\theta_{22} = 0. \quad (3.5)$$

Remark 3.4. There is an alternate way to state conditions (3.5): they say precisely that the 2×2 matrix

$$\Theta(e^{it}) := \begin{pmatrix} \theta_{11}(e^{it}) & \bar{\theta}_{21}(e^{it}) \\ \theta_{12}(e^{it}) & \bar{\theta}_{22}(e^{it}) \end{pmatrix} \quad (3.6)$$

is unitary for almost all t . Moreover, if $\Theta'_1 = \begin{pmatrix} \theta'_{11} \\ \theta'_{12} \end{pmatrix}$, $\Theta'_2 = \begin{pmatrix} \theta'_{21} & \theta'_{22} \end{pmatrix}$ also satisfy (3.5), then $\Theta'_1 = \Omega \Theta_1$, $\Theta'_2 = \Theta_2 \Omega^*$ if and only if $\Theta' = \Omega \Theta$ (where Θ' corresponds to θ'_{ij} through (3.6).)

4 The invariant subspaces

We may now use the information provided by Corollary 3.2 and Lemma 3.3 in order to obtain the desired description of invariant subspaces. The next theorem is the main result of the paper.

Theorem 4.1. *The invariant subspaces of $S \oplus S_*$ acting on $H^2 \oplus H^2$ are the following:*

- (I) *Splitting invariant subspaces; that is,*

$$Y = X \oplus X'$$

with $X \subset H^2$ is invariant to S , $X' \subset H^2_-$ is invariant to S_* .

- (II) *Nonsplitting invariant subspaces. These are of the form*

$$Y = \{(\theta_{21}u_1 + \theta_{22}u_2) \oplus P_-(\bar{\theta}_{11}u_1 + \bar{\theta}_{12}u_2) : u_1, u_2 \in H^2\}, \quad (4.1)$$

where θ_{ij} are functions in the unit ball of H^∞ , such that θ_{11} and θ_{12} are not proportional, and the matrix

$$\Theta(e^{it}) := \begin{pmatrix} \theta_{11}(e^{it}) & \bar{\theta}_{21}(e^{it}) \\ \theta_{12}(e^{it}) & \bar{\theta}_{22}(e^{it}) \end{pmatrix} \quad (4.2)$$

is unitary almost everywhere (equivalently, θ_{ij} satisfy (3.5)).

Two matrices Θ, Θ' define the same invariant subspace if and only if there exists a unitary 2×2 matrix Ω with scalar entries, such that $\Theta = \Omega \Theta'$.

Proof. We take one by one the possibilities displayed in the statement of Lemma 3.3.

Case (1)

We have $\dim \mathcal{F} = 0$, so $\Delta_2 = 0$, and $\Delta_1 = I_{L^2}$, so $Z : L^2 \rightarrow \{0\} \oplus L^2$, $Z(v) = 0 \oplus v$.

$$Y = \{0 \oplus v : v \in L^2\} \ominus (\{0\} \oplus H^2) = \{0\} \oplus H_-^2.$$

The invariant subspace Y is the second component (the space on which acts S^*); it is obviously splitting.

Case (2.1)

Here $\dim \mathcal{F} = 1$, $\Delta_1 = I_{L^2}$, $\Delta_2 = 0$. Z is the same operator as in the previous case. We have

$$Y = \{\Theta_2 u \oplus v : u \in H^2, v \in L^2\} \ominus (\{0\} \oplus H^2) = \Theta_2 H^2 \oplus H_-^2.$$

Case (2.2)

Again $\dim \mathcal{F} = 1$, but $\Delta_1 = 0$, $\Delta_2 = I_{L^2}$. So $Z : L^2 \rightarrow L^2 \oplus \{0\}$, $Zv = \Theta_1 v \oplus 0$, $Z^{-1}(w \oplus 0) = \bar{\Theta}_1 w$. Then

$$Y = \{0 \oplus \bar{\Theta}_1 v : v \in H^2\} \ominus (\{0\} \oplus H^2).$$

Since the projection of $\bar{\Theta}_1 H^2$ onto H_-^2 is $\bar{\Theta}_1 K_{\Theta_1}$, whence

$$Y = \{0\} \oplus \bar{\Theta}_1 K_{\Theta_1}.$$

One sees that both cases (2.1) and (2.2) lead to splitting invariant subspaces.

