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Li-Yorke chaotic eigen set of the backward shift operator on $\ell^2(\mathbb{N})$

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Abstract: In this paper, we prove our main result that the Li-Yorke chaotic eigen set of a positive integer multiple of the backward shift operator on $\ell^2(\mathbb{N})$ is a disk in the complex plane $\mathbb{C}$ and the union of such Li-Yorke chaotic eigen set's is the whole complex plane $\mathbb{C}$.

Keywords: Li-Yorke chaos, eigen set, backward shift, irregular vectors.

1 Introduction

The concept of Li-Yorke chaoticity of a continuous mapping in a topological dynamical system and the concept of irregular vectors of Hilbert space operators are well known in Chaos theory and Operator theory respectively (see [1], [2], [3], [6], [11], [13], [14]). Here, we made an attempt to link these two concepts together and introduce the Li-Yorke chaoticity of an operator in a separable complex Hilbert space. Also we defined the Li-Yorke chaotic eigen set of an operator and use this idea to prove our main result.

In this paper, we denote the unit circle as $\mathbb{T} = \{z \in \mathbb{C} / |z| = 1\}$ and the unit disk as $\mathbb{D} = \{z \in \mathbb{C} / |z| < 1\}$. Also we take X a separable complex Hilbert space and $B(X)$ denotes the space of all bounded linear operators on X .

Definition 1.1. Let $T \in B(X)$. An irregular vector for an operator T is an $x \in X$ such that $\limsup_n \|T^n x\| = \infty$ and $\liminf_n \|T^n x\| = 0$. The operator $T \in B(X)$ is said to be Li-Yorke chaotic if and only if there exists an irregular vector for T . For $T \in B(X)$, the Li-Yorke chaotic eigen set of T , denoted by $LY(T)$, is defined by $LY(T) = \{\lambda \in \mathbb{C} / T - \lambda I \text{ is Li-Yorke chaotic}\}$.

Definition 1.2. ([4]) For Ω a connected open subset of $\mathbb{C}$ and n a positive integer, let $B_n(\Omega)$ denote the operators T in $B(X)$ which satisfy:

- (a) $\Omega \subseteq \sigma(T) = \{\omega \in \mathbb{C} / T - \omega \text{ not invertible}\}$
- (b) $\text{Ran}(T - \omega) = X$ for ω in Ω
- (c) $\bigvee \text{Ker}_{\omega \in \Omega}(T - \omega) = X$
- (d) $\dim \text{Ker}(T - \omega) = n$ for ω in Ω

Lemma 1.3. ([4]) Let $T \in B_n(\Omega)$. If $\Omega \cap \mathbb{T} \neq \emptyset$, then T is Li-Yorke chaotic.

We now state our main result as:

Theorem 1.4. The Li-Yorke chaotic eigen set of the positive integer multiple of backward shift operator on $\ell^2(\mathbb{N})$:

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$$T(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

is $LY(nT) = (n+1)\mathbb{D}$ for any $n > 1$, $n \in \mathbb{N}$.

Proof

By definition, we have $LY(nT) = \{\lambda \in \mathbb{C} / nT - \lambda I \text{ is Li-Yorke chaotic}\}$.

Now, for any $\lambda \in (n+1)\mathbb{D} \setminus \{0\}$, we can prove that $(-\lambda + n\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

For this, let us consider $w = \frac{1}{n} \left(\lambda - \frac{\lambda}{|\lambda|} \right)$.

Then

$$\begin{aligned} |w|^2 &= w\bar{w} \\ &= \frac{1}{n} \left(\lambda - \frac{\lambda}{|\lambda|} \right) \frac{1}{n} \left(\bar{\lambda} - \frac{\bar{\lambda}}{|\bar{\lambda}|} \right) \\ &= \frac{1}{n^2} \left(\lambda\bar{\lambda} - \frac{\lambda\bar{\lambda}}{|\bar{\lambda}|} - \frac{\lambda\bar{\lambda}}{|\lambda|} + \frac{\lambda\bar{\lambda}}{|\lambda||\bar{\lambda}|} \right) \\ &= \frac{1}{n^2} (|\lambda|^2 - 2|\lambda| + 1) \\ &= \frac{1}{n^2} (|\lambda| - 1)^2 \end{aligned}$$

Since $\lambda \in (n+1)\mathbb{D} \setminus \{0\}$, then we have $0 < |\lambda| < n+1$. Hence $(|\lambda| - 1)^2 < n^2$. Thus $|w|^2 < 1$ which implies that $|w| < 1$. Therefore, $w \in \mathbb{D}$ and hence $-\lambda + nw \in -\lambda + n\mathbb{D}$.

But

$$\begin{aligned} -\lambda + nw &= -\lambda + \lambda - \frac{\lambda}{|\lambda|} \\ &= -\frac{\lambda}{|\lambda|} \end{aligned}$$

Therefore, $|\lambda - nw| = \left| \frac{-\lambda}{|\lambda|} \right| = 1$ which implies that $-\lambda + nw \in \mathbb{T}$.

Thus we see that for any $\lambda \in (n+1)\mathbb{D} \setminus \{0\}$, $-\lambda + nw$ belongs to both $-\lambda + n\mathbb{D}$ and $\mathbb{T}$ which shows that $(-\lambda + n\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Now, we have $nT - \lambda I \in B_n(-\lambda + n\mathbb{D})$ and $(-\lambda + n\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. By lemma 1.3, $nT - \lambda I$ is Li-Yorke chaotic. For the case when $\lambda = 0$, nT is Li-Yorke chaotic (see [9, 13]). Thus we have shown that $nT - \lambda I$ is Li-Yorke chaotic for every λ in the disk $(n+1)\mathbb{D}$.

Also, for any $\lambda \notin (n+1)\mathbb{D}$,

$$\begin{aligned} \|nT - \lambda I\| &\geq \|\lambda\| - n\|T\| \\ &\geq n+1 - n \\ &\geq 1 \end{aligned}$$

which shows that $nT - \lambda I$ is not Li-Yorke chaotic.

Thus we have shown that for any $\lambda \in (n+1)\mathbb{D}$, $nT - \lambda I$ is Li-Yorke chaotic and for any $\lambda \notin (n+1)\mathbb{D}$, $nT - \lambda I$ is not Li-Yorke chaotic, which proves that $LY(nT) = (n+1)\mathbb{D}$.

Corollary:- $\bigcup_{n \in \mathbb{N}} LY(nT) = \mathbb{C}$.

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