Case (3)

Here we have $\dim \mathcal{F} = 2$. From (3.5) it follows that $\Theta_1^* \Theta_1 = \Theta_2 \Theta_2^* = I_{\mathbb{C}^2}$ a.e., so $\Delta_1 = 0$, while Δ_2 is a projection a.e.; that is, $\Delta_2 = \Delta_2^2$. Also, Θ unitary a.e. implies that $\Theta \Theta^* = I_{\mathbb{C}^2}$ a.e, which is equivalent to

$$\Theta_1 \Theta_1^* + \Theta_2^* \Theta_2 = I.$$

Therefore

$$\Delta_2 \Theta_1 = \Delta_2^2 \Theta_1 = (I - \Theta_2^* \Theta_2) \Theta_1 = \Theta_1 \Theta_1^* \Theta_1 = \Theta_1.$$

It follows that $Z : L^2 \rightarrow \Delta_2 L^2 \subset L^2(\mathbb{C}^2)$ is defined by $Zw = \Theta_1 w$, and $Z^{-1}(\Delta_2 u) = Z^{-1}(\Delta_2^2 u) = Z^{-1}(\Theta_1 \Theta_1^* u) = \Theta_1^* u$.

Denoting $u \in H^2(\mathbb{C}^2)$ by $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, we have, according to (3.3),

$$Y = \{(\theta_{21}u_1 + \theta_{22}u_2) \oplus P_-(\bar{\theta}_{11}u_1 + \bar{\theta}_{12}u_2) : u \in H^2(\mathbb{C}^2)\}. \quad (4.3)$$

We have again to discuss two cases.

Case (3.1)

Suppose θ_{11}, θ_{12} are proportional; then from (3.5) it follows that $\theta_{11} = \alpha_1 \theta$, $\theta_{12} = \alpha_2 \theta$ for some inner function θ and complex numbers α_i with $|\alpha_1|^2 + |\alpha_2|^2 = 1$. Then again by (3.5) we have $\alpha_1 \theta_{21} + \alpha_2 \theta_{22} = 0$. Therefore Θ defined by (4.2) is

$$\Theta = \begin{pmatrix} \alpha_1 \theta & \bar{\theta}_{21} \\ \alpha_2 \theta & \bar{\theta}_{22} \end{pmatrix}.$$

The 2×2 matrix $\Omega = \begin{pmatrix} \bar{\alpha}_1 & \bar{\alpha}_2 \\ \alpha_2 & -\alpha_1 \end{pmatrix}$ is unitary, and

$$\Omega\theta = \begin{pmatrix} \theta & 0 \\ 0 & \alpha_2\bar{\theta}_{21} - \alpha_1\bar{\theta}_{22} \end{pmatrix}.$$

Since $\Omega\theta$ must be unitary almost everywhere, it follows that $\theta' := \alpha_2\bar{\theta}_{21} - \alpha_1\bar{\theta}_{22}$ is inner. By Corollary 3.2 (iii) and Remark 3.4, the invariant subspace obtained in (4.3) can also be defined by $\theta' = \Omega\theta$, and so

$$Y = \{\theta'u_2 \oplus P_-\bar{\theta}u_1 : u_1, u_2 \in H^2\} = \theta'H^2 \oplus \bar{\theta}K_\theta. \quad (4.4)$$

Note that this case completes the list of splitting invariant subspaces described in Lemma 2.1.

Case (3.2)

The last case appears when θ_{11} and θ_{12} are not proportional, which leads us precisely to the invariant subspaces of type (II). To finish the proof, we have to show that these do not split. We already know the subspaces that split, so we must show that subspaces of type (II) do not coincide with any of them. By Corollary 3.2 (iii) any other factorization producing the same invariant subspace must be of type (3), with the associated invariant subspace given by (4.4). So we must have two inner functions θ, θ' and a 2×2 unitary matrix $\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$ such that

$$\begin{pmatrix} \theta_{11} & \bar{\theta}_{21} \\ \theta_{12} & \bar{\theta}_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \theta & 0 \\ 0 & \bar{\theta}' \end{pmatrix}.$$

It follows that $\theta_{11} = a_{11}\theta$, $\theta_{12} = a_{12}\theta$. So θ_{11} and θ_{12} are proportional, contrary to the assumption. The proof is thus finished. $\square$

Remark 4.2. (i) An alternate compact way to write the nonsplitting subspaces in (4.1) can be obtained if we consider θ as a multiplication operator on $L^2 \oplus L^2$. Then

$$Y = (P_+ \oplus P_-)\theta^*(H^2 \oplus H^2).$$

(ii) The case (I) of Theorem 4.1, that is, the identification of all splitting subspaces, is stated in [1, Theorem 8.1].

Remark 4.3. As noted in Corollary 3.2, the characteristic function of T_Y is the pure part of θ_1 . If we examine the factorizations in Lemma 3.3, we see that θ_1 is pure in cases (1) and (2.1). In case (2.2), θ_1 is pure if it is nonconstant. In case (3), θ_1 is pure, except when $\theta_{11} = t_1$ and $\theta_{12} = t_2$ are constant scalars satisfying $|t_1|^2 + |t_2|^2 = 1$. Consequently, in all these cases the characteristic function of T_Y is θ_1 and $\dim \mathcal{D}_{T_Y} = 1$.

The remaining cases, which lead to $\dim \mathcal{D}_{T_Y} = 0$, are then:

- (2.2), with θ_1 constant of modulus 1. The invariant subspace is $\{0\}$.
- (3), with $\theta_{11} = t_1$ and $\theta_{12} = t_2$ constant scalars satisfying $|t_1|^2 + |t_2|^2 = 1$. Arguing as in the proof of 4.1, Case 3.1, we obtain a 2×2 unitary matrix Ω and an inner function θ' such that

$$\Omega\theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta' \end{pmatrix}.$$

The invariant subspace is $Y = \theta'H^2 \oplus \{0\}$, and the characteristic function of T_Y is the pure part of $\Omega\theta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is $0_{0 \rightarrow 1}$.

Using Remark 4.3, we may identify the reducing subspaces of $S \oplus S_*$.

Theorem 4.4. *The only reducing subspaces for $S \oplus S_*$ are $H^2 \oplus \{0\}$ and $\{0\} \oplus H^2$.*

Proof. Suppose $H^2 \oplus H^2 = Y_1 + Y_2$, with $Y_1 \perp Y_2$ nontrivial and both invariant with respect to $S \oplus S_*$. Then $1 = \dim \mathcal{D}_{S \oplus S_*} = \dim \mathcal{D}_{T_{Y_1}} + \dim \mathcal{D}_{T_{Y_2}}$. We may assume that $\dim \mathcal{D}_{T_{Y_1}} = 0$, $\dim \mathcal{D}_{T_{Y_2}} = 1$. It follows from Remark 4.3 that either $Y_1 = \{0\}$ or $Y_1 = \theta_1 H^2 \oplus \{0\}$ with θ_1 inner. The nontrivial case is the latter; then $Y_2 = K_{\theta_1} \oplus H^2$ is invariant only if $\theta_1 \equiv 1$. $\square$

5 An example

The following example exhibits a whole class of nonsplitting invariant subspaces, for which we may obtain a simpler form than that given by Theorem 4.1.

Example 5.1. Fix $\alpha, a \in \mathbb{D}$, and $\alpha \neq 0$. Denote $b_a(z) = \frac{z-a}{1-\bar{a}z}$, and define the functions θ_{ij} , $i, j = 1, 2$, by

$$\theta_{11}(z) = \theta_{22}(z) = \alpha b_a(z), \quad \theta_{12}(z) = -(1 - |\alpha|^2)^{1/2}, \quad \theta_{21}(z) = (1 - |\alpha|^2)^{1/2}.$$

By Theorem 4.1, we obtain the following nonsplitting subspace:

$$Y = \{((1 - |\alpha|^2)^{1/2} u_1 + \alpha b_a u_2) \oplus P_-(\bar{\alpha} b_a u_1 - (1 - |\alpha|^2)^{1/2} u_2) : u_1, u_2 \in H^2\}.$$

There is a simpler way to write this subspace. First, $P_- u_2 = 0$. Secondly, K_{b_a} is a one dimensional space generated by the reproducing kernel $k_a(z) = \frac{1}{1-\bar{a}z}$, and the orthogonal projection onto K_{b_a} has the formula $P_{K_{b_a}} f = (1 - |a|^2) f(a) k_a$. We have the orthogonal decomposition $H^2 = K_{b_a} \oplus b_a H^2$, according to which $u_1 = (1 - |a|^2) u_1(a) k_a + b_a u'_1$. Therefore

$$(1 - |\alpha|^2)^{1/2} u_1 + \alpha b_a u_2 = (1 - |\alpha|^2)^{1/2} (1 - |a|^2) u_1(a) k_a + b_a ((1 - |\alpha|^2)^{1/2} u'_1 + \alpha u_2)$$

and

$$\begin{aligned} P_-(\bar{\alpha} b_a u_1 - (1 - |\alpha|^2)^{1/2} u_2) &= P_-(\bar{\alpha} b_a (1 - |a|^2) u_1(a) k_a + \bar{\alpha} u'_1 - (1 - |\alpha|^2)^{1/2} u_2) \\ &= P_-(\bar{\alpha} b_a (1 - |a|^2) u_1(a) k_a) = \frac{\bar{\alpha} (1 - |a|^2) u_1(a) \bar{z}}{1 - a\bar{z}}. \end{aligned}$$

If we denote $u = (1 - |\alpha|^2)^{1/2} u_1 + \alpha b_a u_2$, then $u(a) = (1 - |\alpha|^2)^{1/2} u_1(a)$; moreover, if u_1, u_2 are arbitrary functions in H^2 , then u is also an arbitrary function in H^2 . We may therefore write

$$Y = \{u \oplus \frac{\bar{\alpha} (1 - |a|^2) u(a) \bar{z}}{(1 - |\alpha|^2)^{1/2} (1 - a\bar{z})} : u \in H^2\}.$$

It is easy to see that when $\alpha \in \mathbb{D} \setminus \{0\}$, $\frac{\bar{\alpha}(1-|a|^2)}{(1-|\alpha|^2)^{1/2}}$ covers $\mathbb{C} \setminus \{0\}$. Let us denote $\beta = \frac{\bar{\alpha}(1-|a|^2)}{(1-|\alpha|^2)^{1/2}}$; the invariant subspace is then

$$Y = \{u \oplus \beta \frac{u(a) \bar{z}}{1 - a\bar{z}} : u \in H^2\}. \quad (5.1)$$

We have thus obtained in (5.1) a class of nonsplitting invariant subspaces parameterized by the nonzero complex number β . For an appropriate value of this parameter, Y corresponds to the subspace appearing in Example 7.3 of [1] (after taking into account the unitary equivalence implemented by $J : H^2 \rightarrow H^2$).

6 Parametrization of nonsplitting subspaces

The nonsplitting subspaces are the most interesting ones, so it is worth to obtain a more detailed description of this class. Equations 3.5 define Θ_1 and Θ_2 in an implicit manner; we will determine in this last section a parametrization of these two functions.

We start with a pair of nonproportional functions $\theta_{11}, \theta_{12} \in H^\infty$ that satisfy $|\theta_{11}|^2 + |\theta_{12}|^2 = 1$. First, if we denote by g_1, g_2 the outer parts of θ_{11}, θ_{12} respectively, they satisfy $|g_1|^2 + |g_2|^2 = 1$. In fact, this means an outer function g_1 bounded by 1 and subject to the condition $\int (1 - |g_1|^2) > -\infty$, which is equivalent to g_1 not being an extreme point of the unit ball of H^∞ (see [5]). Then g_1 determines g_2 up to a scalar of modulus 1.

Since θ is unitary, $|\theta_{11}(e^{it})| = |\theta_{22}(e^{it})|$ and $|\theta_{12}(e^{it})| = |\theta_{21}(e^{it})|$ almost everywhere. Therefore g_1, g_2 are also the outer parts of θ_{22}, θ_{21} respectively. We may then write

$$\theta_{11} = \alpha_{11}g_1, \quad \theta_{12} = \alpha_{12}g_2, \quad \theta_{21} = \alpha_{21}g_2, \quad \theta_{22} = \alpha_{22}g_1,$$

with α_{ij} inner; from the last formula in (3.5) it follows that

$$\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} = 0.$$

By factoring common inner divisors, let us then write $\alpha_{ij} = \alpha_i\beta_{ij}$, with $(\beta_{i1}, \beta_{i2}) = 1$. It follows then that

$$\beta_{11}\beta_{21} = -\beta_{12}\beta_{22}.$$

Divisibility implies then that $\beta_{22} = \lambda\beta_{11}$ and $\beta_{21} = -\lambda\beta_{12}$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. If we denote, for simplicity, $\beta_1 = \beta_{11}$ and $\beta_2 = \beta_{12}$, we may write

$$Y = \{(\lambda\alpha_2[-\beta_2g_2u_1 + \beta_1g_1u_2]) \oplus P_-(\bar{\alpha}_1[\bar{\beta}_1\bar{g}_1u_1 + \bar{\beta}_2\bar{g}_2u_2])\}.$$

So the nonsplitting invariant subspace Y is determined by the following “free” objects:

- (i) An outer function g_1 bounded by 1 that is not an extreme point of the unit ball of H^∞ .
- (ii) Two arbitrary inner functions α_1, α_2 .
- (iii) Two arbitrary, but coprime inner functions β_1, β_2 .
- (iv) A complex number λ of modulus 1.

To obtain from these parameters θ_{ij} , note first that g_1 determines up to a constant of modulus 1 an outer function g_2 , such that $|g_1|^2 + |g_2|^2 = 1$. Then we have

$$\theta_{11} = \alpha_1\beta_1g_1, \quad \theta_{12} = \alpha_1\beta_2g_2, \quad \theta_{21} = -\lambda\alpha_2\beta_2g_2, \quad \theta_{22} = \lambda\alpha_2\beta_1g_1. \quad (6.1)$$

The condition for two parametrizations to produce the same invariant subspace follows from the last statement of Theorem 4.1. One sees that there is a remarkable richness of nonsplitting subspaces.

References

- [1] M.C. Câmara and W.T. Ross, The dual of the compressed shift, preprint arxiv:2001.02587v1.
- [2] X. Ding and Y. Sang, Dual truncated Toeplitz operators, *J. Math. Anal. Appl.*, 2018, 461, 929–946.
- [3] X. Ding, Y. Sang, and Y. Qin, A theorem of Brown-Halmos type for dual truncated Toeplitz operators, *Ann. Funct. Anal.*, in print.
- [4] X. Ding, Y. Sang, and Y. Qin, Dual truncated Toeplitz C^* -algebras, *Banach J. Math. Anal.*, 2019, 13, 27–292.
- [5] K. de Leeuw and W. Rudin, Extreme points and extremum problems in H_1 , *Pacific J. Math.*, 1958, 8, 467–485.
- [6] B. Sz. Nagy, C. Foias, H. Bercovici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, revised and enlarged edition, 2010, Springer New York Dordrecht Heidelberg London.